ON GRAPHS ISOMORPHIC TO THEIR NEIGHBOUR AND NON-NEIGHBOUR SETS

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ABSTRACT. The paper contains a construction of a universal countable graph, different from the Rado graph, such that for any of its vertices both the neighbourhood and the non-neighbourhood induce subgraphs isomorphic to the whole graph. This solves an open problem proposed by A. Bonato; see Problem 20 in [5]. We supply a construction of several non-isomorphic graphs with the property, and consider tournaments with an analogous property.

1. INTRODUCTION

In 1964, R. Rado [8] introduced his fundamental graph R. He showed that R is universal; that is, any finite or countable graph can be embedded in R as its subgraph, and that R is existentially closed (or e.c.): for any finite disjoint sets of vertices $A, B \subseteq V(R)$ there is a vertex $z \in V(R) \setminus (A \cup B)$, joined to each vertex of A and to no vertex of B. As any two countable existentially closed graphs are isomorphic, the R graph is unique up to an isomorphism countable e.c. graph. The Rado's explicit construction of the graph R proceeds as follows. The vertex set is $V(R) = \mathbb{N}$ (with 0), while the edge set $E(R)$ is given by the condition: for $x, y \in \mathbb{N}$, $x < y$ the vertices x and y are adjacent if and only if, the xth digit in the binary representation of y is equal to 1. Erdős and Rényi [7] proved that with the probability 1, any graph in the probability space $G(\mathbb{N}, p)$ where $p \in (0, 1)$ (any graph having vertex set $\mathbb{N}$ with every two vertices joined independently with a probability p) is isomorphic to R, because it is existentially closed. For this reason, the graph R is also called *the random graph*. The Rado graph has been studied extensively due to its many other interesting properties. There are also many different constructions of the R graph. Surveys can be found in [3, 4].

At the 18th British Combinatorial Conference, Bonato [5] stated the problem: suppose that a graph G has the NN^c property defined by the condition that subgraphs induced by the neighbourhood and by the non-neighbourhood of each vertex of G are isomorphic to G. It is clear that R has this property but it is the only known example of such graph. Which other graphs, if any, have this property? The aim of this paper is to construct a graph not isomorphic to R with the NN^c property. The main construction is contained in Section 2. Section 4 some remarks to the construction and mentions existence of other NN^c graphs. This one leads us to state another open problem of NN^c graphs. The solution to Bonato's problem in case of tournaments is contained in Section 5.

All graphs considered are simple and undirected, except Section 5. For a reference on graph theory, see [6, particularly Chapter 8]. We denote the complement of the graph G by $\overline{G}$. By $G[X]$ we denote the graph induced by the set $X \subseteq V(G)$. Let $N(v)$ or $N^1(v)$ denote the neighbourhood of any vertex v, while $N^c(v)$ or $N^0(v)$ denotes non-neighbourhood of v. For $x, y \in \mathbb{N}$, $b \in \mathbb{N}^+$ let $y_{(b)}(x)$ denote the xth digit of y written in base b. For completeness, when $y < b^x$, let $y_{(b)}(x) = 0$.

Let us recall that for a positive integer n, a graph is *n-existentially closed* or *n-e.c.*, if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B can be empty), there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. A survey on n-e.c. graphs may be found in [1].

2. CONSTRUCTION OF THE GRAPH

The proof that the Rado graph has the NN^c property is simple — the key is the existential closure. As the graph R is e.c., for any of its vertices both the neighbourhood and non-neighbourhood of this vertex also are and so are isomorphic to R. In an analogous way, we start the construction with Definition 2.2

of a certain property, playing the role of the e.c. property. We show that any two graphs having this property are isomorphic and that the graphs induced both by the neighbourhood and by the non-neighbourhood of any of their vertices keep this property. Finally, we give an explicit construction of the graph $\mathfrak{S}$ which has the defined property. In order to increase clarity of the idea's explanation, more difficult proofs are included in Section 3.

The idea of constructing another graph which has the NN^c property is simple. Suppose that a graph G has this property and let $u, v \in V(G)$ be two adjacent vertices. Then each of the intersections $N(u) \cap N(v)$, $N(u) \cap N^c(v)$ and $N^c(u) \cap N(v)$ induces a subgraph of G isomorphic to the whole G. But nothing is known about $N^c(u) \cap N^c(v)$. We intend to construct the graph $\mathfrak{S}$ having the NN^c property, but even not 2-e.c., which for some pairs of adjacent vertices $u, v \in V(\mathfrak{S})$ satisfies condition $N^c(u) \cap N^c(v) = \emptyset$.

The construction will be done by adding vertices to the Rado graph in such way that it disturbs existential closure as explained above but keeps the NN^c property. As both neighbourhood and non-neighbourhood of each vertex induce subgraphs isomorphic to $\mathfrak{S}$, so they must contain an infinite number of pairs of vertices with empty common non-neighbourhood. Therefore, for every finite subset of the vertex set $X \subseteq V(\mathfrak{S})$ we will expect the constructed graph $\mathfrak{S}$ to contain infinite number of pairs of adjacent vertices $u, v \in V(\mathfrak{S})$ for which $N^c(u) \cap N^c(v) = X$.

Definition 2.1. *Let G be a countable graph and $u, v \in V(G)$. We say that the vertex v is* non-perturbating *if for every $w \in V(G)$, $N^c(v) \cap N^c(w)$ is infinite. If $N^c(v) \cap N^c(u)$ is finite, then we say that the 2-set $\{u, v\}$ is* perturbating. *We denote the set of all perturbating 2-sets of vertices by $P(G)$, while the set of non-perturbating vertices is denoted by $Q(G)$.*

Definition 2.2. *Let G be a countable graph. We say that G is* perturbed-existentially closed *(or p.e.c.) if all the following properties holds:*

