# Egyptian fractions and their modern continuation

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# 1 Purpose of the lecture

The first ideas of "number" date from the Upper Paleolithic period. Progress in the understanding of numbers and spatial relations occurred after the transition from gathering food for its production, from the hunted and fisheries to the agriculture. A breakthrough was made in the early second millennium BC in Mesopotamia (mathematical clay tablets) and Egypt (mathematical papyri).

What do we know today about ancient Egyptian mathematics?

Knowledge of Egyptian mathematics comes from the Rhind papyrus and the Moscow papyrus, which are described in many books, such as:

- O. Neugebauer, *Vorlesungen uber geschichte der antiken mathematischen Wissenschaften,* Berlin 1934,
- K. Vogel, *Vorgriechische Mathematik I Vorgeschichte und Agypten,* Hermann 1959,
- A B Chace, L S Bull, H P Manning and R C Archibald, The Rhind Mathematical Papyrus (Oberlin, Ohio, 1927-29),

-in Polish:

- A.P.Juszkiewicz, *Historia matematyki,* Tom I, PWN, Warszawa 1975 (in Polish),

from hundreds of web sites, as well as numerous articles, such as:

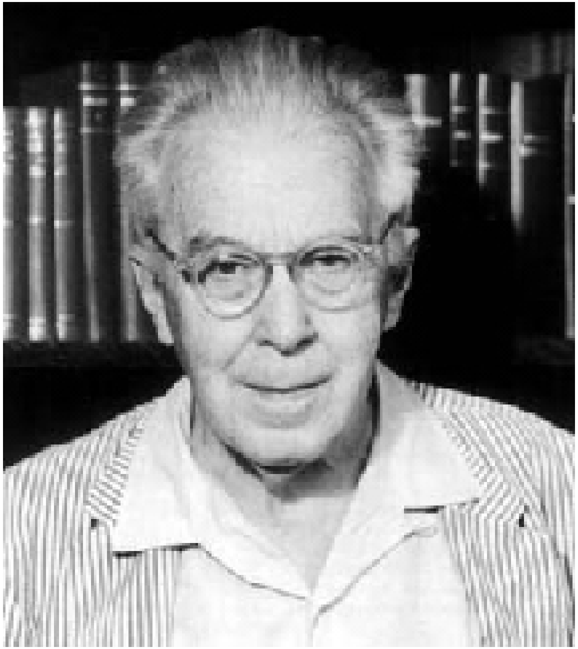
- M H Ahmadi, *On Egyptian fractions*, in Proceedings of the 21st. Annual, Iranian Mathematics Conference, Isfahan, 1990 (Isfahan, 1992), 1-20.

- E M Bruins, *Ancient Egyptian arithmetic: 2/N,* Nederl. Akad. Wetensch. Proc. Ser. A. Indagationes Math. 14 (1952), 81-91.
- R J Gillings, *The Egyptian 2/3 table for fractions*, The Rhind mathematical papyrus (B.M. 10057-8), Austral. J. Sci. 22 (1959), 247-250.
- A E Raik, *On the theory of Egyptian fractions* (Russian), Istor.-Mat. Issled. No. 23 (1978), 181-191; 358.
- C S Rees, *Egyptian fractions*, Math. Chronicle 10 (1-2) (1981/82), 13-30.
- G R Rising, *The Egyptian use of unit fractions for equitable distribution*, Historia Math. 1 (1) (1974), 93-94.
- S A Yanovskaya, *On the theory of Egyptian fractions* (Russian), Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya 1 (1947), 269-282.


It is also worth mentioning a recent very interesting article:


- Ch. Dorsett, *A solution for the Rhind papyrus unit fraction decompositions*, Texas College Mathematics Journal 2008, Volume 5, Number 1, Pages 1-4.


Historically, the first book covering the basic mathematics of ancient Egypt, including fractions, was a book by O. Neugebauer of 1934,



O. Neugebauer

and then by K.Vogel of 1959. The authors noted some patterns in the decompositions of the fraction $\frac{2}{n}$ into a sum of unit fractions (with numerators equal to 1) in the Rhind papyrus.

Szymon Weksler (from University of Lodz) in his work (1968):

- Sz.Weksler, *Decomposition of the fraction $\frac{2}{n}$ into a sum of unit fractions in the mathematics of ancient Egypt*, Zeszyty naukowe Uniwersytetu Łódzkiego, Łódź 1968 (in Polish),

presented a mathematical theory of so-called **regular decompositions** of fractions $\frac{2}{n}$ into sums of unit fractions.

It turns out that all decompositions (except three) of fractions from the **Rhind Papyrus arithmetic table** (shortly: **Rhind Table**) are regular in the sense of Weksler. The three irregular decompositions are better than all the regular ones because they have smaller last denominator. All researchers agree that the ancients regarded a decomposition of the fraction to be better if it had the last denominator smaller.

An insightful and revealing work by Sz.Weksler is written in Polish and is not known or cited in the literature on ancient Egyptian history, mathematics, even by specialists.

In 2006 a MA thesis by F.Fisiak

- Fisiak Marzena, *Unit fractions in Egyptian mathematics and their modern analysis,* Instytut Matematyki Politechniki Łódzkiej, Łódź 2006, (in Polish),

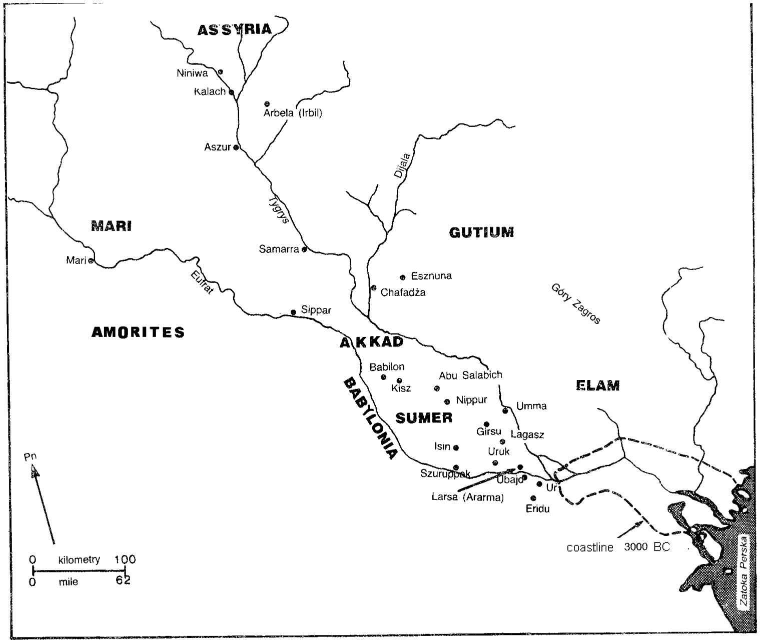
under my supervision presented (in detail) the results of Szymon Weksler for regular decompositions of $\frac{2}{n}$ into sums of unit fractions. The work also provided a computer program to generate regular decompositions.

