TOPOLOGICAL SIZE OF SOME SUBSETS IN CERTAIN CALDERÓN-LOZANOWSKIĬ SPACES

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Abstract. For \( i = 1, 2, 3 \), let \( \varphi_i \) be Young functions, \((\Omega, \mu)\) a (topological) measure space, \( E \) an ideal of \( \mu \)-measurable real-valued functions defined on \( \Omega \) and \( E_{\varphi_i} \) be the corresponding Calderón-Lozanowskiĭ space. Our aim in this paper is to give, under mild conditions, several results on topological size (in the sense of Baire) of the sets 
\[
\{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : |f| \star |g| \in E_{\varphi_3}\}
\]
and 
\[
\{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : \exists x \in V, (f \star g)(x) \text{ is well defined}\}
\]
where \( \star \) denotes the convolution or pointwise product of functions and \( V \) a compact neighbourhood. Our results sharpen and unify the related results obtained in diverse areas during recent thirty years.

1. Introduction

Let \( X \) be a topological vector space of functions such that the product “\( \cdot \)” is defined on \( X \times X \). Then there arises the question whether the product \( f \cdot g \) does not belong to \( X \) for some pair \((f, g)\). In certain cases the solution of this problem is well-known. For example: when is the classical Lebesgue space \( L^p(X) \), for a measure space \((X, \mu)\), closed under pointwise product? An easy application of Hölder’s inequality and a result due essentially to B. Subramanian [41] give the answer. The problem whether the Orlicz space \( L^{\varphi}(X) \), defined on a measure space \((X, \mu)\), with pointwise product is a Banach algebra was studied in [7, 18].

One can consider a quantitative version of this question, namely – is the set of the pairs \((f, g)\) for which \( f \cdot g \) exists small in the sense of Baire category or porosity? The first result of this sort was proved by Balcerzak and Wachowicz in [9], who showed that the set \{ \((f, g) \in L^1[0, 1] \times L^1[0, 1] : f \cdot g \in L^1[0, 1]\} is meager in \( L^1[0, 1] \times L^1[0, 1] \). Jachymski generalized this in [20], by proving that the set of those pairs \((f, g)\) that the product \( f \cdot g \) is in \( L^p(X, \mu) \) is either the whole Cartesian product \( L^p(X, \mu) \times L^p(X, \mu) \), or it is a meager subset, where \( p \geq 1 \). Głąb and Strobin in [13] strengthened this by proving that the set of those pairs \((f, g)\) such that the product \( f \cdot g \) is in \( L^p(X, \mu) \) is either the whole Cartesian product \( L^p(X, \mu) \times L^q(X, \mu) \) or it is a \( \sigma \)-lower porous subset, where \( p \in (0, \infty) \). The similar dichotomies were proved for Orlicz spaces by Akbarbaglu and Maghsoudi in [4] (independently by Strobin in [40]), and for Lorentz spaces by Głąb, Strobin and Yang in [15]. We extend this result to the so-called Calderón-Lozanowskiĭ spaces.

A more subtle and difficult case is what is known as \( L^p \)-conjecture; i.e., is \( L^p(G) \), where \( G \) is a locally compact group with a left Haar measure, closed under convolution product for \( p > 1 \) only if \( G \) is compact? Originated independently by M. Rajagopalan [36] and Z. Żelazko [44], it was an open

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problem until resolved positively by S. Saeki [39] in 1990 after thirty years of struggling. Several authors were involved in proving the $L^p$-conjecture in special cases – it is briefly described in paper of Seaki who gave an extended list of references.

The study of quantitative version of $L^p$-conjecture was initiated by Głab and Strobin in [12], who proved that if $p, q > 1, 1/p + 1/q < 1$, $G$ is a locally compact but not compact group and $V \subset G$ is a compact neighbourhood of the identity, then the set of pairs $(f, g) \in L^p(G) \times L^q(G)$ such that $f * g(x)$ exists for some $x \in V$ is $\sigma$-lower porous. If $p \in (1, 2]$ and $G$ is unimodular, then by the Young inequality $L^p(G) * L^p(G) \subset L^{p/p-1}(G)$. Thus $f * g(x)$ is $\lambda$-a.e. finite for $f, g \in L^p(G)$. Akbarbaglu and Maghsoudi in [1] proved that if $G$ is non-unimodular, locally compact, non-compact group, $V \subset G$ is a compact neighbourhood of the identity, then the set of pairs $(f, g) \in L^p(G) \times L^q(G)$ such that $f * g(x)$ exists for some $x \in V$ is $\sigma$-lower porous. In [5] Akbarbaglu and Maghsoudi proved the same assertion for $p \in (0, 1)$ and $q \in (0, \infty]$. Moreover, in [5] the authors proved that if $G$ is nondiscrete, $1/p + 1/q > 1 + 1/r$ where $p, q \in [1, \infty), r \in [1, \infty], V \subset G$ is a compact neighbourhood of the identity, then the set of those pairs $(f, g) \in L^p(G) \times L^q(G)$ such that $f * g \in L^r(V, \lambda|_V)$ is $\sigma$-lower porous; which solves the old problem of Saeki [39].

Some authors considered the problem whether the Orlicz space $L^\phi$, defined on a locally compact group $G$ with a Haar measure, considered with the convolution product, is a Banach algebra; we will call it, after [3], a Banach-Orlicz algebra. More generally, suppose $\varphi_i, i = 1, 2, 3,$ are Young functions and $L^{\varphi_i}(G)$ are the corresponding Orlicz spaces; then there is a natural question to ask – what needs to be assumed on $\varphi_i$’s to get $f * g \in L^{\varphi_3}(G)$ provided $f \in L^{\varphi_1}(G)$ and $g \in L^{\varphi_2}(G)$? R. O’Neil in [33] examined the convolution operator in the context of Orlicz spaces and proved that if $G$ is a unimodular locally compact group, $\varphi_i, i = 1, 2, 3$ are Young functions satisfying $\varphi_1^{-1}(x) \varphi_2^{-1}(x) \leq x \varphi_3^{-1}(x)$ for $x \geq 0$, then for any $f_i \in L^{\varphi_i}(G)$, $i = 1, 2$, the convolution $f_1 * f_2$ belongs to $L^{\varphi_3}(G)$ and moreover $N_{\varphi_3}(f_1 * f_2) \leq 2N_{\varphi_1}(f_1) N_{\varphi_2}(f_2)$ where $N_{\varphi_i}$ is the Luxemburg norm on $L^{\varphi_i}$. In other words, the convolution map acts from $L^{\varphi_1}(G) \times L^{\varphi_2}(G)$ into $L^{\varphi_3}(G)$. Furthermore, Hudzik, Kamińska and Musielak in [19] undertook the $L^p$-conjecture for Orlicz spaces and proved that if $G$ is abelian then $L^\phi(G)$ is the Banach-Orlicz algebra if and only if $G$ is compact or $\lim_{t \rightarrow 0} \varphi(t)/t > 0$. Kamińska and Musielak in [21] extended this result and gave necessary and sufficient conditions, in terms of Young functions and group $G$, for $L^{\varphi_1}(G) * L^{\varphi_2}(G) \subseteq L^{\varphi_3}(G)$, where $G$ is abelian. Akbarbaglu and Maghsoudi proved in [3] that if $G$ is amenable, $\varphi$ is a $\Delta_2$-regular $N$-function, and $L^{\phi}(G)$ is a Banach-Orlicz algebra, then $G$ is compact, in particular $L^{\phi}(G) \subset L^1(G)$. Some other related results can be found in [37].

Akbarbaglu and Maghsoudi in [2] initiated the study of the quantitative $L^p$-conjecture for Orlicz spaces. They gave sufficient conditions on $\varphi_1$ and $\varphi_2$ so that whenever $G$ is non-unimodular, then the set of those pairs $(f, g) \in L^{\varphi_1}(G) \times L^{\varphi_2}(G)$ for which $f * g$ is well-defined at some point of the fixed neighbourhood of the identity, is $\sigma$-lower porous. The results from [2] shed light on sharpness and necessity of the relation $\varphi_1^{-1}(x) \varphi_2^{-1}(x) \leq x \varphi_3^{-1}(x)$ and unimodularity of the group $G$ for the inclusion $L^{\varphi_1}(G) * L^{\varphi_2}(G) \subseteq L^{\varphi_3}(G)$.

Recently, the analogue quantitative problems for pointwise products have been considered by Akbarbaglu and Maghsoudi, Głab and Strobin, and coauthors for $L^p$-spaces, Orlicz spaces, Lorentz spaces, and the space of continuous functions; see [5, 6, 14, 15, 40].
In this paper we improve of the results known for Orlicz spaces, and put them into the more general setting of the so-called Calderón-Lozanovskiĭ spaces $E_\varphi$ (where $E$ is a Banach ideal space and $\varphi$ a Young function) which are generalizations of Orlicz spaces, Orlicz-Lorentz spaces and contain the $p$-convexification $E^{(p)}$ $(1 \leq p < \infty)$ of $E$. These spaces were introduced by A. P. Calderón in [10] and developed by G. Ja. Lozanovskiĭ in [29, 30]. They play crucial role in the theory of interpolation. There is a lot of basic information on Calderón-Lozanovskiĭ spaces; see, for example, [22, 28, 31]. Also for a recent study on pointwise multipliers of Calderón-Lozanovskiĭ spaces see [24, 25]. Let us mention that the Calderón-Lozanovskiĭ spaces $E_\varphi$ we deal with here are special cases of general Calderón-Lozanovskiĭ spaces $\varphi (E, F)$ with $F = L^\infty$ in this case.

