TOPOLOGICAL SIZE OF SOME SUBSETS IN CERTAIN CALDERÓN-LOZANOWSKIĬ SPACES

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Abstract. For $i = 1, 2, 3$, let φ_i be Young functions, (Ω, μ) a (topological) measure space, E an ideal of μ-measurable real-valued functions defined on Ω and E_{φ_i} be the corresponding Calderón-Lozanowskiĭ space. Our aim in this paper is to give, under mild conditions, several results on topological size (in the sense of Baire) of the sets $\{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : |f| \bigodot |g| \in E_{\varphi_3}\}$ and $\{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : \exists x \in V, (f \bigodot g)(x) \text{ is well defined}\}$ where $\bigodot$ denotes the convolution or pointwise product of functions and V a compact neighbourhood. Our results sharpen and unify the related results obtained in diverse areas during recent thirty years.

1. Introduction

Let X be a topological vector space of functions such that the product "$\cdot$" is defined on $X \times X$. Then there arises the question whether the product $f \cdot g$ does not belong to X for some pair (f, g). In certain cases the solution of this problem is well-known. For example: when is the classical Lebesgue space $L^p(X)$, for a measure space (X, μ), closed under pointwise product? An easy application of Hölder's inequality and a result due essentially to B. Subramanian [41] give the answer. The problem whether the Orlicz space $L^\varphi(X)$, defined on a measure space (X, μ), with pointwise product is a Banach algebra was studied in [7, 18].

One can consider a quantitative version of this question, namely – is the set of the pairs (f, g) for which $f \cdot g$ exists small in the sense of Baire category or porosity? The first result of this sort was proved by Balcerzak and Wachowicz in [9], who showed that the set $\{(f, g) \in L^1[0, 1] \times L^1[0, 1] : f \cdot g \in L^1[0, 1]\}$ is meager in $L^1[0, 1] \times L^1[0, 1]$. Jachymski generalized this in [20], by proving that the set of those pairs (f, g) that the product $f \cdot g$ is in $L^p(X, \mu)$ is either the whole Cartesian product $L^p(X, \mu) \times L^p(X, \mu)$, or it is a meager subset, where $p \geq 1$. Głąb and Strobin in [13] strengthened this by proving that the set of those pairs (f, g) such that the product $f \cdot g$ is in $L^r(X, \mu)$ is either the whole Cartesian product $L^p(X, \mu) \times L^q(X, \mu)$ or it is a σ-lower porous subset, where $p \in (0, \infty]$. The similar dichotomies were proved for Orlicz spaces by Akbarbaglu and Maghsoudi in [4] (independently by Strobin in [40]), and for Lorentz spaces by Głąb, Strobin and Yang in [15]. We extend this result to the so-called Calderón-Lozanowskiĭ spaces.

A more subtle and difficult case is what is known as L^p-conjecture; i.e., is $L^p(G)$, where G is a locally compact group with a left Haar measure, closed under convolution product for $p > 1$ only if G is compact? Originated independently by M. Rajagopalan [36] and Z. Żelazko [44], it was an open

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problem until resolved positively by S. Saeki [39] in 1990 after thirty years of struggling. Several authors were involved in proving the L^p-conjecture in special cases – it is briefly described in paper of Seaki who gave an extended list of references.

The study of quantitative version of L^p-conjecture was initiated by Głąb and Strobin in [12], who proved that if $p, q > 1$, $1/p + 1/q < 1$, G is a locally compact but not compact group and $V \subset G$ is a compact neighbourhood of the identity, then the set of pairs $(f, g) \in L^p(G) \times L^q(G)$ such that $f * g(x)$ exists for some $x \in V$ is σ-lower porous. If $p \in (1, 2]$ and G is unimodular, then by the Young inequality $L^p(G) * L^p(G) \subset L^{\frac{p}{2-p}}(G)$. Thus $f * g(x)$ is λ-a.e. finite for $f, g \in L^p(G)$. Akbarbaglu and Maghsoudi in [1] proved that if G is non-unimodular, locally compact, non-compact group, $V \subset G$ is a compact neighbourhood of the identity, then the set of pairs $(f, g) \in L^p(G) \times L^q(G)$ such that $f * g(x)$ exists for some $x \in V$ is σ-lower porous. In [5] Akbarbaglu and Maghsoudi proved the same assertion for $p \in (0, 1)$ and $q \in (0, \infty]$. Moreover, in [5] the authors proved that if G is nondiscrete, $1/p+1/q > 1+1/r$ where $p, q \in [1, \infty)$, $r \in [1, \infty]$, $V \subset G$ is a compact neighbourhood of the identity, then the set of those pairs $(f, g) \in L^p(G) \times L^q(G)$ such that $f * g \in L^r(V, \lambda|_V)$ is σ-lower porous; which solves the old problem of Saeki [39].

Some authors considered the problem whether the Orlicz space L^φ, defined on a locally compact group G with a Haar measure, considered with the convolution product, is a Banach algebra; we will call it, after [3], a Banach-Orlicz algebra. More generally, suppose φ_i, $i = 1, 2, 3$, are Young functions and $L^{\varphi_i}(G)$ are the corresponding Orlicz spaces; then there is a natural question to ask – what needs to be assumed on φ_i's to get $f * g \in L^{\varphi_3}(G)$ provided $f \in L^{\varphi_1}(G)$ and $g \in L^{\varphi_2}(G)$? R. O'Neil in [33] examined the convolution operator in the context of Orlicz spaces, and proved that if G is a unimodular locally compact group, φ_i, $i = 1, 2, 3$ are Young functions satisfying $\varphi_1^{-1}(x)\varphi_2^{-1}(x) \le x\varphi_3^{-1}(x)$ for $x \ge 0$, then for any $f_i \in L^{\varphi_i}(G)$, $i = 1, 2$, the convolution $f_1 * f_2$ belongs to $L^{\varphi_3}(G)$ and moreover $N_{\varphi_3}(f_1 * f_2) \le 2N_{\varphi_1}(f_1)\, N_{\varphi_2}(f_2)$ where N_{φ_i} is the Luxemburg norm on L^{φ_i}. In other words, the convolution map acts from $L^{\varphi_1}(G) \times L^{\varphi_2}(G)$ into $L^{\varphi_3}(G)$. Furthermore, Hudzik, Kamińska and Musielak in [19] undertook the L^p-conjecture for Orlicz spaces and proved that if G is abelian then $L^\varphi(G)$ is the Banach-Orlicz algebra if and only if G is compact or $\lim_{t\to 0} \varphi(t)/t > 0$. Kamińska and Musielak in [21] extended this result and gave necessary and sufficient conditions, in terms of Young functions and group G, for $L^{\varphi_1}(G)*L^{\varphi_2}(G) \subseteq L^{\varphi_3}(G)$, where G is abelian. Akbarbaglu and Maghsoudi proved in [3] that if G is amenable, φ is a Δ_2-regular N-function, and $L^\varphi(G)$ is a Banach-Orlicz algebra, then G is compact, in particular $L^\varphi(G) \subset L^1(G)$. Some other related results can be found in [37].

Akbarbaglu and Maghsoudi in [2] initiated the study of the quantitative L^p-conjecture for Orlicz spaces. They gave sufficient conditions on φ_1 and φ_2 so that whenever G is non-unimodular, then the set of those pairs $(f, g) \in L^{\varphi_1}(G) \times L^{\varphi_2}(G)$ for which $f * g$ is well-defined at some point of the fixed neighbourhood of the identity, is σ-lower porous. The results from [2] shed light on sharpness and necessity of the relation $\varphi_1^{-1}(x)\varphi_2^{-1}(x) \le x\varphi_3^{-1}(x)$ and unimodularity of the group G for the inclusion $L^{\varphi_1}(G) * L^{\varphi_2}(G) \subseteq L^{\varphi_3}(G)$.

Recently, the analogue quantitative problems for pointwise products have been considered by Akbarbaglu and Maghsoudi, Głąb and Strobin, and coauthors for L^p-spaces, Orlicz spaces, Lorentz spaces, and the space of continuous functions; see [5, 6, 14, 15, 40].

In this paper we improve of the results known for Orlicz spaces, and put them into the more general setting of the so-called Calderón-Lozanovskiĭ spaces E_φ (where E is a Banach ideal space and φ a Young function) which are generalizations of Orlicz spaces, Orlicz-Lorentz spaces and contain the p-convexification $E^{(p)}$ $(1 \leq p < \infty)$ of E. These spaces were introduced by A. P. Calderón in [10] and developed by G. Ja. Lozanovskiĭ in [29, 30]. They play crucial role in the theory of interpolation. There is a lot of basic information on Calderón-Lozanovskiĭ spaces; see, for example, [22, 28, 31]. Also for a recent study on pointwise multipliers of Calderón-Lozanovskiĭ spaces see [24, 25]. Let us mention that the Calderón-Lozanovskiĭ spaces E_φ we deal with here are special cases of general Calderón-Lozanovskiĭ spaces $\varrho(E, F)$ with $F = L^\infty$ in this case.

The paper is organized as follows: In Section 2 we give some necessary definitions and notations concerning Orlicz and Calderón-Lozanovskiĭ spaces. In Section 3 we consider two Calderón-Lozanovskiĭ spaces on a locally compact group with the Haar measure. In this section, various results concerning the size of pairs belonging to the product of the two Calderón-Lozanovskiĭ spaces for which the convolution multiplication exists or is in another Calderón-Lozanovskiĭ space are given. These results generalize and sharpen the results for L^p and Orlicz spaces known in literature. Finally, in Section 4 the problem of pointwise product in Calderón-Lozanovskiĭ spaces is studied.

2. **Preliminaries**

We need to recall some necessary definitions from abstract harmonic analysis, and Orlicz and Calderón-Lozanovskiĭ spaces.

Throughout this paper, let G denote a locally compact group with a fixed left Haar measure λ. Also, let $\mathcal{M}_\lambda$ denotes the σ-algebra of all Haar measurable sets and $L^0(G)$ denote the set of all (equivalence classes of) λ-measurable complex-valued functions on G. For measurable functions f and g on G, the *convolution*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\, d\lambda(y)$$

is defined at each point $x \in G$ for which the function $y \mapsto f(y)g(y^{-1}x)$ is Haar integrable.

For each $x \in G$, the formula $\lambda_x(A) = \lambda(Ax)$ defines a left invariant regular Borel measure λ_x on G. Thus, the uniqueness of the left Haar measure implies that for each $x \in G$ there is a positive number, say $\Delta(x)$, such that $\lambda_x = \Delta(x)\lambda$. The function $\Delta : G \to (0, \infty)$ defined in this way is called the modular function of G. It is clear that Δ is a continuous homomorphism on G. Moreover, for every measurable subset A of G,

$$\lambda(A^{-1}) = \int_A \Delta(x^{-1}) d\lambda(x);$$

for more details see [11] or [17]. The group G is called *unimodular* whenever $\Delta = 1$. In this case, the left Haar measure and the right Haar measure coincide.

