ALGEBRAIC STRUCTURES IN THE SETS OF SURJECTIVE FUNCTIONS

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Abstract. In the paper we construct several algebraic structures (vector spaces, algebras and free algebras) inside sets of different types of surjective functions. Among many results we prove that: the set of everywhere but not strongly everywhere surjective complex functions is strongly $\mathfrak{c}$-algebrable and that its $2^{\mathfrak{c}}$-algebrability is consistent with ZFC; under CH the set of everywhere surjective complex functions which are Sierpiński-Zygmund in the sense of continuous but not Borel functions is strongly $\mathfrak{c}$-algebrable; the set of Jones complex functions is strongly $2^{\mathfrak{c}}$-algebrable.

1. Introduction

For some time now, many mathematicians have been looking at the largeness of some sets by constructing algebraic structures inside them. This approach is called *algebrability.* A comprehensive description of this concept as well as numerous examples and some general techniques can be found in the surveys [10, 15].

Following R. Aron, A. Bartoszewicz, S. Głąb, V. Gurariy, D. Pérez-García, J.B. Seoane-Sepúlveda, [4, 5, 6, 12] let us recall the following notions:

Definition 1.1. *Let κ be a cardinal number.*

(1) *Let $\mathcal{L}$ be a vector space and $A \subseteq \mathcal{L}$. We say that A is* κ-lineable *if $A \cup \{0\}$ contains a κ-dimensional subspace of $\mathcal{L}$;*
(2) *Let $\mathcal{L}$ be a commutative algebra and $A \subseteq \mathcal{L}$. We say that A is* κ-algebrable *if $A \cup \{0\}$ contains a κ-generated subalgebra B of $\mathcal{L}$ (i.e. the minimal cardinality of the system of generators of B is κ).*
(3) *Let $\mathcal{L}$ be a commutative algebra and $A \subseteq \mathcal{L}$. We say that A is* strongly κ-algebrable *if $A \cup \{0\}$ contains a κ-generated subalgebra B that is isomorphic to a free algebra.*

Fact 1.2. *Observe that the set $X = \{x_\alpha : \alpha < \kappa\}$ is the set of free generators of some free algebra if and only if the set $\tilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent; equivalently for any $k \in \mathbb{N}$, any nonzero polynomial P in k variables without a constant term and any distinct $x_{\alpha_1}, ..., x_{\alpha_k} \in X$, we have that $P(x_{\alpha_1}, ..., x_{\alpha_k})$ is nonzero.*

This paper is devoted to the investigation of algebrability properties of several classes of surjective functions. In the sequel we take into our considerations the following ones: *everywhere surjective type functions* (section 3), *Sierpiński-Zygmund functions* (section 4) and *Jones functions* (section 5). In the paper the symbol $\mathbb{K}$ stands for the set $\mathbb{R}$ or $\mathbb{C}$. We use a standard set theoretical notion. In particular, we identify ordinal number α with the set of all ordinals $\beta < \alpha$. Cardinal numbers are those ordinals α which are not equipotent with any $\beta < \alpha$. A cardinal number κ is called regular, if it cannot be decomposed into less than κ sets of cardinality less than κ. Moreover, to indicate the difference between the sets of natural numbers with or without 0 we use standard notation, i.e. $\mathbb{N} = \{1, 2, 3, ...\}$ and $\omega = \{0, 1, 2, ...\}$ (it should be mentioned here that ω is also identified with the first infinite cardinal).

2. The general method

We start with the simple, but in the view of further results, useful observation. It is a foundation of a powerful method whose particular case is the so-called *exponential like function method.*

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Theorem 2.1. *Let κ be a cardinal number, $\mathcal{A} \subseteq \mathbb{K}^{\mathbb{K}}$ be a κ-generated algebra (resp. free algebra, vector space) and $\mathcal{G} \subseteq \mathbb{K}^{\mathbb{K}}$. Assume that there exists a function $F : \mathbb{K} \to \mathbb{K}$ such that $f \circ F \in \mathcal{G} \setminus \{0\}$ for every $f \in \mathcal{A} \setminus \{0\}$. Then $\mathcal{G}$ is κ-algebrable (resp. strongly κ-algebrable, κ-lineable).*

Proof. Observe that a function $h : \mathcal{A} \to \mathcal{G}$ defined by $h(f) = f \circ F$ is a morphism of structures. Hence, if $\mathcal{A}$ is κ-generated algebra (resp. free algebra or vector space), then $h[\mathcal{A}]$ has the same property. □

It turns out that algebra $\mathcal{A}$, which is very useful in several cases, is the $\mathfrak{c}$-generated free algebra of the so-called exponential like functions. In 2013 M. Balcerzak et al. (see [7]) introduced the following notion.

Definition 2.2 ([7])**.** *We say that a function $f : \mathbb{R} \to \mathbb{R}$ is* exponential like *(of rank $m \in \mathbb{N}$), whenever for $x \in \mathbb{R}$*

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x},$$

for some distinct nonzero real numbers $\beta_1, ..., \beta_m$ and some nonzero real numbers $a_1, ..., a_m$ (let us denote the set of all exponential like functions $f : \mathbb{R} \to \mathbb{R}$ by $\mathcal{EXP}(\mathbb{R})$).

In this setting, actually using the fact that $\mathcal{EXP}(\mathbb{R})$ is strongly $\mathfrak{c}$-algebrable, they proved the following.

Theorem 2.3 ([7])**.** *Let X be a nonempty set and $\mathcal{G} \subseteq \mathbb{R}^X$. Assume that there exists a function $F : X \to \mathbb{R}$ such that $f \circ F \in \mathcal{G} \setminus \{0\}$ for every $f \in \mathcal{EXP}(\mathbb{R})$. Then $\mathcal{G}$ is strongly $\mathfrak{c}$-algebrable.*

It is a simple observation, looking at the proof of Theorem 2.3, that this result is a particular case of Theorem 2.1. Many applications of Theorem 2.3 can be found in the paper by A. Bartoszewicz et al. ([11]).

In the next sections, while applying Theorem 2.1, it will be used many times a $\mathfrak{c}$-generated free algebra $\mathcal{A} \subseteq \mathbb{C}^{\mathbb{C}}$ such that any $f \in \mathcal{A} \setminus \{0\}$ is surjective and is *"$\leq \omega$-to-one"* mapping (i.e. the preimage $f^{-1}(\{x\})$ of any singleton is of cardinality at most ω). The construction of such an algebra $\mathcal{A}$ is possible, thanks to the results of N. Albuquerque et al. (cf. [1]). We recall the necessary notion and lemmas, which are well-known facts of the complex analysis.