The paper is organized as follows: In Section 2 we give some necessary definitions and notations concerning Orlicz and Calderón-Lozanovskiĭ spaces. In Section 3 we consider two Calderón-Lozanovskiĭ spaces on a locally compact group with the Haar measure. In this section, various results concerning the size of pairs belonging to the product of the two Calderón-Lozanovskiĭ spaces for which the convolution multiplication exists or is in another Calderón-Lozanovskiĭ space are given. These results generalize and sharpen the results for $L^p$ and Orlicz spaces known in literature. Finally, in Section 4 the problem of pointwise product in Calderón-Lozanovskiĭ spaces is studied.

2. Preliminaries

We need to recall some necessary definitions from abstract harmonic analysis, and Orlicz and Calderón-Lozanovskiĭ spaces.

Throughout this paper, let $G$ denote a locally compact group with a fixed left Haar measure $\lambda$. Also, let $\mathcal{M}_\lambda$ denotes the $\sigma$-algebra of all Haar measurable sets and $L^0(G)$ denote the set of all (equivalence classes of) $\lambda$-measurable complex-valued functions on $G$. For measurable functions $f$ and $g$ on $G$, the convolution

$$(f \ast g)(x) = \int_G f(y)g(y^{-1}x) \, d\lambda(y)$$

is defined at each point $x \in G$ for which the function $y \mapsto f(y)g(y^{-1}x)$ is Haar integrable.

For each $x \in G$, the formula $\lambda_x(A) = \lambda(Ax)$ defines a left invariant regular Borel measure $\lambda_x$ on $G$. Thus, the uniqueness of the left Haar measure implies that for each $x \in G$ there is a positive number, say $\Delta(x)$, such that $\lambda_x = \Delta(x)\lambda$. The function $\Delta : G \to (0, \infty)$ defined in this way is called the modular function of $G$. It is clear that $\Delta$ is a continuous homomorphism on $G$. Moreover, for every measurable subset $A$ of $G$,

$$\lambda(A^{-1}) = \int_A \Delta(x^{-1})d\lambda(x);$$

for more details see [11] or [17]. The group $G$ is called unimodular whenever $\Delta = 1$. In this case, the left Haar measure and the right Haar measure coincide.

For $1 \leq p \leq \infty$, classical Lebesgue spaces on $G$ with respect to the Haar measure $\lambda$ will be denoted by $L^p(G)$ with the norm $\| \cdot \|_p$ defined in the usual way.

Orlicz spaces have been thoroughly investigated from the point of view of functional analysis. We refer to two excellent books [26] and [38] for more details. Also [31, 32, 42] provide some useful information on the subject.
A function \( \varphi : \mathbb{R} \to [0, \infty) \) is called a Young function if \( \varphi \) is convex, even, left continuous on \((0, \infty)\) with \( \varphi(0) = 0 \); we also assume that \( \varphi \) is neither identically zero nor identically infinite on \((0, \infty)\). As an elementary example of a Young function we can consider \( \varphi(x) = |x|^p/p \), for \( p > 1 \). For any Young function \( \varphi \) we define:

\[
a_\varphi = \sup\{ x \in \mathbb{R} : \varphi(x) = 0 \} \quad \text{and} \quad b_\varphi = \sup\{ x \in \mathbb{R} : \varphi(x) < \infty \}.
\]

A Young function \( \varphi \) is called finite if \( b_\varphi = \infty \). It is easy to observe that it is continuous on \([0, b_\varphi)\), nondecreasing on \([0, \infty)\) and strictly increasing on \([a_\varphi, b_\varphi)\).

We also need an inverse of a Young function \( \varphi \). For a Young function \( \varphi \) and \( y \in [0, \infty) \) let

\[
\varphi^{-1}(y) = \sup\{ x \geq 0 : \varphi(x) \leq y \}.
\]

The following lists basic properties of Young functions and their inverses. We skip an easy proof.

**Lemma 2.1.** In the above frame:

1. For all \( x \geq 0 \), \( \varphi(\varphi^{-1}(x)) \leq x \).
2. If \( \varphi(x) < \infty \) then \( x \leq \varphi^{-1}(\varphi(x)) \).
3. If \( x \in [0, \varphi(b_\varphi)] \), then \( \varphi(\varphi^{-1}(x)) = x \).
4. If \( x \in [a_\varphi, b_\varphi] \), then \( x = \varphi^{-1}(\varphi(x)) \).

Let \((\Omega, \Sigma, \mu)\) be a measure space and \( \varphi \) be a Young function. For each \( f \in L^0(\Omega) \), the space of all (equivalence classes of) \( \mu \)-measurable (complex-valued) functions defined on \( \Omega \), we define

\[
\varrho_\varphi(f) = \int_\Omega \varphi(|f(x)|) \, d\mu(x).
\]

Then the Orlicz space \( L^\varphi(\Omega) \) is defined by

\[
L^\varphi(\Omega) = \{ f \in L^0(\Omega) : \varrho_\varphi(af) < \infty, \text{ for some } a > 0 \}.
\]

The Orlicz space \( L^\varphi(\Omega) \) is a Banach space under the norm \( N_\varphi(\cdot) \), called Luxemburg norm, defined for \( f \in L^\varphi(\Omega) \) by

\[
N_\varphi(f) = \inf\{ k > 0 : \varrho_\varphi(f/k) \leq 1 \}.
\]

It is well-known that

1. \( N_\varphi(f) \leq 1 \) if and only if \( \varrho_\varphi(f) \leq 1 \),

and if \( 0 < \mu(F) < \infty \) then

2. \( N_\varphi(\chi_F) = \left[ \varphi^{-1}\left( \frac{1}{\mu(F)} \right) \right]^{-1} \);

see Corollary 3.4.7 in [38]. Here \( \chi_A \) denotes the characteristic function of a subset \( A \).

Now let us give definitions concerning our main object in this paper, namely Calderón-Lozanowski spaces which are defined in the similar way as Orlicz spaces, and they share common properties. Let \((\Omega, \Sigma, \mu)\) be a complete \( \sigma \)-finite measure space. A Banach space \( E = (E, \| \cdot \|_E) \) is called a Banach ideal space on \( \Omega \) if \( E \) is a linear subspace of \( L^0(\Omega) \) and satisfies the ideal property, that is if \( f \in E, g \in L^0(\Omega) \) and \( |g(t)| \leq |f(t)| \) for \( \mu \)-almost all \( t \in \Omega \), then \( g \in E \) and \( \|g\|_E \leq \|f\|_E \).
We distinguish two cases - real and complex ones. Namely, if $E$ consists only of real functions, then we consider $E$ as a real Banach space. In the second case, when $E$ consists also of complex functions, then we consider $E$ as a complex space.

**Remark 2.2.** Assume that $E$ is complex and let $E^\mathbb{R} = \{\text{Re}(f) : f \in E\}$. Then it is easy to see that $(E^\mathbb{R}, \| \cdot \|_E)$ is a (real) Banach ideal space and $E = \{f + ig : f, g \in E^\mathbb{R}\}$.

Note that most proofs presented later will automatically work for both cases (in some places we will write $\text{Re}(f)$, $\text{Re}(g)$ etc., but in the real case we just have $\text{Re}(f) = f$, $\text{Re}(g) = g$ etc.). We will emphasize the cases when some changes appear.

For a given Banach ideal space $E$ on $\Omega$ and a Young function $\varphi$, let $I_{\varphi} : L^0(\Omega) \to [0, \infty]$ be a semimodular defined by

$$I_{\varphi}(f) = \begin{cases} \|\varphi(|f|)\|_E & \text{if } \varphi(|f|) \in E, \\ \infty & \text{otherwise.} \end{cases}$$

The Calderón-Lozanowskiǐ space $E_{\varphi}$ is the space

$$E_{\varphi} = \{f \in L^0(\Omega) : I_{\varphi}(cf) < \infty \text{ for some } c > 0\}$$

with the Luxemburg norm

$$\|f\|_{E_{\varphi}} = \inf\{c > 0 : I_{\varphi}(f/c) \leq 1\}.$$ 

If $E = L^1(\Omega)$, then $E_{\varphi}$ is the Orlicz space $L^\varphi(\Omega)$ equipped with the Luxemburg norm. If $E$ is a Lorentz function (sequence) space, then $E_{\varphi}$ is the corresponding Orlicz-Lorentz function (sequence) space equipped with the Luxemburg norm. Also, if $\varphi(t) = t^p$, $1 \leq p < \infty$, then $E_{\varphi}$ is in this case the $p$-convexification $E^{(p)}$ of $E$ with the norm $\|f\|_{E^{(p)}} = \|f^p\|_E^{1/p}$. Finally, if $\varphi(t) = 0$ for $t \in [0, 1]$ and $\varphi(t) = \infty$ otherwise, then $E_{\varphi} = L^\infty(\Omega)$ and the corresponding norms are equal.

The following result links the Luxemburg norm $\| \cdot \|_{E_{\varphi}}$ with the original norm $\| \cdot \|_E$ on a Banach ideal space $E$.