For $1 \leq p \leq \infty$, classical Lebesgue spaces on G with respect to the Haar measure λ will be denoted by $L^p(G)$ with the norm $\|\cdot\|_p$ defined in the usual way.

Orlicz spaces have been thoroughly investigated from the point of view of functional analysis. We refer to two excellent books [26] and [38] for more details. Also [31, 32, 42] provide some useful information on the subject.

A function $\varphi : \mathbb{R} \to [0,\infty]$ is called a *Young function* if φ is convex, even, left continuous on $(0,\infty)$ with $\varphi(0) = 0$; we also assume that φ is neither identically zero nor identically infinite on $(0,\infty)$. As an elementary example of a Young function we can consider $\varphi(x) = |x|^p/p$, for $p > 1$. For any Young function φ we define:

$$a_\varphi = \sup\{x \in \mathbb{R} : \varphi(x) = 0\} \quad \text{and} \quad b_\varphi = \sup\{x \in \mathbb{R} : \varphi(x) < \infty\}.$$

A Young function φ is called *finite* if $b_\varphi = \infty$. It is easy to observe that it is continuous on $[0, b_\varphi)$, nondecreasing on $[0,\infty)$ and strictly increasing on $[a_\varphi, b_\varphi]$.

We also need an inverse of a Young function φ. For a Young function φ and $y \in [0,\infty)$ let

$$\varphi^{-1}(y) = \sup\{x \geq 0 : \varphi(x) \leq y\}.$$

The following lists basic properties of Young functions and their inverses. We skip an easy proof.

Lemma 2.1. *In the above frame:*

(1.) *For all $x \geq 0$, $\varphi\big(\varphi^{-1}(x)\big) \leq x$.*
(2.) *If $\varphi(x) < \infty$ then $x \leq \varphi^{-1}\big(\varphi(x)\big)$.*
(3.) *If $x \in [0, \varphi(b_\varphi)]$, then $\varphi\big(\varphi^{-1}(x)\big) = x$.*
(4.) *If $x \in [a_\varphi, b_\varphi]$, then $x = \varphi^{-1}\big(\varphi(x)\big)$.*

Let (Ω, Σ, μ) be a measure space and φ be a Young function. For each $f \in L^0(\Omega)$, the space of all (equivalence classes of) μ-measurable (complex-valued) functions defined on Ω, we define

$$\varrho_\varphi(f) = \int_\Omega \varphi(|f(x)|)\, d\mu(x).$$

Then the *Orlicz space* $L^\varphi(\Omega)$ is defined by

$$L^\varphi(\Omega) = \{f \in L^0(\Omega) : \varrho_\varphi(af) < \infty, \text{ for some } a > 0\}.$$

The Orlicz space $L^\varphi(\Omega)$ is a Banach space under the norm $N_\varphi(\cdot)$, called *Luxemburg norm*, defined for $f \in L^\varphi(\Omega)$ by

$$N_\varphi(f) = \inf\{k > 0 : \varrho_\varphi(f/k) \leq 1\}.$$

It is well-known that

$$N_\varphi(f) \leq 1 \quad \text{if and only if} \quad \varrho_\varphi(f) \leq 1, \tag{1}$$

and if $0 < \mu(F) < \infty$ then

$$N_\varphi(\chi_F) = \left[\varphi^{-1}\left(\frac{1}{\mu(F)}\right)\right]^{-1}; \tag{2}$$

see Corollary 3.4.7 in [38]. Here χ_A denotes the characteristic function of a subset A.

Now let us give definitions concerning our main object in this paper, namely Calderón-Lozanowskiĭ spaces which are defined in the similar way as Orlicz spaces, and they share common properties. Let (Ω, Σ, μ) be a complete σ-finite measure space. A Banach space $E = (E, \|\cdot\|_E)$ is called *a Banach ideal space* on Ω if E is a linear subspace of $L^0(\Omega)$ and satisfies the ideal property, that is if $f \in E, g \in L^0(\Omega)$ and $|g(t)| \leq |f(t)|$ for μ-almost all $t \in \Omega$, then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

We distinguish two cases - real and complex ones. Namely, if E consists only of real functions, then we consider E as a real Banach space. In the second case, when E consists also of complex functions, then we consider E as a complex space.

Remark 2.2. Assume that E is complex and let $E^{\mathbb{R}} = \{\mathrm{Re}(f) : f \in E\}$. Then it is easy to see that $(E^{\mathbb{R}}, \|\cdot\|_E)$ is a (real) Banach ideal space and $E = \{f + ig : f, g \in E^{\mathbb{R}}\}$.

Note that most proofs presented later will automatically work for both cases (in some places we will write $\mathrm{Re}(f)$, $\mathrm{Re}(g)$ etc., but in the real case we just have $\mathrm{Re}(f) = f$, $\mathrm{Re}(g) = g$ etc.). We will emphasize the cases when some changes appear.

For a given Banach ideal space E on Ω and a Young function φ, let $I_\varphi : L^0(\Omega) \to [0, \infty]$ be a semimodular defined by

$$I_\varphi(f) = \begin{cases} \|\varphi(|f|)\|_E & \text{if} \quad \varphi(|f|) \in E, \\ \infty & \text{otherwise.} \end{cases}$$

The *Calderón-Lozanowskiĭ* space E_φ is the space

$$E_\varphi = \{f \in L^0(\Omega) : I_\varphi(cf) < \infty \text{ for some } c > 0\}$$

with the *Luxemburg norm*

$$\|f\|_{E\varphi} = \inf\{c > 0 : I_\varphi(f/c) \le 1\}.$$

If $E = L^1(\Omega)$, then E_φ is the Orlicz space $L^\varphi(\Omega)$ equipped with the Luxemburg norm. If E is a Lorentz function (sequence) space, then E_φ is the corresponding Orlicz-Lorentz function (sequence) space equipped with the Luxemburg norm. Also, if $\varphi(t) = t^p$, $1 \le p < \infty$, then E_φ is in this case the p convexification $E^{(p)}$ of E with the norm $\|f\|_{E^{(p)}} = \||f|^p\|_E^{1/p}$. Finally, if $\varphi(t) = 0$ for $t \in [0, 1]$ and $\varphi(t) = \infty$ otherwise, then $E_\varphi = L^\infty(\Omega)$ and the corresponding norms are equal.

The following result links the Luxemburg norm $\|\cdot\|_{E_\varphi}$ with the original norm $\|\cdot\|_E$ on a Banach ideal space E.

Lemma 2.3. *Let (Ω, Σ, μ) be a measure space and E be a Banach ideal space. If $A \subseteq \Omega$ is such that $0 < \mu(A) < \infty$, then (we assume here $1/0 = \infty$)*

$$\|\chi_A\|_{E\varphi} = \left[\varphi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)\right]^{-1}.$$

Proof. By the definition of Luxemburg norm we have

$$\|\chi_A\|_{E\varphi} = \inf\{t > 0 : \|\varphi(\chi_A/t)\|_E \le 1\}.$$

Then (we assume here that $1/\infty = 0$)

$$\begin{aligned} \|\varphi(\chi_A/t)\|_E \le 1 &\iff \varphi(1/t)\|\chi_A\|_E \le 1 \iff \varphi(1/t) \le 1/\|\chi_A\|_E \\ &\iff 1/t \le \varphi^{-1}(\|\chi_A\|_E) \iff t \ge \left[\varphi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)\right]^{-1}. \end{aligned}$$

□

We finish this section with the notion of porosity from [12]; for more details see also [43]. Let X be a metric space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x,r)$. For a given number $0 < c \leq 1$, a subset M of X is called *c-lower porous* if

$$\liminf_{R \to 0+} \frac{\gamma(x,M,R)}{R} \geq \frac{c}{2}$$

for all $x \in M$, where

$$\gamma(x,M,R) = \sup\{r \geq 0 : \exists\, z \in X, B(z,r) \subseteq B(x,R) \setminus M\}.$$

It is clear that M is c-lower porous if and only if

$$\forall x \in M, \forall \alpha \in (0,c/2), \exists\, r_0 > 0, \forall r \in (0,r_0), \exists\, z \in X, B(z,\alpha r) \subseteq B(x,r) \setminus M.$$

A set is called *σ-c-lower porous* if it is a countable union of c-lower porous sets with the same constant $c > 0$. It is easy to see that a σ-c-lower porous set is meager, and the notion of σ-porosity is stronger than that of meagerness.

3. SUBSETS RELATED TO CONVOLUTION PRODUCT

Throughout this section let G be a locally compact group with a fixed left Haar measure λ, and E be a Banach ideal in $L^0(G)$ which satisfies additionally the conditions:

(a) if $f_n \nearrow f$ for some nonnegative (real) functions $f_n \in E$, $n \in \mathbb{N}$ and $f \in L^0(G)$, then $||f_n||_E \to ||f||_E$ provided $f \in E$, and $||f_n||_E \to \infty$ if $f \notin E$.

(b) if $V \subset G$ and $\lambda(V) < \infty$, then $\chi_V \in E$;

(c) if $V \subset G$ and $\lambda(V) < \infty$, then there is $C_V < \infty$ such that $\int_V |f| d\lambda \leq C_V ||f||_E$ for every $f \in E$.

Remark 3.1. If a Banach ideal E consists of real functions, then the above conditions mean that E is a *Banach function space* according to [8, Definitions 1.1.1 and 1.1.3]. Indeed, it is enough to take $\varrho(f) := \begin{cases} ||f||_E, & f \in E \\ \infty, & f \notin E \end{cases}$. Moreover, it is easy to see that if E is complex, then in such case, the real space $E^{\mathbb{R}}$ (see Remark 2.2) is a Banach function space.

Nonnegative (real) functions $f, g \in L^0(G)$ are called *equimeasurable*, if for every $t \geq 0$,

$$\lambda(\{x \in G : |f(x)| > t\}) = \lambda(\{x \in G : |g(x)| > t\}).$$

We additionally assume that:

$$\text{for every equimeasurable real functions } f, g \in E,\ \|f\|_E = \|g\|_E \tag{3}$$

Remark 3.2. In the case when E is real, condition (3) means that E is so-called *rearrangement-invariant space* [8, Definition 2.4.1]. If E is complex, then $E^{\mathbb{R}}$ is rearrangement-invariant space.

