Lie algebroids: spectral sequences and signature

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Abstract. It is proved that for any transitive Lie algebroid L on a compact oriented connected manifold with unimodular isotropy Lie algebras and trivial monodromy the cohomology algebra is a Poincaré algebra with trivial signature. Examples of such algebroids are algebroids on simply connected manifolds, algebroids such that the outer automorphism group of the isotropy Lie algebra is equal to its inner automorphism group, or such that the adjoint Lie algebra bundle $\mathbf{g}$ induces a trivial homology bundle $H^*(\mathbf{g})$ in the category of flat bundles.

Bibliography: 27 titles.

§1. Introduction

1.1. Category of Lie algebroids.

1.1.1. Lie algebroids appear as infinitesimal objects of Lie groupoids, principal fibre bundles, vector bundles ([1], see also [2]–[4]), TC-foliations and non-closed Lie subgroups ([5], see also [6], [7]), Poisson manifolds (see [8]), and so on. Their algebraic analogues are known as Lie pseudoalgebras [9], also called Lie–Reinhardt algebras [10].

1.1.2. A *Lie algebroid* on a manifold M is a triple $L = (L, [\![\cdot , \cdot]\!], \gamma_L)$, where L is a vector bundle on M with the space of cross-sections $(\operatorname{Sec} L, [\![\cdot , \cdot]\!])$ endowed with a structure of a Lie $\mathbb{R}$-algebra, the map $\gamma_L \colon L \to TM$ (the *anchor*) is a linear homomorphism of vector bundles, and the following Leibniz condition is satisfied:

$$[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + \gamma_L(\xi)(f) \cdot \eta, \qquad f \in C^\infty(M), \quad \xi, \eta \in \operatorname{Sec} L.$$

The anchor is bracket-preserving: $\gamma_L \circ [\![\xi, \eta]\!] = [\gamma_L \circ \xi, \gamma_L \circ \eta]$ [9], [11].

A Lie algebroid is said to be *transitive* if γ_L is an epimorphism. For a transitive Lie algebroid we have the Atiyah exact sequence

$$0 \longrightarrow \mathbf{g} \hookrightarrow L \xrightarrow{\gamma_L} TM \longrightarrow 0,$$

in which $\mathbf{g} := \ker \gamma_L$ is a fibre bundle that is a Lie algebra bundle (LAB for brevity), called the *adjoint* of L. The fibre $\mathbf{g}_x$ of the bundle $\mathbf{g}$ at a point $x \in M$ is a Lie algebra with commutator

$$[v, w] = [\![\xi, \eta]\!](x), \qquad \xi, \eta \in \operatorname{Sec} L, \quad \xi(x) = v, \quad \eta(x) = w, \quad v, w \in \mathbf{g}_x.$$

The Lie algebra $\mathbf{g}_x$ is called the *isotropy Lie algebra of L at $x \in M$*. Clearly,

$$\operatorname{rank} L = \dim M + \dim \mathbf{g}_x.$$

Each transitive Lie algebroid L over a contractible manifold M is isomorphic to the trivial Lie algebroid ([12], [13]) $TM \times \mathbf{g}$ with natural projection pr_1 as the anchor and bracket given by

$$[\![(X, \sigma), (Y, \eta)]\!] = \big([X, Y], X(\eta) - Y(\sigma) + [\sigma, \eta]\big),$$

$X, Y \in \mathfrak{X}(M)$, $\sigma, \eta \in C^\infty(M; \mathbf{g})$.

If $\gamma_L = 0$, then L is said to be *totally non-transitive*. For example, if $\mathbf{g}$ is an LAB adjoint to a transitive Lie algebroid, then $\mathbf{g}$ is a totally non-transitive Lie algebroid.