**Lemma 2.3.** Let $(\Omega, \Sigma, \mu)$ be a measure space and $E$ be a Banach ideal space. If $A \subseteq \Omega$ is such that $0 < \mu(A) < \infty$, then (we assume here $1/0 = \infty$)

$$\|\chi_A\|_{E_{\varphi}} = \varphi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)^{-1}.$$ 

**Proof.** By the definition of Luxemburg norm we have

$$\|\chi_A\|_{E_{\varphi}} = \inf\{t > 0 : \|\varphi(\chi_A/t)\|_E \leq 1\}.$$ 

Then (we assume here that $1/\infty = 0$)

$$\|\varphi(\chi_A/t)\|_E \leq 1 \iff \varphi(1/t)\|\chi_A\|_E \leq 1 \iff \varphi(1/t) \leq 1/\|\chi_A\|_E$$

$$\iff 1/t \leq \varphi^{-1}(\|\chi_A\|_E) \iff t \geq \varphi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)^{-1}.$$
We finish this section with the notion of porosity from [12]; for more details see also [43]. Let $X$ be a metric space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. For a given number $0 < c \leq 1$, a subset $M$ of $X$ is called $c$-lower porous if
\[
\liminf_{R \to 0^+} \frac{\gamma(x, M, R)}{R} \geq \frac{c}{2}
\]
for all $x \in M$, where
\[
\gamma(x, M, R) = \sup \{r \geq 0 : \exists z \in X, B(z, r) \subseteq B(x, R) \setminus M\}.
\]
It is clear that $M$ is $c$-lower porous if and only if
\[
\forall x \in M, \forall \alpha \in (0, c/2), \exists r_0 > 0, \forall r \in (0, r_0), \exists z \in X, B(z, \alpha r) \subseteq B(x, r) \setminus M.
\]
A set is called $\sigma$-$c$-lower porous if it is a countable union of $c$-lower porous sets with the same constant $c > 0$. It is easy to see that a $\sigma$-$c$-lower porous set is meager, and the notion of $\sigma$-porosity is stronger than that of meagerness.

3. SUBSETS RELATED TO CONVOLUTION PRODUCT

Throughout this section let $G$ be a locally compact group with a fixed left Haar measure $\lambda$, and $E$ be a Banach ideal in $L^0(G)$ which satisfies additionally the conditions:

(a) if $f_n \not\to f$ for some nonnegative (real) functions $f_n \in E$, $n \in \mathbb{N}$ and $f \in L^0(G)$, then $\|f_n\|_E \to \|f\|_E$ provided $f \in E$, and $\|f_n\|_E \to \infty$ if $f \not\in E$.

(b) if $V \subset G$ and $\lambda(V) < \infty$, then $\chi_V \in E$;

(c) if $V \subset G$ and $\lambda(V) < \infty$, then there is $C_V < \infty$ such that $\int_V |f|d\lambda \leq C_V \|f\|_E$ for every $f \in E$.

Remark 3.1. If a Banach ideal $E$ consists of real functions, then the above conditions mean that $E$ is a Banach function space according to [8, Definitions 1.1.1 and 1.1.3]. Indeed, it is enough to take
\[
g(f) := \begin{cases} 
\|f\|_E, & f \in E \\
\infty, & f \not\in E
\end{cases}.
\]
Moreover, it is easy to see that if $E$ is complex, then in such case, the real space $E^\mathbb{R}$ (see Remark 2.2) is a Banach function space.

Nonnegative (real) functions $f, g \in L^0(G)$ are called equimeasurable, if for every $t \geq 0$,
\[
\lambda\{x \in G : |f(x)| > t\} = \lambda\{x \in G : |g(x)| > t\}.
\]
We additionally assume that:

(3) for every equimeasurable real functions $f, g \in E$, $\|f\|_E = \|g\|_E$

Remark 3.2. In the case when $E$ is real, condition (3) means that $E$ is so-called rearrangement-invariant space [8, Definition 2.4.1]. If $E$ is complex, then $E^\mathbb{R}$ is rearrangement-invariant space.

Observe that if $U, V \in \mathcal{M}_\lambda$ and $\lambda(V) = \lambda(U)$, then $\chi_V$ and $\chi_U$ are equimeasurable, hence $\|\chi_V\|_E = \|\chi_U\|_E$. Thus there exists a function $\xi_E : [0, \infty) \to [0, \infty)$ such that for every measurable $V \subset G$ with $\lambda(V) < \infty$
\[
\xi_E(\lambda(V)) := \|\chi_V\|_E.
\]
The function $\xi_E$ (which is uniquely determined just on the range of $\lambda$), is called the fundamental function of $E$ [8, Definition 2.5.1].

**Remark 3.3.** Note that the Haar measure is resonant in the sense of [8] (see [8, Theorem 2.2.7]), so the settings of [8, Definition 2.5.1] are satisfied. Moreover, using the properties of the Haar measure, we see that:

1. If $G$ is discrete and infinite, then for some $c > 0$, $\lambda(\{x\}) = c$ for all $x \in G$. Hence $\xi_E$ is determined on $\{kc : k \in \mathbb{N} \cup \{0\}\}$. Then we assume that $\xi_E$ is defined on the rest linearly.

2. If $G$ is not discrete and not compact, then for every $t \in [0, \infty)$ there exists $V \subset G$ such that $\lambda(V) = t$. In particular, $\xi_E$ is uniquely determined on the whole $[0, \infty)$.

3. If $G$ is compact but not discrete, then the image of $\lambda$ equals the interval $[0, \lambda(G)]$. Hence $\xi_E$ is uniquely determined on this interval. In this case we choose $\xi_E(t) := \xi_E(\lambda(G))$ for the rest $t > \lambda(G)$.

Now, [8, Corollary 2.5.3] give us also the following properties. Note that part (3.) follows from (2.) in below:

**Lemma 3.4.** The fundamental function $\xi_E$ satisfies the following:

1. $\xi_E$ is nondecreasing and continuous except (perhaps) at origin and $\xi_E(0) = 0$.
2. The mapping $t \to \frac{\xi_E(t)}{t}$ is nonincreasing.
3. For $K := \lim_{t \to \infty} \frac{\xi_E(t)}{t}$ and every $\varepsilon > 0$ there is $t_0$ such that $\xi_E(t) \leq (K + \varepsilon)t$ for every $t \geq t_0$.

Finally, we make another assumptions:

4. The fundamental function $\xi_E$ is continuous at 0, that is, $\lim_{t \to 0} \xi_E(t) = 0$

5. The fundamental function $\xi_E$ is unbounded, that is, $\lim_{t \to \infty} \xi_E(t) = \infty$.

**Remark 3.5.** In view of Remark 3.3, we see that if $G$ is discrete, then $\xi_E$ is continuous at 0. Also by [8, Corollary 2.5.5] we know that if $E$ is separable, then $\xi_E$ is continuous at 0. Let us remark that the condition (5) is also natural, for example it (and also the earlier ones) is satisfied when $G$ is not compact and $E = L^p(G)$ where $\|\cdot\|$ is the $L^p$-norm.

Now for every $s \in [0, \infty)$, set

$$\xi_E^{-1}(s) := \sup\{t \in [0, \infty) : \xi_E(t) \leq s\}.$$ 

We will use the following simple facts. Observe that the second part of (1.) in below follows from the condition (4) above.

**Lemma 3.6.**

1. The function $\xi_E^{-1} : [0, \infty) \to [0, \infty]$ is nondecreasing and $\lim_{t \to 0} \xi_E^{-1}(t) = 0$;
2. For every $t \in [0, \infty)$, $\xi_E^{-1}(\xi_E(t)) \geq t$.

Let us summarize our assumptions on the Banach ideal $E$: we assume that it is chosen such that (a) – (b), (3), (4) and (5) are satisfied (in particular, $G$ is assumed to be non-compact), which, in the language of [8], means that $E$ or $E^R$ is a rearrangement-invariant Banach function space such that its fundamental function $\xi_E$ is continuous at 0 and unbounded. In particular, $E$ can be taken
as many Lebesgue spaces $L^p$ (or even Orlicz spaces $L^\varphi$). We refer the interest reader to [8] and to [23] for further discussion on the topic.

Next, let $\varphi_1, \varphi_2$ be Young functions. We equip the product of Calderón-Lozanowski spaces $E_{\varphi_1} \times E_{\varphi_2}$ with the complete norm

$$
\|(f,g)\|_{\varphi_1,\varphi_2} = \max \{ \|f\|_{E_{\varphi_1}}, \|g\|_{E_{\varphi_2}} \} \quad (f \in E_{\varphi_1}, g \in E_{\varphi_2}).
$$

We commence with some definitions which we need to state our first result.

A locally compact group $G$ satisfies a condition $(\ast)$ if

for every compact neighbourhood $V$ of the identity element of $G$, there exist $\alpha > 1$ and a strictly increasing sequence $(p_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\lambda(V^{2p_n}) < \alpha \lambda(V^{p_n})$.

We say that $G$ has polynomial growth if for every compact neighbourhood $V$ of the identity element of $G$, there exists $d \in \mathbb{N}$ such that

$$
\limsup_{n \to \infty} \frac{\lambda(V^n)}{n^d} < \infty.
$$

Let us recall that according to [35, Proposition 16.28], every locally compact group $G$ having polynomial growth satisfies the condition $(\ast)$. Also, by [34, Corollary 6.18], locally compact abelian groups and nilpotent groups have polynomial growth. Moreover, by [34, Proposition 6.6, 6.9] every polynomially growing group is unimodular.

Let us remark that for any Young function $\varphi$, we write $f * g \in L^\varphi(G)$ to mean that $|f| * |g| < \infty \lambda$-almost everywhere, $f * g$ is Haar measurable on the set $\{ x \in G : |f * g|(x) < \infty \}$ and $N_\varphi(f * g) < \infty$.

**Theorem 3.7.** Let $G$ be a locally compact group that satisfies the condition $(\ast)$ and let $\varphi_i$, $i = 1, 2, 3$ be Young functions with $\varphi_i(b_{\varphi_i}) > 0$, for $i = 1, 2, 3$ and

$$
(6) \quad \liminf_{x \to 0} \frac{\varphi_1^{-1}(x) \varphi_2^{-1}(x)}{x \varphi_3^{-1}(x)} = \infty.
$$

If $G$ is non-compact, then the set

$$
F = \{ (f, g) \in E_{\varphi_1} \times E_{\varphi_2} : |f| * |g| \in E_{\varphi_3} \}
$$

is of first category in $E_{\varphi_1} \times E_{\varphi_2}$.