Observe that if $U, V \in \mathcal{M}_\lambda$ and $\lambda(V) = \lambda(U)$, then χ_V and χ_U are equimeasurable, hence $\|\chi_V\|_E = \|\chi_U\|_E$. Thus there exists a function $\xi_E : [0,\infty) \to [0,\infty)$ such that for every measurable $V \subset G$ with $\lambda(V) < \infty$

$$\xi_E(\lambda(V)) := \|\chi_V\|_E.$$

The function ξ_E (which is uniquely determined just on the range of λ), is called the *fundamental function of* E [8, Definition 2.5.1].

Remark 3.3. Note that the Haar measure is *resonant* in the sense of [8] (see [8, Theorem 2.2.7]), so the settings of [8, Definition 2.5.1] are satisfied. Moreover, using the properties of the Haar measure, we see that:

1. If G is discrete and infinite, then for some $c > 0$, $\lambda(\{x\}) = c$ for all $x \in G$. Hence ξ_E is determined on $\{kc : k \in \mathbb{N} \cup \{0\}\}$. Then we assume that ξ_E is defined on the rest linearly.
2. If G is not discrete and not compact, then for every $t \in [0, \infty)$ there exists $V \subset G$ such that $\lambda(V) = t$. In particular, ξ_E is uniquely determined on the whole $[0, \infty)$.
3. If G is compact but not discrete, then the image of λ equals the interval $[0, \lambda(G)]$. Hence ξ_E is uniquely determined on this interval. In this case we choose $\xi_E(t) := \xi_E(\lambda(G))$ for the rest $t > \lambda(G)$.

Now, [8, Corollary 2.5.3] give us also the following properties. Note that part (3.) follows from (2.) in below:

Lemma 3.4. *The fundamental function ξ_E satisfies the following:*
(1.) ξ_E is nondecreasing and continuous except (perhaps) at origin and $\xi_E(0) = 0$.
(2.) The mapping $t \to \frac{\xi_E(t)}{t}$ is nonincreasing.
(3.) For $K := \lim_{t\to\infty} \frac{\xi_E(t)}{t}$ and every $\varepsilon > 0$ there is t_0 such that $\xi_E(t) \leq (K + \varepsilon)t$ for every $t \geq t_0$.

Finally, we make another assumptions:

$$\text{The fundamental function } \xi_E \text{ is continuous at 0, that is, } \lim_{t\to 0} \xi_E(t) = 0 \tag{4}$$

$$\text{The fundamental function } \xi_E \text{ is unbounded, that is, } \lim_{t\to\infty} \xi_E(t) = \infty. \tag{5}$$

Remark 3.5. In view of Remark 3.3, we see that if G is discrete, then ξ_E is continuous at 0. Also by [8, Corollary 2.5.5] we know that if E is separable, then ξ_E is continuous at 0. Let us remark that the condition (5) is also natural, for example it (and also the earlier ones) is satisfied when G is not compact and $E = L^p(G)$ where $\|\cdot\|$ is the L^p-norm.

Now for every $s \in [0, \infty)$, set

$$\xi_E^{-1}(s) := \sup\{t \in [0, \infty) : \xi_E(t) \leq s\}.$$

We will use the following simple facts. Observe that the second part of (1.) in below follows from the condition (4) above.

Lemma 3.6.
(1.) The function $\xi_E^{-1} : [0, \infty) \to [0, \infty]$ is nondecreasing and $\lim_{t\to 0} \xi_E^{-1}(t) = 0$;
(2.) For every $t \in [0, \infty)$, $\xi_E^{-1}(\xi_E(t)) \geq t$.

Let us summarize our assumptions on the Banach ideal E: we assume that it is chosen such that $(a) - (b)$, (3), (4) and (5) are satisfied (in particular, G is assumed to be non-compact), which, in the language of [8], means that E or $E^{\mathbb{R}}$ is a rearrangement-invariant Banach function space such that its fundamental function ξ_E is continuous at 0 and unbounded. In particular, E can be taken as many Lebesgue spaces L^p (or even Orlicz spaces L^φ). We refer the interest reader to [8] and to [23] for further discussion on the topic.

Next, let φ_1, φ_2 be Young functions. We equip the product of Calderón-Lozanowskiĭ spaces $E_{\varphi_1} \times E_{\varphi_2}$ with the complete norm

$$\|(f,g)\|_{\varphi_1,\varphi_2} = \max\left\{\|f\|_{E_{\varphi_1}}, \|g\|_{E_{\varphi_2}}\right\} \quad \big(f \in E_{\varphi_1}, g \in E_{\varphi_2}\big).$$

We commence with some definitions which we need to state our first result.

A locally compact group G satisfies a condition $(\star)$ if

> for every compact neighbourhood V of the identity element of G, there exist $\alpha > 1$ and a strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\lambda\big(V^{2p_n}\big) < \alpha\lambda(V^{p_n})$.

We say that G has *polynomial growth* if for every compact neighbourhood V of the identity element of G, there exists $d \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \frac{\lambda(V^n)}{n^d} < \infty.$$

Let us recall that according to [35, Proposition 16.28], every locally compact group G having polynomial growth satisfies the condition $(\star)$. Also, by [34, Corollary 6.18], locally compact abelian groups and nilpotent groups have polynomial growth. Moreover, by [34, Proposition 6.6, 6.9] every polynomially growing group is unimodular.

Let us remark that for any Young function φ, we write $f * g \in L^\varphi(G)$ to mean that $|f| * |g| < \infty$ λ-almost everywhere, $f*g$ is Haar measurable on the set $\big\{x \in G : |f*g|(x) < \infty\big\}$ and $N_\varphi(f*g) < \infty$.

Theorem 3.7. *Let G be a locally compact group that satisfies the condition $(\star)$ and let φ_i, $i = 1,2,3$ be Young functions with $\varphi_i(b_{\varphi_i}) > 0$, for $i = 1,2,3$ and*

$$\liminf_{x\to 0} \frac{\varphi_1^{-1}(x)\varphi_2^{-1}(x)}{x\varphi_3^{-1}(x)} = \infty. \tag{6}$$

If G is non-compact, then the set

$$F = \{(f,g) \in E_{\varphi_1} \times E_{\varphi_2} : |f| * |g| \in E_{\varphi_3}\}.$$

is of first category in $E_{\varphi_1} \times E_{\varphi_2}$.

Remark 3.8. Let us remark that the above theorem is an extension of the first part of [21, Theorem 11], where it is shown that $F \neq E_{\varphi_1} \times E_{\varphi_2}$ in the case of Orlicz spaces E_{φ_i} (observe that if $(f,g) \in (E_{\varphi_1} \times E_{\varphi_2}) \setminus F$, then also $(|f|,|g|) \in (E_{\varphi_1} \times E_{\varphi_2}) \setminus F$). In fact, it just a partial extension since we additionally assume that $\varphi_i(b_{\varphi_i}) < \infty$ for $i = 1,2,3$. However, note that [21, Theorem 11] is restricted to abelian groups but Theorem 3.7 can be applied for a wider class of groups, in particular, nilpotent groups.

Proof. We will write $\|\cdot\|$ instead of $\|\cdot\|_E$. For any natural number n, put

$$F_n = \big\{(f,g) \in E_{\varphi_1} \times E_{\varphi_2} : \big\||f| * |g|\big\|_{E_{\varphi_3}} < n\big\}.$$

So $F = \bigcup_{n\in\mathbb{N}} F_n$. The proof will be complete if we show that for each $n \in \mathbb{N}$, F_n is nowhere dense.

Fix $n \in \mathbb{N}$, $R > 0$ and $(f,g) \in F_n$. There is $0 < t_n < \min\{\varphi_i(b_{\varphi_i}) : i = 1,2,3\}$ such that for $0 < t \le t_n$,

$$\frac{R^2}{288(K+1)} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{t\varphi_3^{-1}(t)} > n. \tag{7}$$

Since G is not compact, there are compact neighbourhoods of the identity element of G with as big but finite measure as needed. Hence, by Lemma 3.4(3.) and (5) we can find a compact symmetric neighbourhood V of the identity element of G such that:

$$\frac{1}{\|\chi_V\|} < t_n \quad \text{and} \quad \|\chi_V\| \le (K+1)\lambda(V). \tag{8}$$

Since G satisfies the condition ($\star$), for such compact neighbourhood V, there are $\alpha > 1$ and a sequence $(p_k)_{k\in\mathbb{N}}$ with $\lambda\big(V^{2p_k}\big) < \alpha\lambda(V^{p_k})$ (in particular, $\lambda(V^{p_k}) < \infty$ for all $k \in \mathbb{N}$).

Now consider two cases:

Case 1. $\lim_{k\to\infty} \lambda(V^{p_k}) = \infty$. By (6) we choose $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ we have

$$\frac{R^2}{72}\frac{\|\chi_{V^{p_k}}\|}{K+1}\varphi_1{}^{-1}\left(\frac{1}{\alpha\|\chi_{V^{p_k}}\|}\right)\varphi_2{}^{-1}\left(\frac{1}{\alpha\|\chi_{V^{p_k}}\|}\right) > n\alpha\varphi_3{}^{-1}\left(\frac{1}{\alpha\|\chi_{V^{p_k}}\|}\right). \tag{9}$$

Case 2. The increasing sequence $\big(\lambda(V^{p_k})\big)_{k\in\mathbb{N}}$ is bounded from above, so it is convergent. Hence, by (7), (8) and the fact that $||\chi_V|| \le ||\chi_{V^{p_k}}||$, there is $k_0 \in \mathbb{N}$ such that for $k \ge k_0$, $\lambda(V^{2p_k}) \le 2\lambda(V^{p_k})$ and

$$\frac{R^2}{72}\frac{\|\chi_{V^{p_k}}\|}{K+1}\varphi_1{}^{-1}\left(\frac{1}{2\|\chi_{V^{p_k}}\|}\right)\varphi_2{}^{-1}\left(\frac{1}{2\|\chi_{V^{p_k}}\|}\right) > 2n\varphi_3{}^{-1}\left(\frac{1}{2\|\chi_{V^{p_k}}\|}\right). \tag{10}$$

Now let $A = V^{p_{k_0}}$ and $B = V^{2p_{k_0}}$. We proceed with Case 1. and Case 2. together (if Case 2. holds, then in the next computations α equals 2). By Lemma 3.4(1.), (2.), we have

$$\|\chi_B\| = \xi_E\big(\lambda(B)\big) \le \xi_E\big(\alpha\lambda(A)\big) \le \alpha\xi_E\big(\lambda(A)\big) = \alpha\|\chi_A\|. \tag{11}$$

Now let $r < R/6$ be such that

$$\lambda(A) - S\xi_E^{-1}\left(\frac{6r}{R}\|\chi_B\|\right) - \xi_E^{-1}\left(\frac{6r}{R}\|\chi_A\|\right) \ge \frac{1}{2}\lambda(A), \tag{12}$$

where $S := \sup_{x\in B}\Delta(x^{-1})$ (such an r exists by Lemma 3.6(1.)).