A splitting $\lambda \colon TM \to L$ of the Atiyah sequence $0 \to \mathbf{g} \to L \underset{\lambda}{\overset{}{\rightleftarrows}} TM \to 0$ is called a *connection in the Lie algebroid L*. For the Lie algebroid of a principal fibre bundle P there exists a one-to-one correspondence between connections in the algebroid and usual connections in P. Each connection $\lambda \colon TM \to L$ determines a covariant derivative ∇ in the adjoint bundle $\mathbf{g}$ by the formula $\nabla_X \sigma = [\![\lambda X, \sigma]\!]$ and a curvature form $\Omega \in \Omega^2(M; \mathbf{g})$ by the formula $\Omega(X, Y) = \lambda[X, Y] - [\![\lambda X, \lambda Y]\!]$. If the isotropy Lie algebras are Abelian, then the covariant derivative ∇ in $\mathbf{g}$ is flat and independent of one's choice of the connection λ; it is said to be *characteristic for the Abelian Lie algebroid L*.

1.1.3. Lie algebroids form a category with strong and with non-strong homomorphisms. (Recall that a strong homomorphism of vector bundles is a homomorphism identical on the base manifold.)

Let L and L' be two Lie algebroids on a manifold M. A strong homomorphism $H \colon L' \to L$ of vector bundles (that is, a homomorphism over $\operatorname{id}_M \colon M \to M$) is called a *(strong) homomorphism of Lie algebroids* if $\gamma \circ H = \gamma'$ and the map $\operatorname{Sec} H \colon \operatorname{Sec} L' \to \operatorname{Sec} L$ is a homomorphism of Lie algebras.

Let $(L, [\![\,\cdot\,, \cdot\,]\!], \gamma)$ and $(L', [\![\,\cdot\,, \cdot\,]\!]', \gamma')$ be two Lie algebroids on manifolds M and M', respectively. By a (in general *non-strong*) *homomorphism between these algebroids*

$$H \colon (L', [\![\,\cdot\,, \cdot\,]\!]', \gamma') \to (L, [\![\,\cdot\,, \cdot\,]\!], \gamma)$$

we mean a homomorphism of vector bundles $H \colon L' \to L$ over a map $f \colon M' \to M$, say, such that

(a) $\gamma \circ H = f_* \circ \gamma'$;

(b) let $\xi, \xi' \in \operatorname{Sec} L'$ be two arbitrary cross-sections of L' and let $\eta_1, \ldots, \eta_n$ be cross-sections of L forming a basis of $\operatorname{Sec} L$ over an open subset U of M. Then $H \circ \xi$ and $H \circ \xi'$ can be expressed in the following form (over $f^{-1}[U]$):

$$H \circ \xi_{f^{-1}[U]} = \sum_i f^i \cdot (\eta_i \circ f)_{f^{-1}[U]},$$

$$H \circ \xi'_{f^{-1}[U]} = \sum_j f'^j \cdot (\eta_j \circ f)_{f^{-1}[U]}$$

for suitable $f^i, f'^j \in C^\infty(M')$. We postulate that

$$H \circ [\![\xi, \xi']\!]_{f^{-1}[U]} = \left(\sum_{i,j} f^i \cdot f'^j [\![\eta_i, \eta_j]\!] \circ f + \sum_j [(\gamma' \circ \xi)(f'^j) - (\gamma' \circ \xi')(f^j)] \cdot \eta_j \circ f \right)_{f^{-1}[U]}.$$

The tangent map $f_* \colon TM' \to TM$ to a C^∞-map $f \colon M' \to M$ of manifolds is an example of a (non-strong) homomorphism of regular Lie algebroids.

The above, quite difficult and non-intuitive, definition is 'evident' from the point of view of Lie groupoids. The concept of a non-strong homomorphism of Lie groupoids is obvious: it must be compatible with the source, the target, and partial multiplication. Passing to infinitesimal objects (Lie algebroids) we obtain just a non-strong homomorphism of Lie algebroids as defined above.

In the case of Lie algebroids L and L' on the same manifold M the concept of homomorphism $H \colon L' \to L$ over the identity map $\mathrm{id}_M \colon M \to M$ (that is, of so-called strong homomorphism) coincides with the above-described one.