**The purpose of this lecture** is to present these results and also put forward some hypotheses that relate of the Rhind Table and results of Weksler.

## 2 Historical overview

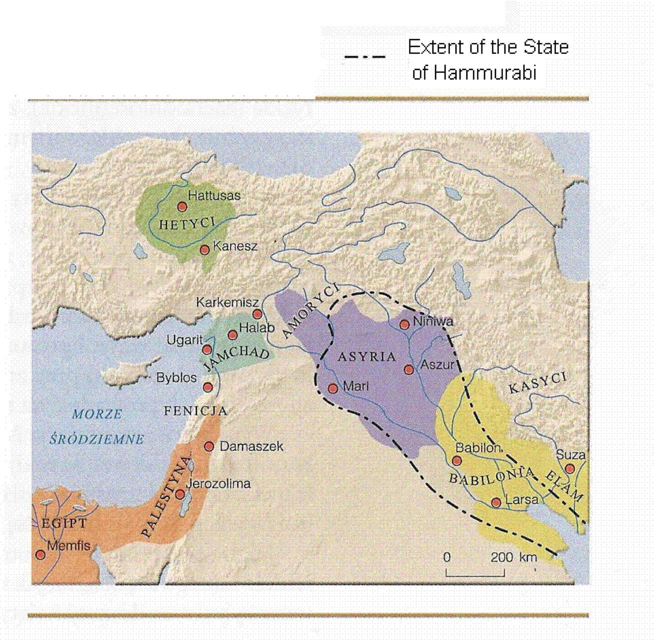
The oldest mathematical texts known today (Egyptian and Babylonian) date from the beginning of the second millennium BC. In Egypt, mathematical texts were written on fragile papyrus, sometimes on skin, so only those texts were preserved, which were deposited in pyramids. Babylonian texts were written on clay tablets, far more durable.

The beginning of the second millennium BC in Egypt was a period of Middle Kingdom (about 2060 – 1802 BC), XI and XII dynasty. It was preceded by the period of Old Kingdom (about 2686-2181 BC), III – VI dynasty, and the First Intermediate Period.




In 1930 in the ruins of the Zimri-Lim palace in Mari (Tell Hariri today) a huge archive of clay tablets was discovered. Mari was the main residence of the West Semitic nomadic tribe called Amorites (Sumerian: Martu, Akkadian: Amurrūm, Egyptian: Amar), from which the First Dynasty of Babylon derives (1894-1595 BC, after the Amorites took control of Sumerian state), with their most prominent representative Hammurabi.

The following map shows the State of Hammurabi




Extent of the State of Hammurabi

The archive covers the years 1810 – 1760 BC and informs, i.a. about the political manoeuvres of Hammurabi and his rivals.

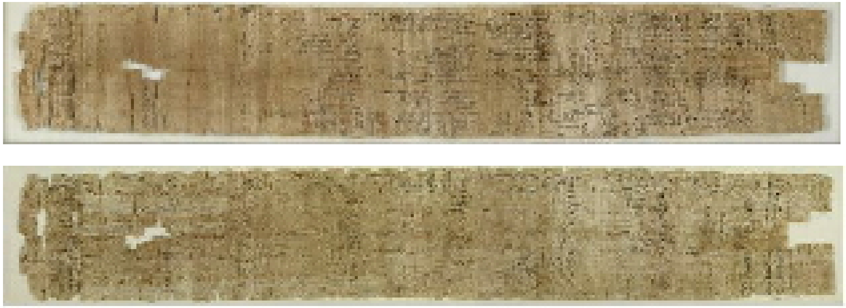
Those clay tablets also inform about contacts with Egyptian pharaohs of the XII dynasty (circa 1991 – 1802 BC). It was the time when mathematical clay tablets in Mesopotamia and mathematical papyri in Egypt were made. In the nineteenth century BC the original of the Rhind papyrus was manufactured [probably it does not exist any more], which 200 years later was copied by Ahmes (the copy is now known as the Rhind papyrus). The two powers, Babylon and Egypt, knew each other's scientific achievements as evidenced by similarities in the problems and equations.

It is of interest that the Rhind papyrus was made during the Hyksos Dynasty in Egypt (about 1674 to 1535 BC) of West-Semitic origin just as the First Dynasty of Babylon in Mesopotamia.

During the Old Kingdom Egyptians used hieroglyphs - pictorials, in which each figure represented a word or syllable. During the Middle Kingdom hieroglyphic writing was replaced by the simpler hieratic writing, in which every hieroglyph was turned into a few characteristic lines, and hieroglyphics were used only on extremely solemn occasions. In the New Kingdom the so-called condensed demotic writing appeared.

Egyptians usually wrote from right to left, in vertical lines.

Let us return to one of the oldest mathematical documents of the world, the so-called "Rhind papyrus," often called "Ahmes papyrus."



This papyrus was discovered around 1858 by a scientific expedition working in Upper Egypt (Luxor today). It come into possession of a Scottish antiquarian Alexander Henry Rhind, and therefore it is often called the Rhind papyrus. In 1864 it was bought by the British Museum.

Dr Richard Parkinson from the British Museum before the Rhind papyrus

Difficulties, which were related to reading it, were overcome by A.Eisenlohr, an Egyptologist, and M.Cantor a historian of mathematics.

Papyrus was first translated and published in print in 1877, it begins with the words:

> "Accurate reckoning for inquiring into things, and the knowledge of all things, mysteries...all secrets. This book was copied in regnal year 33, month 4 of Akhet, under the majesty of the King of Upper and Lower Egypt, Awserre, given life, from an ancient copy made in the time of the King of Upper and Lower Egypt Nimaatre. The scribe Ahmose writes this copy."