   (i) *For every vertex $v \in V(G)$ there is at most one vertex $v' \in V(G)$, such that $\{v, v'\} \in P(G)$. Moreover, the vertices v and v' are adjacent.*
   (ii) *In the set $P(G)$, the relation R such that $\{v, v'\}R\{u, u'\}$ holds if and only if there is $v \in N^c(u) \cap N^c(u')$ or $v' \in N^c(u) \cap N^c(u')$ generates a partial order $\prec$. Moreover, for every $\{u, u'\} \in P(G)$ the set $\big\{ \{v, v'\} \in P(G) \colon \{v, v'\} \prec \{u, u'\} \big\}$ is finite.*
   (iii) *For any finite disjoint sets of vertices $U, U' \subseteq V(G)$ such that U' contains no perturbating 2-set, there exists a non-perturbating vertex $u \in Q(G) \setminus (U \cup U')$ which satisfies:*
$$u \in \bigcap_{v \in U} N(v) \cap \bigcap_{v \in U'} N^c(v).$$
   (iv) *For any finite set of vertices $U \subseteq V(G)$ and every partitions of the set U into two disjoint pieces $A \cup A' = U$ and $B \cup B' = U$, such that both A' and B' contain no perturbating 2-set, there exists a perturbating 2-set of vertices $\{u, u'\} \in P(G)$, disjoint with U, which satisfies all three below:*
$$u \in \bigcap_{v \in A} N(v) \cap \bigcap_{v \in A'} N^c(v), \quad u' \in \bigcap_{v \in B} N(v) \cap \bigcap_{v \in B'} N^c(v), \quad and \ N^c(u) \cap N^c(u') = A' \cap B'.$$

Considering *(iii)*, one can notice that for any two vertices u, v of the p.e.c. graph both sets $N(u) \cap N(v)$ and $N(u) \cap N^c(v)$ are infinite. From *(iv)* it follows that the p.e.c. graph is not e.c. Nevertheless, it follows from *(iii)* that such graph contains the e.c. graph as an induced subgraph. More precisely, if from each perturbating 2-set of vertices at most one vertex is taken, then a subgraph induced by these vertices together with all non-perturbating vertices will be existentially closed.

Theorem 2.3. *If G is a p.e.c. graph and $v \in V(G)$, then both $G[N(v)]$ and $G[N^c(v)]$ are p.e.c.*

Theorem 2.4. *If G and H are p.e.c. graphs, then G and H are isomorphic.*

In order to prove that the given property leads us to another example of an NN^c graph we still need to show that there exists a p.e.c. graph. It will be done by the explicit construction, quite analogous to the Rado's construction, of course more complicated. To satisfy the p.e.c. property we need to consider representations of natural numbers in four bases given by coprime numbers. We will use 2, 3, 5 and 7. Let the graph $\mathfrak{S}$ be defined as follows. For ease of notation, in the following definition and below, we use logical connectives (such as $\wedge$ for "and", $\vee$ for "or") to describe the edges of the graph $\mathfrak{S}$.

Definition 2.5. *Let $\mathfrak{S} = (V(\mathfrak{S}), E(\mathfrak{S}))$ be a graph with the vertex set $V(\mathfrak{S}) = \mathbb{N}^+$ and the edge set $E(\mathfrak{S})$ defined by the following conditions.*
For $n, v \in \mathbb{N}^+$, $v < 3n$, the vertices v and $u = 3n$ are adjacent if and only if one has:

    (i) $v \not\equiv 2 \pmod 3 \wedge u_{(2)}(v) = 1$,
    (ii) $v \equiv 2 \pmod 3 \wedge (u_{(2)}(v-1) = 0 \vee u_{(3)}(v) = 1)$.

For $n \in \mathbb{N}, v \in \mathbb{N}^+$, $v < 3n+1$, the vertices v and $u = 3n+1$ are adjacent if and only if one has:

    (iii) $v \equiv 0 \pmod 3 \wedge u_{(2)}(v) = 1 \wedge (u+1)_{(7)}(v) \neq 1$,
    (iv) $v \equiv 1 \pmod 3 \wedge ((u_{(2)}(v) = 1 \wedge (u+1)_{(7)}(v) \neq 1) \vee (u+1)_{(7)}(v+1) = 1)$,
    (v) $v \equiv 2 \pmod 3 \wedge (u+1)_{(7)}(v) \neq 1 \wedge (u_{(2)}(v-1) = 0 \vee u_{(3)}(v) = 1 \vee (u+1)_{(7)}(v-1) = 1)$.

For $n \in \mathbb{N}, v \in \mathbb{N}^+$, $v < 3n+2$, the vertices v and $u = 3n+2$ are adjacent if and only if one has:

    (vi) $v \equiv 0 \pmod 3 \wedge u_{(7)}(v) \neq 1 \wedge ((u-1)_{(2)}(v) = 0 \vee u_{(3)}(v) = 1)$,
    (vii) $v \equiv 1 \pmod 3 \wedge ((((u-1)_{(2)}(v) = 0 \vee u_{(3)}(v) = 1) \wedge u_{(7)}(v) \neq 1) \vee u_{(7)}(v+1) = 1 \vee v = u-1)$,
    (viii) $v \equiv 2 \pmod 3 \wedge u_{(7)}(v) \neq 1 \wedge (((u-1)_{(2)}(v-1) = 1 \wedge ((u-1)_{(3)}(v) \neq 1 \vee u_{(3)}(v-1) \neq 1)) \vee$
        $\vee u_{(5)}(v) = 1 \vee u_{(7)}(v-1) = 1)$.

Please note that, as vertex set is $\mathbb{N}^+$, the zeroth digit in all considered bases is free — it does not affect adjacency. From this construction it follows that $Q(\mathfrak{S}) = \{3n \colon n \in \mathbb{N}^+\}$, while $P(\mathfrak{S}) = \big\{\{3n+1, 3n+2\} \colon n \in \mathbb{N}\big\}$. It is shown in the proof of Theorem 2.6.

Theorem 2.6. *The graph $\mathfrak{S}$ is perturbed-existentially closed.*

Corollary 2.7. *The graph $\mathfrak{S}$ has the NN^c property; that is, for every vertex $v \in V(\mathfrak{S})$ both $\mathfrak{S}[N(v)]$ and $\mathfrak{S}[N^c(v)]$ are isomorphic to $\mathfrak{S}$.*

The main result presented in this paper is now just the conclusion of Theorems 2.6, 2.3 and 2.4. As $\mathfrak{S}$ is p.e.c., then graphs induced both by the neighbourhood and by the non-neighbourhood of any vertex also are p.e.c., and so are isomorphic to $\mathfrak{S}$.

3. Proofs of theorems

This section contains proofs of theorems stated in Section 2. Starting with proof of Theorem 2.3 we formulate a lemma that the only perturbating 2-sets in neighbourhood or non-neighbourhood of any vertex of a p.e.c. graph are those which are perturbating 2-sets in the whole graph.