Definition 2.4. *The* (growth) order *$\rho(f)$ of an entire function $f \in \mathcal{H}(\mathbb{C})$ (by the symbol $\mathcal{H}(\mathbb{C})$, we denote the algebra of all entire functions from $\mathbb{C}$ to $\mathbb{C}$) is the infimum of all positive real numbers α with the following property:* $\max\{|f(z)| : |z| = r\} < e^{r^\alpha}$ *for all $r > r(\alpha)$, for some $r(\alpha) > 0$.*

Remark 2.5 ([1])**.**
(1) *The order of a constant function is* 0.
(2) *For every $\alpha \in (0, 1)$ there is $f_\alpha \in \mathcal{H}(\mathbb{C})$ such that $\rho(f) = \alpha$.*
(3) *Every nonconstant entire function f with $\infty > \rho(f) \notin \mathbb{N}$ is surjective.*

The authors of [1] proved the following fact:

Fact 2.6 (Lemma 2.4, [1])**.** *Let $f_1, ..., f_n \in \mathcal{H}(\mathbb{C})$ be such that $\rho(f_i) \neq \rho(f_j)$ whenever $i \neq j$. Then $\rho(P(f_1, ..., f_n)) = \max_{k \in \{1,...,n\}} \rho(f_k)$, for every nonconstant polynomial P in n complex variables.*

Having these tools, we may prove the following (this fact was not pointed out by the authors but can be easily deduced from the proof of Theorem 2.6 (see [1])).

Theorem 2.7. *There exists a $\mathfrak{c}$-generated free algebra $\mathcal{A}$, whose every nonzero member is a surjective and "$\leq \omega$-to-one" entire mapping.*

Proof. Let $\mathcal{F} = \{f_\alpha : \alpha \in (0, 1)\} \subseteq \mathcal{H}(\mathbb{C})$, where $\rho(f_\alpha) = \alpha$ for $\alpha \in (0, 1)$ (the existence of f_α follows from Remark 2.5). Consider the algebra $\mathcal{A}$ generated by $\mathcal{F}$. We will show that $\mathcal{A}$ has the desired properties.

Let $f \in \mathcal{A} \setminus \{0\}$. Then there is a nonzero polynomial in n variables without a constant term and $\alpha_1, ..., \alpha_n \in (0, 1)$ such that $f = P(f_{\alpha_1}, ..., f_{\alpha_n})$. By Fact 2.6, $\rho(f) = \max\{\alpha_1, ...\alpha_n\} \in (0, 1)$. Therefore f is surjective and $\mathcal{A}$ is a $\mathfrak{c}$-generated free algebra.

Let $f \in \mathcal{A} \setminus \{0\}$ and suppose that f is not "$\leq \omega$-to-one". Then there is an uncountable set $A \subseteq \mathbb{C}$ such that $f|A$ is constant. Then A has an accumulation point and by a well-known property of entire functions, f is constant, which is impossible because f is surjective. $\square$

3. Everywhere, but not strongly everywhere surjective functions

This section is devoted to the algebrability properties of the different classes of everywhere surjective functions.

Definition 3.1. *A function $f : \mathbb{K} \to \mathbb{K}$ is called:*

(1) everywhere surjective *($f \in \mathcal{ES}(\mathbb{K})$), if it takes every value on every nonempty open set (i.e. for every $y \in \mathbb{K}$ and every nonempty open set I, $f[I] \ni y$);*
(2) strongly everywhere surjective *($f \in \mathcal{SES}(\mathbb{K})$), if it takes every value $\mathfrak{c}$ times on every nonempty open set (i.e. for every $y \in \mathbb{K}$ and every nonempty open set I the set $\{x \in I : f(x) = y\}$ is of cardinality $\mathfrak{c}$);*
(3) perfectly everywhere surjective *($f \in \mathcal{PES}(\mathbb{K})$), if $f[P] = \mathbb{K}$ for any perfect set $P \subseteq \mathbb{K}$.*

These classes have been considered in the context of lineability and algebrability by many authors (see [4, 5, 14, 29, 30, 32]). In particular, in [29, Theorem 2.6, 2.7, 2.8] it was proved that the set $\mathcal{PES}(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable, the set $\mathcal{SES}(\mathbb{R}) \setminus \mathcal{PES}(\mathbb{R})$ is also $2^{\mathfrak{c}}$-lineable and that $\mathcal{PES}(\mathbb{C})$ is $\mathfrak{c}$-algebrable. Moreover, in [14] the authors showed that the latter set is $2^{\mathfrak{c}}$-algebrable. On the other hand A. Bartoszewicz et al. proved that $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable (see [9]). Moreover, A. Bartoszewicz et al. showed (see [13]) that these results can be strengthened by proving strong $2^{\mathfrak{c}}$-algebrability of the sets $\mathcal{PES}(\mathbb{C})$ and $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C})$. Hence, the last open problem connected with the above classes is the algebrability of the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ (it is known that this set is $\mathfrak{c}$-lineable, see [29]). As an application of Theorem 2.1, in particular, we prove strong $\mathfrak{c}$-algebrability of the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$.

Let us generalize the notion of everywhere surjective functions in the following direction.

Definition 3.2. *Let κ be a cardinal number. We say that a function $f : \mathbb{K} \to \mathbb{K}$ is* everywhere κ surjective *($f \in \mathcal{ES}_\kappa(\mathbb{K})$), if it takes every value at least κ times on every nonempty open set (i.e. for every $y \in \mathbb{K}$ and every nonempty open set I, $|\{x \in I : f(x) = y\}| \geq \kappa$).*

Remark 3.3. *It is easy to see that*

(1) $\mathcal{ES}(\mathbb{K}) = \mathcal{ES}_\kappa(\mathbb{K})$ *for every* $\kappa \leq \omega, \kappa > 0$;
(2) $\mathcal{ES}_\kappa(\mathbb{K}) \subseteq \mathcal{ES}_\lambda(\mathbb{K})$ *for every* $\kappa \geq \lambda$;
(3) $\mathcal{SES}(\mathbb{K}) = \mathcal{ES}_{\mathfrak{c}}(\mathbb{K})$;
(4) $\mathcal{ES}_\kappa(\mathbb{K}) = \emptyset$ *for every* $\kappa > \mathfrak{c}$.

It is known (see [3]), that there exists a function $F \in \mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ such that $|F^{-1}(\{y\})| = \omega$, for any $y \in \mathbb{C}$. In fact the construction from the paper [3] gives an example of a function from $\mathcal{ES}(\mathbb{K}) \setminus \mathcal{ES}_{\omega_1}(\mathbb{K})$ (this is obvious since the construction works in ZFC).

This construction can be easily generalized to the obtain following one (to the end of this section, we assume that κ, λ are cardinal numbers such that $\omega \leq \lambda \leq \mathfrak{c}$ and $\lambda < \kappa$).