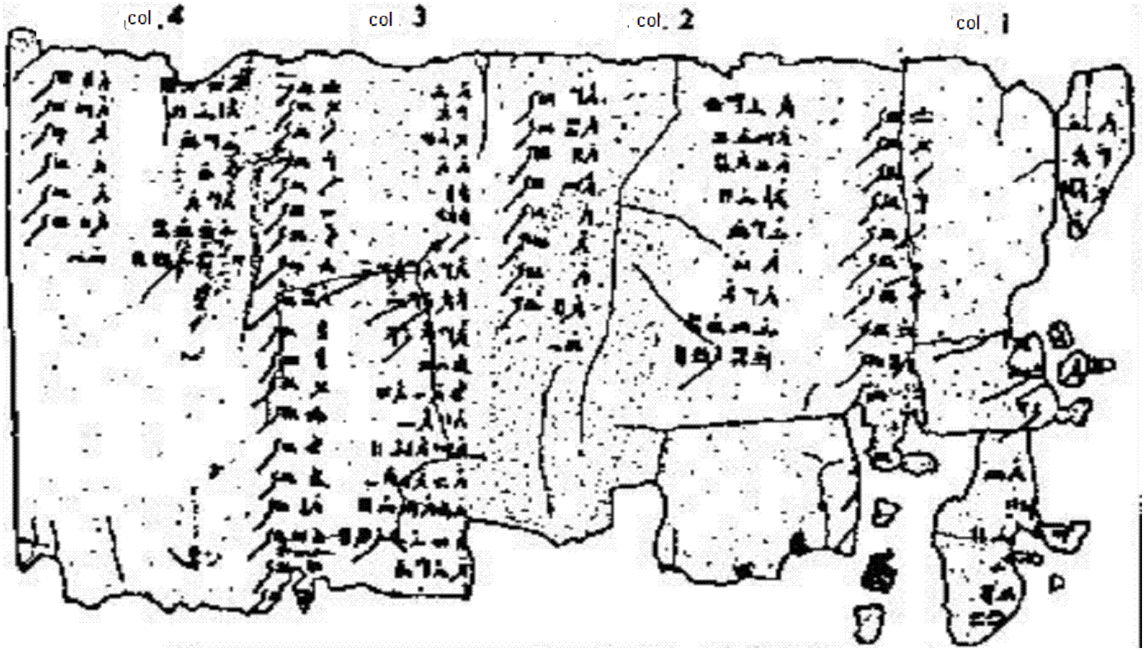
This information appears on the official website of the British Museum. There you can read that

"The Rhind Mathematical Papyrus is also important as a historical document, since the copyist noted that he was writing in year 33 of the reign of Apophis, the penultimate king of the Hyksos Fifteenth Dynasty (about 1650 – 1550 BC) and was copied after an original of the Twelfth Dynasty (about 1985 – 1795 BC)."

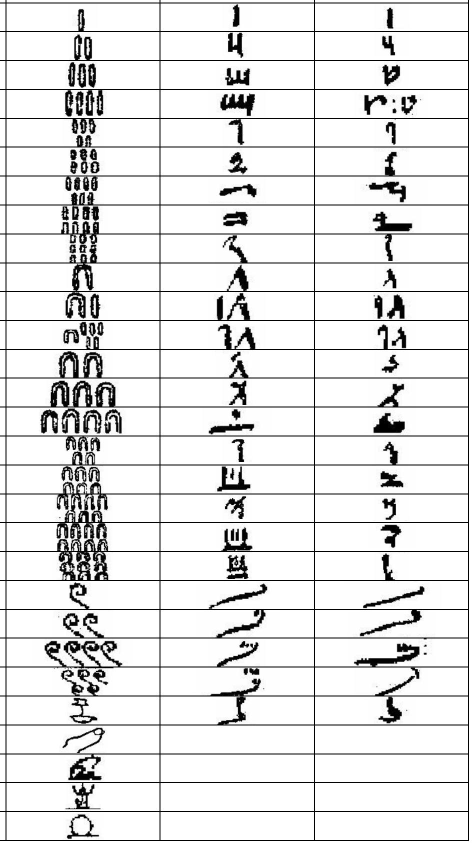
Papyrus has the shape of ribbons of length of nearly 5.25 m and width of 33 cm and contains probably everything that in that time was known to the Egyptians in arithmetic and geometry. It is written in hieratic characters, used in daily life, on papyrus.

# 3 Presentation of the Rhind Table of Egyptian fractions

A table of fractions at the beginning of the Rhind papyrus shows fractions of the form $\frac{2}{n}$ for odd integers from $n = 3$ to $n = 101$ as sums of two, three or four **different** unit fractions. Here is a fragment of the papyrus containing the table of decomposition of fractions.




It is easier to understand the meaning and use of Egyptian fractions by writing them in the earlier hieroglyphic writing, and not in hieratic writing because of the more "natural" signs of small numbers (certainly hieroglyphic writing was known to Ahmes and the original was written in hieroglyphic). The table below presents digits and greater numbers in hieroglyphic, hieratic and demotic writting:

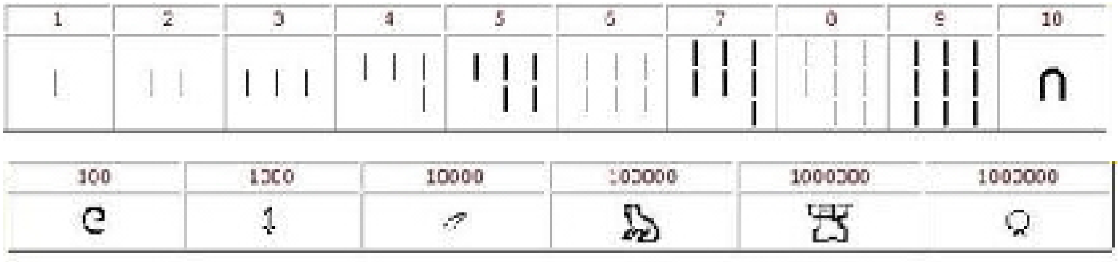


| | Hieroglify | Cyfry hieratyczne | Cyfry demotyczne |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |
| 8 | | | |
| **9** | | | |
| **10** | | | |
| **11** | | | |
| **15** | | | |
| **20** | | | |
| **30** | | | |
| **40** | | | |
| **50** | | | |
| **60** | | | |
| **70** | | | |
| **80** | | | |
| **90** | | | |
| **100** | | | |
| **200** | | | |
| **400** | | | |
| **500** | | | |
| **1000** | | | |
| **10000** | | | |
| **100000** | | | |
| **1000000** | | | |
| **10000000** | | | |

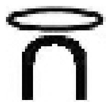
numbers / hieroglyphic / hieratic / demotic

Ancient Egyptians knew and used large numbers. This is evidenced by a document from the beginning of the First Dynasty, that is, about 3000 BC.