Lemma 3.1. *Let G be a p.e.c. graph and $v \in V(G)$ any of its vertices, let $\alpha \in \{0, 1\}$. For any vertices $x, y \in N^\alpha(v)$ the set $N^\alpha(v) \cap N^c(x) \cap N^c(y)$ is finite if and only if $N^c(x) \cap N^c(y)$ is finite.*

Proof. For a contradiction assume that there exist two vertices $x, y \in N^\alpha(v)$ such that $N^c(x) \cap N^c(y)$ is infinite but the set $W = N^\alpha(v) \cap N^c(x) \cap N^c(y)$ is finite. Now if $\alpha = 0$ let $U = W$ and $U' = \{v, x, y\}$, else let $U = W \cup \{v\}$ and $U' = \{x, y\}$. Note that any two vertices from U' have infinite common non-neighbourhood — if either x or y is such vertex that has finite common with v non-neighbourhood, then $x, y \in N(v)$ and so $\alpha = 1$ and one has $v \notin U'$. According to Definition 2.2*(iii)* there exists a vertex $u \in \bigcap_{w \in U} N(w) \cap \bigcap_{w \in U'} N^c(w)$, different from any of the vertices from $\{v, x, y\} \cup W$. As in particular one has $u \in N^\alpha(v) \cap N^c(x) \cap N^c(y) = W$, we obtain a contradiction. $\square$

Proof of Theorem 2.3. Let G be a p.e.c. graph and $v \in V(G)$ any of its vertices. Let $\alpha \in \{0, 1\}$. To prove the theorem one should show all conditions of the p.e.c. property (Definition 2.2) for $G[N^\alpha(v)]$.

The condition *(i)* follows automatically from Lemma 3.1. Satisfaction of the condition *(ii)* also can be obtained directly from the lemma, as inducing a subgraph by a vertex set does not disturb adjacency in this subgraph.

Now let us prove the condition *(iii)*. Let U, U' be any finite disjoint subsets of $N^\alpha(v)$, such that U' contains no perturbating 2-set — according to Lemma 3.1 it is enough to check it in the graph G only. Now let us enlarge either set U or respectively U' by the vertex v. In the case when such vertex $z \in V(G)$ that $\{v, z\} \in P(G)$ exists, note that from the condition *(i)*, one has $z \in N(v)$, hence, if $z \in U'$ then α must be 1 and vertex v will enlarge the U set. From the condition *(iii)* of the p.e.c. property of the whole graph G there exists a non-perturbating vertex $u \in \bigcap_{w \in U} N(w) \cap \bigcap_{w \in U'} N^c(w)$ and additionally there is $u \in N^\alpha(v)$. Since the obtained vertex u from $V(G)$ is non-perturbating, according to Lemma 3.1, it remains so in the graph $G[N^\alpha(v)]$.

Proof of the condition *(iv)* is analogous to the proof of the *(iii)*. If $\alpha = 1$ it is enough to enlarge all three sets U, A and B by the vertex v and to keep the sets A' and B', while $\alpha = 0$ it is enough to enlarge all three sets U, A' and B' by the vertex v and to keep sets A and B unchanged. Then the 2-set of vertices, obtained according to the condition *(iv)* of the whole graph G, satisfies the condition for $G[N^\alpha(v)]$. $\square$

The following lemma is useful for showing that any two p.e.c. graphs are isomorphic. As a result of the suitable partition into sequences, the construction of the isomorphism is made more clear.

Lemma 3.2. *If G is a p.e.c. graph, then there exists a partition of the vertex set $V(G)$ into three sequences $\{x_i\}_{i=1}^\infty$, $\{y_i\}_{i=1}^\infty$ and $\{z_i\}_{i=1}^\infty$, satisfying for $i \in \mathbb{N}^+$ conditions:*

$$x_i \in Q(G) \quad and \quad N^c(y_i) \cap N^c(z_i) \subseteq \bigcup_{j=1}^{i-1}\{x_j, y_j, z_j\}.$$

Proof. Let G be any p.e.c. graph. Let us partition vertices from $V(G)$ into sequences: $Q(G) = \{x_i\}_{i=1}^\infty$ and $P(G) = \{\{a_i, b_i\}\}_{i=1}^\infty$. This is possible because the vertex set is countable, so it follows from the condition *(i)* that this sequences may contain every vertex once, while conditions *(iii)* and *(iv)* of Definition 2.2 implies that both sets $Q(G)$ and $P(G)$ are infinite.

The sequence $\{(a_i, b_i)\}$ should be renumbered according to the second condition given in the lemma. We construct the sequence $\{(y_i, z_i)\}_{i=1}^\infty$ as a suitably renumbered version of the sequence $\{(a_i, b_i)\}$. Define $W_1 = \{k \in \mathbb{N}^+ \colon N^c(a_k) \cap N^c(b_k) = \emptyset\}$. From Definition 2.2*(iv)*, the set W_1 is nonempty. Therefore, let $y_1 = a_{\min(W_1)}, z_1 = b_{\min(W_1)}$.

Now, for any $j \in \mathbb{N}^+$ let us assume that we have already defined sets $W_i \subseteq \mathbb{N}$ as well as y_i and z_i, for $i \le j$, such that there is:

$$W_i = \left\{ k \in \mathbb{N} \colon N^c(a_k) \cap N^c(b_k) \subseteq \bigcup_{m=1}^{i-1}\{x_m, y_m, z_m\} \wedge \{a_k, b_k\} \neq \{y_m, z_m\}, m < i \right\},$$

$$y_i = a_{\min(W_i)}, \ z_i = b_{\min(W_i)}.$$

We use the above to define W_{j+1}, y_{j+1} and z_{j+1}.

It follows from Definition 2.2*(iv)* that, for every $j \in \mathbb{N}^+$, the set W_j is nonempty and one has $W_j \setminus \{\min(W_j)\} \subsetneq W_{j+1}$. Therefore, every element of W_j eventually occurs in the constructed renumbering. Hence, in order to prove that the sequence $\{(y_i, z_i)\}_{i=1}^\infty$ is a renumbering of $\{(a_i, b_i)\}_{i=1}^\infty$, it remains to show that for every $n \in \mathbb{N}^+$ there exists $m \in \mathbb{N}^+$, such that $n \in \bigcup_{i=1}^m W_i$.