**Remark 3.8.** Let us remark that the above theorem is an extension of the first part of [21, Theorem 11], where it is shown that $F \neq E_{\varphi_1} \times E_{\varphi_2}$ in the case of Orlicz spaces $E_{\varphi_i}$ (observe that if $(f, g) \in (E_{\varphi_1} \times E_{\varphi_2}) \setminus F$, then also $(|f|, |g|) \in (E_{\varphi_1} \times E_{\varphi_2}) \setminus F$). In fact, it just a partial extension since we additionally assume that $\varphi_i(b_{\varphi_i}) < \infty$ for $i = 1, 2, 3$. However, note that [21, Theorem 11] is restricted to abelian groups but Theorem 3.7 can be applied for a wider class of groups, in particular, nilpotent groups.

**Proof.** We will write $\| \cdot \|$ instead of $\| \cdot \|_E$. For any natural number $n$, put

$$
F_n = \{ (f, g) \in E_{\varphi_1} \times E_{\varphi_2} : \| |f| * |g| \|_{E_{\varphi_3}} < n \}.
$$

So $F = \bigcup_{n \in \mathbb{N}} F_n$. The proof will be complete if we show that for each $n \in \mathbb{N}$, $F_n$ is nowhere dense.
Fix \( n \in \mathbb{N}, \, R > 0 \) and \((f, g) \in F_n\). There is \( 0 < t_n < \min\{\varphi_i(b_{\varphi_i}) : i = 1, 2, 3\} \) such that for \( 0 < t \leq t_n \),
\[
\frac{R^2}{288(K + 1)} \frac{\varphi_1^{-1}(t) \varphi_2^{-1}(t)}{t \varphi_3^{-1}(t)} > n.
\]
Since \( G \) is not compact, there are compact neighbourhoods of the identity element of \( G \) with as big but finite measure as needed. Hence, by Lemma 3.4(3.) and (5) we can find a compact symmetric neighbourhood \( V \) of the identity element of \( G \) such that:
\[
\frac{1}{\|\chi_V\|} < t_n \quad \text{and} \quad \|\chi_V\| \leq (K + 1)\lambda(V).
\]
Since \( G \) satisfies the condition (\( \ast \)), for such compact neighbourhood \( V \), there are \( \alpha > 1 \) and a sequence \((p_k)_{k \in \mathbb{N}}\) with \( \lambda(V^{2p_k}) < \alpha \lambda(V^{p_k}) \) (in particular, \( \lambda(V^{p_k}) < \infty \) for all \( k \in \mathbb{N} \)).

Now consider two cases:

Case 1. \( \lim_{k \to \infty} \lambda(V^{p_k}) = \infty \). By (6) we choose \( k_0 \in \mathbb{N} \) such that for every \( k \geq k_0 \) we have
\[
\frac{R^2}{72} \frac{\|\chi_{V^{p_k}}\|}{K + 1} \varphi_1^{-1} \left( \frac{1}{\alpha \|\chi_{V^{p_k}}\|} \right) \varphi_2^{-1} \left( \frac{1}{\alpha \|\chi_{V^{p_k}}\|} \right) > n \alpha \varphi_3^{-1} \left( \frac{1}{\alpha \|\chi_{V^{p_k}}\|} \right).
\]

Case 2. The increasing sequence \((\lambda(V^{p_k}))_{k \in \mathbb{N}}\) is bounded from above, so it is convergent. Hence, by (7), (8) and the fact that \( \|\chi_V\| \leq \|\chi_{V^{p_k}}\| \), there is \( k_0 \in \mathbb{N} \) such that for \( k \geq k_0 \), \( \lambda(V^{2p_k}) \leq 2\lambda(V^{p_k}) \) and
\[
\frac{R^2}{72} \frac{\|\chi_{V^{p_k}}\|}{K + 1} \varphi_1^{-1} \left( \frac{1}{2 \|\chi_{V^{p_k}}\|} \right) \varphi_2^{-1} \left( \frac{1}{2 \|\chi_{V^{p_k}}\|} \right) > 2n \varphi_3^{-1} \left( \frac{1}{2 \|\chi_{V^{p_k}}\|} \right).
\]

Now let \( A = V^{p_{k_0}} \) and \( B = V^{2p_{k_0}} \). We proceed with Case 1. and Case 2. together (if Case 2. holds, then in the next computations \( \alpha \) equals 2). By Lemma 3.4(1.), (2.), we have
\[
\|\chi_B\| = \xi E(\lambda(B)) \leq \xi E(\alpha \lambda(A)) \leq \alpha \xi E(\lambda(A)) = \alpha \|\chi_A\|.
\]
Now let \( r < R/6 \) be such that
\[
\lambda(A) - S \xi E^{-1} \left( \frac{6r}{R} \|\chi_B\| \right) - \xi E^{-1} \left( \frac{6r}{R} \|\chi_A\| \right) \geq \frac{1}{2} \lambda(A),
\]
where \( S := \sup_{x \in B} \Delta(x^{-1}) \) (such an \( r \) exists by Lemma 3.6(1.)).

Define
\[
M_f := \frac{R}{3 \|\chi_A\|_{E_{\varphi_1}}} = \frac{R}{3} \varphi_1^{-1} \left( \frac{1}{\|\chi_A\|} \right), \quad M_g := \frac{R}{3 \|\chi_B\|_{E_{\varphi_2}}} = \frac{R}{3} \varphi_2^{-1} \left( \frac{1}{\|\chi_B\|} \right)
\]
and
\[
\tilde{f}(y) := \begin{cases} f(y) & y \notin A \\ f(y) + M_f & \text{Re}(f(y)) \geq 0, \ y \in A \\ f(y) - M_f & \text{Re}(f(y)) < 0, \ y \in A, \end{cases}
\]
\[
\tilde{g}(y) := \begin{cases} g(y) & y \notin B \\ g(y) + M_g & \text{Re}(g(y)) \geq 0, \ y \in B \\ g(y) - M_g & \text{Re}(g(y)) < 0, \ y \in B. \end{cases}
\]
Then
\[
\|f - \tilde{f}\|_{E_{\varphi_1}} = \|M_f \chi_A\|_{E_{\varphi_1}} = R/3 < R,
\]
and similarly
\[ \|g - \tilde{g}\|_{E_{\varphi_2}} < \|Mg\chi_B\|_{E_{\varphi_2}} = R/3 < R. \]
Hence \( B((\tilde{f}, \tilde{g}), r) \subset B((f, g), R) \). It remains to show that \( B((\tilde{f}, \tilde{g}), r) \cap F_n = \emptyset \).

Let \((h, k) \in B((\tilde{f}, \tilde{g}), r)\). Put
\[ A_1 := \{ x \in A : |h(x)| \leq M_f/2 \}, \quad B_1 := \{ x \in B : |k(x)| \leq M_g/2 \} \]
and \( A_2 = A \setminus A_1 \) and \( B_2 = B \setminus B_1 \).

Then
\[ r \geq \|h - \tilde{f}\|_{E_{\varphi_1}} \geq \|(h - \tilde{f})\chi_A\|_{E_{\varphi_1}} \geq \frac{1}{2} M_f \|\chi_A\|_{E_{\varphi_1}} \]
so by Lemma 2.1, we have
\[
\frac{1}{\|\chi_A\|} \geq \varphi_1 \left( \varphi_1^{-1} \left( \frac{1}{\|\chi_A\|} \right) \right) \geq \varphi_1 \left( \frac{M_f}{2r} \right) = \frac{R}{6r} \varphi_1^{-1} \left( \frac{1}{\|\chi_A\|} \right).
\]
Hence
\[ \|\chi_A\| \leq \frac{6r}{R} \|\chi_A\|. \]
In the same way we can show that
\[ \|\chi_B\| \leq \frac{6r}{R} \|\chi_B\|. \]

Using
\[ \lambda(B_1^{-1}) = \int_{B_1} \Delta(x^{-1})d\lambda(x) \leq S \int_{B_1} d\lambda(x) = S\lambda(B_1), \]
and Lemma 3.6, we have that
\[
\lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) \geq \lambda(A) - S\lambda(B_1) - \lambda(A_1) \\
\geq \lambda(A) - S\xi_E^{-1}(\|\chi_B\|) - \xi_E^{-1}(\|\chi_A\|) \\
\geq \lambda(A) - S\xi_E^{-1} \left( \frac{6r}{R} \|\chi_B\| \right) - \xi_E^{-1} \left( \frac{6r}{R} \|\chi_A\| \right),
\]
which in view of (12) gives us
\[ \lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) \geq \frac{1}{2} \lambda(A). \quad (14) \]

Take \( z \in A \) and consider the set
\[ H_z = A_2 \cap zB_2^{-1}. \]
Then \( H_z \subset A_2 \) and \( H_z^{-1}z \subset B_2 \). Also by (14),
\[
\lambda(H_z) = \lambda(A_2 \cap zB_2^{-1}) = \lambda(z^{-1}A_2 \cap B_2^{-1}) = \lambda(z^{-1}A_2 \cap B_2^{-1}) = \\
= \lambda(z^{-1}A_2) - \lambda(z^{-1}A_2 \setminus B_2^{-1}) \geq \lambda(A_2) - \lambda(B_2^{-1}) \]
\[ \geq \lambda(A) - \lambda(A_1) - \lambda(B_1^{-1}) \geq \frac{1}{2} \lambda(A) \quad (14) \]
\[ \geq \frac{1}{2} \lambda(A). \]