The Egyptian system of writing numbers was based on the number 10. The numbers appear in hieroglyphic writing thus:

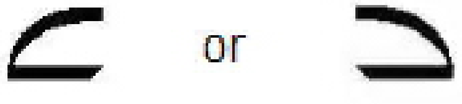



In addition to symbols for integers, Egyptians also had special symbols for fractions of the form $\frac{1}{n}$ and the fraction $\frac{2}{3}$. To write fraction, they used the same hieroglyphics as for natural numbers, adorning them with an oval placed above or by the sick, indicating reciprocal. For example, the hieroglyph

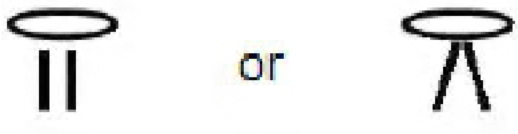


should be read as $\frac{1}{10}$. In other words, the oval above a hieroglyph is the same as exponent:$-1$ today.

The fraction $\frac{1}{2}$ had a special hieroglyphic form:




In addition of fractions with unit numerator, the ancients used the fraction $\frac{2}{3}$, which had its own hieroglyph form:




Egyptians did not use the general form of rational fractions $\frac{m}{n}$ (did not have a hieroglyph for such a fraction). Division $\frac{m}{n}$ was represented as multiplication $m \cdot \frac{1}{n}$ based on representation of $m$ in the form of a sum of several $2's$ and possibly a 1, for example

$$7 \cdot \frac{1}{5} = (2+2+2+1) \cdot \frac{1}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{1}{5}.$$

This method required the knowledge of decompositions of the fraction $\frac{2}{n}$ into unit fractions for odd $n$ - which justifies placing a table of such decompositions at the beginning of the Rhind papyrus. When $n$ is even the fraction $\frac{2}{n}$ is simplified by 2 and becomes a simple fraction, so there was no need to put it in the table.

Elementary use is illustrated by the following example: (from the Rhind table we read off: $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$ and $\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$)

$$
\begin{aligned}
7 \cdot \frac{1}{5} &= (2+2+2+1) \cdot \frac{1}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{1}{5} \\
&= \left(\frac{2}{5} + \frac{2}{5}\right) + \frac{2}{5} + \frac{1}{5} = 2 \cdot \frac{2}{5} + \frac{2}{5} + \frac{1}{5} \\
&= 2 \cdot \left(\frac{1}{3} + \frac{1}{15}\right) + \left(\frac{1}{3} + \frac{1}{15}\right) + \frac{1}{5} \\
&= \frac{2}{3} + \frac{2}{15} + \left(\frac{1}{3} + \frac{1}{15}\right) + \frac{1}{5} \\
&= 1 + \left(\frac{1}{10} + \frac{1}{30}\right) + \frac{1}{15} + \frac{1}{5} \\
&= 1 + \frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{30}.
\end{aligned}
$$

We give the decomposition into a sum of simple fractions with different denominators.

**Remark 1** *In what follows, by a decomposition into unit fractions, we shall always mean a decomposition with different denominators.*

A table of decompositions of fractions $\frac{2}{n}$ into sums of unit fractions of Rhind papyrus is in modern notation as follows:

| | | |
|---|---|---|
| 2/3 = 1/2 + 1/6 | 2/5 = 1/3 + 1/15 | 2/7 = 1/4 + 1/28 |
| 2/9 = 1/6 + 1/18 | 2/11 = 1/6 + 1/66 | 2/13 = 1/8 + 1/52 + 1/104 |
| 2/15 = 1/10 + 1/30 | 2/17 = 1/12 + 1/51 + 1/68 | 2/19 = 1/12 + 1/76 + 1/114 |
| 2/21= 1/14 + 1/42 | 2/23 = 1/12 + 1/276 | 2/25 = 1/15 + 1/75 |
| 2/27 = 1/18 + 1/54 | 2/29 = 1/24 + 1/58 + 1/174 + 1/232 | 2/31 = 1/20 + 1/124 + 1/155 |
| 2/33 = 1/22 + 1/66 | 2/35 = 1/30 + 1/42 | 2/37 = 1/24 + 1/111 + 1/296 |
| 2/39 = 1/26 + 1/78 | 2/41 = 1/24 + 1/246 + 1/328 | 2/43 = 1/42 + 1/86 + 1/129 + 1/301 |
| 2/45 = 1/30 + 1/90 | 2/47 = 1/30 + 1/141 + 1/470 | 2/49 = 1/28 + 1/196 |
| 2/51 = 1/34 + 1/102 | 2/53 = 1/30 + 1/318 + 1/795 | 2/55 = 1/30 + 1/330 |
| 2/57 = 1/38 + 1/114 | 2/59 = 1/36 + 1/236 + 1/531 | 2/61 = 1/40 + 1/244 + 1/488 + 1/610 |
| 2/63 = 1/42 + 1/126 | 2/65 = 1/39 + 1/195 | 2/67 = 1/40 + 1/335 + 1/536 |
| 2/69 = 1/46 + 1/138 | 2/71 = 1/40 + 1/568 + 1/710 | 2/73 = 1/60 + 1/219 + 1/292 + 1/365 |
| 2/75 = 1/50 + 1/150 | 2/77 = 1/44 + 1/308 | 2/79 = 1/60 + 1/237 + 1/316 + 1/790 |
| 2/81 = 1/54 + 1/162 | 2/83 = 1/60 + 1/332 + 1/415 + 1/498 | 2/85 = 1/51 + 1/255 |
| 2/87 = 1/58 + 1/174 | 2/89 = 1/60 + 1/356 + 1/534 + 1/890 | 2/91 = 1/70 + 1/130 |
| 2/93 = 1/62 + 1/186 | 2/95 = 1/60 + 1/380 + 1/570 | 2/97 = 1/56 + 1/679 + 1/776 |
| 2/99 = 1/66 + 1/198 | 2/101 = 1/101 + 1/202 + 1/303 + 1/606 | |