1.1.4. Let $L' = (L', [\![\cdot, \cdot]\!]', \gamma')$ and $L = (L, [\![\cdot, \cdot]\!], \gamma)$ be two Lie algebroids on manifolds M' and M, respectively. The *Cartesian product* of L' and L is the Lie algebroid $(L' \times L, [\![\cdot, \cdot]\!]^\times, \gamma' \times \gamma)$ over the manifold $M' \times M$, where $L' \times L$ is the Cartesian product of vector bundles, and for $\overline{\xi} = (\xi^1, \xi^2), \overline{\eta} = (\eta^1, \eta^2) \in \mathrm{Sec}(L' \times L)$ the bracket

$$[\![\overline{\xi}, \overline{\eta}]\!]^\times = ([\![\overline{\xi}, \overline{\eta}]\!]^1, [\![\overline{\xi}, \overline{\eta}]\!]^2)$$

is given by the formulae

$$[\![\overline{\xi}, \overline{\eta}]\!]^1_{(x,y)} = [\![\xi^1(\cdot, y), \eta^1(\cdot, y)]\!]'_x + (\gamma \circ \xi^2)_{(x,y)}(\eta^1(x, \cdot)) - (\gamma \circ \eta^2)_{(x,y)}(\xi^1(x, \cdot)),$$

$$[\![\overline{\xi}, \overline{\eta}]\!]^2_{(x,y)} = [\![\xi^2(x, \cdot), \eta^2(x, \cdot)]\!]_y + (\gamma' \circ \xi^1)_{(x,y)}(\eta^2(\cdot, y)) - (\gamma' \circ \eta^1)_{(x,y)}(\xi^2(\cdot, y)).$$

Clearly, the bundle isomorphism $TM \times TN = T(M \times N)$ is an isomorphism of Lie algebroids.

1.1.5. Let L and L' be regular Lie algebroids on manifolds M and M', respectively, and let $\phi, \phi' \colon L' \to L$ be homomorphisms of Lie algebroids. By a *homotopy joining* ϕ *to* ϕ' we mean a homomorphism of Lie algebroids

$$H \colon T\mathbb{R} \times L' \to L$$

such that $H(\theta_0, \cdot) = \phi$ and $H(\theta_1, \cdot) = \phi'$, where θ_0 and θ_1 are vectors zero tangent to $\mathbb{R}$ at 0 and 1, respectively [14]. This definition can also be applied to homomorphisms of Lie algebras. A homotopy H induces a chain homotopy operator and, as a consequence, the equality in cohomology $\phi^\# = \phi'^\#$ [14].

1.1.6. We associate with a Lie algebroid L the cohomology algebra $H_L(M)$ [15] defined via the DG-algebra $(\Omega_L(M), d_L)$ of L-differential forms (with real coefficients), where

$$\Omega_L(M) = \mathrm{Sec} \wedge L^*, \qquad d_L \colon \Omega_L^*(M) \to \Omega_L^{*+1}(M),$$

$$(d_L \Theta)(\xi_0, \ldots, \xi_k) = \sum_{j=0}^{k} (-1)^j (\gamma_L \circ \xi_j)(\Theta(\xi_0, \ldots, \widehat{\xi}_j, \ldots, \xi_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \Theta([\![\xi_i, \xi_j]\!], \xi_0, \ldots, \widehat{\xi}_i, \ldots, \widehat{\xi}_j, \ldots, \xi_k),$$

$$\Theta \in \Omega_L^k(M), \quad \xi_i \in \mathrm{Sec}\, L.$$

Here L^* is the space dual to L while '$*$' in Ω_L^* denotes the gradation on differential forms.

The exterior differential d_L gives rise to the cohomology algebra

$$H_L(M) = H(\Omega_L(M), d_L).$$

For the trivial Lie algebroid TM — the tangent bundle of the manifold M — the differential d_{TM} is the usual de Rham differential d_M of differential forms on M, whereas for $L = \mathfrak{g}$, a Lie algebra, the differential $d_\mathfrak{g}$ is the usual Chevalley–Eilenberg differential $d_\mathfrak{g} = \delta_\mathfrak{g}$. An arbitrary (in general, non-strong) homomorphism $H \colon L' \to L$ over a map $f \colon M' \to M$ determines a pullback of differential forms $H^* \colon \Omega_L(M) \to \Omega_{L'}(M')$ by the formula

$$H^*(\Theta)_{x'}(v_1, \ldots, v_k) = \Theta_{f(x')}(H(v_1), \ldots, H(v_k)).$$

The pullback of differential forms commutes with exterior derivatives producing a homomorphism on cohomology $H^\# \colon H_L(M) \to H_{L'}(M')$.