Thus, we have shown that
\[ \lambda(K) \geq \frac{1}{2} \lambda(A). \]
Finally, for \( z \in A \) we have by (8), (9), (10), (11), (15) and (13) (recall that we set \( \alpha := 2 \) if Case 2. holds)

\[
|h| \ast |k|(z) \geq \int_{H_z} |h(y)||k(y^{-1}z)| \, d\lambda(y) \geq \lambda(H_z) \frac{M_f M_g}{4}
\]

\[
\geq R^2 \frac{T}{72} \varphi_1^{-1} \left( \frac{1}{\| \chi_A \|} \right) \varphi_2^{-1} \left( \frac{1}{\| \chi_B \|} \right) \lambda(A)
\]

\[
\geq R^2 \frac{T}{72} \varphi_1^{-1} \left( \frac{1}{\alpha \| \chi_A \|} \right) \varphi_2^{-1} \left( \frac{1}{\alpha \| \chi_A \|} \right) \lambda(A)
\]

\[
\geq n \alpha \varphi_3^{-1} \left( \frac{1}{\alpha \| \chi_A \|} \right)
\]

and hence

\[
\left\| \varphi_3 \left( \frac{|h| \ast |k|}{n} \right) \right\| \geq \left\| \varphi_3 \left( \frac{|h| \ast |k|}{n} \right) \chi_A \right\|
\]

\[
\geq \left\| \varphi_3 \left( \frac{\alpha \varphi_3^{-1} \left( \frac{1}{\alpha \| \chi_A \|} \right)}{\alpha \| \chi_A \|} \right) \chi_A \right\|
\]

\[
\geq \left\| \varphi_3 \left( \frac{\varphi_3^{-1} \left( \frac{1}{\alpha \| \chi_A \|} \right)}{\alpha \| \chi_A \|} \right) \chi_A \right\| = 1.
\]

So \( \| |h| \ast |k| \|_{F_{\varphi_3}} \geq n \) and \((h, k) \notin F_n\).

\[\square\]

A locally compact group \( G \) satisfies a condition (\( \ast \ast \)) if

- for every compact neighbourhood \( V \) of the identity element of \( G \) there exist \( \kappa > 1 \) and
- a sequence \((U_n)_{n \in \mathbb{N}}\) contained in \( V \) with \( \lim_{n \to \infty} \lambda(U_n) = 0 \) and \( \lambda(U_n^{-1} U_n) \leq \kappa \lambda(U_n) \).

In [17] there are examples of groups fulfilling the condition (\( \ast \ast \)). For instance groups containing an open subgroup of the form \( \mathbb{R}^a \times \mathbb{T}^b \times F \), where \( a, b \) are positive integers and \( F \) is a finite group, satisfy the condition (\( \ast \ast \)).

**Theorem 3.9.** Let \( G \) be a locally compact group that satisfies the condition (\( \ast \ast \)), and let \( \varphi_i, i = 1, 2, 3 \) be Young functions such that

\[
\liminf_{x \to \infty} \frac{\varphi_1^{-1}(x) \varphi_2^{-1}(x)}{x \varphi_3^{-1}(x)} = \infty.
\]

Then the set

\[
F = \{ (f, g) \in L^{\varphi_1}(G) \times L^{\varphi_2}(G) : |f| \ast |g| \in L^{\varphi_3}(G) \}
\]

is of first category in \( L^{\varphi_1}(G) \times L^{\varphi_2}(G) \).

**Remark 3.10.** Note that the above result is a topological strengthening of [21, Theorem 14] with a slightly weaker condition.

**Proof.** The proof is similar to Theorem 3.7 with required modifications. In particular, we should define here \( A := U_n \) and \( B := U_n^{-1} U_n \) (observe that \( U_n^{-1} U_n \) is symmetric) for sufficiently large \( n \).

In the sequel we generalize the main result of [19] from abelian locally compact groups to amenable ones in the context of Orlicz spaces. Let us recall that amenability of a locally compact group \( G \) can be equivalently define using the so-called Leptin condition, namely \( G \) is amenable whenever for every compact subset \( U \) of \( G \) and any \( \epsilon > 0 \) there exists a compact subset \( V \) in \( G \) of positive measure such that \( \lambda(U V) < (1 + \epsilon) \lambda(V) \); see Theorem 7.9 and Proposition 7.11 in [35]. It should be pointed
out that by Propositions 12.1 and 12.2 in [35] every compact and abelian locally compact group is amenable.

**Theorem 3.11.** Let $G$ be an amenable locally compact group, $\varphi$ a Young function with $\lim_{t \to 0} \varphi(t)/t = 0$, $\varphi(b_\varphi) > 0$ and $\psi$ be a Young function with $\psi(b_\psi) = \infty$. If $G$ is non-compact, then the set

$$F = \{(f, g) \in E_\varphi \times E_\psi : \|f \ast g\|_{E_\psi} \in E_\psi\}$$

is of first category in $E_\varphi \times E_\psi$.

**Proof.** For any natural number $n$, put

$$F_n = \{(f, g) \in E_\varphi \times E_\psi : \|f \ast g\|_{E_\psi} < n\}$$

So, $F = \bigcup_{n \in \mathbb{N}} F_n$. We will show that for each $n \in \mathbb{N}$, $F_n$ is nowhere dense. This will complete the proof.

Fix a natural number $n \in \mathbb{N}$. Let $(f, g) \in F_n$ and $R > 0$. Note that the assumption $\lim_{t \to 0} \varphi(t)/t = 0$ implies that $\lim_{t \to 0} \varphi^{-1}(t)/t = \infty$. Now by non-compactness of $G$, Lemma 3.4(3.) and (4), we can choose a large enough compact symmetric neighbourhood $V$ of the identity of $G$ such that $\|\chi_V\| \leq (K + 1)\lambda(V)$, $\|\chi_V\|\varphi(b_\varphi) > 1$ and

$$\frac{R^2}{72} \frac{\|\chi_V\|\varphi^{-1}(1/\|\chi_V\|)}{K + 1} > 2n.$$

Since $G$ is amenable, there is a compact set $C$ with $0 < \lambda(C) < \infty$ such that $\lambda(V C) < 2\lambda(C)$. Then, setting $B := V C$, we have by Lemma 3.4(1.),(2.),

$$\|\chi_B\| = \xi_E(\lambda(B)) \leq \xi_E(2\lambda(C)) \leq 2\xi_E(\lambda(C)) = 2\|\chi_C\|.$$

Now let $r < R/6$ be such that

$$\lambda(V) - S\xi_E^{-1}\left(\frac{6r}{R}\|\chi_B\|\right) - \xi_E^{-1}\left(\frac{6r}{R}\|\chi_V\|\right) \geq \frac{1}{2}\lambda(V),$$

where $S = \max\{\Delta(x^{-1}) : x \in B\}$ (the existence is guaranteed by Lemma 3.6(1.)).

Next, define $M_f$, $M_g$ and functions $\tilde{f}$ and $\tilde{g}$ on $G$ as in Theorem 3.7 (for sets $V$, $B$ and functions $\varphi$, $\psi$, respectively). Then

$$\|f - \tilde{f}\|_{E_\varphi} = \|g - \tilde{g}\|_{E_\psi} = R/3.$$  

Hence $B((\tilde{f}, \tilde{g}), r) \subseteq B((f, g), R)$ and it remains only to be proved that $B((\tilde{f}, \tilde{g}), r) \cap F_n = \emptyset$. Take $(h, k) \in B((\tilde{f}, \tilde{g}), r)$.

Put

$$V_1 := \{x \in V : |h(x)| < M_f/2\}, \quad B_1 := \{x \in B : |k(x)| < M_g/2\}.$$

Then, proceeding similarly as in the proof of Theorem 3.7, we get

$$\|\chi_{V_1}\| \leq \frac{6r}{R}\|\chi_V\| \quad \text{and} \quad \|\chi_{B_1}\| \leq \frac{6r}{R}\|\chi_B\|.$$  

and by (17),

$$\lambda(V) - \lambda(V_1) - \lambda(B_1^{-1}) \geq \frac{1}{2}\lambda(V).$$

The above inequalities also show that sets $V_2 := V \setminus V_1$ and $B_2 := B \setminus B_1$ are of positive measure and hence non-empty. Now let $z \in C$ be an arbitrary element and define a set $H_z = V_2 \cap (zB_2^{-1})$. 


It can be easily seen that we have $z^{-1}V \subseteq B^{-1}$, and thus $z^{-1}V_2 \subseteq B^{-1}$. Hence (see the proof of Theorem 3.7) $\lambda(H_z) \geq \frac{1}{2} \lambda(V)$. Also, $H_z \subseteq V_2$ and $H_z^{-1}z \subseteq B_2$. Finally, we conclude

$$|h| * |k|(z) \geq \int_{H_z} |h(y)||k(y^{-1}z)|d\lambda(y) \geq \frac{R^2}{72} \lambda(V) \phi^{-1} \left( \frac{1}{\|\chi_V\|} \right) \psi^{-1} \left( \frac{1}{\|\chi_B\|} \right) \geq$$

$$\frac{R^2}{72} \lambda(V) \phi^{-1} \left( \frac{1}{\|\chi_V\|} \right) \psi^{-1} \left( \frac{1}{2\|\chi_C\|} \right) \geq 2n\psi^{-1} \left( \frac{1}{2\|\chi_C\|} \right),$$

whence

$$\left\| \psi \left( \frac{|h| * |k|}{n} \right) \right\| \geq \left\| \psi \left( 2\psi^{-1} \left( \frac{1}{2\|\chi_C\|} \right) \right) \chi_C \right\| \geq 1.$$

Therefore $\left\| |h| * |k| \right\|_{E_{\psi}} \geq n$, which ends the proof. \(\square\)

**Remark 3.12.** The amenability hypothesis cannot be dropped in Theorem 3.11 because in [27] R.A. Kunze and E.M. Stein show that the multiplication group of real matrices with determinant 1, $G = SL(2, \mathbb{R})$, satisfies $L^p(G) * L^2(G) \subset L^2(G)$ for $1 \leq p < 2$.