For $n$ divisible by 3 the decompositions were obtained using the following formula:

$$\frac{2}{3k} = \frac{1}{k} \cdot \frac{2}{3} = \frac{1}{k}\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{2k} + \frac{1}{6k}$$

for $k = 3, 5, \ldots, 33$. Decompositions for the composite number $n = k \cdot n_1$ are obtained (except in two cases $\frac{2}{35}$ and $\frac{2}{95}$) with similar decomposition for the factor $n_1$ by multiplying the denominators of the components under consideration by $k$. For example

$$\frac{2}{25} = \frac{1}{5} \cdot \frac{2}{5} = \frac{1}{5} \cdot \left(\frac{1}{3} + \frac{1}{15}\right) = \frac{1}{15} + \frac{1}{75},$$

$$\frac{2}{95} = \frac{1}{5} \cdot \frac{2}{19} = \frac{1}{5} \cdot \left(\frac{1}{12} + \frac{1}{76} + \frac{1}{114}\right) = \frac{1}{60} + \frac{1}{380} + \frac{1}{570}. \tag{1}$$

In any case (except $\frac{2}{45}$ and $\frac{2}{75}$) the number $n_1$ can be prime. For the fraction $\frac{2}{45}$ we choose the form $\frac{2}{45} = \frac{1}{5} \cdot \frac{2}{9}$ ($n_1 = 9$) and for $\frac{2}{75}$ the form $\frac{2}{75} = \frac{1}{5} \cdot \frac{2}{15}$ ($n_1 = 15$) and next we use the Rhind decomposition:.

$$\frac{2}{45} = \frac{1}{5} \cdot \frac{2}{9} = \frac{1}{5} \cdot \left(\frac{1}{6} + \frac{1}{18}\right) = \frac{1}{30} + \frac{1}{90}$$
$$\frac{2}{75} = \frac{1}{5} \cdot \frac{2}{15} = \frac{1}{5} \cdot \left(\frac{1}{10} + \frac{1}{30}\right) = \frac{1}{50} + \frac{1}{150}.$$

Of course, the decompositions for $\frac{2}{9}$ and $\frac{2}{15}$ were obtained using prime factors. In addition, note that using $\frac{2}{45} = \frac{1}{9} \cdot \frac{2}{5}$ and $\frac{2}{75} = \frac{1}{15} \cdot \frac{2}{5}$ (with $n_1$ prime) we obtain a "worse" decomposition because the last denominators are larger (in calculations, smaller denominators are more favourable):

$$\frac{2}{45} = \frac{1}{9} \cdot \frac{2}{5} = \frac{1}{9} \cdot \left(\frac{1}{3} + \frac{1}{15}\right) = \frac{1}{27} + \frac{1}{135}$$
$$\frac{2}{75} = \frac{1}{15} \cdot \frac{2}{5} = \frac{1}{15} \cdot \left(\frac{1}{3} + \frac{1}{15}\right) = \frac{1}{45} + \frac{1}{225}.$$

The previously mentioned fractions $\frac{2}{35}$ and $\frac{2}{91}$ are in the Rhind Table decomposed as follows:

$$\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$$
$$\frac{2}{91} = \frac{1}{70} + \frac{1}{130}.$$

From the decomposition $35 = 5 \cdot 7$ by this method we obtain the decompositions

$$\frac{2}{35} = \frac{1}{5} \cdot \frac{2}{7} = \frac{1}{5} \cdot \left(\frac{1}{4} + \frac{1}{28}\right) = \frac{1}{20} + \frac{1}{140}$$
$$\frac{2}{35} = \frac{1}{7} \cdot \frac{2}{5} = \frac{1}{7} \cdot \left(\frac{1}{3} + \frac{1}{15}\right) = \frac{1}{21} + \frac{1}{105}.$$

And from $91 = 7 \cdot 13$ we obtain

$$\frac{2}{91} = \frac{1}{13} \cdot \frac{2}{7} = \frac{1}{13} \cdot \left(\frac{1}{4} + \frac{1}{28}\right) = \frac{1}{52} + \frac{1}{364}$$
$$\frac{2}{91} = \frac{1}{7} \cdot \frac{2}{13} = \frac{1}{7} \cdot \left(\frac{1}{8} + \frac{1}{52} + \frac{1}{104}\right) = \frac{1}{56} + \frac{1}{364} + \frac{1}{728}.$$

We observe, however, that the decompositions of the Rhind papyrus have smaller last denominators. The above two denominators obey the rule indicated by K.Vogel,

$$\frac{2}{p \cdot q} = \frac{1}{p \cdot \frac{p+q}{2}} + \frac{1}{q \cdot \frac{p+q}{2}},$$

which has been used also elsewhere in the papyrus.

Summing up, the selection criterion of the decomposition of a given fraction $\frac{2}{n}$ is determined by the decomposition of fractions $\frac{2}{n_1}$ for $n_1$ prime. Therefore it remains to consider the decompositions of $\frac{2}{n}$ from the Rhind Table only for $n$ prime. The decomposition of $\frac{2}{n}$ for $n$ prime into a sum of unit fractions is of course not unique, e.g.

$$\frac{2}{5} = \frac{1}{3}+\frac{1}{15} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20},$$
$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28} = \frac{1}{6} + \frac{1}{14} + \frac{1}{21},$$
$$\frac{2}{13} = \frac{1}{7} + \frac{1}{91} = \frac{1}{8}+\frac{1}{52}+\frac{1}{104} = \frac{1}{12} + \frac{1}{26} + \frac{1}{39} + \frac{1}{156}.$$

Therefore, researchers have long put up fundamental questions about the Rhind Table:

- Which criteria were used by the ancients to select a decomposition?
- Can one give an algorithm producing the distributions of the Rhind Table ?
- Is there any regularity in the decompositions of fractions from the papyrus for $n$ prime?

There were many attempts to answer these questions. Recently the paper

- Ch. Dorsett, *A solution for the Rhind papyrus unit fraction decompositions*, Texas College Mathematics Journal 2008, Volume 5, Number 1, Pages 1-4 gives a way of finding decompositions of $\frac{2}{n}$ into a sum of unit fractions (which at first does not give a uniquely determined result) but always gives the decomposition from the Rhind Table. The method consists in finding a number $p$ and an odd number $o$ such that

$$n + o = 2p$$

and

$$\frac{2}{n} = \frac{n+o}{np}$$

and then decomposing $o$ into a sum of decreasing divisors of $p$. Second, the author looks for a method of choosing $p$ and $o$ to get the decomposition from the Rhind Table.

I want to present Szymon Weksler concept of regular decomposition for $n$ prime. It is not unique but it is interesting that all decompositions from the Rhind papyrus except one are regular in this sense [and this one is so much better than regular, it gives less last denominator (it is favourable in calculations)].