Lemma 3.4. *There exists a Lebesgue measurable function $F : \mathbb{K} \to \mathbb{K}$ that has the Baire property, such that for any $y \in \mathbb{K}$ and any nonempty open set I, the set $\{x \in I : F(x) = y\}$ is of cardinality λ. In particular $F \in \mathcal{ES}_\lambda(\mathbb{K}) \setminus \mathcal{ES}_\kappa(\mathbb{K})$*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable base of open sets in $\mathbb{K}$. By induction, we can define a sequence $(C_n)_{n\in\mathbb{N}}$ of Cantor-like sets (i.e. perfect, nowhere dense sets and of Lebesgue measure zero) such that $C_n \subseteq U_n \setminus \bigcup_{k<n} C_k$, for $n \in \mathbb{N}$. As any perfect set contains $\mathfrak{c}$ disjoint perfect subsets, for any $n \in \mathbb{N}$ let $\{C_n^\alpha : \alpha < \lambda\}$ be a family of disjoint perfects subsets of C_n. Let $g_n^\alpha : C_n^\alpha \to \mathbb{K}$ be a bijection for $n \in \mathbb{N}$ and $\alpha < \lambda$.

Let us define

$$F(x) = \begin{cases} g_n^\alpha(x), \text{ when } x \in C_n^\alpha \\ x, \text{ otherwise.} \end{cases}$$

It is easy to see that F has the required property. Moreover, as $\bigcup_{n\in\mathbb{N}} \bigcup_{\alpha<\lambda} C_n^\alpha$ is null and meager, F is Lebesgue measurable and has the Baire property. $\square$

The above together with Theorem 2.1 allow us to state:

Theorem 3.5. *The set $\mathcal{ES}_\lambda(\mathbb{C})\setminus\mathcal{ES}_\kappa(\mathbb{C})$ is strongly $\mathfrak{c}$-algebrable. In particular, the set $\mathcal{ES}(\mathbb{C})\setminus\mathcal{SES}(\mathbb{C})$ is strongly $\mathfrak{c}$-algebrable. The same holds in the class of Lebesgue measurable functions that have the Baire property.*

Proof. Let $\mathcal{A}$ be a $\mathfrak{c}$-generated free algebra as in Theorem 2.7 and let $F\in\mathcal{ES}_\lambda(\mathbb{C})\setminus\mathcal{ES}_\kappa(\mathbb{C})$ be as in Lemma 3.4. We will show that $f\circ F\in\mathcal{ES}_\lambda(\mathbb{C})\setminus\mathcal{ES}_\kappa(\mathbb{C})$, for any $f\in\mathcal{A}\setminus\{0\}$. Let $f\in\mathcal{A}\setminus\{0\}$ and let $I\subseteq\mathbb{C}$ be nonempty and open. We have that $(f\circ F)[I]=f[F[I]]=f[\mathbb{C}]=\mathbb{C}$, since f is surjective. Moreover, $|(f\circ F)^{-1}(\{y\})|=|F^{-1}(f^{-1}(\{y\}))|=\lambda<\kappa$, for any $y\in\mathbb{C}$, as a countable union of sets of cardinality λ. Moreover, $f\circ F\in\mathcal{ES}_\lambda(\mathbb{C})$ by surjectivity of f. Thus, $f\circ F\in\mathcal{ES}_\lambda(\mathbb{C})\setminus\mathcal{ES}_\kappa(\mathbb{C})$. By Theorem 2.1, the first part of the assertion has been proved. Moreover, since any $f\in\mathcal{A}$ is continuous and F is Lebesgue measurable and has the Baire property, the composition $f\circ F$ is Lebesgue measurable and has the Baire property as well. This ends the proof. $\square$

On the other hand, since $|\mathcal{ES}_\lambda(\mathbb{K})\setminus\mathcal{ES}_\kappa(\mathbb{K})|=2^{\mathfrak{c}}$, a natural question is what can we say about algebrability of $\mathcal{ES}_\lambda(\mathbb{K})\setminus\mathcal{ES}_\kappa(\mathbb{K})$ at a higher level? We give a partial answer to this question.

Definition 3.6. *Let $\mathcal{A},\mathcal{I}\subseteq\mathcal{P}(\mathbb{K})$. We say that a family $\mathcal{A}$ is $\mathcal{I}$-*independent*, whenever $A_1^{\varepsilon(1)}\cap...\cap A_n^{\varepsilon(n)}\in\mathcal{I}$, for any distinct $A_1,...,A_n\in\mathcal{A}$, any $\varepsilon\in\{0,1\}^n$ and $n\in\mathbb{N}$ (where $A^0=\mathbb{K}\setminus A$ and $A^1=A$).*

A slight modification of the construction in the proof of Lemma 3.4 can give us the following one (we leave this result without a proof, as it is a simple transfinite construction).

Lemma 3.7. *Let $B\subseteq\mathbb{K}$ be $\mathfrak{c}$-*dense* in $\mathbb{K}$ (i.e. it intersects any nonempty open set on the set of cardinality $\mathfrak{c}$). There exists a function $g:B\to\mathbb{K}$ such that for any nonempty open set $I\subseteq\mathbb{K}$ and any $y\in\mathbb{K}$ the set $\{x\in I\cap A: g(x)=y\}$ is of cardinality λ.*

Theorem 3.8. *The set of Lebesgue measurable functions with the Baire property that belong to $\mathcal{ES}_\lambda(\mathbb{C})\setminus\mathcal{ES}_\kappa(\mathbb{C})$ is 2^λ-algebrable.*

Proof. Let $\{C_n^\alpha:n\in\mathbb{N},\alpha<\lambda\}$ be a family of Cantor-like sets constructed in the same way as in the proof of Lemma 3.4. For $\alpha<\lambda$ let us put $B_\alpha=\bigcup_{n\in\mathbb{N}}C_n^\alpha$. Clearly each set B_α is $\mathfrak{c}$-dense in $\mathbb{C}$. By a well-known theorem of G. Fichtenholz and L. Kantorovich (see [27] and [33, Lemma 7.7]) there exists $[\lambda]^\lambda$-independent family $\{N_\xi:\xi<2^\lambda\}$ in λ, where $[\lambda]^\lambda$ stands for the family of all subsets of λ that have cardinality λ. For $\xi<2^\lambda$, let $B^\xi=\bigcup_{\alpha\in N_\xi}B_\alpha$. For any $\alpha<\lambda$ let us define $g_\alpha:B_\alpha\to\mathbb{C}$ as in Lemma 3.7, i.e. such that $|g_\alpha^{-1}(\{y\})\cap I|=\lambda$ for any nonempty open set $I\subseteq\mathbb{C}$ and any $y\in\mathbb{C}$. Let $\mathcal{F}=\{f_\xi:\xi<2^\lambda\}$ where $f_\xi:\mathbb{C}\to\mathbb{C}$ and

$$f_\xi(x)=\begin{cases}g_\alpha(x), \text{ when } x\in B_\alpha,\alpha\in N_\xi\\ 0, \text{ elsewhere.}\end{cases}$$