**Theorem 3.13.** Assume that $G$ is a non-compact but locally compact group and $\phi, \psi$ are Young functions with $\phi(b_\phi) > 0$, $\psi(b_\psi) > 0$, satisfying

$$\liminf_{x \to 0} \frac{\phi^{-1}(x)\psi^{-1}(x)}{x} = \infty.$$

(1.) If $E$ is a real space, then for every compact set $V$ with $\lambda(V) > 0$, the set

$$F_V = \{(f, g) \in E_\phi \times E_\psi : f \otslash g(x) \text{ is well defined in some point } x \in V\}$$

is of first category in $E_\phi \times E_\psi$.

(2.) If $E$ is complex, then for every compact set $V$ with $\lambda(V) > 0$, the set

$$F'_V = \{(f, g) \in E_\phi \times E_\psi : |f| \otslash |g|(x) \text{ is finite at some point } x \in V\}$$

is of first category in $E_\phi \times E_\psi$.

**Proof.** We prove (1.).

Notice first that $F_V = \bigcup_{n \in \mathbb{N}} (F^+_n \cup F^-_n)$, where

$$F^+_n = \{(f, g) \in E_\phi \times E_\psi : \exists x \in V, \forall H \in \mathcal{M}_\lambda, \int_H f(y)g(y^{-1}x)d\lambda(y) < n\},$$

$$F^-_n = \{(f, g) \in E_\phi \times E_\psi : \exists x \in V, \forall H \in \mathcal{M}_\lambda, \int_H f(y)g(y^{-1}x)d\lambda(y) > -n\}.$$
Observe that
\[ \|\chi_{B^{-1}}\| = \xi(E) (\lambda(B^{-1})) = \xi_E \left( \sum_{k=1}^{l} \lambda(V^2) \Delta(a_k^{-1}) \right) = \xi_E \left( \frac{\lambda(V^2)}{\lambda(V)} \sum_{k=1}^{l} \lambda(V) \Delta(a_k^{-1}) \right) \leq \lambda(V^2) \xi_E \left( \sum_{k=1}^{l} \lambda(V) \Delta(a_k^{-1}) \right) = \lambda(V^2) \xi_E(\lambda(A)) = \frac{\lambda(V^2)}{\lambda(V)} \|\chi_A\| \]

This, together with the fact that \( \lambda(B) \leq \lambda(B^{-1}) \) (because \( \Delta(a_k^{-1}) \geq 1 \)), implies that
\[ \frac{R^2}{72} \lambda(A) \varphi^{-1} \left( \frac{1}{\|\chi_A\|} \right) \psi^{-1} \left( \frac{1}{\|\chi_B\|} \right) > n. \]

Now let \( r < R/6 \) be such that
\[ \lambda(A) - S \xi_E^{-1} \left( \frac{6r}{R} \|\chi_B\| \right) - \xi_E^{-1} \left( \frac{6r}{R} \|\chi_A\| \right) \geq \frac{1}{2} \lambda(A), \]

where \( S := \sup_{x \in B} \Delta(x^{-1}). \)

Define \( M_f, M_g \) and functions \( \tilde{f}, \tilde{g} \) on \( G \) as in Theorem 3.7 (for sets \( A, B \) and functions \( \varphi, \psi \), respectively). Then
\[ \|f - \tilde{f}\|_{E_{\varphi}} = \|g - \tilde{g}\|_{E_{\psi}} = R/3 < R. \]

Hence \( B((\tilde{f}, \tilde{g}), r) \subset B((f, g), R) \). It remains to be shown that \( B((\tilde{f}, \tilde{g}), r) \cap F_n^+ = \emptyset \). For this reason, take \( (h, k) \in B((\tilde{f}, \tilde{g}), r) \). Set
\[ A_1 = \{x \in A : \|h(x)\| \leq M_f/2\}, \quad B_1 = \{x \in B : \|k(x)\| \leq M_g/2\} \]

and \( A_2 = A \setminus A_1 \) and \( B_2 = B \setminus B_1 \).

Then
\[ \|\chi_{A_1}\| \leq \frac{6r}{R} \|\chi_A\| \quad \text{and} \quad \|\chi_{B_1}\| \leq \frac{6r}{R} \|\chi_B\|. \]

Also, in view of (18) we get
\[ \lambda(A) - \lambda(B_1^{-1}) - \lambda(A_1) \geq \frac{1}{2} \lambda(A). \]

Take \( z \in V \) and consider the set
\[ H = A_2 \cap zB_2^{-1}. \]

Then \( H \subset A_2 \) and \( H^{-1}z \subset B_2 \). Also, by (19),
\[ \lambda(H) \geq \frac{1}{2} \lambda(A). \]

Finally, we have
\[ \int_H h(y)k(y^{-1}z) \ d\lambda(y) \geq \frac{M_f M_g}{4} \lambda(H) = \frac{R^2}{72} \lambda(A) \varphi^{-1} \left( \frac{1}{\|\chi_A\|} \right) \psi^{-1} \left( \frac{1}{\|\chi_B\|} \right) > n. \]

Therefore \( (h, k) \notin F_n^+ \). Hence we proved (1).

The proof of (2.) is essentially the same - we just have to consider sets
\[ F'_n := \{(f, g) \in E_{\varphi} \times E_{\psi} : \exists x \in V, \forall H \in \mathcal{M}, \int_H |f(y)||g(y^{-1}x)|d\lambda(y) < n\}. \]

\[ \square \]
Theorem 3.14. Assume that \( G \) is a non-unimodular locally compact group and \( \varphi, \psi \) are Young functions with \( \lim_{t \to 0} \varphi(t)/t = 0 \), \( \varphi(b_\varphi) > 0 \) and \( \psi(b_\psi) > 0 \).

(1.) If \( E \) is real, then for every compact set \( V \) with \( \lambda(V) > 0 \), the set

\[
F_V = \{(f, g) \in E_\varphi \times E_\psi : f * g \text{ is well defined in some point } x \in V\}
\]
is of first category in \( E_\varphi \times E_\psi \).

(2.) If \( E \) is complex, then for every compact set \( V \) with \( \lambda(V) > 0 \), the set

\[
F'_V = \{(f, g) \in E_\varphi \times E_\psi : |f| \cdot |g|(x) \text{ is finite at some point } x \in V\}
\]
is of first category in \( E_\varphi \times E_\psi \).

Proof. Again, we will just prove (1.). Proceeding as in the previous proof, we will show that each set

\[
F^+_n = \{(f, g) \in E_\varphi \times E_\psi : \exists x \in V, \forall H \in \mathcal{M}_\lambda, \int_H f(y)g(y^{-1}x)d\lambda(y) < n\}
\]
is nowhere dense.

We can assume that \( V \) is symmetric, contains the identity element and \( \frac{1}{\|\chi_{V^2}\|} < \psi(b_\psi) \). Fix a natural number \( n \in \mathbb{N} \). Take \( (f, g) \in E_\varphi \times E_\psi \) and \( R > 0 \). Since \( \lim_{t \to 0} \varphi^{-1}(t)/t = \infty \), there is \( \varphi(b_\varphi) > t_0 > 0 \) such that for any \( 0 < t \leq t_0 \),

\[
(20) \quad \frac{R^2}{72(K+1)} \varphi^{-1}(t) \psi^{-1}\left(\frac{1}{\|\chi_{V^2}\|}\right) > n.
\]

Since \( G \) is not unimodular, there is \( b \in G \) such that \( \Delta(b) > (\sup_{x \in V} \Delta(x))^4 + 1 \). This implies that for every distinct \( m, k \in \mathbb{N} \), \( Vb^m \cap Vb^k = \emptyset \) and \( b^{-m}V^2 \cap b^{-k}V^2 = \emptyset \). Also, since \( \xi_E(s) \to \infty \) as \( s \to \infty \), we can take \( k_0 \in \mathbb{N} \) such that for every \( k \geq k_0 \) we have \( \|\chi_{Vb^k}\| \geq 1/t_0 \) and \( \|\chi_{Vb^k}\| \leq (K+1)\lambda(Vb^k) \).

Define

\[
A := Vb^{k_0}, \quad B := b^{-k_0}V^2.
\]

Then \( \|\chi_A\| \geq 1/t_0 \) and \( \lambda(B) = \lambda(V^2) \). So by (20) we have

\[
(21) \quad \frac{R^2}{72} \varphi^{-1}\left(\frac{1}{\|\chi_A\|}\right) \psi^{-1}\left(\frac{1}{\|\chi_B\|}\right) > n.
\]

Now let \( r < R/6 \) be such that

\[
(22) \quad \lambda(A) - S\xi^{-1}_E\left(\frac{6r}{R}\|\chi_B\|\right) - \xi^{-1}_E\left(\frac{6r}{R}\|\chi_A\|\right) \geq \frac{1}{2} \lambda(A),
\]

where \( S := \sup_{x \in B} \Delta(x^{-1}) \).

Define \( M_f, M_g \) and functions \( \tilde{f} \) and \( \tilde{g} \) on \( G \) as in Theorem 3.7.

Then

\[
\|f - \tilde{f}\|_{E_\varphi} = \|g - \tilde{g}\|_{E_\psi} = R/3 < R.
\]

Hence \( B((\tilde{f}, \tilde{g}), r) \subset B((f, g), R) \). It remains only to be shown that \( B((\tilde{f}, \tilde{g}), r) \cap F^+_n = \emptyset \).

Let \( (h, k) \in B((\tilde{f}, \tilde{g}), r) \). Put

\[
A_1 = \{x \in A : |h(x)| \leq M_f/2\}, \quad B_1 = \{x \in B : |k(x)| \leq M_g/2\}
\]

and \( A_2 = A \setminus A_1 \) and \( B_2 = B \setminus B_1 \).