On a web page that no longer exists, there was a program to find all possible decompositions of the fraction $\frac{2}{n}$ into unit fractions with denominators not exceeding a given number $N$, eg, for $\frac{2}{17}$ and $N = 250$ there are 5 possible decompositions as sums of three unit fractions with different denominators:

$2/17 = 1/10 + 1/85 + 1/\underline{\mathbf{170}}$ – regular in the sense of Sz.Weksler
$2/17 = 1/10 + 1/90 + 1/153$
$2/17 = 1/12 + 1/34 + 1/\underline{\mathbf{204}}$ – regular in the sense of Sz.Weksler
$2/17 = 1/12 + 1/36 + 1/153$
$2/17 = 1/12 + 1/51 + 1/\underline{\mathbf{68}}$ – Rhind and regular in the sense of Sz.Weksler (smallest last denominator)

# 4 Regularity in the sense of Sz.Weksler

## 4.1 Definition of regularity in the sense of Sz.Weksler

**Definition 2 (Sz.Weksler)** *A $(p+1)$-term decomposition*

$$\frac{2}{n} = \frac{1}{x} + \sum_{j=1}^{p} \frac{1}{ny_j}$$

*is called regular if*

$$x \in \left(\frac{n}{2}, n\right), \;\; y_j < y_{j+1}, \;\; j = 1, ..., p-1; \;\; x = \mathrm{LCM}\left(y_j, ..., y_p\right).$$

**Remark 3** *All decompositions from the Rhind Table for $n$ prime except $n = 101$ are regular (among them there are 2 -, 3 - and 4-term decompositions).*

In the MA thesis (in Polish)

- M.Fisiak, *Unit fractions in Egyptian mathematics and their modern analysis*, Instytut Matematyki Politechniki Łódzkiej, Łódź 2006,

in addition to accurately reporting on the results of Sz.Weksler, a program is given to generate all regular 2 -, 3-, 4-, 5-, and 6-term decompositions.

## 4.2 2-term regular decompositions

By definition, 2-term regular decompositions of $\frac{2}{n}$ are of the form $\frac{2}{n} = \frac{1}{x} + \frac{1}{nx}$, where $x \in \left(\frac{n}{2}, n\right)$.

**Theorem 4 (Sz.Weksler)** *For a prime number $n \geq 3$ there is exactly one regular 2-term decomposition of $\frac{2}{n}$ and it is of the form*

$$\frac{2}{n} = \frac{1}{x} + \frac{1}{nx}$$

*for*

$$x = \frac{n+1}{2}.$$

**Theorem 5 (Sz.Weksler)** *If $n$ is a prime, and $\frac{2}{n} = \frac{1}{x} + \frac{1}{ny_1}$, where $x < n$, then $y_1 = x$ and the decomposition is regular.*

**Theorem 6 (Sz.Weksler)** *If $n$ is a prime then there is exactly one decomposition*

$$\frac{2}{n} = \frac{1}{x} + \frac{1}{z}, \qquad \text{where } x < z.$$

**Corollary 7** *If $n$ is a prime then every 2-terrm decomposition $\frac{2}{n} = \frac{1}{x} + \frac{1}{z}$, where $x < z$, is regular. In particular, all 2-term decompositions of $\frac{2}{n}$ from the Rhind Table for $n$ prime must be regular, because there are no other decompositions with different denominators. So the ancients do not deserve the credit for the fact that the 2-term decompositions from the Rhind Table for $n$ prime are regular.*

In twenty Rhind fractions the decompositions have more than 2 terms, 19 of them involve prime numbers. There remains the fraction $\frac{2}{95}$ with composite denominator and regular 3-term decomposition $\frac{2}{95} = \frac{1}{60} + \frac{1}{380} + \frac{1}{570} = \frac{1}{60} + \frac{1}{95 \cdot 4} + \frac{1}{95 \cdot 6}$, considered before.

## 4.3 Regular 3-term decompositions

Let us recall: by definition, for $p = 2$, a 3-term regular decomposition of $\frac{2}{n}$ is of the form

$$\frac{2}{n} = \frac{1}{x} + \frac{1}{ny_1} + \frac{1}{ny_2} \tag{2}$$

where $x \in \left(\frac{n}{2}, n\right)$, $y_1 < y_2$ and $x = \text{LCM}\left(y_1, y_2\right)$.

**Theorem 8 (Sz.Weksler)** *For a prime $n > 3$, all regular 3-term decompositions (2) are obtained by assuming that*

$$x = d \cdot \text{LCM}\left(\lambda_1, \lambda_2\right), \quad y_1 = d \cdot \lambda_1, \quad y_2 = d \cdot \lambda_2,$$

*where $d, \lambda_1, \lambda_2,\ \lambda_1 < \lambda_2$, satisfy the following equations*

$$\text{LCM}\left(\lambda_1, \lambda_2\right) \cdot \left[2d - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right] = n \tag{3}$$

$$\mathrm{GCD}\,(\lambda_1, \lambda_2) = 1, \quad d \cdot \mathrm{LCM}\,(\lambda_1, \lambda_2) \in \left(\frac{n}{2}, n\right).$$

**Proposition 9 (Sz.Weksler)** *Let $d, \lambda_1, \lambda_2$ be natural numbers. Suppose that $\mathrm{GCD}\,(\lambda_1, \lambda_2) = 1$ and $x = d \cdot \mathrm{LCM}\,(\lambda_1, \lambda_2)$, $y_1 = d \cdot \lambda_1$, $y_2 = d \cdot \lambda_2$. Then equality (3) is equivalent to:*

$$2x - (\lambda_1 + \lambda_2) = n.$$

The above proposition is used in the proof of the following theorem.

**Theorem 10 (Sz.Weksler)** *(A) There is no 3-term regular decomposition for $\frac{2}{3}$. (B) If $n \geq 5$ is prime then there exists at least one 3-term regular decomposition of $\frac{2}{n}$, and their number is finite.*

**Remark 11** *In the Rhind Table all 3-term decompositions of $\frac{2}{n}$ are regular in the sense of Sz.Weksler !*

### 4.4 Regular 4-term decompositions

Let us recall: by definition, for $p = 3$, a 4-term regular decomposition of $\frac{2}{n}$ is of the form

$$\frac{2}{n} = \frac{1}{x} + \frac{1}{ny_1} + \frac{1}{ny_2} + \frac{1}{ny_3} \tag{4}$$

where $x \in \left(\frac{n}{2}, n\right)$, $y_1 < y_2 < y_3$, $x = \mathrm{LCM}\,(y_1, y_2, y_3)$.