We will show that $Alg(\mathcal{F})\subseteq(\mathcal{ES}_\lambda(\mathbb{C})\setminus\mathcal{ES}_\kappa(\mathbb{C}))\cup\{0\}$. Let $n\in\mathbb{N},f_{\xi_1},...,f_{\xi_n}\in\mathcal{F}$ and P be a polynomial in n variables without a constant term. Note that $\mathbb{C}=\bigcup_{\varepsilon\in\{0,1\}^n}(B^{\xi_1})^{\varepsilon(1)}\cap...\cap(B^{\xi_n})^{\varepsilon(n)}$. For $\varepsilon\in\{0,1\}^n$ let us set $P_\varepsilon(x)=P(\varepsilon(1)\cdot x,...,\varepsilon(n)\cdot x)$. We have two possibilities:

- $P_\varepsilon=0$ for every $\varepsilon\in\{0,1\}^n$. Then $P(f_{\xi_1},...,f_{\xi_n})=0$, or
- there is $\varepsilon\in\{0,1\}^n$ with $P_\varepsilon\neq0$. Let $\alpha\in N_{\xi_1}^{\varepsilon(1)}\cap...\cap N_{\xi_n}^{\varepsilon(n)}$, then $P(f_{\xi_1},...,f_{\xi_n})|B_\alpha=P_\varepsilon\circ g_\alpha$, and for any nonempty open set $I\subseteq\mathbb{C}$ and $y\in\mathbb{C}$ we have inequalities

$$|\{x\in I:P(f_{\xi_1},...,f_{\xi_n})(x)=y\}|\geq|\{x\in I\cap B_\alpha:P_\varepsilon\circ g_\alpha(x)=y\}|=$$

$$=|\{x\in I\cap B_\alpha:g_\alpha(x)\in P_\varepsilon^{-1}(\{y\})\}|=\lambda.$$

On the other hand, for any $y\in\mathbb{C}\setminus\{0\}$ we have that

$$(P(f_{\xi_1},...,f_{\xi_n}))^{-1}(\{y\})\subseteq\bigcup_{\varepsilon\in\{0,1\}^n}\ \bigcup_{\alpha\in N_{\xi_1}^{\varepsilon(1)}\cap...\cap N_{\xi_n}^{\varepsilon(n)}}(P_\varepsilon\circ g_\alpha)^{-1}(\{y\})$$

is a union of λ many sets each of cardinality λ, i.e. it is of cardinality $\lambda<\kappa$.

It is easy to see that any element of $Alg(\mathcal{F})$ is a Lebesgue measurable function with the Baire property (as it is equal to 0 on a co-meager and co-null set). $\square$

Corollary 3.9. *If $\omega_1 < \mathfrak{c}$ then by taking $\lambda = \omega_1$ and $\kappa = \mathfrak{c}$ we obtain 2^{ω_1}-algebrability of the set $\mathcal{ES}_{\omega_1}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ and as a consequence 2^{ω_1}-algebrability of $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$.*

Corollary 3.10. *It is consistent with ZFC that the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable.*

Proof. It is the set theoretical folklore that conditions $\omega_1 < \mathfrak{c}$ and $2^{\omega_1} = 2^{\mathfrak{c}}$ are consistent with ZFC, cf. Easton's Theorem, [33, Theorem 15.18]. $\square$

Let us note that $\mathfrak{c}^+$-lineability of the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ was proved under CH by K. Potka in [38] and again very recently in ZFC by C. Ciesielski et al. in [19].

Let us state two problems that are still unsolved:

Problem 3.11.
(1) *Is the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ strongly $2^{\mathfrak{c}}$-algebrable (in ZFC or in some model)?*
(2) *Is it $2^{\mathfrak{c}}$-algebrable in ZFC?*

At the end of this section let us state the following observations and comments. It is easy to see that any perfectly everywhere surjective function is neither Lebesgue measurable, nor has the Baire property, nor is s-measurable (where s denotes the σ-field of Marczewski sets, see [36]). Recently, G. Araújo et al. in [2] proved that the set of real, measurable, everywhere surjective functions is $\mathfrak{c}$-lineable and asked whether it is $2^{\mathfrak{c}}$-lineable (obviously as a set of real surjective functions it cannot be algebrable). Looking at the proof of Theorem 3.8 and simply taking $\mathbb{R}$ instead of $\mathbb{C}$ and linear combinations instead of polynomials, one can obtain the following.

Theorem 3.12. *The set of Lebesgue measurable functions with the Baire property that belong to $\mathcal{ES}_{\lambda}(\mathbb{R}) \setminus \mathcal{ES}_{\kappa}(\mathbb{R})$ is 2^{λ}-lineable.*

The following corollary for $\lambda = \mathfrak{c}$ answers the problem cited above (see [2, Remark 3.3]).

Corollary 3.13. *The set of Lebesgue measurable, strongly everywhere surjective real functions with the Baire property is $2^{\mathfrak{c}}$-lineable.*

An alternative proof of this fact and of even a stronger result can be deduced from the work [13] of A. Bartoszewicz, S. Głąb and A. Paszkiewicz. It is enough to repeat the proofs of Theorems 2.1, 3.6 and 3.7 in [13], in the case of real functions using linear combinations instead of polynomials to get the following theorems.

Theorem 3.14. *The set of all Lebesgue and s- measurable real functions with the Baire property that belong to $\mathcal{SES}(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable.*

Theorem 3.15. *The sets:*

- *of functions from $\mathcal{SES}(\mathbb{R})$ which have the Baire property but are neither Lebesgue nor s-measurable*
- *of functions from $\mathcal{SES}(\mathbb{R})$ which are Lebesgue measurable but neither have the Baire property nor are s- measurable*

are $2^{\mathfrak{c}}$-lineable.

4. Sierpiński-Zygmund functions

In 1923, W. Sierpiński and A. Zygmund (see [39]) constructed an example of a function $f : \mathbb{R} \to \mathbb{R}$ with a discontinuous restriction $f|Z$ for any set $Z \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$. Let Φ be a family of functions defined on subsets of $\mathbb{K}$ and with values in $\mathbb{K}$. By $\mathcal{SZ}(\Phi) \subseteq \mathbb{K}^{\mathbb{K}}$ we denote the family of all functions $f : \mathbb{K} \to \mathbb{K}$ with $f|Z \notin \Phi$ for any $Z \in [\mathbb{K}]^{\mathfrak{c}}$ (the symbol $[\mathbb{K}]^{\mathfrak{c}}$ stands for the family of all subsets of $\mathbb{K}$ that have cardinality $\mathfrak{c}$). Such a family occurs in the literature for two families: $\Phi = \mathcal{C}$ of all continuous functions (see [8], [20], [21] or [24]) or $\Phi = \mathcal{B}or$ of all Borel functions (see [11], [14], [29], [31] or [35]) both in the real or complex case. In this setting, the function constructed by W. Sierpiński and A. Zygmund is a real function from $\mathcal{SZ}(\mathcal{C})$. Actually, it is not difficult to observe that the definition of $\mathcal{SZ}(\Phi)$ can be generalized for functions between arbitrary perfect Polish spaces. On the other hand, while considering algebrability properties of $\mathcal{SZ}(\Phi)$ we restrict our considerations to the functions from $\mathbb{K}^{\mathbb{K}}$ or $\mathbb{K}^X$ where X is a perfect Polish space.