Hence

\[
\|\chi_{A_1}\| \leq \frac{6r}{R}\|\chi_A\| \quad \text{and} \quad \|\chi_{B_1}\| \leq \frac{6r}{R}\|\chi_B\|.
\]
Also, in view of (22), we get
\begin{equation}
\lambda(A) - \lambda(B_i^{-1}) - \lambda(A_1) \geq \frac{1}{2} \lambda(A) \tag{23}
\end{equation}

Take \( z \in V \) and consider the set
\[ H = A_2 \cap zB_2^{-1}. \]
Then \( H \subset A_2 \) and \( H^{-1}z \subset B_2 \). Also by (23),
\[ \lambda(H) \geq \frac{1}{2} \lambda(A) \]

Finally,
\[ \int_H h(y)k(y^{-1}z) \, d\lambda(y) \geq \frac{M_f M_g}{4} \lambda(H) = \frac{R^2}{72} \varphi^{-1} \left( \frac{1}{\|\chi_A\|} \right) \psi^{-1} \left( \frac{1}{\|\chi_B\|} \right) \lambda(A) \geq n, \]
so \((h, k) \notin F_n^+\). \( \square \)

4. SUBSETS RELATED TO POINTWISE PRODUCT

In this section we study a similar problem for Calderón-Lozanowski spaces under pointwise multiplication. As it may be expected in this case we encounter less difficulties and we can make much less assumptions.

Let \((\Omega, \Sigma, \mu)\) be a complete \( \sigma \)-finite measure space and \( E \) be a Banach ideal in \( L^0(\Omega) \). Also let \( \Sigma_+ = \{ A \in \Sigma : 0 < \mu(A) < \infty \} \). By our assumptions, \( \Sigma_+ \neq \emptyset \). Additionally, we assume that:

for every \( A \in \Sigma_+ \), \( \chi_A \in E \).

We start this section with the following lemma.

**Lemma 4.1.** Let \( E_{\varphi_1}, E_{\varphi_2} \) and \( E_{\varphi_3} \) be Calderón-Lozanowski spaces, \( A \in \Sigma_+ \), \( s_1, s_2 > 0 \), \( C \in (0, 1) \), and \( f_i, g_i \in E_{\varphi_i} \), \( i = 1, 2 \), be such that \( |g_i(x)| \geq 1 \) for \( i = 1, 2 \) and \( x \in A \). Assume that
\[ \infty > \|\chi_A\| > \frac{1}{\varphi_1 (\frac{1-C}{s_1})} + \frac{1}{\varphi_2 (\frac{1-C}{s_2})}, \]
and \( \| (f_i - g_i)\chi_A \|_{E_{\varphi_i}} \leq s_i \) for \( i = 1, 2 \). Then
\[ \|f_1 \cdot f_2\|_{E_{\varphi_3}} \geq \frac{C^2}{\varphi_3^{-1} \left( \|\chi_A\| - \frac{1}{\varphi_1 (\frac{1-C}{s_1})} - \frac{1}{\varphi_2 (\frac{1-C}{s_2})} \right)^{-1}}. \]

**Proof.** Let \( A_i = \{ x \in A : |f_i(x)| < C|g_i(x)| \} \), \( i = 1, 2 \). If \( x \in A_i \), then
\[ |f_i(x) - g_i(x)| \geq |g_i(x)| - |f_i(x)| \geq (1-C)|g_i(x)| \geq (1-C). \]

We will prove that
\begin{equation}
\|\chi_A\| \leq \frac{1}{\varphi_i (\frac{1-C}{s_i})}. \tag{24}
\end{equation}
It holds true if \( \|\chi_A\| = 0 \). Assume that \( \|\chi_A\| > 0 \). Then
\[ s_i \geq \| (f_i - g_i)\chi_A \|_{E_{\varphi_i}} \geq (1-C)\|\chi_A\|_{E_{\varphi_i}} = \frac{1-C}{\varphi_i^{-1} \left( \|\chi_A\| \right)} \cdot \]

Thus
\[ \frac{1}{\|\chi_A\|} \geq \varphi_i \left( \frac{1-C}{s_i} \right) \]
and consequently we obtain (24). Note that
\[ \chi_A = \chi_A \setminus (A_1 \cup A_2) + \chi_A \cap (A_1 \cup A_2) = \chi_A \setminus (A_1 \cup A_2) + \chi_A \setminus A_2 \leq \chi_A \setminus (A_1 \cup A_2) + \chi_A_1 + \chi_{A_2}. \]
Therefore
\[ \|\chi_A\| \leq \|\chi_A \setminus (A_1 \cup A_2) + \chi_A_1 + \chi_{A_2}\| \leq \|\chi_A \setminus (A_1 \cup A_2)\| + \|\chi_A_1\| + \|\chi_{A_2}\| \]
and consequently
\[ \|\chi_{A \setminus (A_1 \cup A_2)}\| \geq \|\chi_A\| - \|\chi_A_1\| - \|\chi_{A_2}\| \geq \|\chi_A\| - \frac{1}{\varphi_1 \left( \frac{1-C}{s_1} \right)} - \frac{1}{\varphi_2 \left( \frac{1-C}{s_2} \right)}. \]
Since \(|f_i(x)| \geq C\) for \(x \in A \setminus A_i\), we obtain
\[ \|f_1 \cdot f_2\|_{E_{\varphi_3}} \geq \|C^{2}\chi_{A \setminus (A_1 \cup A_2)}\|_{E_{\varphi_3}} \]
\[ = \frac{C^2}{\varphi_3^{-1} \left( \|\chi_{A \setminus (A_1 \cup A_2)}\|^{-1} \right)} \]
\[ \geq \frac{C^2}{\varphi_3^{-1} \left( \left( \frac{1}{\|\chi_A\|} - \frac{1}{\varphi_1 \left( \frac{1-C}{s_1} \right)} - \frac{1}{\varphi_2 \left( \frac{1-C}{s_2} \right)} \right)^{-1} \right)}. \]

The following theorem generalizes Theorem 2.4 in [4] and Theorem 8 in [40].

**Theorem 4.2.** Let \(E_{\varphi_1}, E_{\varphi_2}, E_{\varphi_3}\) be Calderón-Lozanowskiĭ spaces with \(\Sigma_+ \neq \emptyset\). Assume that \(a_{\varphi_3} = 0\) and for any \(\varepsilon > 0\) there is \(A \in \Sigma_+\) such that \(\frac{1}{\|\chi_A\|} \leq \min\{\varphi_1(b_{\varphi_1}), \varphi_2(b_{\varphi_2})\}\) and

\[ \frac{\|\chi_A\|_{E_{\varphi_1}} \cdot \|\chi_A\|_{E_{\varphi_2}}}{\|\chi_A\|_{E_{\varphi_3}}} = \frac{\varphi_3^{-1} \left( \frac{1}{\|\chi_A\|} \right)}{\varphi_1^{-1} \left( \frac{1}{\|\chi_A\|} \right) \cdot \varphi_2^{-1} \left( \frac{1}{\|\chi_A\|} \right)} \leq \varepsilon. \]

Then the set \(F = \{(f_1, f_2) \in E_{\varphi_1} \times E_{\varphi_2} : f_1 \cdot f_2 \in E_{\varphi_3}\}\) is \(\sigma - \frac{2}{3}\)-lower porous.

**Proof.** We will show that for any \(n \in \mathbb{N}\), the set \(F_n = \{(f_1, f_2) \in E_{\varphi_1} \times E_{\varphi_2} : \|f_1 \cdot f_2\|_{E_{\varphi_3}} < n\}\) is \(\frac{2}{3}\)-lower porous. Let \(\delta \in (0, 1/3)\). Then \(\frac{1-\delta}{\delta} > 2\), and therefore there is a \(C \in (0, 1)\) with \(\frac{(1-C)(1-\delta)}{\delta} > 2\).

Let \(k > 1\) be a real number such that
\[ \frac{(1-C)(1-\delta)}{\delta} = 2k. \]

Let \(R > 0\) and \(A \in \Sigma_+\) be such that \(\frac{1}{\|\chi_A\|} \leq \min\{\varphi_1(b_{\varphi_1}), \varphi_2(b_{\varphi_2})\}\) and

\[ \frac{(C(1-\delta)R)^2}{\|\chi_A\|^2} \geq \frac{\varphi_3^{-1} \left( \frac{1}{\|\chi_A\|} \right)}{\varphi_1^{-1} \left( \frac{1}{\|\chi_A\|} \right) \cdot \varphi_2^{-1} \left( \frac{1}{\|\chi_A\|} \right)}. \]
Put \( t = \frac{1}{\|x_A\|} \). Since \( \varphi_i \) is convex, we have \( \varphi_i(2k\varphi_i^{-1}(t)) \geq 2kt \) for \( i = 1, 2 \). Thus by (26)

\[
\begin{align*}
\frac{1}{t} - \frac{1}{\varphi_1 \left( \frac{(1-C)(1-\delta)}{\delta} \varphi_1^{-1}(t) \right)} - \frac{1}{\varphi_2 \left( \frac{(1-C)(1-\delta)}{\delta} \varphi_2^{-1}(t) \right)} = \frac{1}{t} - \frac{1}{\varphi_1(2k\varphi_1^{-1}(t))} - \frac{1}{\varphi_2(2k\varphi_2^{-1}(t))} \\
\geq \frac{1}{t} - \frac{1}{kt} = \frac{k-1}{kt}.
\end{align*}
\]