In analogy to the theorem for 3-term regular decompositions we get:

**Theorem 12 (Sz.Weksler)** *For a prime $n \geq 13$, all regular 4-term decompositions (4) are obtained by assuming that:*

$$x = d \cdot \mathrm{LCM}(\lambda_1, \lambda_2, \lambda_3), \quad y_1 = d \cdot \lambda_1, \quad y_2 = d \cdot \lambda_2,\ y_3 = d \cdot \lambda_3,$$

*where $d, \lambda_1, \lambda_2, \lambda_3,\ \lambda_1 < \lambda_2 < \lambda_3$, satisfy the following equations:*

$$\mathrm{LCM}(\lambda_1, \lambda_2, \lambda_3) \cdot \left[2d - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)\right] = n$$

$$\mathrm{GCD}(\lambda_1, \lambda_2, \lambda_3) = 1, \qquad d \cdot \mathrm{LCM}(\lambda_1, \lambda_2, \lambda_3) \in \left(\frac{n}{2}, n\right).$$

**Theorem 13 (Sz.Weksler)** *For primes $n < 13$ there is no 4-term regular decomposition of $\frac{2}{n}$.*

In the thesis by M. Fisiak the regular decomposition

$$\frac{2}{9} = \frac{1}{8} + \frac{1}{9 \cdot 2} + \frac{1}{9 \cdot 4} + \frac{1}{9 \cdot 8}$$

is found, and $n = 9$ is the only odd number smallest than 13 for which there is a 4-term regular decomposition.

**Theorem 14 (Sz.Weksler)** *For every prime $n \geq 13$ there exists at least one 4-term regular decomposition of $\frac{2}{n}$, and their number is finite.*

**Remark 15** *In the Rhind Table there are eight 4-term decompositions and all relate to prime numbers. All of these decompositions, except one (the last* $\frac{2}{101} = \frac{1}{101} + \frac{1}{202} + \frac{1}{303} + \frac{1}{606}$*), are regular! Why is this last decomposition in the Rhind Table irregular? Well, because all the 4-term regular decompositions have a much bigger last denominator. The smallest last denominator has the regular decomposition*

$$\frac{2}{101} = \frac{1}{60} + \frac{1}{101 \cdot 6} + \frac{1}{101 \cdot 12} + \frac{1}{101 \cdot 15}.$$

*We now give 2- and 3-term regular decompositions of* $\frac{2}{101}$*:*

$$\frac{2}{101} = \frac{1}{51} + \frac{1}{101 \cdot 51},$$
$$\frac{2}{101} = \frac{1}{56} + \frac{1}{101 \cdot 8} + \frac{1}{101 \cdot 14}.$$

*We observe that the criterion of the smallest last denominator is employed here:* $606 < 101 \cdot 14 < 101 \cdot 15 < 101 \cdot 51$. *The above Rhind decomposition obeys the more general rule*

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{n \cdot 2} + \frac{1}{n \cdot 3} + \frac{1}{n \cdot 6}.$$

In summary, in the Rhind Table all 3- and 4-term decompositions for $n$ prime (except the last one, $\frac{1}{101}$) are regular and among those 19 regular cases only in 4 cases there is a slight derogation from the principle of the smallest last denominators. The derogation concerns only the fractions

$$\frac{2}{13} = \frac{1}{8} + \frac{1}{13 \cdot 4} + \frac{1}{13 \cdot 8}$$ – Rhind and regular in the sense of Sz.Weksler.

$$\frac{2}{13} = \frac{1}{10} + \frac{1}{13 \cdot 2} + \frac{1}{13 \cdot 5}$$ – regular in the sense of Sz.Weksler and "the best"

"the best" =the smallest last denominator

$$\frac{2}{61} = \frac{1}{40} + \frac{1}{61 \cdot 4} + \frac{1}{61 \cdot 8} + \frac{1}{61 \cdot 10} \quad \text{– Rhind and regular in the sense of Sz.Weksler.}$$
$$\frac{2}{61} = \frac{1}{45} + \frac{1}{61 \cdot 3} + \frac{1}{61 \cdot 5} + \frac{1}{61 \cdot 9} \quad \text{– regular in the sense of Sz.Weksler and "the best"}$$

$$\frac{2}{71} = \frac{1}{40} + \frac{1}{71 \cdot 8} + \frac{1}{71 \cdot 10} \quad \text{– Rhind and regular in the sense of Sz.Weksler.}$$
$$\frac{2}{71} = \frac{1}{42} + \frac{1}{71 \cdot 6} + \frac{1}{71 \cdot 7} \quad \text{– regular in the sense of Sz.Weksler and "the best"}$$

$$\frac{2}{89} = \frac{1}{60} + \frac{1}{89 \cdot 4} + \frac{1}{89 \cdot 6} + \frac{1}{89 \cdot 10} \quad \text{– Rhind and regular in the sense of Sz.Weksler.}$$
$$\frac{2}{89} = \frac{1}{63} + \frac{1}{89 \cdot 3} + \frac{1}{89 \cdot 7} + \frac{1}{89 \cdot 9} \quad \text{– regular in the sense of Sz.Weksler and "the best".}$$

# 5 Conclusion for Egyptian fractions

Decompositions of Egyptians fractions in the Rhind Table have the following properties:

- the denominator of the first (except for $n = 101$), the largest component of the decomposition is contained in the interval $\left(\frac{n}{2}, n\right)$,
- for $n$ prime (except $n = 101$) the denominator of the first fraction is the LCM of the quotients of the remaining denominators by $n$,

- all decompositions from the Rhind Table (except three cases, $\frac{2}{35}=\frac{1}{30}+\frac{1}{42}$, $\frac{2}{91}=\frac{1}{70}+\frac{1}{130}$, $\frac{2}{101}=\frac{1}{101}+\frac{1}{202}+\frac{1}{303}+\frac{1}{606}$) are regular. The three irregular decompositions are "better" than any regular decomposition because they have smaller the last denominator. These irregular decompositions can of course be obtained by the method of Dorsett:

$$\frac{2}{35}=\frac{2}{5\cdot 7}=\frac{35+5\cdot 5}{30\cdot 35}=\frac{1}{30}+\frac{1}{42}\quad /\ o=5\cdot 5,\ p=\frac{35+o}{2}=30,$$
$$\frac{2}{91}=\frac{2}{7\cdot 13}=\frac{91+7\cdot 7}{91\cdot 70}=\frac{1}{70}+\frac{1}{130}\quad /\ o=7\cdot 7,\ p=\frac{91+o}{2}=70,$$
$$\frac{2}{101}=\frac{6+3+2+1}{101\cdot 6}\quad /\ p=6=1\cdot 2\cdot 3,\ o=1+2+3.$$

- decompositions for $n$ composite (except in two cases 35 and 95) are obtained from the corresponding decompositions for prime numbers [it remains a mystery how ancients came to those decompositions: whether they used previously obtained decompositions for smaller prime numbers or came up with the method given by Dorsett].