In both cases (i.e. for $\Phi = \mathcal{C}$ and $\Phi = \mathcal{B}or$) the authors call a function $f \in \mathcal{SZ}(\Phi)$ simply *Sierpiński-Zygmund* function, so one could expect that $\mathcal{SZ}(\mathcal{C}) = \mathcal{SZ}(\mathcal{B}or)$. In the context of algebrability only the set $\mathcal{SZ}(\mathcal{B}or)$ was taken into consideration. It was done for the first time by J.L. Gámez-Merino et al. in [29], where the authors proved that this set is $\mathfrak{c}^+$-lineable and also $\mathfrak{c}$-algebrable. Moreover, A. Bartoszewicz et al. in [14] showed the following.

Theorem 4.1 ([10, 14])**.** *The set $\mathcal{SZ}(\mathcal{B}or)$ is strongly κ-algebrable, if there exists an almost disjoint family in $\mathfrak{c}$ of cardinality κ (a family $\{A_\alpha : \alpha < \kappa\} \subseteq [\mathfrak{c}]^{\mathfrak{c}}$ is called* almost disjoint *in $\mathfrak{c}$, provided that for any distinct $\alpha, \beta < \kappa$ we have $|A_\alpha \cap A_\beta| < \mathfrak{c}$).*

It should be mentioned here that the original proof of this theorem had a small gap (it was correct whenever $\mathfrak{c}$ is a regular cardinal) but the correct argumentation can be found in [10]. On the other hand, by the fact that any additive group $A \subseteq \mathcal{SZ}(\mathcal{B}or) \cup \{0\}$ generates an almost disjoint family in $\mathfrak{c}$ (it is enough to consider the graphs as subsets of $\mathbb{K}^2$), Theorem 4.1 is in fact a characterization of κ-algebrability of $\mathcal{SZ}(\mathcal{B}or)$ (as well as κ-lineability of $\mathcal{SZ}(\mathcal{B}or)$). Notice also that the same characterizations hold for the family $\mathcal{SZ}(\mathcal{C})$. It is known that in ZFC there is an almost disjoint family in $\mathfrak{c}$ of cardinality greater than $\mathfrak{c}$. However, J.L. Gámez-Merino and J.B. Seoane-Sepúlveda (in [31]) showed that there is a model of ZFC in which there is **no** almost disjoint family in $\mathfrak{c}$ of cardinality $2^{\mathfrak{c}}$ (it had been known earlier as folklore but it was probably impossible to find this result in the literature).

Let us come back to the problem of the equality $\mathcal{SZ}(\mathcal{C}) = \mathcal{SZ}(\mathcal{B}or)$. Although the inclusion $\mathcal{SZ}(\mathcal{B}or) \subseteq \mathcal{SZ}(\mathcal{C})$ is clear, the opposite one can be false. Let us recall some terminology that is useful in the next theorems.

Definition 4.2. *Let X, Y be topological spaces. A function $f : X \to Y$ is called:*

- countably continuous *if there exists a countable partition $(X_n)_{n\in\mathbb{N}}$ of X such that the restriction of f to any X_n is continuous;*
- $<\mathfrak{c}$-continuous *if there exists a cardinal $\kappa < \mathfrak{c}$ and a partition $(X_\alpha)_{\alpha<\kappa}$ of X such that the restriction of f to any X_α is continuous.*

Observe that these notions coincide under CH. The following result characterizes $<\mathfrak{c}$-continuity under the assumption that $\mathfrak{c}$ is a regular cardinal. (Note that this observation generalizes a characterization of countable continuity formulated (without the proof) for $X = Y = \mathbb{R}$ by U. Darji in [25].) To the end of this section X is a perfect Polish space.

Proposition 4.3. *If $f : X \to \mathbb{K}$ satisfies the following condition*

$$(*) \qquad \text{for every set } U \in [X]^{\mathfrak{c}} \text{ there is } Z \in [U]^{\mathfrak{c}} \text{ with } f|Z \in \mathcal{C},$$

then f is $<\mathfrak{c}$-continuous. If additionally $\mathfrak{c}$ is a regular cardinal then every $<\mathfrak{c}$-continuous function $f : X \to \mathbb{K}$ satisfies the condition $(*)$.

Proof. Suppose that $f : X \to \mathbb{K}$ is not $<\mathfrak{c}$-continuous. Let $\mathcal{C}_{G_\delta} = \{g_\alpha : \alpha < \mathfrak{c}\}$ be a family of all continuous functions $g : A \to \mathbb{K}$ with A being a G_δ set in X. For each $\alpha < \mathfrak{c}$ choose $x_\alpha \in X \setminus \{x_\beta : \beta < \alpha\}$ such that $(x_\alpha, f(x_\alpha)) \notin \bigcup_{\beta<\alpha} g_\beta$. This is possible because f is not $<\mathfrak{c}$-continuous, so $f \nsubseteq \bigcup_{\beta<\alpha}\{(x_\beta, f(x_\beta))\} \cup \bigcup_{\beta<\alpha} g_\beta$. Then $U = \{x_\alpha : \alpha < \mathfrak{c}\} \in [X]^{\mathfrak{c}}$ and $|(f|U) \cap g| < \mathfrak{c}$ for each $g \in \mathcal{C}_{G_\delta}$, so $f|U$ is $\mathcal{SZ}(\mathcal{C})$ (see [39] and cf. [20], [21]). Hence $f|Z \notin \mathcal{C}$ for each $Z \in [U]^{\mathfrak{c}}$ and we reach a contradiction.