Since \( \varphi_3^{-1} \) is concave and increasing, using (27), (28) and the fact that \( \varphi_3^{-1}(0) = 0 \) (which follows from \( a_{\varphi_3} = 0 \)), we obtain

\[
\frac{(C(1-\delta)R)^2}{n} > \frac{k}{k-1} \frac{\varphi_3^{-1}(t)}{\varphi_1^{-1}(t) \cdot \varphi_2^{-1}(t)} \geq \frac{\varphi_3^{-1}(tk)}{\varphi_1^{-1}(t) \cdot \varphi_2^{-1}(t)}
\]

Hence

\[
\frac{(C(1-\delta)R)^2}{n} \varphi_3^{-1} \left( \left( \frac{1}{t} - \frac{1}{\varphi_1 \left( \frac{(1-C)(1-\delta)}{\delta} \varphi_1^{-1}(t) \right)} - \frac{1}{\varphi_2 \left( \frac{(1-C)(1-\delta)}{\delta} \varphi_2^{-1}(t) \right)} \right)^{-1} \right) > n.
\]

Put \( M_i = (1-\delta)R \varphi_i^{-1}(t), i = 1, 2 \). Then \( \|M_i x_A\|_{E_{\varphi_i}} = \frac{M_i}{\varphi_i(t)} = (1-\delta)R \). Let

\[
\tilde{f}_i(y) := \begin{cases} 
  f_i(y) & y \notin A \\
  f_i(y) + M_i & \text{Re}(f_i(y)) \geq 0, y \in A \\
  f_i(y) - M_i & \text{Re}(f_i(y)) < 0, y \in A.
\end{cases}
\]

We have

\[
\|f_i - \tilde{f}_i\|_{E_{\varphi_i}} = \|M_i x_A\|_{E_{\varphi_i}} = (1-\delta)R.
\]

Hence \( B((\tilde{f}_1, \tilde{f}_2), \delta R) \subset B((f_1, f_2), R) \). We will show that \( B((\tilde{f}_1, \tilde{f}_2), \delta R) \cap F_n = \emptyset \). Let \((h_1, h_2) \in B((\tilde{f}_1, \tilde{f}_2), \delta R) \). Then

\[
\delta R \geq \|h_i - \tilde{f}_i\|_{E_{\varphi_i}} \geq \|(h_i - \tilde{f}_i) x_A\|_{E_{\varphi_i}} = M_i \left\| \frac{h_i}{M_i} - \frac{\tilde{f}_i}{M_i} \right\|_{x_A}.
\]
Note that \( \left| \frac{f(x)}{M_i} \right| \geq 1 \) for \( x \in A \). Finally

\[
\|h_1 \cdot h_2\|_{E_{\varphi_3}} = M_1 \cdot M_2 \frac{\|h_1 / M_1 \cdot h_2 / M_2\|_{E_{\varphi_3}}}{C^2}
\]

\[
\geq M_1 \cdot M_2 \cdot \varphi^{-1}_3 \left( \left( \|X\| - \varphi_1 \left( \frac{1}{\pi^2 M_1} \right) \right)^{-1} - \frac{1}{\varphi_2 \left( \frac{1}{\pi^2 M_2} \right)} \right)
\]

\[
= \varphi^{-1}_3 \left( \left( \|X\| - \frac{1}{\varphi_1 \left( \frac{1}{\pi^2} \right) \varphi^{-1}_1(t)} \right)^{-1} - \frac{1}{\varphi_2 \left( \frac{1}{\pi^2} \varphi^{-1}_1(t) \right)} \right)
\]

\[
> n.
\]

Where the first inequality follows from Lemma 4.1 used for \( A, C, f_i := \frac{h_i}{M_i}, g_i := \frac{f_i}{M_i}, s_i := \frac{\delta R}{M_i} \). Therefore \( (h_1, h_2) \notin F_n \).

A Banach ideal space \( E \) is called order continuous if for every \( f \in E \) and every sequence \( \{A_n\} \) satisfying \( A_n \downarrow 0 \) (that is \( A_n \supset A_{n+1} \) and \( \mu (\cap_{n=1}^{\infty} A_n) = 0 \)), we have that \( \|f \chi_{A_n}\|_E \downarrow 0 \). It is easy to see that, in the setting of the previous section, the order continuity of \( E \) implies the continuity of the fundamental function \( \xi_E \) at 0.

**Theorem 4.3.** Let \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) be Young functions with \( b_{\varphi_3} = \infty \) and \( E \) be a Banach ideal space with order continuous norm. If there exists \( (h, k) \in E_{\varphi_1} \times E_{\varphi_2} \) such that \( h \cdot k \notin E_{\varphi_3} \), then the set

\[
F := \{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : f \cdot g \in E_{\varphi_3}\}
\]

is of the first category in \( E_{\varphi_1} \times E_{\varphi_2} \).

**Proof.** For every \( u, v > 0 \), define \( F_u^v := \{(f, g) \in E_{\varphi_1} \times E_{\varphi_2} : I_{\varphi_3}(vf \cdot g) < u\} \). Since \( F = \bigcup_{u,v \in \mathbb{N}} F_u^v \), we only have to show that for every \( u, v > 0 \), \( F_u^v \) is nowhere dense. Fix \( u, v > 0 \) and let \( (f, g) \in E_{\varphi_1} \times E_{\varphi_2} \) and \( R > 0 \). Set

\[
\tilde{f}(y) := \begin{cases} f(y) + \frac{R}{2\|h\|_{E_{\varphi_1}}} & \text{Re}(f(y)) \geq 0 \\ f(y) - \frac{R}{2\|h\|_{E_{\varphi_1}}} & \text{Re}(f(y)) < 0 \end{cases}
\]

\[
\tilde{g}(y) := \begin{cases} g(y) + \frac{R}{2\|k\|_{E_{\varphi_1}}} & \text{Re}(g(y)) \geq 0 \\ g(y) - \frac{R}{2\|k\|_{E_{\varphi_1}}} & \text{Re}(g(y)) < 0 \end{cases}
\]

Then \( \tilde{f} \in E_{\varphi_1} \) and \( \tilde{g} \in E_{\varphi_2} \). Also, obviously \( \|\tilde{f} - f\|_{E_{\varphi_1}} = \frac{R}{2} = \|\tilde{g} - g\|_{E_{\varphi_2}} \) and \( \tilde{f} \cdot \tilde{g} \notin E_{\varphi_3} \). Hence \( I_{\varphi_3}(\frac{u}{4} \tilde{f} \cdot \tilde{g}) = \infty \). Now for every \( n \in \mathbb{N} \), put

\[
A_n := \left\{ x \in \Omega : n > |\tilde{f}(x)| > n^{-1} \right\} \cap \left\{ x \in \Omega : n > |\tilde{g}(x)| > n^{-1} \right\}.
\]

Since \( \tilde{f} \in E_{\varphi_1} \) and \( \tilde{g} \in E_{\varphi_2} \), we have that \( \|\chi_{A_n}\|_E < \infty \) for each \( n \in \mathbb{N} \). Also, if we put \( A := \bigcup_{n \in \mathbb{N}} A_n = \{x \in \Omega : \infty > |\tilde{f}(x) \cdot \tilde{g}(x)| > 0\} \), then \( I_{\varphi_3}(\frac{u}{4} \tilde{f} \cdot \tilde{g} \cdot \chi_A) = \infty \). Therefore, since every element of \( E \) has an order continuous norm and \( b_{\varphi_3} = \infty \), there exists \( m \in \mathbb{N} \) such that

\[
\infty > I_{\varphi_3}(\frac{u}{4} \tilde{f} \cdot \tilde{g} \cdot \chi_{A_m}) > u.
\]
By the order continuity of $E$, there exists $\delta > 0$ such that for every measurable subset $B \subset A_m$ with $\mu(B) \leq \delta$, we have
\begin{equation}
I_{\varphi_3} \left( \frac{u}{4} \tilde{f} \cdot \tilde{g} \cdot \chi_{A_m \setminus B} \right) > u.
\end{equation}

Now choose $r \in (0, \frac{1}{2}R)$ such that for every $i = 1, 2$ we have
\begin{equation}
\varphi_i \left( \frac{1}{2mr} \right) > \frac{2}{\delta}.
\end{equation}

Assume that $(d, l) \in E_{\varphi_1} \times E_{\varphi_2}$ is such that $(d, l) \in B((\tilde{f}, \tilde{g}), r)$. Put
\begin{align*}
C & := \left\{ x \in A_m : |d(x)| \leq \frac{1}{2} |\tilde{f}(x)| \right\}, \\
D & := \left\{ x \in A_m : |l(x)| \leq \frac{1}{2} |\tilde{g}(x)| \right\}.
\end{align*}

Then we have
\begin{equation*}
r > \|f - d\|_{E_{\varphi_1}} \geq \left\| \frac{1}{2} \tilde{f} \chi_C \right\|_{E_{\varphi_1}} \geq \left\| \frac{1}{2m} \chi_C \right\|_{E_{\varphi_1}} = \frac{1}{2m\varphi_1^{-1}} \left( \frac{1}{\|\chi_C\|_E} \right),
\end{equation*}

provided that $\mu(C) > 0$. Hence by (32), $\|\chi_C\|_E < \frac{\delta}{2}$. Similarly, $\|\chi_D\|_E < \frac{\delta}{2}$. Finally, by (31) we get
\begin{equation*}
I_{\varphi_3}(vd \cdot l) \geq I_{\varphi_3}(vd \cdot l \cdot \chi_{A_m \setminus (C \cup D)}) \geq I_{\varphi_3} \left( \frac{u}{4} \tilde{f} \cdot \tilde{g} \cdot \chi_{A_m \setminus (C \cup D)} \right) > u.
\end{equation*}

Hence $B((\tilde{f}, \tilde{g}), r) \subset B((f, g), R)$ and $B((\tilde{f}, \tilde{g}), r) \cap F_u = \emptyset$. This proves our claim. \qed

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