Let $n \in \mathbb{N}^+$ be any fixed positive integer, while $\prec$ denotes the partial order generated according to Definition 2.2*(ii)*. Let $U = \left\{\{u, v\} \colon \{u, v\} \prec \{a_n, b_n\}\right\}$ and

$$S = \bigcup_{\{u,v\} \in U \cup \{a_n, b_n\}} \left(N^c(u) \cap N^c(v)\right).$$

It is clear that $N^c(a_n) \cap N^c(b_n) \subseteq S$. Moreover, the set U, and so also the set S, is finite. Therefore, let us define

$$s = \max\left(k \colon x_k \in S\right) + \sum_{\{a_k, b_k\} \in U} k.$$

Due to the construction, one has $n \in \bigcup_{i=1}^m W_i$, if only m satisfies both:

$$S \cap Q(G) \subseteq \bigcup_{i=1}^{m-1}\{x_i\} \quad \text{and} \quad U \subseteq \bigcup_{i=1}^{m-1}\{y_i, z_i\}.$$

The first condition is true for any m such that $m > m_0 = \max\left(k \colon x_k \in S\right)$, so it is enough to show that there exists an m which satisfies the second condition. It is worth noting that, for $p > m_0$, the set W_p will contain indices of all the minimal (due to the relation $\prec$) elements from the set $U \setminus \bigcup_{k=1}^{p-1}\{y_k, z_k\}$. Notice, that every k occurs in at most k subsequent sets W_i. Then, by a simple and so omitted induction, all the elements from the set U occur in the sequence $\{(y_i, z_i)\}$ in the position not exceeding s. Hence, for $m > s$, one has $n \in \bigcup_{i=1}^m W_i$. $\square$

Proof of Theorem 2.4. Let G and H be p.e.c. graphs. To show that G and H are isomorphic, we construct an isomorphism $f \colon V(H) \to V(G)$. The construction will be done using a back-and-forth argument.

Let us partition vertices of the graphs into sequences ordered as described in Lemma 3.2. Let $Q(H) = \{x_i\}_{i=1}^{\infty}$, $P(H) = \{y_i, z_i\}_{i=1}^{\infty}$ and $Q(G) = \{c_i\}_{i=1}^{\infty}$, $P(G) = \{a_i, b_i\}_{i=1}^{\infty}$ respectively. For every $i \in \mathbb{N}^+$ there is:

$$\left(N^c(y_i) \cap N^c(z_i) \subseteq \bigcup_{j=1}^{i-1}\{x_j, y_j, z_j\}\right) \text{ and } \left(N^c(a_i) \cap N^c(b_i) \subseteq \bigcup_{j=1}^{i-1}\{a_j, b_j, c_j\}\right).$$

To simplify the discussion we introduce an auxiliary non-decreasing sequence of sets $\{\Phi\}_{n=1}^{\infty} \subseteq 2^{V(H)}$. Let us start constructing the isomorphism with $f(y_1) = a_1$, $f(z_1) = b_1$ and $\Phi_1 = \{a_1, b_1\}$.

Let us assume that, for any $n \in \mathbb{N}^+$, at the nth stage of the constructing of the isomorphism, values of the function f for all $v \in \Phi_n$ are already defined such way that $f \colon \Phi_n \to f(\Phi_n)$ is a bijection. Moreover Φ_n contains all x_i, y_i and z_i where $i < n$ and y_n and z_n, while $f(\Phi_n)$ contains all a_i, b_i, $i \leq n$ as well as c_i, $i < n$. Furthermore, assume that the set Φ_n has all six properties:

(i) $y_i \in \Phi_n$ if and only if $z_i \in \Phi_n$, for $i \in \mathbb{N}^+$,
(ii) $a_i \in f(\Phi_n)$ if and only if $b_i \in f(\Phi_n)$, for $i \in \mathbb{N}^+$,
(iii) for all $u, v \in \Phi_n$ $\{u, v\} \in P(H)$ implies $N^c(u) \cap N^c(v) \subseteq \Phi_n$,
(iv) for all $u, v \in f(\Phi_n)$ $\{u, v\} \in P(G)$ implies $N^c(u) \cap N^c(v) \subseteq f(\Phi_n)$,
(v) for all $u, v \in \Phi_n$ $\{u, v\} \in P(H)$ implies $f(\{u, v\}) \in P(G)$,
(vi) for all $u, v \in f(\Phi_n)$ $\{u, v\} \in P(G)$ implies $f^{-1}(\{u, v\}) \in P(H)$.

We intend to extend f and to find a suitable Φ_{n+1} in four steps:

$$\Phi_n \subseteq \Phi_n^1 \subseteq \ldots \subseteq \Phi_n^4 = \Phi_{n+1},$$

to satisfy $x_n \in \Phi_n^1$, $\{y_{n+1}, z_{n+1}\} \subseteq \Phi_n^2$, $c_n \in f(\Phi_n^3)$ and finally $\{a_{n+1}, b_{n+1}\} \subseteq f(\Phi_n^4)$.

To start with, let x_n be the next element of the sequence. If $x_n \in \Phi_n$, then it is enough to set $\Phi_n^1 = \Phi_n$. If not let us define sets:

$$U = \left\{v \in f(\Phi_n) \colon \{f^{-1}(v), x_n\} \in E(H)\right\} \quad \text{and} \quad U' = f(\Phi_n) \setminus U.$$

Note that Φ_n contains no 2-set of vertices with a finite common non-neighbourhood containing vertex x_n, and so $f^{-1}(U')$ and then U' contain no perturbating 2-set. From Definition 2.2*(iii)*, the graph G contains a vertex $u \in Q(G) \setminus f(\Phi_n)$, such that adjacency between the set Φ_n and the vertex x_n are just the same as those between $f(\Phi_n)$ with u. Let $f(x_n) = u$ and put $\Phi_n^1 = \Phi_n \cup \{x_n\}$. Note that Φ_n^1 keeps the assumed properties.