Define

$$M_f := \frac{R}{3\|\chi_A\|_{E_{\varphi_1}}} = \frac{R}{3}\varphi_1^{-1}\left(\frac{1}{\|\chi_A\|}\right), \quad M_g := \frac{R}{3\|\chi_B\|_{E_{\varphi_2}}} = \frac{R}{3}\varphi_2^{-1}\left(\frac{1}{||\chi_B||}\right) \tag{13}$$

and

$$\widetilde{f}(y) := \begin{cases} f(y) & y \notin A \\ f(y) + M_f & \mathrm{Re}\big(f(y)\big) \ge 0,\ y \in A \\ f(y) - M_f & \mathrm{Re}\big(f(y)\big) < 0,\ y \in A, \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} g(y) & y \notin B \\ g(y) + M_g & \mathrm{Re}\big(g(y)\big) \ge 0,\ y \in B \\ g(y) - M_g & \mathrm{Re}\big(g(y)\big) < 0,\ y \in B. \end{cases}$$

Then

$$\|f - \tilde{f}\|_{E_{\varphi_1}} = \|M_f\chi_A\|_{E_{\varphi_1}} = R/3 < R,$$

and similarly

$$\|g - \tilde{g}\|_{E_{\varphi_2}} < \|M_g \chi_B\|_{E_{\varphi_2}} = R/3 < R.$$

Hence $B\big((\tilde{f}, \tilde{g}), r\big) \subset B\big((f, g), R\big)$. It remains to show that $B\big((\tilde{f}, \tilde{g}), r\big) \cap F_n = \emptyset$.

Let $(h, k) \in B\big((\tilde{f}, \tilde{g}), r\big)$. Put

$$A_1 := \{x \in A : |h(x)| \le M_f/2\}, \quad B_1 := \{x \in B : |k(x)| \le M_g/2\}$$

and $A_2 = A \setminus A_1$ and $B_2 = B \setminus B_1$.

Then

$$r \ge \|h - \tilde{f}\|_{E_{\varphi_1}} \ge \|(h - \tilde{f})\chi_{A_1}\|_{E_{\varphi_1}} \ge \frac{1}{2} M_f \|\chi_{A_1}\|_{E_{\varphi_1}} \overset{Lem. 2.3}{=} \frac{M_f}{2\varphi_1^{-1}\left(\frac{1}{\|\chi_{A_1}\|}\right)},$$

so by Lemma 2.1, we have

$$\begin{aligned}\frac{1}{\|\chi_{A_1}\|} &\ge \varphi_1\left(\varphi_1^{-1}\left(\frac{1}{\|\chi_{A_1}\|}\right)\right) \ge \varphi_1\left(\frac{M_f}{2r}\right) = \varphi_1\left(\frac{R}{6r}\varphi_1^{-1}\left(\frac{1}{\|\chi_A\|}\right)\right) \\ &\ge \frac{R}{6r}\varphi_1\left(\varphi_1^{-1}\left(\frac{1}{\|\chi_A\|}\right)\right) = \frac{R}{6r}\frac{1}{\|\chi_A\|}.\end{aligned}$$

Hence

$$\|\chi_{A_1}\| \le \frac{6r}{R}\|\chi_A\|.$$

In the same way we can show that

$$\|\chi_{B_1}\| \le \frac{6r}{R}\|\chi_B\|.$$

Using

$$\lambda(B_1^{-1}) = \int_{B_1} \Delta(x^{-1}) d\lambda(x) \le S \int_{B_1} d\lambda(x) = S\lambda(B_1),$$

and Lemma 3.6, we have that

$$\begin{aligned}\lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) &\ge \lambda(A) - S\lambda(B_1) - \lambda(A_1) \\ &\ge \lambda(A) - S\xi_E^{-1}(\|\chi_{B_1}\|) - \xi_E^{-1}(\|\chi_{A_1}\|) \\ &\ge \lambda(A) - S\xi_E^{-1}\left(\frac{6r}{R}\|\chi_B\|\right) - \xi_E^{-1}\left(\frac{6r}{R}\|\chi_A\|\right),\end{aligned}$$

which in view of (12) gives us

$$\lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) \ge \frac{1}{2}\lambda(A). \tag{14}$$

Take $z \in A$ and consider the set

$$H_z = A_2 \cap z B_2^{-1}.$$

Then $H_z \subset A_2$ and $H_z^{-1} z \subset B_2$. Also by (14),

$$\begin{aligned}\lambda(H_z) &= \lambda(A_2 \cap z B_2^{-1}) = \lambda\big(z(z^{-1}A_2 \cap B_2^{-1})\big) = \lambda(z^{-1}A_2 \cap B_2^{-1}) = \\ &= \lambda(z^{-1}A_2) - \lambda(z^{-1}A_2 \setminus B_2^{-1}) \ge \lambda(A_2) - \lambda(B^{-1} \setminus B_2^{-1}) \qquad (15) \\ &= \lambda(A) - \lambda(A_1) - \lambda(B_1^{-1}) \overset{(14)}{\ge} \frac{1}{2}\lambda(A)\end{aligned}$$

Finally, for $z \in A$ we have by (8), (9), (10), (11), (15) and (13) (recall that we set $\alpha := 2$ if Case 2. holds)

$$\begin{aligned}|h| * |k|(z) &\geq \int_{H_z} |h(y)||k(y^{-1}z)|\, d\lambda(y) \geq \lambda(H_z)\frac{M_f M_g}{4} \\ &\overset{(15),(13)}{\geq} \frac{R^2}{72}\varphi_1^{-1}\left(\frac{1}{\|\chi_A\|}\right)\varphi_2^{-1}\left(\frac{1}{\|\chi_B\|}\right)\lambda(A) \\ &\geq \frac{R^2}{72}\varphi_1^{-1}\left(\frac{1}{\alpha\|\chi_A\|}\right)\varphi_2^{-1}\left(\frac{1}{\alpha\|\chi_A\|}\right)\lambda(A) \\ &\overset{(8),(9),(10),(11)}{>} n\alpha\varphi_3^{-1}\left(\frac{1}{\alpha\|\chi_A\|}\right)\end{aligned}$$

and hence

$$\begin{aligned}\left\|\varphi_3\left(\frac{|h| * |k|}{n}\right)\right\| &\geq \left\|\varphi_3\left(\frac{|h| * |k|}{n}\right)\chi_A\right\| \\ &\geq \left\|\varphi_3\left(\alpha\varphi_3^{-1}\left(\frac{1}{\alpha\|\chi_A\|}\right)\right)\chi_A\right\| \\ &\geq \left\|\varphi_3\left(\varphi_3^{-1}\left(\frac{1}{\|\chi_A\|}\right)\right)\chi_A\right\| = 1.\end{aligned}$$

So $\||h| * |k|\|_{E_{\varphi_3}} \geq n$ and $(h, k) \notin F_n$. □

A locally compact group G satisfies a condition $(\star\star)$ if

> for every compact neighbourhood V of the identity element of G there exist $\kappa > 1$ and a sequence $(U_n)_{n\in\mathbb{N}}$ contained in V with $\lim_{n\to\infty} \lambda(U_n) = 0$ and $\lambda(U_n^{-1}U_n) \leq \kappa\lambda(U_n)$.

In [17] there are examples of groups fulfilling the condition $(\star\star)$. For instance groups containing an open subgroup of the form $\mathbb{R}^a \times \mathbb{T}^b \times F$, where a, b are positive integers and F is a finite group, satisfy the condition $(\star\star)$.

Theorem 3.9. *Let G be a locally compact group that satisfies the condition $(\star\star)$, and let φ_i, $i = 1, 2, 3$ be Young functions such that*

$$\liminf_{x\to\infty} \frac{\varphi_1^{-1}(x)\varphi_2^{-1}(x)}{x\varphi_3^{-1}(x)} = \infty.$$

Then the set

$$F = \{(f, g) \in L^{\varphi_1}(G) \times L^{\varphi_2}(G) : |f| * |g| \in L^{\varphi_3}(G)\}.$$

is of first category in $L^{\varphi_1}(G) \times L^{\varphi_2}(G)$.

Remark 3.10. Note that the above result is a topological strengthening of [21, Theorem 14] with a slightly weaker condition.

Proof. The proof is similar to Theorem 3.7 with required modifications. In particular, we should define here $A := U_n$ and $B := U_n^{-1}U_n$ (observe that $U_n^{-1}U_n$ is symmetric) for sufficiently large n. □

In the sequel we generalize the main result of [19] from abelian locally compact groups to amenable ones in the context of Orlicz spaces. Let us recall that *amenability* of a locally compact group G can be equivalently define using the so-called Leptin condition, namely G is *amenable* whenever for every compact subset U of G and any $\epsilon > 0$ there exists a compact subset V in G of positive measure such that $\lambda(UV) < (1 + \epsilon)\lambda(V)$; see Theorem 7.9 and Proposition 7.11 in [35]. It should be pointed out that by Propositions 12.1 and 12.2 in [35] every compact and abelian locally compact group is amenable.

Theorem 3.11. *Let G be an amenable locally compact group, φ a Young function with $\lim_{t\to 0}\varphi(t)/t = 0$, $\varphi(b_\varphi) > 0$ and ψ be a Young function with $\psi(b_\psi) = \infty$. If G is non-compact, then the set*

$$F = \big\{(f, g) \in E_\varphi \times E_\psi : |f| * |g| \in E_\psi\big\}$$

is of first category in $E_\varphi \times E_\psi$.

Proof. For any natural number n, put

$$F_n = \big\{(f, g) \in E_\varphi \times E_\psi : \big\||f| * |g|\big\|_{E_\psi} < n\big\}$$

So, $F = \bigcup_{n\in\mathbb{N}} F_n$. We will show that for each $n \in \mathbb{N}$, F_n is nowhere dense. This will complete the proof.