Now assume that $\mathfrak{c}$ is regular and that $f : X \to \mathbb{K}$ is $<\mathfrak{c}$-continuous, so $X = \bigcup_{\alpha<\kappa} X_\alpha$, where $\kappa < \mathfrak{c}$ and $f|X_\alpha \in \mathcal{C}$ for each $\alpha < \kappa$. Fix $U \in [X]^{\mathfrak{c}}$, then there is $\alpha < \kappa$ for which $Z = U \cap X_\alpha$ is of size $\mathfrak{c}$ and $f|Z$ is continuous. $\square$

To state the next result we need to recall the definitions of some cardinal invariants:

- $\operatorname{dec}(\mathcal{B}or, \mathcal{C})$ denotes the minimal cardinal κ such that for every Borel function $f : X \to \mathbb{K}$ there is a partition $(X_\alpha)_{\alpha<\kappa}$ of X, with $f|X_\alpha \in \mathcal{C}$ for all $\alpha < \kappa$. This cardinal has been defined (in a more general case) by J. Cichoń, M. Morayne, J. Pawlikowski and S. Solecki in [17] (cf. [18]).

- $\operatorname{cov}(\mathcal{M})$ (*covering of category*) denotes the minimal cardinality of a cover of X with meager sets.
- $\mathfrak{d}$ (*dominating number*) denotes the minimal cardinality of a dominating family $D \subseteq \omega^\omega$.

Recall that $\operatorname{cov}(\mathcal{M}) \leq \operatorname{dec}(\mathcal{B}or, \mathcal{C}) \leq \mathfrak{d}$, (see [17, Theorem 5.7] and [16, Theorem 4.3], cf. [18, Theorem 4.1]). Hence, the cardinal $\operatorname{dec}(\mathcal{B}or, \mathcal{C})$ can be different in different models of ZFC.

Theorem 4.4. *Consider $\mathcal{SZ}(\mathcal{C}), \mathcal{SZ}(\mathcal{B}or)$ as subsets of $\mathbb{K}^X$.*

(1) *If $\mathfrak{c}$ is a successor cardinal and $\operatorname{dec}(\mathcal{B}or, \mathcal{C}) = \mathfrak{c}$, then $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or) \neq \emptyset$.*
(2) *If $\mathfrak{c}$ is a regular cardinal and $\operatorname{dec}(\mathcal{B}or, \mathcal{C}) < \mathfrak{c}$, then $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or) = \emptyset$.*

Proof. Ad (1): By the assumption, there exists a Borel function $f_0 : X \to \mathbb{K}$ which is not $< \mathfrak{c}$-continuous (see also [34], [16] or [18]). By Proposition 4.3, there exists a set $U \in [X]^{\mathfrak{c}}$ such that $f_0|Z \notin \mathcal{C}$ for any $Z \in [U]^{\mathfrak{c}}$, so $f_0|U \in \mathcal{SZ}(\mathcal{C})$. Now, using a standard method of transfinite induction we can extend $f_0|U$ to a $\mathcal{SZ}(\mathcal{C})$ function f defined on the set X (see [21, Lemma 5]). It easy to observe that $f|U$ is Borel, so $f \notin \mathcal{SZ}(\mathcal{B}or)$.
Ad (2): Assume that $\mathfrak{c}$ is a regular cardinal and $\operatorname{dec}(\mathcal{B}or, \mathcal{C}) = \kappa < \mathfrak{c}$. Then each Borel function $f : X \to \mathbb{K}$ can by covered by κ continuous functions. Now, if $f \notin \mathcal{SZ}(\mathcal{B}or)$, then there is $B \in [X]^{\mathfrak{c}}$ with $f|B \in \mathcal{B}or$. Let $\tilde{f} : X \to \mathbb{K}$ be a Borel extension of $f|B$. Since $\tilde{f}$ can be covered by $\kappa < \mathfrak{c}$ continuous functions, $f|B$ has the same property, and by the regularity of $\mathfrak{c}$, $f|B_0 \in \mathcal{C}$ for some $B_0 \in [B]^{\mathfrak{c}}$. Hence $f \notin \mathcal{SZ}(\mathcal{C})$. □

Corollary 4.5. *The equality $\mathcal{SZ}(\mathcal{B}or) = \mathcal{SZ}(\mathcal{C})$ is independent with ZFC.*

Proof. First, let us observe that the condition $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or) \neq \emptyset$ is consistent with ZFC. In fact, it holds in every model of ZFC with $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ and $\mathfrak{c}$ being a successor cardinal.

Now, the equality $\mathcal{SZ}(\mathcal{B}or) = \mathcal{SZ}(\mathcal{C})$ holds in any model of ZFC in which $\mathfrak{c}$ is a regular cardinal and $\mathfrak{d} < \mathfrak{c}$, so e.g. in the iterated perfect set model (or any model of ZFC in which the Covering Property Axiom CPA holds, see [23] (see also [22, Theorem 3.1])). □

Corollary 4.6. *The family of all real or complex functions $f \in \mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or) \subseteq \mathbb{K}^{\mathbb{K}}$ is either empty or strongly $\mathfrak{c}$-algebrable.*

Proof. Suppose $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or) \neq \emptyset$. Fix $F \in \mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$.
In the case $\mathbb{K} = \mathbb{R}$, let us fix $f \in \mathcal{EXP}(\mathbb{R}) \setminus \{0\}$. It is easy to see that $f \circ F \in \mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$ (cf. [20, Theorem 4.1]). Hence applying Theorem 2.3 finishes the proof for $\mathbb{K} = \mathbb{R}$.
In the case $\mathbb{K} = \mathbb{C}$, let us fix $f \in \mathcal{A} \setminus \{0\}$ where $\mathcal{A}$ is an $\mathfrak{c}$-generated free algebra of entire, "$\leq \omega$-to-one" surjective mappings (see Theorem 2.7). We will show, that $f \circ F \in \mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$. Following [20, Theorem 4.1], it is enough to show that any $f \in \mathcal{A}$ fulfills the following condition:

every choice function $g : \mathbb{C} \to \mathbb{C}$ of f (i.e. $g(y) \in f^{-1}(\{y\})$ for $y \in \mathbb{C}$) satisfies the following:

$$(\star) \qquad \forall_{X \in [\mathbb{C}]^{\mathfrak{c}}} \exists_{Y \in [X]^{\mathfrak{c}}} \ g|Y \text{ is continuous.}$$

Let $f \in \mathcal{A}$ and $g : \mathbb{C} \to \mathbb{C}$ be its choice function. Let us fix a set $X \in [\mathbb{C}]^{\mathfrak{c}}$. Since f is an entire function, the set $A = \{x \in \mathbb{C} : f'(x) = 0\}$ is countable and for any $x \in \mathbb{C} \setminus A$ there is an open set $U_x \ni x$ such that $f|U_x$ is invertible and $(f|U_x)^{-1}$ is continuous. By the Lindelöf property for the Polish space $\mathbb{C} \setminus A$, the open cover $\{U_x : x \in \mathbb{C} \setminus A\}$ contains a countable subcover, i.e. there is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements from $\mathbb{C} \setminus A$ such that $\mathbb{C} \setminus A \subseteq \bigcup_{n \in \mathbb{N}} U_{x_n}$. As $g[X]$ is of cardinality $\mathfrak{c}$, there is $n_0 \in \mathbb{N}$ such that $|g[X] \cap U_{x_{n_0}}| = \mathfrak{c}$. Clearly for $Y = g^{-1}(g[X] \cap U_{x_{n_0}})$ we have that $Y \in [X]^{\mathfrak{c}}$ and $g|Y = (f|U_{n_0})^{-1}|Y$ i.e. $g|Y$ is continuous. Hence $f \circ F \in \mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$ and application of Theorem 2.1 finishes the proof for $\mathbb{K} = \mathbb{C}$.