Similarly, let y_{n+1} and z_{n+1} be the next elements of the sequences. If $y_{n+1} \in \Phi_n^1$ (then also $z_{n+1} \in \Phi_n^1$), it is enough to set $\Phi_n^2 = \Phi_n^1$. If not, then let us proceed as follows. We denote by A and B the subsets:

$$A = \left\{v \in f(\Phi_n^1) \colon \{f^{-1}(v), y_{n+1}\} \in E(H)\right\},$$
$$B = \left\{v \in f(\Phi_n^1) \colon \{f^{-1}(v), z_{n+1}\} \in E(H)\right\},$$

and put $A' = f(\Phi_n^1) \setminus A$ and $B' = f(\Phi_n^1) \setminus B$. Note that the assumption for Φ_n, kept by the set Φ_n^1, implies that neither A' nor B' contains any perturbating 2-set. Moreover, as a result of the ordering of the sequences, given by Lemma 3.2, one has $N^c(y_{n+1}) \cap N^c(z_{n+1}) \subseteq \Phi_n \subseteq \Phi_n^1$. Hence, there is:

$$N^c(y_{n+1}) \cap N^c(z_{n+1}) = f^{-1}(A') \cap f^{-1}(B') = f^{-1}(A' \cap B').$$

By the application of the condition *(iv)* of the p.e.c. property for the graph G, we find a proper pair (u, u') to set $f(y_{n+1}) = u$, $f(z_{n+1}) = u'$. Let then $\Phi_n^2 = \Phi_n^1 \cup \{y_{n+1}, z_{n+1}\}$. Note that Φ_n^2 keeps the assumed properties.

Next, we should find correct arguments of the function f for which it has values c_n, a_{n+1} and b_{n+1}. We may achieve this by analogous construction, changing roles of the graphs G and H; that means, by construction of conditions due to adjacency in the graph G, with all vertices already used, and then by finding a suitable vertex or a 2-set of vertices, respectively, in the graph H. Details of these steps are omitted. At the end of the nth stage it is enough to put $\Phi_{n+1} = \Phi_n^4$. Note that now Φ_{n+1} contains all y_i, z_i for $i \leq n+1$ and x_i for $i \leq n$, while $f(\Phi_{n+1})$ — all a_i, b_i, $i \leq n+1$ as well as c_i, $i \leq n$ and $f \colon \Phi_{n+1} \to f(\Phi_{n+1})$ is a bijection. Moreover, Φ_{n+1} keeps the properties assumed for Φ_n.

To recapitulate, by the above inductive construction, the function $f\colon V(H) \to V(G)$ has been obtained. It is a bijection (this naturally follows from the construction that f is an injection) and preserves adjacency. Hence, the function f is an isomorphism. $\qquad\square$

Lemma 3.3. *Let k be any positive integer and x, y, z, w be non-negative integers such that $x < 2^k$, $y < 3^k$, $z < 5^k$, $w < 7^k$. There exists a positive integer $n \in \mathbb{N}^+$ such that $n \equiv x \pmod{2^k}$, $n \equiv y \pmod{3^k}$, $n \equiv z \pmod{5^k}$, $n \equiv w \pmod{7^k}$ and $n - w$ is written in base 7 without using digit 1.*

The above lemma is used in the proof that graph $\mathfrak{S}$ is p.e.c. Observe that the first k less significant digits of n written in base 7 are determined by the value of w, while the rest is the same as in $n - w$.

Proof. As 2^k, 3^k and 5^k are pairwise coprime, then from the Chinese remainder theorem, there exists a number $p \in \mathbb{N}$, $p < 30^k$ which satisfies $p \equiv x \pmod{2^k}$, $p \equiv y \pmod{3^k}$ and $p \equiv z \pmod{5^k}$. We intend to find an $r \in \mathbb{N}$ such that $r \equiv 0 \pmod{7^k}$ and $r \equiv p - w \pmod{30^k}$ and r is written in base 7 without using digit 1.

As 7 and 30^k are coprime, then according to the Fermat–Euler theorem one has $7^{\varphi(30^k)} \equiv 1 \pmod{30^k}$. Hence, we expect r to be a number which written in base 7 has non-zero digits only at positions which are multiples of $\varphi(30^k)$ as follows $r_{(7)} = a_1 00\ldots 00 a_2 00\ldots 00 a_3 00\ldots\ldots 000$. Then one has $r \equiv \sum_i a_i \pmod{30^k}$. Note that any integer between 2 and $30^k + 1$ can be presented as a sum of numbers from the set $\{2, 3, 4, 5, 6\}$. Hence, such number r exists. Setting $n = r + w$ we are done. $\qquad\square$

Proof of Theorem 2.6. To prove that the graph $\mathfrak{S}$ is p.e.c., we should show that it satisfies all conditions of Definition 2.2. First, according to Definition 2.2*(i)*, we show that $P(\mathfrak{S}) = \big\{\{3k+1, 3k+2\}\colon k \in \mathbb{N}\big\}$. For any other 2-set of vertices, say $\{x, y\}$, such that $x < y$ and $y \neq x + 1$ or $x \not\equiv 1 \pmod 3$, let us consider a subset of $N^c(x) \cap N^c(y)$, say U_{xy}, of suitably large multiples of 3. Elements of the set U_{xy} are then controlled by Definition 2.5*(i)(ii)* only. According to congruence of both x and y to $2 \pmod 3$, there are four cases to consider. As all the cases are similar, we show only one — when $x \not\equiv 2 \pmod 3$ and $y \equiv 2 \pmod 3$. Then set U_{xy} is defined as follows:

$$U_{xy} = \{z \in N^+\colon z \equiv 0 \pmod 3 \wedge z \geq 3^y \wedge z_{(2)}(x) = 0 \wedge z_{(2)}(y-1) = 1 \wedge z_{(3)}(y) \neq 1\}.$$

Note that in this case one has $x < y - 1$. Hence, as there is no dependency between fixed digits, the set U_{xy} is infinite. On the other hand, for any $k \in \mathbb{N}$ the number of elements in $N^c(3k+1) \cap N^c(3k+2)$ is bounded from above by the number of occurrences of digit 1 in the representation of $3k + 2$ in base 7. We have proved $P(\mathfrak{S}) = \big\{\{3k+1, 3k+2\}\colon k \in \mathbb{N}\big\}$. From this one it easily follows that $Q(\mathfrak{S})$ is defined by $Q(\mathfrak{S}) = \{3k\colon k \in \mathbb{N}\}$.