Fix a natural number $n \in \mathbb{N}$. Let $(f, g) \in F_n$ and $R > 0$. Note that the assumption $\lim_{t\to 0}\varphi(t)/t = 0$ implies that $\lim_{t\to 0}\varphi^{-1}(t)/t = \infty$. Now by non-compactness of G, Lemma 3.4(3.) and (4), we can choose a large enough compact symmetric neighbourhood V of the identity of G such that $\|\chi_V\| \le (K + 1)\lambda(V)$, $\|\chi_V\|\varphi(b_\varphi) > 1$ and

$$\frac{R^2}{72}\frac{\|\chi_V\|}{K+1}\varphi^{-1}\left(\frac{1}{\|\chi_V\|}\right) > 2n. \tag{16}$$

Since G is amenable, there is a compact set C with $0 < \lambda(C) < \infty$ such that $\lambda(VC) < 2\lambda(C)$. Then, setting $B := VC$, we have by Lemma 3.4(1.),(2.),

$$\|\chi_B\| = \xi_E\big(\lambda(B)\big) \le \xi_E\big(2\lambda(C)\big) \le 2\xi_E(\lambda(C)) = 2\|\chi_C\|.$$

Now let $r < R/6$ be such that

$$\lambda(V) - S\xi_E^{-1}\left(\frac{6r}{R}\|\chi_B\|\right) - \xi_E^{-1}\left(\frac{6r}{R}\|\chi_V\|\right) \ge \frac{1}{2}\lambda(V), \tag{17}$$

where $S = \max\{\Delta(x^{-1}) : x \in B\}$ (the existence is guaranteed by Lemma 3.6(1.)).

Next, define M_f, M_g and functions $\widetilde{f}$ and $\widetilde{g}$ on G as in Theorem 3.7 (for sets V, B and functions φ, ψ, respectively). Then

$$\|f - \tilde{f}\|_{E_\varphi} = \|g - \tilde{g}\|_{E_\psi} = R/3.$$

Hence $B\big((\widetilde{f}, \widetilde{g}), r\big) \subseteq B\big((f, g), R\big)$ and it remains only to be proved that $B\big((\widetilde{f}, \widetilde{g}), r\big) \cap F_n = \emptyset$. Take $(h, k) \in B\big((\widetilde{f}, \widetilde{g}), r\big)$.

Put

$$V_1 := \big\{x \in V : |h(x)| < M_f/2\big\}, \quad B_1 := \big\{x \in B : |k(x)| < M_g/2\big\}.$$

Then, proceeding similarly as in the proof of Theorem 3.7, we get

$$\|\chi_{V_1}\| \le \frac{6r}{R}\|\chi_V\| \quad \text{and} \quad \|\chi_{B_1}\| \le \frac{6r}{R}\|\chi_B\|.$$

and by (17),

$$\lambda(V) - \lambda(V_1) - \lambda(B_1^{-1}) \ge \frac{1}{2}\lambda(V).$$

The above inequalities also show that sets $V_2 := V \setminus V_1$ and $B_2 := B \setminus B_1$ are of positive measure and hence non-empty. Now let $z \in C$ be an arbitrary element and define a set $H_z = V_2 \cap (zB_2^{-1})$.

It can be easily seen that we have $z^{-1}V \subseteq B^{-1}$, and thus $z^{-1}V_2 \subseteq B^{-1}$. Hence (see the proof of Theorem 3.7) $\lambda(H_z) \geq \frac{1}{2}\lambda(V)$. Also, $H_z \subseteq V_2$ and $H_z^{-1}z \subseteq B_2$. Finally, we conclude

$$|h| * |k|(z) \geq \int_{H_z} |h(y)||k(y^{-1}z)|d\lambda(y) \geq \frac{R^2}{72}\lambda(V)\varphi^{-1}\left(\frac{1}{\|\chi_V\|}\right)\psi^{-1}\left(\frac{1}{\|\chi_B\|}\right) \geq$$

$$\frac{R^2}{72}\lambda(V)\varphi^{-1}\left(\frac{1}{\|\chi_V\|}\right)\psi^{-1}\left(\frac{1}{2\|\chi_C\|}\right) \overset{(16)}{\geq} 2n\psi^{-1}\left(\frac{1}{2\|\chi_C\|}\right),$$

whence

$$\left\|\psi\left(\frac{|h| * |k|}{n}\right)\right\| \geq \left\|\psi\left(2\psi^{-1}\left(\frac{1}{2\|\chi_C\|}\right)\right)\chi_C\right\| \geq 1.$$

Therefore $\big\||h| * |k|\big\|_{E_\psi} \geq n$, which ends the proof. ☐

Remark 3.12. The amenability hypothesis cannot be dropped in Theorem 3.11 because in [27] R.A. Kunze and E.M. Stein show that the multiplication group of real matrices with determinant 1, $G = SL(2,\mathbb{R})$, satisfies $L^p(G) * L^2(G) \subset L^2(G)$ for $1 \leq p < 2$.

Theorem 3.13. *Assume that G is a non-compact but locally compact group and φ, ψ are Young functions with $\varphi(b_\varphi) > 0$, $\psi(b_\psi) > 0$, satisfying*

$$\liminf_{x\to 0} \frac{\varphi^{-1}(x)\psi^{-1}(x)}{x} = \infty.$$

(1.) If E is a real space, then for every compact set V with $\lambda(V) > 0$, the set

$$F_V = \big\{(f, g) \in E_\varphi \times E_\psi : f * g(x) \text{ is well defined in some point } x \in V\big\}$$

is of first category in $E_\varphi \times E_\psi$.
(2.) If E is complex, then for every compact set V with $\lambda(V) > 0$, the set

$$F'_V = \big\{(f, g) \in E_\varphi \times E_\psi : |f| * |g|(x) \text{ is finite at some point } x \in V\big\}$$

is of first category in $E_\varphi \times E_\psi$.

Proof. We prove (1.).
Notice first that $F_V = \bigcup_{n\in\mathbb{N}}(F_n^+ \cup F_n^-)$, where

$$F_n^+ = \left\{(f, g) \in E_\varphi \times E_\psi : \exists\, x \in V,\ \forall H \in \mathcal{M}_\lambda,\ \int_H f(y)g(y^{-1}x)d\lambda(y) < n\right\},$$

$$F_n^- = \left\{(f, g) \in E_\varphi \times E_\psi : \exists\, x \in V,\ \forall H \in \mathcal{M}_\lambda,\ \int_H f(y)g(y^{-1}x)d\lambda(y) > -n\right\}.$$

We will prove that for every $n \in \mathbb{N}$, F_n^+ is nowhere dense (the case of F_n^- is the same).

We can assume that V is symmetric and contains the identity element. Since G is not compact, we can choose a sequence $(a_n)_n \subset G$ such that for $n \neq m$, $a_nV^2 \cap a_mV^2 = \emptyset = Va_n^{-1} \cap Va_m^{-1}$, and $\Delta(a_n) \leq 1$.

Fix $n \in \mathbb{N}$. Take $(f, g) \in E_\varphi \times E_\psi$ and $R > 0$. Pick a natural number ℓ such that if $A = \bigcup_{k=1}^{\ell} Va_k^{-1}$ and $B = \bigcup_{k=1}^{\ell} a_kV^2$, then $\|\chi_A\| \leq (K + 1)\lambda(A)$, $\|\chi_A\|\varphi(b_\varphi) > 1$, $\|\chi_B\|\psi(b_\psi) > 1$ and

$$\frac{R^2}{72}\frac{\lambda(V)}{\lambda(V^2)}\frac{\|\chi_A\|}{K+1}\varphi^{-1}\left(\frac{1}{\|\chi_A\|}\right)\psi^{-1}\left(\frac{1}{\|\chi_A\|}\right) > n.$$

Observe that

$$\|\chi_{B^{-1}}\| = \xi_E\left(\lambda(B^{-1})\right) = \xi_E\left(\sum_{k=1}^{l}\lambda(V^2)\Delta(a_k^{-1})\right) = \xi_E\left(\frac{\lambda(V^2)}{\lambda(V)}\sum_{k=1}^{l}\lambda(V)\Delta(a_k^{-1})\right) \leq$$

$$\frac{\lambda(V^2)}{\lambda(V)}\xi_E\left(\sum_{k=1}^{l}\lambda(V)\Delta(a_k^{-1})\right) = \frac{\lambda(V^2)}{\lambda(V)}\xi_E(\lambda(A)) = \frac{\lambda(V^2)}{\lambda(V)}||\chi_A||$$

This, together with the fact that $\lambda(B) \leq \lambda(B^{-1})$ (because $\Delta(a_k^{-1}) \geq 1$), implies that

$$\frac{R^2}{72}\lambda(A)\varphi^{-1}\left(\frac{1}{\|\chi_A\|}\right)\psi^{-1}\left(\frac{1}{\|\chi_B\|}\right) > n.$$

Now let $r < R/6$ be such that

$$\lambda(A) - S\xi_E^{-1}\left(\frac{6r}{R}\|\chi_B\|\right) - \xi_E^{-1}\left(\frac{6r}{R}\|\chi_A\|\right) \geq \frac{1}{2}\lambda(A), \tag{18}$$

where $S := \sup_{x\in B}\Delta(x^{-1})$.

Define M_f, M_g and functions $\widetilde{f}$, $\widetilde{g}$ on G as in Theorem 3.7 (for sets A, B and functions φ, ψ, respectively). Then

$$\|f - \widetilde{f}\|_{E_\varphi} = \|g - \widetilde{g}\|_{E_\psi} = R/3 < R.$$

Hence $B\big((\widetilde{f},\widetilde{g}), r\big) \subset B\big((f,g), R\big)$. It remains to be shown that $B\big((\widetilde{f},\widetilde{g}), r\big) \cap F_n^+ = \emptyset$. For this reason, take $(h,k) \in B\big((\widetilde{f},\widetilde{g}), r\big)$. Set

$$A_1 = \{x \in A : |h(x)| \leq M_f/2\}, \quad B_1 = \{x \in B : |k(x)| \leq M_g/2\}$$

and $A_2 = A \setminus A_1$ and $B_2 = B \setminus B_1$.

Then

$$\|\chi_{A_1}\| \leq \frac{6r}{R}\|\chi_A\| \quad \text{and} \quad \|\chi_{B_1}\| \leq \frac{6r}{R}\|\chi_B\|.$$

Also, in view of (18) we get

$$\lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) \geq \frac{1}{2}\lambda(A) \tag{19}$$

Take $z \in V$ and consider the set

$$H = A_2 \cap zB_2^{-1}.$$

Then $H \subset A_2$ and $H^{-1}z \subset B_2$. Also, by (19),

$$\lambda(H) \geq \frac{1}{2}\lambda(A).$$

Finally, we have

$$\int_H h(y)k(y^{-1}z)\,d\lambda(y) \geq \frac{M_f M_g}{4}\lambda(H) = \frac{R^2}{72}\lambda(A)\varphi^{-1}\left(\frac{1}{\|\chi_A\|}\right)\psi^{-1}\left(\frac{1}{\|\chi_B\|}\right) > n.$$

Therefore $(h,k) \notin F_n^+$. Hence we proved (1.).