□

Corollary 4.7. *Assume CH. The family $\mathcal{ES}(\mathbb{C}) \cap (\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)) \subseteq \mathbb{C}^{\mathbb{C}}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. We need to construct an example $F \in \mathcal{ES}(\mathbb{C})$ which belongs to the class $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$. Let $(C_n)_{n \in \mathbb{N}}$ be a countable family of pairwise disjoint Cantor sets such that each non-empty open set U contains some C_n, as in the proof of Lemma 3.4. Let $F_1 : C_1 \to \mathbb{C}$ belong to the class $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$.

Now, CH implies (in fact the equality $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ plays a key role here) that for each $n > 1$ we can choose a $\mathcal{SZ}(\mathcal{C})$ surjection $F_n : C_n \to \mathbb{C}$ (cf [21, Lemma 5]). Then $\bigcup_{n\geq 1} F_n$ is a $\mathcal{SZ}(\mathcal{C})$ function, and using a standard method of transfinite induction we can extend it to a $\mathcal{SZ}(\mathcal{C})$ function F defined on the set $\mathbb{C}$ (see [21, Lemma 5]). It easy to observe that $F[U] = \mathbb{C}$ for any non-empty open set $U \subseteq \mathbb{C}$, hence $F \in \mathcal{ES}(\mathbb{C})$. Moreover, $F|C_1 \notin \mathcal{SZ}(\mathcal{B}or)$, so $F \notin \mathcal{SZ}(\mathcal{B}or)$. Now applying Theorem 2.1 to the algebra $\mathcal{A}$ from Theorem 2.7 gives the assertion. □

Corollary 4.8. *Assume CH. The family $\mathcal{ES}(\mathbb{R}) \cap (\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)) \subseteq \mathbb{R}^{\mathbb{R}}$ is $\mathfrak{c}$-lineable.*

Proof. One can define $F \in \mathcal{ES}(\mathbb{R}) \cap (\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or))$ in the similar way as it was done in the proof of Corollary 4.7. Let $\mathcal{A} \subseteq \mathbb{R}^{\mathbb{R}}$ be a $\mathfrak{c}$-dimensional vector space of continuous, "$<\omega$-to-one" surjections (for example $\mathcal{A} = span\{e^{ax} - e^{-ax} : a \in \mathbb{R}\}$, see [15]). It is easy to see that an application of Theorem 2.1 to a vector space $\mathcal{A}$ finishes the proof. □

Note that $\mathcal{SZ}(\mathcal{C}) \cap \mathcal{SES}(\mathbb{C}) = \emptyset$, thus Corollary 4.7 implies (under CH) the strong $\mathfrak{c}$-algebrability of the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$. Finally, note that an example of a $\mathcal{SZ}(\mathcal{C})$ surjection cannot be constructed in ZFC (see [8]).

Clearly, the family $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$ is either empty or it is of cardinality $2^{\mathfrak{c}}$. Using analogous methods as in proof of [29, Theorem 5.6] one can observe that under CH, the family $\mathcal{SZ}(\mathcal{C}) \setminus \mathcal{SZ}(\mathcal{B}or)$ is $\mathfrak{c}^+$-lineable.

Problem 4.9. *Assume CH. Is the family $\mathcal{SZ}(\mathcal{B}or) \setminus \mathcal{SZ}(\mathcal{C})$ strongly $2^{\mathfrak{c}}$-algebrable ($2^{\mathfrak{c}}$-algebrable or $2^{\mathfrak{c}}$-lineable)? Note that CH implies the existence of an almost disjoint family in $\mathfrak{c}$ of cardinality $2^{\mathfrak{c}}$.*

5. Jones functions

This section is devoted to the following class of functions.

Definition 5.1. *A function $f : \mathbb{K} \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is called* a Jones function *($f \in \mathcal{J}(\mathbb{K})$), if for every closed set $K \subseteq \mathbb{K} \times \mathbb{K}$ with an uncountable projection on the x-axis, we have $f \cap K \neq \emptyset$.*

It is easy to see that the following diagram holds (by an arrow we mean a strict inclusion)

$$\mathcal{J}(\mathbb{K}) \to \mathcal{PES}(\mathbb{K}) \to \mathcal{ES}_{\kappa}(\mathbb{K}) \to \mathcal{ES}_{\lambda}(\mathbb{K}),$$

for $\omega \leq \lambda < \kappa \leq \mathfrak{c}$.

In [28] J.L. Gámez-Merino proved that the set $\mathcal{J}(\mathbb{C})$ is $2^{\mathfrak{c}}$-lineable. Moreover, T. Natkaniec (see [37]) showed that $\mathcal{J}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable and that both sets $\mathcal{J}(\mathbb{C})$ and $\mathcal{J}(\mathbb{R})$ contain large sets of free generators (i.e. of cardinality $2^{\mathfrak{c}}$). Unfortunately, contrary to what was claimed in [37], the algebra generated by those generators is not contained in $\mathcal{J}(\mathbb{C})$. Hence, the problem whether the set $\mathcal{J}(\mathbb{C})$ is **strongly $2^{\mathfrak{c}}$-algebrable** remained unsolved.

A positive answer to this problem is our goal in this section, but firstly we recall some necessary notions. By an *ultrafilter* on the set ω we mean any maximal nontrivial family of subsets of ω that is closed under taking supersets and finite intersections. Endowing ω with the discrete topology, by $\beta\omega$ we denote its Stone-Čech compactification. It is well-known that $\beta\omega$ can be viewed as the set of all ultrafilters on ω, hence $|\beta\omega| = 2^{\mathfrak{c}}$ (for details we refer to [26]).

For every $n \in \mathbb{N}$ let us fix a set $\mathcal{H}^n$ of surjective functions $h : \mathbb{K}^n \to \mathbb{K}$ with $|\mathcal{H}^n| \leq \mathfrak{c}$.