The condition *(ii)* follows from the fact, that due to Definition 2.5, for any 2-set of vertices $\{u, u'\} \in P(\mathfrak{S})$ and any $v \in \mathbb{N}^+$, if there is $v \in N^c(u) \cap N^c(u')$ then both $v < u$ and $v < u'$. Actually, without loss of generality let us assume that $u = u' - 1 \equiv 1 \pmod 3$, and fix any $n \in \mathbb{N}$ such that $u < 3n$. From Definition 2.5*(i)(ii)* it is obvious that u and $3n$ or $u+1$ and $3n$ are adjacent. Considering 2.5*(iv)(v)* we have that u and $3n + 1$ are adjacent if:

$$\big((3n+1)_{(2)}(u) = 1 \wedge (3n+2)_{(7)}(u) \neq 1\big) \vee (3n+2)_{(7)}(u+1) = 1.$$

On the other hand, $u+1$ and $3n+1$ are adjacent if, in particular, the above condition is false. Analogously, considering 2.5*(vii)(viii)* we have that u and $3n + 2$ are adjacent if in particular:

$$\Big(\big((3n+1)_{(2)}(u) = 0 \vee (3n+2)_{(3)} = 1\big) \wedge (3n+2)_{(7)}(u) \neq 1\Big) \vee (3n+2)_{(7)}(u+1) = 1,$$

while $u+1$ and $3n+2$ are if the above condition is false.

For any perturbating 2-sets of vertices, $\{u, u+1\}, \{v, v+1\} \in P(\mathfrak{S})$, there occurs $v \in N^c(u) \cap N^c(u+1)$ or $v+1 \in N^c(u) \cap N^c(u+1)$ only when $v < u$. Hence, the relation R, described in Definition 2.2*(ii)*, is a subrelation of the relation $\leq$ on minor elements of pairs, which of course is the linear order on the set $P(\mathfrak{S})$ and satisfies the condition that for any element the number of elements which are in relation $\leq$ with it is finite. The relation $\prec$, generated by R, is a subrelation of the relation $\leq$, satisfying reflexivity and transitivity, so it is a partial order and also satisfies the condition that number of elements which are in the relation $\prec$ with a given one is finite.

For the condition *(iii)* of the p.e.c. property, let us consider sets U and U' as shown in Definition 2.2. Let us denote $n = |U \cup U'|$ and build an increasing sequence $\{v_i\}_{i=1}^n$ composed of all vertices from

$U \cup U'$. Next, let us build a sequence $\{\epsilon_i\}_{i=1}^n$ putting for $i = 1, \ldots, n$, $\epsilon_i = 1$ if $v_i \in U$, or else $\epsilon_i = 0$. It is easy to notice that one has:

$$\bigcap_{v \in U} N(v) \cap \bigcap_{v \in U'} N^c(v) = \bigcap_{i=1}^n N^{\epsilon_i}(v_i).$$

The condition that the set U' contains no perturbating 2-set, according to the definition of $P(\mathfrak{S})$, means that for any i, $1 < i \le n$, if there occur $v_i \equiv 2 \pmod 3$ and $v_{i-1} = v_i - 1$, then $\epsilon_{i-1} = 1$ or $\epsilon_i = 1$.

Of course, if there exists such vertex u as in the condition being proved, it should satisfy $u \equiv 0 \pmod 3$. Let us assume, for simplification, that $u > v_n$. The looked for u, according to Definition 2.5$(i)(ii)$ has a part of digits in base 2 and 3 determined by the sequences $\{v_i\}$ and $\{\epsilon_i\}$. In order to find such u, let us construct, following the definition, two numbers x_2 and x_3 having $v_n + 1$ digits in base 2 and 3 respectively. Note that we permit to have digit zero on leading digits.

To start with, let us put digit zero in x_2 and x_3 everywhere. For $i = 1, \ldots, n$ successively, let us proceed as follows. If $v_i \not\equiv 2 \pmod 3$, then let us set $(x_2)_{(2)}(v_i) = \epsilon_i$. If both $v_i \equiv 2 \pmod 3$ and $\epsilon_i = 0$, let us set $(x_2)_{(2)}(v_i - 1) = 1$ and $(x_3)_{(3)}(v_i) = 0$. Notice that if $v_{i-1} = v_i - 1$, then one has $\epsilon_{i-1} = 1$. Finally, when both $v_i \equiv 2 \pmod 3$ and $\epsilon_i = 1$, then if additionally $v_{i-1} \ne v_i - 1$ or $\epsilon_{i-1} = 0$, the $(v_i - 1)$-th digit of $(x_2)_{(2)}$ is zero, otherwise that means when both $v_{i-1} = v_i - 1$ and $\epsilon_{i-1} = 1$, as $(x_2)_{(2)}(v_i - 1)$ should be 1, then let us set $(x_3)_{(3)}(v_i) = 1$.

The vertex u may be the smallest number which has first, less significant, $v_n + 1$ digits in base 2 like x_2 and first $v_n + 1$ digits in base 3 like x_3, increased by 6^{v_n+1} to obtain $u > v_n$. As 2^{v_n+1} and 3^{v_n+1} are coprime, such number of course exists, for instance according to the Chinese remainder theorem.

To prove the condition (iv), let us consider any finite set $U \subseteq V(\mathfrak{S})$ and any proper partitions $A \cup A'$ and $B \cup B'$ of the set U — all as shown in Definition 2.2(iv). Let us denote $n = |U|$ and build an increasing sequence $\{v_i\}_{i=1}^n$ composed of all vertices from U. Next, let us build sequences $\{\epsilon_i\}_{i=1}^n$ and $\{\delta_i\}_{i=1}^n$ setting for $i = 1, \ldots, n$, $\epsilon_i = 1$ if $v_i \in A$ or else $\epsilon_i = 0$, and respectively $\delta_i = 1$ if $v_i \in B$ or else $\delta_i = 0$. It is easy to notice that one has:

$$\bigcap_{v \in A} N(v) \cap \bigcap_{v \in A'} N^c(v) = \bigcap_{i=1}^n N^{\epsilon_i}(v_i) \qquad \text{and} \qquad \bigcap_{v \in B} N(v) \cap \bigcap_{v \in B'} N^c(v) = \bigcap_{i=1}^n N^{\delta_i}(v_i).$$

It follows from the definition of the set $P(\mathfrak{S})$, that for any i, $1 < i \le n$, if it occurs $v_i \equiv 2 \pmod 3$ and $v_{i-1} = v_i - 1$, then $\epsilon_{i-1} = 1$ or $\epsilon_i = 1$ and similarly $\delta_{i-1} = 1$ or $\delta_i = 1$. It also follows that if there exist such u and u' as in the condition being proved, one has $\max(u, u') = \min(u, u') + 1 \equiv 2 \pmod 3$. For simplicity, let us assume that both $u' > u$ and $u > v_n$.