The proof of (2.) is essentially the same - we just have to consider sets

$$F_n' := \left\{(f,g) \in E_\varphi \times E_\psi : \exists\, x \in V,\ \forall H \in \mathcal{M}_\lambda,\ \int_H |f(y)||g(y^{-1}x)|d\lambda(y) < n\right\}.$$

□

Theorem 3.14. *Assume that G is a non-unimodular locally compact group and φ, ψ are Young functions with $\lim_{t\to 0}\varphi(t)/t = 0$, $\varphi(b_\varphi) > 0$ and $\psi(b_\psi) > 0$.*
(1.) If E is real, then for every compact set V with $\lambda(V) > 0$, the set

$$F_V = \big\{(f,g) \in E_\varphi \times E_\psi : f * g \text{ is well defined in some point } x \in V\big\}$$

is of first category in $E_\varphi \times E_\psi$.
(2.) If E is complex, then for every compact set V with $\lambda(V) > 0$, the set

$$F'_V = \big\{(f,g) \in E_\varphi \times E_\psi : |f| * |g|(x) \text{ is finite at some point } x \in V\big\}$$

is of first category in $E_\varphi \times E_\psi$.

Proof. Again, we will just prove (1.). Proceeding as in the previous proof, we will show that each set

$$F_n^+ = \left\{(f,g) \in E_\varphi \times E_\psi : \exists\, x \in V,\ \forall H \in \mathcal{M}_\lambda,\ \int_H f(y)g(y^{-1}x)d\lambda(y) < n\right\}.$$

is nowhere dense.

We can assume that V is symmetric, contains the identity element and $\frac{1}{\|\chi_{V^2}\|} < \psi(b_\psi)$. Fix a natural number $n \in \mathbb{N}$. Take $(f,g) \in E_\varphi \times E_\psi$ and $R > 0$. Since $\lim_{t\to 0}\varphi^{-1}(t)/t = \infty$, there is $\varphi(b_\varphi) > t_0 > 0$ such that for any $0 < t \le t_0$,

$$\frac{R^2}{72(K+1)}\frac{\varphi^{-1}(t)}{t}\psi^{-1}\left(\frac{1}{\|\chi_{V^2}\|}\right) > n. \tag{20}$$

Since G is not unimodular, there is $b \in G$ such that $\Delta(b) > (\sup_{x\in V}\Delta(x))^4 + 1$. This implies that for every distinct $m,k \in \mathbb{N}$, $Vb^m \cap Vb^k = \emptyset$ and $b^{-m}V^2 \cap b^{-k}V^2 = \emptyset$. Also, since $\xi_E(s) \to \infty$ as $s \to \infty$, we can take $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ we have $\|\chi_{Vb^k}\| \ge 1/t_0$ and $\|\chi_{Vb^k}\| \le (K+1)\lambda(Vb^k)$.

Define

$$A := Vb^{k_0}, \quad B := b^{-k_0}V^2.$$

Then $\|\chi_A\| \ge 1/t_0$ and $\lambda(B) = \lambda(V^2)$. So by (20) we have

$$\frac{R^2}{72}\varphi^{-1}\left(\frac{1}{\|\chi_A\|}\right)\psi^{-1}\left(\frac{1}{\|\chi_B\|}\right)\lambda(A) > n. \tag{21}$$

Now let $r < R/6$ be such that

$$\lambda(A) - S\xi_E^{-1}\left(\frac{6r}{R}\|\chi_B\|\right) - \xi_E^{-1}\left(\frac{6r}{R}\|\chi_A\|\right) \ge \frac{1}{2}\lambda(A), \tag{22}$$

where $S := \sup_{x\in B}\Delta(x^{-1})$.

Define M_f, M_g and functions $\widetilde{f}$ and $\widetilde{g}$ on G as in Theorem 3.7.

Then

$$\|f - \widetilde{f}\|_{E_\varphi} = \|g - \widetilde{g}\|_{E_\psi} = R/3 < R.$$

Hence $B\big((\widetilde{f},\widetilde{g}), r\big) \subset B\big((f,g), R\big)$. It remains only to be shown that $B\big((\widetilde{f},\widetilde{g}), r\big) \cap F_n^+ = \emptyset$.

Let $(h,k) \in B\big((\widetilde{f},\widetilde{g}), r\big)$. Put

$$A_1 = \{x \in A : |h(x)| \le M_f/2\}, \quad B_1 = \{x \in B : |k(x)| \le M_g/2\}$$

and $A_2 = A \setminus A_1$ and $B_2 = B \setminus B_1$.

Hence

$$\|\chi_{A_1}\| \le \frac{6r}{R}\|\chi_A\| \quad \text{and} \quad \|\chi_{B_1}\| \le \frac{6r}{R}\|\chi_B\|.$$

Also, in view of (22), we get

$$\lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) \geq \frac{1}{2}\lambda(A) \tag{23}$$

Take $z \in V$ and consider the set

$$H = A_2 \cap z B_2^{-1}.$$

Then $H \subset A_2$ and $H^{-1}z \subset B_2$. Also by (23),

$$\lambda(H) \geq \frac{1}{2}\lambda(A)$$

Finally,

$$\int_H h(y)k(y^{-1}z)\, d\lambda(y) \geq \frac{M_f M_g}{4}\lambda(H) = \frac{R^2}{72}\varphi^{-1}\left(\frac{1}{\|\chi_A\|}\right)\psi^{-1}\left(\frac{1}{\|\chi_B\|}\right)\lambda(A) \overset{(21)}{>} n,$$

so $(h,k) \notin F_n^+$. □

4. SUBSETS RELATED TO POINTWISE PRODUCT

In this section we study a similar problem for Calderón-Lozanowskiĭ spaces under pointwise multiplication. As it may be expected in this case we encounter less difficulties and we can make much less assumptions.

Let (Ω, Σ, μ) be a complete σ-finite measure space and E be a Banach ideal in $L^0(\Omega)$. Also let $\Sigma_+ = \{A \in \Sigma : 0 < \mu(A) < \infty\}$. By our assumptions, $\Sigma_+ \neq \emptyset$. Additionally, we assume that:

$$\text{for every } A \in \Sigma_+,\ \chi_A \in E.$$

We start this section with the following lemma.

Lemma 4.1. *Let E_{φ_1}, E_{φ_2} and E_{φ_3} be Calderón-Lozanowskiĭ spaces, $A \in \Sigma_+$, $s_1, s_2 > 0$, $C \in (0,1)$, and $f_i, g_i, \in E_{\varphi_i}$, $i = 1,2$, be such that $|g_i(x)| \geq 1$ for $i = 1,2$ and $x \in A$. Assume that*

$$\infty > \|\chi_A\| > \frac{1}{\varphi_1\left(\frac{1-C}{s_1}\right)} + \frac{1}{\varphi_2\left(\frac{1-C}{s_2}\right)}$$

and $\|(f_i - g_i)\chi_A\|_{E_{\varphi_i}} \leq s_i$ for $i = 1,2$. Then

$$\|f_1 \cdot f_2\|_{E_{\varphi_3}} \geq \frac{C^2}{\varphi_3^{-1}\left(\left(\|\chi_A\| - \frac{1}{\varphi_1\left(\frac{1-C}{s_1}\right)} - \frac{1}{\varphi_2\left(\frac{1-C}{s_2}\right)}\right)^{-1}\right)}.$$

Proof. Let $A_i = \{x \in A : |f_i(x)| < C|g_i(x)|\}$, $i = 1,2$. If $x \in A_i$, then

$$|f_i(x) - g_i(x)| \geq \big||g_i(x)| - |f_i(x)|\big| \geq (1-C)|g_i(x)| \geq (1-C).$$

We will prove that

$$\|\chi_{A_i}\| \leq \frac{1}{\varphi_i\left(\frac{1-C}{s_i}\right)}. \tag{24}$$

It holds true if $\|\chi_{A_i}\| = 0$. Assume that $\|\chi_{A_i}\| > 0$. Then

$$s_i \geq \|(f_i - g_i)\chi_{A_i}\|_{E_{\varphi_i}} \geq (1-C)\|\chi_{A_i}\|_{E_{\varphi_i}} = \frac{1-C}{\varphi_i^{-1}\left(\frac{1}{\|\chi_{A_i}\|}\right)}.$$

Thus

$$\frac{1}{\|\chi_{A_i}\|} \geq \varphi_i\left(\frac{1-C}{s_i}\right)$$

and consequently we obtain (24). Note that

$$\chi_A = \chi_{A\setminus(A_1\cup A_2)} + \chi_{A\cap(A_1\cup A_2)} = \chi_{A\setminus(A_1\cup A_2)} + \chi_{A_1\cup A_2} \leq \chi_{A\setminus(A_1\cup A_2)} + \chi_{A_1} + \chi_{A_2}.$$

Therefore

$$\|\chi_A\| \leq \|\chi_{A\setminus(A_1\cup A_2)} + \chi_{A_1} + \chi_{A_2}\| \leq \|\chi_{A\setminus(A_1\cup A_2)}\| + \|\chi_{A_1}\| + \|\chi_{A_2}\|$$

and consequently

$$\|\chi_{A\setminus(A_1\cup A_2)}\| \geq \|\chi_A\| - \|\chi_{A_1}\| - \|\chi_{A_2}\| \overset{(24)}{\geq} \|\chi_A\| - \frac{1}{\varphi_1\left(\frac{1-C}{s_1}\right)} - \frac{1}{\varphi_2\left(\frac{1-C}{s_2}\right)}.$$

Since $|f_i(x)| \geq C$ for $x \in A \setminus A_i$, we obtain

$$\begin{aligned}\|f_1 \cdot f_2\|_{E_{\varphi_3}} &\geq \|C^2 \chi_{A\setminus(A_1\cup A_2)}\|_{E_{\varphi_3}} \\ &= \frac{C^2}{\varphi_3^{-1}\left(\|\chi_{A\setminus(A_1\cup A_2)}\|^{-1}\right)} \\ &\geq \frac{C^2}{\varphi_3^{-1}\left(\left(\|\chi_A\| - \frac{1}{\varphi_1\left(\frac{1-C}{s_1}\right)} - \frac{1}{\varphi_2\left(\frac{1-C}{s_2}\right)}\right)^{-1}\right)}.\end{aligned}$$

□

The following theorem generalizes Theorem 2.4 in [4] and Theorem 8 in [40].