# 6 Hypotheses concerning 5- and 6-term regular decompositions for $\frac{2}{n}$, for $n$ prime

Consideration of 5- and 6-term decompositions of $\frac{2}{n}$ can be regarded in a sense a contemporary continuation of the study of Egyptian fractions.

For $k$-term decompositions with $k=3$ and $k=4$ Sz.Weksler discovered the following rule:

- there is a positive integer $N$ such that for prime $n < N$, there is no $k$-term decomposition of $\frac{2}{n}$ and for $n \geq N$ there is at least one, and their number is finite (for $k = 3,\ N = 5$, for $k = 4,\ N = 13$).

The computer program presented in M. Fisiak's thesis allows one to verify the hypothesis of the existence of such a number $N$ for $k = 5$ and $k = 6$: it turns out that for $k = 5$, no such $N$ exists.

## 6.1 5-term regular decompositions

Let $k = 5$, and let $n$ be odd.

**Theorem 16** *For odd $n < 17$ there is no 5-term regular decomposition of $\frac{2}{n}$.*

The proof is based on a computer program by Mrs. M.Fisiak.

**Example 17** *Examples of 5- term regular decompositions of $\frac{2}{n}$ for $n \geq 17$ :*

- $\frac{2}{17} = \frac{1}{16} + \frac{1}{17 \cdot 2} + \frac{1}{17 \cdot 4} + \frac{1}{17 \cdot 8} + \frac{1}{17 \cdot 16}$ *and this is the only solution,*
- $\frac{2}{19}$ *has no regular decomposition,*
- $\frac{2}{21} = \frac{1}{18} + \frac{1}{21 \cdot 2} + \frac{1}{21 \cdot 6} + \frac{1}{21 \cdot 9} + \frac{1}{21 \cdot 18}$ *and this is the only solution,*
- $\frac{2}{23} = \frac{1}{20} + \frac{1}{23 \cdot 2} + \frac{1}{23 \cdot 5} + \frac{1}{23 \cdot 10} + \frac{1}{23 \cdot 20}$ *and this is the only solution,*

- $\frac{2}{25} = \begin{cases} \frac{1}{24} + \frac{1}{25\cdot 2} + \frac{1}{25\cdot 4} + \frac{1}{25\cdot 6} + \frac{1}{25\cdot 24} \\ \frac{1}{24} + \frac{1}{25\cdot 2} + \frac{1}{25\cdot 4} + \frac{1}{25\cdot 8} + \frac{1}{25\cdot 12} \\ \frac{1}{24} + \frac{1}{25\cdot 2} + \frac{1}{25\cdot 3} + \frac{1}{25\cdot 12} + \frac{1}{25\cdot 24} \end{cases}$ *there are three solutions,*
- $\frac{2}{27} = \begin{cases} \frac{1}{24} + \frac{1}{27\cdot 2} + \frac{1}{27\cdot 4} + \frac{1}{27\cdot 12} + \frac{1}{27\cdot 24} \\ \frac{1}{24} + \frac{1}{27\cdot 2} + \frac{1}{27\cdot 6} + \frac{1}{27\cdot 8} + \frac{1}{27\cdot 12} \\ \frac{1}{24} + \frac{1}{27\cdot 3} + \frac{1}{27\cdot 4} + \frac{1}{27\cdot 6} + \frac{1}{27\cdot 8} \end{cases}$ *there are three solutions, etc.*

This supports the hypothesis:

**Conjecture 18** *For every prime $n \geq 23$ there exists at least one 5-trrm regular decomposition of $\frac{2}{n}$, and their number is finite.*

A similar situation is for 6-terrm regular decompositions.

## 6.2 6-term decompositions

Let $k = 6$, and let $n$ be odd.

**Theorem 19** *For odd $n < 25$ there is no 6-term regular decomposition of $\frac{2}{n}$.*

The proof is based on a computer program by Mrs. M.Fisiak.

**Example 20** *Examples of 6- term regular decompositions of $\frac{2}{n}$ for $n \geq 25$ :*

- $\frac{2}{25} = \frac{1}{24} + \frac{1}{25\cdot 3} + \frac{1}{25\cdot 4} + \frac{1}{25\cdot 6} + \frac{1}{25\cdot 8} + \frac{1}{25\cdot 12}$ *and this is the only solution,*

- $\frac{2}{27} = \frac{1}{24} + \frac{1}{27 \cdot 3} + \frac{1}{27 \cdot 4} + \frac{1}{27 \cdot 6} + \frac{1}{27 \cdot 12} + \frac{1}{27 \cdot 24}$ *and this is the only solution,*
- $\frac{2}{29}$ *- has no regular decomposition,*
- $\frac{2}{31} = \frac{1}{30} + \frac{1}{31 \cdot 2} + \frac{1}{31 \cdot 5} + \frac{1}{31 \cdot 6} + \frac{1}{31 \cdot 15} + \frac{1}{31 \cdot 30}$ *and this is the only solution,*
- $\frac{2}{33} = \begin{cases} \frac{1}{32} + \frac{1}{33 \cdot 2} + \frac{1}{33 \cdot 4} + \frac{1}{33 \cdot 8} + \frac{1}{33 \cdot 16} + \frac{1}{33 \cdot 32} \\ \frac{1}{30} + \frac{1}{33 \cdot 2} + \frac{1}{33 \cdot 5} + \frac{1}{33 \cdot 10} + \frac{1}{33 \cdot 15} + \frac{1}{33 \cdot 30} \end{cases}$ *there are two solutions,*
- *for* $\frac{2}{35}$ *and and* $\frac{2}{37}$ *there are single decompositions, etc.*

This supports the hypothesis:

**Conjecture 21** *For every prime* $n \geq 31$ *there exists at least one* 6*-term regular decomposition of* $\frac{2}{n}$*, and their number is finite.*

THANK YOU VERY MUCH FOR YOUR ATTENTION