The numbers u and $u' = u + 1$, according to Definition 2.5, will have a part of digits in bases 2, 3, 5 and 7 determined by the values of the sequences v_i, ϵ_i and δ_i. In order to find proper u and u', let us construct, according to the definition, auxiliary numbers x_2, x_3, x_5 and x_7 having $v_n + 1$ digits in base 2, 3, 5 and 7 respectively. As the least significant digit does not affect adjacency, let us set the zeroth digit of $(x_2)_{(2)}$ as 1, of $(x_3)_{(3)}$ as 2, of $(x_5)_{(5)}$ as 1 and of $(x_7)_{(7)}$ as 1. Then subtraction of 1 does not change any digit of $(x_j)_{(j)}$ except the least significant one.

The construction of x_7 is elementary. For every $i \le n$ such that $\epsilon_i = \delta_i = 0$ let us set v_i-th digit of $(x_7)_{(7)}$ as 1 and fill x_7 with zeros elsewhere. The construction of x_2, x_3 and x_5 is shown in Table 1. Note that any unfixed digit should be zero.

If $v_{i-1} \ne v_i - 1$, we proceed as the last three rows of Table 1 show (as if it occurred both $\epsilon_{i-1} = 1$ and $\delta_{i-1} = 1$). If both $v_{i-1} = v_i - 1$ and $\epsilon_{i-1} = \delta_{i-1} = 0$, we need to do nothing — then one has $\epsilon_i = \delta_i = 1$ and adjacency follows from conditions on the representation of u' in base 7.

Finally, the number u' should be a number which has the least significant $v_n + 1$ digits in base 2, 3, 5 and 7 as x_2, x_3, x_5 and x_7, respectively. Moreover, u' written in base 7 should have no digits 1, except written in x_7. According to Lemma 3.3, such number exists, while of course $u = u' - 1$. $\qquad \square$

4. Remarks on the construction

It easily follows from the construction of the graph $\mathfrak{S}$ that another example of the NN^c graph is the complement $\overline{\mathfrak{S}}$. Note, for the graph $\overline{\mathfrak{S}}$ a 2-set of non-adjacent vertices with finite common neighbourhood is the equivalent of a perturbating 2-set.

Let $i \leq n$. If $v_i \not\equiv 2 \pmod 3$ then:

ϵ_i	δ_i	$(x_2)_{(2)}(v_i)$	$(x_3)_{(3)}(v_i)$
0	1	0	0
1	0	1	0
1	1	1	1

When $v_i \equiv 2 \pmod 3$ and $v_{i-1} = v_i - 1$ then:

ϵ_{i-1}	δ_{i-1}	ϵ_i	δ_i	$(x_2)_{(2)}(v_i-1)$	$(x_3)_{(3)}(v_i-1)$	$(x_3)_{(3)}(v_i)$	$(x_5)_{(5)}(v_i)$
0	1	1	0	0	0	0	0
0	1	1	1	0	0	0	1
1	0	0	1	1	0	0	0
1	0	1	1	1	0	1	0
1	1	0	1	1	1	0	0
1	1	1	0	1	1	1	0
1	1	1	1	1	1	1	1

TABLE 1. Construction of x_2, x_3 and x_5

Theorem 4.1. *The graph $\overline{\mathfrak{S}}$ is NN^c and is not isomorphic to the graph $\mathfrak{S}$.*

Proof. Since the graph $\mathfrak{S}$ is NN^c, it is obvious that $\overline{\mathfrak{S}}$ is also. Considering Definition 2.2(i)(iii) one can notice that if for any $\alpha, \beta \in \{0,1\}$ and for any vertices $u, v \in V(\mathfrak{S})$ the set $N^\alpha(u) \cap N^\beta(v)$ is finite, then both $\alpha = \beta = 0$ and vertices u and v are adjacent. While the same holds in the graph $\overline{\mathfrak{S}}$ only when both $\alpha = \beta = 1$ and $u, v \in V(\overline{\mathfrak{S}})$ are non-adjacent. It is enough to show that $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ are not isomorphic. □

In a similar way we can construct other graphs with NN^c property, not isomorphic to the Rado graph and to the graphs shown above. We can exchange perturbating 2-sets to perturbating n-sets, for any $n \in \mathbb{N}^+, n > 2$, obtaining this way graphs with both NN^c and $(n-1)$-e.c. (but not n-e.c.) properties. However, all similarly constructed graphs contain the Rado graph as an induced subgraph.

Problem 4.2. *Determine if every graph with NN^c property is universal; that is, contains an induced subgraph isomorphic to the Rado graph.*

5. The tournament case

The n-e.c. property of undirected graphs can be naturally transferred to tournaments by treating the out- and in-neighbourhood of any vertex of a tournament in a similar way as the neighbourhood and non-neighbourhood of any vertex of an undirected graph. Note that the relation of adjacency is now anti-symmetric. For $n \in \mathbb{N}^+$, a tournament is n-e.c. if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B may be empty) there is a vertex z not in $A \cup B$ such that there is an arc from z to each vertex of A and there is an arc from each vertex of B to z. A tournament is e.c. if it is n-e.c. for every $n \in \mathbb{N}^+$. The random tournament T_∞ is defined analogously to the Rado graph — it is unique (up to isomorphism) countable e.c. tournament. An analogous explicit construction of the tournament T_∞ also works. The vertex set is $V(T_\infty) = \mathbb{N}$, while the edge set $E(T_\infty)$ is given by condition: for $x, y \in \mathbb{N}, x < y$ the vertex x dominates y if and only if one has $y_{(2)}(x) = 1$. See [1, 2] for more background on n-e.c. and e.c. tournaments.

Bonato's problem on infinite graphs has the natural equivalent in the case of tournaments. Suppose that a tournament T has the property (let us define it as the N^+N^- property) that subtournaments induced by out- and in-neighbourhood of each vertex of T are isomorphic to T. It is clear that T_∞ has this property, but which other tournaments have it?

Now we describe how to adjust our construction of the NN^c graph to obtain a N^+N^- tournament, non-isomorphic to T_∞. Let us denote by $N^+(v)$ and $N^-(v)$ the out- and in-neighbourhoods of a vertex v, respectively. Suppose that a tournament T has the N^+N^- property and let $u, v \in V(T)$ be two vertices, such that u dominates v. Then all three $N^+(u) \cap N^+(v)$, $N^+(u) \cap N^-(v)$ and $N^-(u) \cap N^-(v)$ induce a subtournament of T isomorphic to the whole T.