Theorem 4.2. *Let E_{φ_1}, E_{φ_2}, E_{φ_3} be Calderón-Lozanowskiǐ spaces with $\Sigma_+ \neq \emptyset$. Assume that $a_{\varphi_3} = 0$ and for any $\varepsilon > 0$ there is $A \in \Sigma_+$ such that $\frac{1}{\|\chi_A\|_E} \leq \min\{\varphi_1(b_{\varphi_1}), \varphi_2(b_{\varphi_2})\}$ and*

$$\text{(25)} \qquad \frac{\|\chi_A\|_{E_{\varphi_1}} \cdot \|\chi_A\|_{E_{\varphi_2}}}{\|\chi_A\|_{E_{\varphi_3}}} = \frac{\varphi_3^{-1}\left(\frac{1}{\|\chi_A\|}\right)}{\varphi_1^{-1}\left(\frac{1}{\|\chi_A\|}\right) \cdot \varphi_2^{-1}\left(\frac{1}{\|\chi_A\|}\right)} \leq \varepsilon.$$

Then the set $F = \{(f_1, f_2) \in E_{\varphi_1} \times E_{\varphi_2} : f_1 \cdot f_2 \in E_{\varphi_3}\}$ is $\sigma - \frac{2}{3}$-lower porous.

Proof. We will show that for any $n \in \mathbb{N}$, the set $F_n = \{(f_1, f_2) \in E_{\varphi_1} \times E_{\varphi_2} : \|f_1 \cdot f_2\|_{E_{\varphi_3}} < n\}$ is $\frac{2}{3}$-lower porous. Let $\delta \in (0, 1/3)$. Then $\frac{1-\delta}{\delta} > 2$, and therefore there is a $C \in (0, 1)$ with $\frac{(1-C)(1-\delta)}{\delta} > 2$. Let $k > 1$ be a real number such that

$$\text{(26)} \qquad \frac{(1-C)(1-\delta)}{\delta} = 2k.$$

Let $R > 0$ and $A \in \Sigma_+$ be such that $\frac{1}{\|\chi_A\|_E} \leq \min\{\varphi_1(b_{\varphi_1}), \varphi_2(b_{\varphi_2})\}$ and

$$\text{(27)} \qquad \frac{(C(1-\delta)R)^2}{\frac{nk}{k-1}} > \frac{\varphi_3^{-1}\left(\frac{1}{\|\chi_A\|}\right)}{\varphi_1^{-1}\left(\frac{1}{\|\chi_A\|}\right) \cdot \varphi_2^{-1}\left(\frac{1}{\|\chi_A\|}\right)}.$$

Put $t = \frac{1}{\|\chi_A\|}$. Since φ_i is convex, we have $\varphi_i(2k\varphi_i^{-1}(t)) \geq 2kt$ for $i = 1, 2$. Thus by (26)

$$\frac{1}{t} - \frac{1}{\varphi_1\left(\frac{(1-C)(1-\delta)}{\delta}\varphi_1^{-1}(t)\right)} - \frac{1}{\varphi_2\left(\frac{(1-C)(1-\delta)}{\delta}\varphi_2^{-1}(t)\right)} = \frac{1}{t} - \frac{1}{\varphi_1(2k\varphi_1^{-1}(t))} - \frac{1}{\varphi_2(2k\varphi_2^{-1}(t))} \tag{28}$$

$$\geq \frac{1}{t} - \frac{1}{kt} = \frac{k-1}{kt}. \tag{29}$$

Since φ_3^{-1} is concave and increasing, using (27), (28) and the fact that $\varphi_3^{-1}(0) = 0$ (which follows from $a_{\varphi_3} = 0$), we obtain

$$\frac{(C(1-\delta)R)^2}{n} \overset{(27)}{>} \frac{\frac{k}{k-1}\varphi_3^{-1}(t)}{\varphi_1^{-1}(t)\cdot\varphi_2^{-1}(t)} \geq \frac{\varphi_3^{-1}(\frac{tk}{k-1})}{\varphi_1^{-1}(t)\cdot\varphi_2^{-1}(t)}$$
$$\overset{(28)}{\geq} \frac{\varphi_3^{-1}\left(\left(\frac{1}{t} - \frac{1}{\varphi_1\left(\frac{(1-C)(1-\delta)}{\delta}\cdot\varphi_1^{-1}(t)\right)} - \frac{1}{\varphi_2\left(\frac{(1-C)(1-\delta)}{\delta}\cdot\varphi_2^{-1}(t)\right)}\right)^{-1}\right)}{\varphi_1^{-1}(t)\cdot\varphi_2^{-1}(t)}.$$

Hence

$$\frac{(C(1-\delta)R)^2\cdot\varphi_1^{-1}(t)\cdot\varphi_2^{-1}(t)}{\varphi_3^{-1}\left(\left(\frac{1}{t} - \frac{1}{\varphi_1\left(\frac{(1-C)(1-\delta)}{\delta}\cdot\varphi_1^{-1}(t)\right)} - \frac{1}{\varphi_2\left(\frac{(1-C)(1-\delta)}{\delta}\cdot\varphi_2^{-1}(t)\right)}\right)^{-1}\right)} > n. \tag{30}$$

Put $M_i = (1-\delta)R\varphi_i^{-1}(t)$, $i = 1, 2$. Then $\|M_i\chi_A\|_{E_{\varphi_i}} = \frac{M_i}{\varphi_i^{-1}(t)} = (1-\delta)R$. Let

$$\widetilde{f}_i(y) := \begin{cases} f_i(y) & y \notin A \\ f_i(y) + M_i & \mathrm{Re}\big(f_i(y)\big) \geq 0,\ y \in A \\ f_i(y) - M_i & \mathrm{Re}\big(f_i(y)\big) < 0,\ y \in A. \end{cases}$$

We have

$$\|f_i - \tilde{f}_i\|_{E_{\varphi_i}} = \|M_i\chi_A\|_{E_{\varphi_i}} = (1-\delta)R.$$

Hence $B((\tilde{f}_1, \tilde{f}_2), \delta R) \subset B((f_1, f_2), R)$. We will show that $B((\tilde{f}_1, \tilde{f}_2), \delta R) \cap F_n = \emptyset$. Let $(h_1, h_2) \in B((\tilde{f}_1, \tilde{f}_2), \delta R)$. Then

$$\delta R \geq \|h_i - \tilde{f}_i\|_{E_{\varphi_i}} \geq \|(h_i - \tilde{f}_i)\chi_A\|_{E_{\varphi_i}} = M_i\left\|\left(\frac{h_i}{M_i} - \frac{\tilde{f}_i}{M_i}\right)\chi_A\right\|_{E_{\varphi_i}}.$$

Note that $\left|\frac{\tilde{f}_i(x)}{M_i}\right| \geq 1$ for $x \in A$. Finally

$$\begin{aligned}
\|h_1 \cdot h_2\|_{E_{\varphi_3}} &= M_1 \cdot M_2 \left\| \frac{h_1}{M_1} \cdot \frac{h_2}{M_2} \right\|_{E_{\varphi_3}} \\
&\geq M_1 \cdot M_2 \cdot \frac{C^2}{\varphi_3^{-1}\left(\left(\|\chi_A\| - \frac{1}{\varphi_1\left(\frac{1-C}{\delta R} M_1\right)} - \frac{1}{\varphi_2\left(\frac{1-C}{\delta R} M_2\right)} \right)^{-1} \right)} \\
&= \frac{(C(1-\delta)R)^2 \varphi_1^{-1}(t) \cdot \varphi_2^{-1}(t)}{\varphi_3^{-1}\left(\left(\|\chi_A\| - \frac{1}{\varphi_1\left(\frac{(1-C)(1-\delta)}{\delta} \varphi_1^{-1}(t)\right)} - \frac{1}{\varphi_2\left(\frac{(1-C)(1-\delta)}{\delta} \varphi_1^{-1}(t)\right)} \right)^{-1} \right)} \\
&\overset{(30)}{>} n.
\end{aligned}$$

Where the first inequality follows from Lemma 4.1 used for A, C, $f_i := \frac{h_i}{M_i}$, $g_i := \frac{\tilde{f}_i}{M_i}$, $s_i := \frac{\delta R}{M_i}$. Therefore $(h_1, h_2) \notin F_n$. ☐

A Banach ideal space E is called *order continuous* if for every $f \in E$ and every sequence $\{A_n\}$ satisfying $A_n \downarrow \emptyset$ (that is $A_n \supset A_{n+1}$ and $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$), we have that $\|f\chi_{A_n}\|_E \downarrow 0$. It is easy to see that, in the setting of the previous section, the order continuity of E implies the continuity of the fundamental function ξ_E at 0.

Theorem 4.3. *Let φ_1, φ_2 and φ_3 be Young functions with $b_{\varphi_3} = \infty$ and E be a Banach ideal space with order continuous norm. If there exists $(h, k) \in E_{\varphi_1} \times E_{\varphi_2}$ such that $h \cdot k \notin E_{\varphi_3}$, then the set*

$$F := \{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : f \cdot g \in E_{\varphi_3}\}$$

is of the first category in $E_{\varphi_1} \times E_{\varphi_2}$.