Theorem 5.2. *There is a family $\{f_\xi : \xi < 2^{\mathfrak{c}}\} \subseteq \mathbb{K}^{\mathbb{K}}$ such that for any $h \in \mathcal{H}^n$ and any distinct ordinals $\xi_1 < \xi_2 < \cdots < \xi_n < 2^{\mathfrak{c}}$ we have $h(f_{\xi_1}, \ldots, f_{\xi_n}) \in \mathcal{J}(\mathbb{K})$.*

Proof. Let $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}^n \times n^{\omega}$ (we use a standard set theoretical notation, i.e. $n^{\omega} = \{0, ..., n-1\}^{\omega}$). Let us denote by $\mathcal{K}$ the family of all closed sets $K \subseteq \mathbb{K} \times \mathbb{K}$ with uncountable projection on the x-axis. As $|\mathcal{K}| = |\mathcal{H}| = \mathfrak{c}$, let $\{(K_\alpha, h_\alpha, p_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{K} \times \mathcal{H}$. Let $\{r_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathbb{K}$. Inductively we define a function $\xi : \mathfrak{c} \to \mathfrak{c}$ in the following way. Let $\xi(0)$ be the smallest ordinal such that there is $y \in \mathbb{K}$ with $(r_0, y) \in K_{\xi(0)}$. In general for $\alpha < \mathfrak{c}$, let $\xi(\alpha)$ be the smallest ordinal among $\mathfrak{c} \setminus \{\xi(\beta) : \beta < \alpha\}$ such that there is $y \in \mathbb{K}$ with $(r_\alpha, y) \in K_{\xi(\alpha)}$. Obviously, by the definition ξ is injective. We will show that ξ is a bijection. Suppose on the contrary that $\xi[\mathfrak{c}] \neq \mathfrak{c}$, i.e. there is $\beta < \mathfrak{c}$ that is not in the range of ξ. Since $K_\beta \in \mathcal{K}$, its projection on the x-axis as well as the set $X = \{\alpha < \mathfrak{c} : \text{there is } y \in \mathbb{K} \text{ with } (r_\alpha, y) \in K_\beta\}$, is of cardinality $\mathfrak{c}$. Hence, ξ maps injectively X into β and we reach a contradiction.

Observe that the family $\{(r_\alpha, K_{\xi(\alpha)}, h_{\xi(\alpha)}, p_{\xi(\alpha)}) : \alpha < \mathfrak{c}\}$ is such that

(i) for any triple $(K, h, p) \in \mathcal{K} \times \mathcal{H}$, there is exactly one $\alpha < \mathfrak{c}$ with $(K_{\xi(\alpha)}, h_{\xi(\alpha)}, p_{\xi(\alpha)}) = (K, h, p)$,

(ii) for any $\alpha < \mathfrak{c}$ there is $y \in \mathbb{K}$ with $(r_\alpha, y) \in K_{\xi(\alpha)}$,

(iii) $\{r_\alpha : \alpha < \mathfrak{c}\}$ is an enumeration of $\mathbb{K}$.

Hence, without the loss of generality, we may assume that the family $\{(r_\alpha, K_\alpha, h_\alpha, p_\alpha) : \alpha < \mathfrak{c}\}$ has properties (i)-(iii).

For any ordinal $\alpha < \mathfrak{c}$, let n be the unique natural number such that $(h_\alpha, p_\alpha) \in \mathcal{H}^n \times n^\omega$. By (ii) there is $y \in \mathbb{K}$ with $(r_\alpha, y) \in K_\alpha$. Since $h_\alpha : \mathbb{K}^n \to \mathbb{K}$ is surjective, we can choose a vector $\vec{x}_\alpha \in \mathbb{K}^n$ such that $h_\alpha(\vec{x}_\alpha) = y$ (the vector $\vec{x}_\alpha$ will be thought of as a function $\vec{x}_\alpha : n \to \mathbb{K}$). Observe also that the map $p_\alpha \in n^\omega$ possesses a continuous extension $\overline{p_\alpha} : \beta\omega \to n$ to the Stone-Čech compactification of ω.

Now to each ultrafilter $\mathcal{U} \in \beta\omega$ let us assign a function $f_\mathcal{U} : \mathbb{K} \to \mathbb{K}$ defined by the formula $f_\mathcal{U}(r_\alpha) = \vec{x}_\alpha \circ \overline{p_\alpha}(\mathcal{U})$. We claim that the indexed family of functions $\{f_\mathcal{U} : \mathcal{U} \in \beta\omega\} \subseteq \mathbb{K}^\mathbb{K}$ is the desired one. Let $\mathcal{U}_0, ..., \mathcal{U}_{n-1} \in \beta\omega$ be distinct, let $h \in \mathcal{H}^n$ and let $K \in \mathcal{K}$. We will show that $h(f_{\mathcal{U}_0}, ..., f_{\mathcal{U}_{n-1}}) \cap K \neq \emptyset$. Since the ultrafilters $\mathcal{U}_0, ..., \mathcal{U}_{n-1}$ are distinct, we can find a partition $\omega = U_0 \cup ... \cup U_{n-1}$ such that $U_i \in \mathcal{U}_j$ if and only if $i = j$. This partition determines a function $p \in n^\omega$ such that $p^{-1}(i) = U_i$ for every $i \in n$. Then its extension $\overline{p} : \beta\omega \to n$ has the property $\overline{p}(\mathcal{U}_i) = i$ for every $i \in n$. For the triple (K, h, p), by (i), there is exactly one $\alpha < \mathfrak{c}$ such that $(K_\alpha, h_\alpha, p_\alpha) = (K, h, p)$. We have that

$$h(f_{\mathcal{U}_0}, ..., f_{\mathcal{U}_{n-1}})(r_\alpha) = h_\alpha(f_{\mathcal{U}_0}(r_\alpha), ..., f_{\mathcal{U}_{n-1}}(r_\alpha)) = h_\alpha(\vec{x}_\alpha \circ \overline{p_\alpha}(\mathcal{U}_0), ..., \vec{x}_\alpha \circ \overline{p_\alpha}(\mathcal{U}_{n-1})) =$$
$$= h_\alpha(\vec{x}_\alpha(0), ..., \vec{x}_\alpha(n-1)) = h_\alpha(\vec{x}_\alpha) = y,$$

where $y \in \mathbb{K}$ is such that $(r_\alpha, y) \in K_\alpha$ by the choice of the vector $\vec{x}_\alpha$. Hence, $h(f_{\mathcal{U}_0}, ..., f_{\mathcal{U}_{n-1}}) \cap K \neq \emptyset$. □

Observe that in particular by taking $\mathcal{H}^n$ as the family of all nonzero polynomials in n complex variables without a constant term, we obtain the following.

Corollary 5.3. *The set $\mathcal{J}(\mathbb{C})$ is strongly $2^\mathfrak{c}$-algebrable.*

On the other hand, to obtain $2^\mathfrak{c}$-lineability of $\mathcal{J}(\mathbb{C})$ (or $\mathcal{J}(\mathbb{R})$), one can take $\mathcal{H}^n$ as the family of all nonzero polynomials of the $1st$ degree in n complex (or real) variables without a constant term.

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