Definition 5.1. *Let T be a countable tournament and $u, v \in V(T)$ any of its vertices, such that u dominates v. If $N^-(u) \cap N^+(v)$ is finite, we say that the pair (u, v) is* perturbating. *We denote the set of all perturbating pairs of vertices by $P(T)$.*
If for any vertex $w \in V(T)$ the set $P(T)$ does not contain any pair with the vertex w, we say that the vertex w is non-perturbating. *We denote the set of all non-perturbating vertices by $Q(T)$.*

Definition 5.2. *Let T be a countable tournament. We say that T is* perturbed-existentially closed *(or p.e.c.) if and only if there occurs:*
- (i) *For every vertex $v \in V(T)$ there is at most one vertex $v' \in V(T)$, such that $(v, v') \in P(T)$ or $(v', v) \in P(T)$.*
- (ii) *In the set $P(T)$, the relation R, such that $(v, v')R(u, u')$ holds if and only if there is $v \in N^-(u) \cap N^+(u')$ or $v' \in N^-(u) \cap N^+(u')$, generates a partial order $\prec$. Moreover, for every $(u, u') \in P(T)$ the set $\big\{(v, v') \in P(T) \colon (v, v') \prec (u, u')\big\}$ is finite.*
- (iii) *For any finite, disjoint sets of vertices $U, U' \subseteq V(T)$, such that $U' \times U$ contains no perturbating pair, there exists a non-perturbating vertex $u \in Q(T) \setminus (U \cup U')$ such that $u \in \bigcap_{v \in U} N^+(v) \cap \bigcap_{v \in U'} N^-(v)$.*
- (iv) *For any finite set of vertices $U \subseteq V(T)$ and every partitions of the set U into two disjoint pieces $A \cup A' = U$ and $B \cup B' = U$ such that both $A' \times A$ and $B' \times B$ contain no perturbating pair, there exists a perturbating pair of vertices $(u, u') \in P(T)$, such that $\{u, u'\} \cap U = \emptyset$, which satisfies all three below:*

$$u \in \bigcap_{v \in A} N^+(v) \cap \bigcap_{v \in A'} N^-(v), \quad u' \in \bigcap_{v \in B} N^+(v) \cap \bigcap_{v \in B'} N^-(v), \quad \text{and } N^-(u) \cap N^+(u') = A \cap B'.$$

With the p.e.c. property of tournaments already defined, we are ready to formulate theorems on the isomorphism and N^+N^- property. Next, there is given the explicit construction of the p.e.c. tournament. As the proofs are analogous to the undirected case, they are omitted.

Theorem 5.3. *Let T be any p.e.c. tournament and $v \in V(T)$ any vertex. Then both $T[N^+(v)]$ and $T[N^-(v)]$ are p.e.c..*

Theorem 5.4. *Let T_1, T_2 be any p.e.c. tournament. Then T_1 and T_2 are isomorphic.*

Definition 5.5. *Let $\mathfrak{T} = (V(\mathfrak{T}), A(\mathfrak{T}))$ be a tournament with the vertex set $V(\mathfrak{T}) = \mathbb{N}^+$ and the edge set $E(\mathfrak{T})$ defined by the following conditions.*
For $n, v \in \mathbb{N}^+$, $v < 3n$, vertex v dominates vertex $u = 3n$ if and only if one has:
- (i) $v \not\equiv 2 \pmod 3 \wedge u_{(2)}(v) = 1,$
- (ii) $v \equiv 2 \pmod 3 \wedge u_{(2)}(v-1) = 1 \wedge u_{(3)}(v) = 1.$

For $n \in \mathbb{N}, v \in \mathbb{N}^+$, such that $v < 3n + 1$, vertex v dominates vertex $u = 3n + 1$ if and only if one has:
- (iii) $v \equiv 0 \pmod 3 \wedge (u_{(2)}(v) = 1 \vee (u+1)_{(7)}(v) = 1),$
- (iv) $v \equiv 1 \pmod 3 \wedge (u_{(2)}(v) = 1 \vee (u+1)_{(7)}(v) = 1 \vee (u+1)_{(7)}(v+1) = 1),$
- (v) $v \equiv 2 \pmod 3 \wedge ((u_{(2)}(v-1) = 1 \wedge u_{(3)}(v) = 1 \wedge (u+1)_{(7)}(v-1) \neq 1) \vee (u+1)_{(7)}(v) = 1).$

For $n \in \mathbb{N}, v \in \mathbb{N}^+$, such that $v < 3n + 2$, vertex v dominates vertex $u = 3n + 2$ if and only if one has:
- (vi) $v \equiv 0 \pmod 3 \wedge u_{(7)}(v) \neq 1 \wedge ((u-1)_{(2)}(v) = 1 \vee u_{(3)}(v) = 1),$
- (vii) $v \equiv 1 \pmod 3 \wedge u_{(7)}(v) \neq 1 \wedge ((u-1)_{(2)}(v) = 1 \vee u_{(3)}(v) = 1 \vee u_{(7)}(v+1) \vee v = u - 1),$
- (viii) $v \equiv 2 \pmod 3 \wedge u_{(7)}(v) \neq 1 \wedge u_{(7)}(v-1) \neq 1 \wedge (((u-1)_{(2)}(v-1) = 1 \wedge (u-1)_{(3)}(v) = 1) \vee$
  $\vee ((u-1)_{(2)}(v-1) = 0 \wedge u_{(3)}(v-1) \neq 1) \vee u_{(5)}(v) = 1).$

One can prove that from this construction it follows that $Q(\mathfrak{T}) = \{3n \colon n \in \mathbb{N}^+\}$, while $P(\mathfrak{T}) = \big\{(3n+1, 3n+2) \colon n \in \mathbb{N}\big\}$.

Theorem 5.6. *The tournament $\mathfrak{T}$ is perturbed-existentially closed.*

Corollary 5.7. *The tournament $\mathfrak{T}$ has the N^+N^- property, that means for every vertex $v \in V(\mathfrak{T})$ both $\mathfrak{T}[N^+(v)]$ and $\mathfrak{T}[N^-(v)]$ are isomorphic to $\mathfrak{T}$.*

It is worth noting one difference between the tournament case and the undirected one. The complement $\overline{\mathfrak{T}}$ is isomorphic to $\mathfrak{T}$, which follows by the p.e.c. property. The isomorphism between $\mathfrak{T}$ and $\overline{\mathfrak{T}}$ is an automorphism of $\mathfrak{T}$ which transposes vertices in every perturbating pair.

6. Acknowledgements

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