Proof. For every $u, v > 0$, define $F_u^v := \{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : I_{\varphi_3}(vf \cdot g) < u\}$. Since $F = \bigcup_{u,v \in \mathbb{N}} F_u^v$, we only have to show that for every $u, v > 0$, F_u^v is nowhere dense. Fix $u, v > 0$ and let $(f, g) \in E_{\varphi_1} \times E_{\varphi_2}$ and $R > 0$. Set

$$\widetilde{f}(y) := \begin{cases} f(y) + \frac{R}{2\|h\|_{E_{\varphi_1}}} & \mathrm{Re}\big(f(y)\big) \geq 0 \\ f(y) - \frac{R}{2\|h\|_{E_{\varphi_1}}} & \mathrm{Re}\big(f(y)\big) < 0 \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} g(y) + \frac{R}{2\|k\|_{E_{\varphi_1}}} & \mathrm{Re}\big(g(y)\big) \geq 0 \\ g(y) - \frac{R}{2\|k\|_{E_{\varphi_1}}} & \mathrm{Re}\big(g(y)\big) < 0 \end{cases}$$

Then $\widetilde{f} \in E_{\varphi_1}$ and $\widetilde{g} \in E_{\varphi_2}$. Also, obviously $\|\widetilde{f} - f\|_{E_{\varphi_1}} = \frac{R}{2} = \|\widetilde{g} - g\|_{E_{\varphi_2}}$ and $\widetilde{f} \cdot \widetilde{g} \notin E_{\varphi_3}$. Hence $I_{\varphi_3}(\frac{v}{4}\widetilde{f} \cdot \widetilde{g}) = \infty$. Now for every $n \in \mathbb{N}$, put

$$A_n := \left\{x \in \Omega : n > |\widetilde{f}(x)| > n^{-1}\right\} \cap \left\{x \in \Omega : n > |\widetilde{g}(x)| > n^{-1}\right\}.$$

Since $\widetilde{f} \in E_{\varphi_1}$ and $\widetilde{g} \in E_{\varphi_2}$, we have that $\|\chi_{A_n}\|_E < \infty$ for each $n \in \mathbb{N}$. Also, if we put $A := \bigcup_{n \in \mathbb{N}} A_n = \{x \in \Omega : \infty > |\widetilde{f}(x) \cdot \widetilde{g}(x)| > 0\}$, then $I_{\varphi_3}(\frac{v}{4}\widetilde{f} \cdot \widetilde{g} \cdot \chi_A) = \infty$. Therefore, since every element of E has an order continuous norm and $b_{\varphi_3} = \infty$, there exists $m \in \mathbb{N}$ such that

$$\infty > I_{\varphi_3}\left(\frac{v}{4}\widetilde{f} \cdot \widetilde{g} \cdot \chi_{A_m}\right) > u.$$

By the order continuity of E, there exists $\delta > 0$ such that for every measurable subset $B \subset A_m$ with $\mu(B) \leq \delta$, we have

$$I_{\varphi_3}\left(\frac{v}{4}\widetilde{f} \cdot \widetilde{g} \cdot \chi_{A_m \setminus B}\right) > u. \tag{31}$$

Now choose $r \in (0, \frac{1}{2}R)$ such that for every $i = 1, 2$ we have

$$\varphi_i\left(\frac{1}{2mr}\right) > \frac{2}{\delta}. \tag{32}$$

Assume that $(d, l) \in E_{\varphi_1} \times E_{\varphi_2}$ is such that $(d, l) \in B\big((\widetilde{f}, \widetilde{g}), r\big)$. Put

$$C := \left\{x \in A_m : |d(x)| \leq \frac{1}{2}|\widetilde{f}(x)|\right\},$$
$$D := \left\{x \in A_m : |l(x)| \leq \frac{1}{2}|\widetilde{g}(x)|\right\}.$$

Then we have

$$r > \|\widetilde{f} - d\|_{E_{\varphi_1}} \geq \left\|\frac{1}{2}\widetilde{f}\chi_C\right\|_{E_{\varphi_1}} \geq \left\|\frac{1}{2m}\chi_C\right\|_{E_{\varphi_1}} = \frac{1}{2m\varphi_1^{-1}\left(\frac{1}{\|\chi_C\|_E}\right)},$$

provided that $\mu(C) > 0$. Hence by (32), $\|\chi_C\|_E < \frac{\delta}{2}$. Similarly, $\|\chi_D\|_E < \frac{\delta}{2}$. Finally, by (31) we get

$$I_{\varphi_3}(vd \cdot l) \geq I_{\varphi_3}(vd \cdot l \cdot \chi_{A_m \setminus (C \cup D)}) \geq I_{\varphi_3}\left(\frac{v}{4}\widetilde{f} \cdot \widetilde{g} \cdot \chi_{A_m \setminus (C \cup D)}\right) \overset{(31)}{>} u.$$

Hence $B\big((\widetilde{f}, \widetilde{g}), r\big) \subset B\big((f, g), R\big)$ and $B\big((\widetilde{f}, \widetilde{g}), r\big) \cap F_u^v = \emptyset$. This proves our claim. □

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References

[1] I. Akbarbaglu and S. Maghsoudi, An answer to a question on the convolution of functions, *Arch. Math. (Basel)*, **98** (2012), no. 6, 545–553.

[2] I. Akbarbaglu and S. Maghsoudi, On certain porous sets in the Orlicz space of a locally compact group, *Colloq. Math.* **129** (2012), no.1, 99–111.

[3] I. Akbarbaglu and S. Maghsoudi, Banach-Orlicz algebras on a locally compact group, *Mediterr. J. Math.* **10** (2013), no. 4, 1937–1947.

[4] I. Akbarbaglu and S. Maghsoudi, Porosity and products of Orlicz spaces, *Positivity*, **17** (2013), no.4, 1009–1020.

[5] I. Akbarbaglu and S. Maghsoudi, Porosity of certain subsets of Lebesgue spaces on locally compact groups, *Bull. Aust. Math. Soc.* **88** (2013), no. 1, 113–122.

[6] I. Akbarbaglu, S. Maghsoudi and J. B. Seoane-Sepúlveda, Porous sets and lineability of continuous functions on locally compact groups, *J. Math. Anal. Appl.* **406** (2013), no.1, 211–218.

[7] R. Arens, M. Goldberg and W. A. J. Luxemburg, Multiplicativity factors for Orlicz space function norms, *J. Math. Anal. Appl.* **177** (1993), no. 2, 386–411.

[8] C. Bennett, R. Sharpley, *Interpolation of operators*. Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988. xiv+469 pp.

[9] M. Balcerzak and A. Wachowicz, Some examples of meager sets in Banach spaces, *Real Anal. Exchange*, **26** (2000/01), no. 2, 877–884.

[10] A. P. Calderón, Intermediate spaces and interpolation, *Studia Math.* **24** (1964), 113–190.

[11] G. B. Folland, A course in abstract harmonic analysis. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995. x+276 pp.

[12] S. Głąb and F. Strobin, Porosity and the L^p-conjecture, *Arch. Math. (Basel)*, **95** (2010), no.6, 583–592.

[13] S. Głąb and F. Strobin, Dichotomies for L^p spaces, *J. Math. Anal. Appl.*, **368** (2010), no.1, 382–390.

[14] S. Głąb and F. Strobin, Dichotomies for $C_0(X)$ and $C_b(X)$ spaces, *Czechoslovak Math. J.* **63(136)** (2013), no.1, 91–105.

[15] S. Głąb, F. Strobin and C. W. Yang, Dichotomies for Lorentz spaces, *Cent. Eur. J. Math.* **11** (2013), no.7, 1228–1242.

[16] L. Grafakos, Classical Fourier analysis. Second edition, Graduate Texts in Mathematics, 249 Springer, New York, 2008.

[17] E. Hewitt and K. A. Ross, Abstract harmonic analysis I, II, Springer-Verlag, New York, 1963–1970.

[18] H. Hudzik, Orlicz spaces of essentially bounded functions and Banach-Orlicz algebras, *Arch. Math. (Basel)*, **44** (1985), no. 6, 535–538.

[19] H. Hudzik, A. Kamińska and J. Musielak, On some Banach algebras given by a modular, *Colloq. Math. Soc. J. Bolyai, Alfred Haar Memorial Conference, Budapest* **49** (1985), 445–463.

[20] J. Jachymski, A nonlinear Banach-Steinhaus theorem and some meager sets in Banach spaces, *Studia Math.* **170** (2005), no.3, 303–320.

[21] A. Kamińska and J. Musielak, On convolution operator in Orlicz spaces, *Rev. Mat. Complut.* **2** (1989), 157–178.

[22] P. Kolwicz and K. Leśnik, Topological and geometrical structure of Calderón-Lozanovskiĭ construction, *Math. Inequal. Appl.* **13** (2010), no. 1, 175–196.

[23] S. G. Krein, Y. Petunin, E. M. Semenov, *Interpolation of linear operators.* (Translated from the Russian by J. Szucs). Translations of Mathematical Monographs, 54. American Mathematical Society, Providence, R.I., 1982. xii+375 pp.

[24] P. Kolwicz, K. Leśnik and L. Maligranda, Pointwise multipliers of Calderón-Lozanovskiĭ spaces, *Math. Nachr.* **286** (2013), no. 8–9, 876–907.

[25] P. Kolwicz, K. Leśnik and L. Maligranda, Pointwise products of some Banach function spaces and factorization, *J. Funct. Anal.* **266** (2014), no. 2, 616–659.

[26] M. A. Krasnosel'skii and Ya. B. Rutickii, Convex functions and Orlicz spaces, Noordhoff, Netherlands, 1961.

[27] R. A. Kunze and E. M. Stein, Uniformly bounded representations and harmonic analysis of the 2×2 real unimodular group, *Amer. J. Math.*, **82** (1960), 1–62.

[28] J. Lindestrauss and L. Tzafriri, Classical Banach spaces II, Springer-Verlag, New York, 1979.

[29] G. Ja. Lozanovoskiĭ, On reflexive spaces generalizing the reflexive space of Orlicz, *Dokl. Akad. Nauk. SSSR*, **163** (1965), 573–576.

[30] G. Ja. Lozanovoskiĭ, Certain Banach lattices, IV, *Sibirsk. Math. Zh.*, **14** (1973), 140–155.

[31] L. Maligranda, Orlicz spaces and interpolation, Sem. Math. Univ. of Campinas, Campinas SP., Brazil, 1989.

[32] J. Musielak, Orlicz spaces and modular spaces. Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983.

[33] R. O'Neil, Fractional integration in Orlicz spaces, I, *Trans. Amer. Math. Soc.*, **115** (1965), 300–328.

[34] A. L. T. Paterson, Amenability, Mathematical surveys and monographs; no. 29, American Mathematical Society, Providence, RI, 1988.

[35] J. P. Pier, Amenable locally compact group, John Wiley and Sons, New York, 1984.

[36] M. Rajagopalan, L_p-conjecture for locally compact groups, I, *Trans. Amer. Math. Soc.*, **125** (1966), 216–222.

[37] M. M. Rao, Convolutions of vector fields, III, Amenability and spectral properties,. Real and stochastic analysis, 375–401, Trends Math., Birkhäuser, Boston, 2004.

[38] M. M. Rao, and Z. D. Ren, Theory of Orlicz spaces, Marcel Dekker, New York, 1991.

[39] S. Saeki, The L^p-conjecture and Young's inequality, *Illinois J. Math.*, **34** (1990), 614–627.

[40] F. Strobin, Dichotomies for Orlicz spaces, *Math. Slovaca*, to appear.

[41] B. Subramanian, On the inclusion $L^p(\mu) \subseteq L^q(\mu)$, *Amer. Math. Monthly*, **85** (1978), no. 6, 479–481.

[42] A. C. Zaanen, Riesz spaces, II, North-Holland Mathematical Library, 30. North-Holland Publishing Co., Amsterdam, 1983.

[43] L. Zajíček, On σ-porous sets in abstract spaces, *Abstr. Appl. Anal.*, **5** (2005), 509–534,

[44] W. Żelazko, On the algebras L_p of locally compact groups, *Colloq. Math.*, **8** (1961), 115–120.

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