1.2. Lie functors. The so-called Lie functors act from many categories of differential objects mentioned at the beginning, such as the categories of Lie groupoids, principal bundles, vector bundles, TC-foliations, non-closed Lie subgroups, and so on, in the category of Lie algebroids. They generalize the Lie functor for Lie groups.

Example 1. The Lie algebroid $A(P)$ of a G-principal bundle $P = P(M, G)$. There exist three distinct equivalent definitions of $A(P)$ [2]. One definition of $A(P)$ is based on the vector bundle TP/G introduced by Atiyah [16] and is as follows: $A(P) = TP/G$ is the space of orbits of the right action of G on TP given by the differentials of right translations (see [3], [12], [2]). The anchor is $\gamma([v]) = \pi_*(v)$. The bracket is constructed on the basis of the following observation: for each cross-section $\eta \in \operatorname{Sec} A(P)$ there exists a unique C^∞ right-invariant vector field $\eta' \in \mathfrak{X}^R(P)$ such that $[\eta'(z)] = \eta(\pi z)$, $z \in P$, and the map $\operatorname{Sec} A(P) \to \mathfrak{X}^R(P)$, $\eta \mapsto \eta'$, is an isomorphism of C^∞-modules. The bracket $[\![\xi, \eta]\!]$ for $\xi, \eta \in \operatorname{Sec} A(P)$ is defined so that $[\![\xi, \eta]\!]' = [\xi', \eta']$. One says that the transitive Lie algebroid arises from the principal bundle. A transitive Lie algebroid is said to be *integrable* if it arises from a principal bundle.

The Lie algebroid $A(P)$ has a simpler structure than the principal bundle P. Namely, non-isomorphic principal bundles can possess isomorphic Lie algebroids. For example, there exists a non-trivial principal bundle for which the Lie algebroid is trivial (the non-trivial Spin(3)-structure of the trivial principal bundle $\mathbb{R}(5) \times SO(3)$ has this property [2]).

Example 2. The general form of a transitive Lie algebroid. Each transitive Lie algebroid has — up to an isomorphism — the form described below. This was demonstrated by Mackenzie [12] and, independently, by Kubarski [2].

Let $(\mathbf{g}, \nabla, \Omega_b)$ be a system consisting of an LAB $\mathbf{g}$ on M, a covariant derivative ∇ in $\mathbf{g}$, and a 2-form $\Omega_b \in \Omega^2(M, \mathbf{g})$ on M with values in $\mathbf{g}$ satisfying the following conditions:

(1) $\nabla^2 \sigma = -[\Omega_b, \sigma]$, $\sigma \in \operatorname{Sec} \mathbf{g}$;
(2) $\nabla_X [\sigma, \eta] = [\nabla_X \sigma, \eta] + [\sigma, \nabla_X \eta]$, $X \in \mathfrak{X}(M)$, $\sigma, \eta \in \operatorname{Sec} \mathbf{g}$;
(3) $\nabla \Omega_b = 0$.

Then $TM \oplus \mathbf{g}$ is a transitive Lie algebroid with bracket defined by the formula

$$[\![(X, \sigma), (Y, \eta)]\!] = ([X, Y], -\Omega_b(X, Y) + \nabla_X \eta - \nabla_X \sigma + [\sigma, \eta])$$

and anchor that is the projection onto the first component.

As Mackenzie [12] noted, this result leads to the first (*algebraic*) step of the solution of the long-standing Weil problem: find out when a 2-form $\Omega_b \in \Omega^2(M; \mathbf{g})$ is the curvature tensor of a connection in a principal bundle $P(M, G)$ over M with Ad-associated Lie algebra bundle $\mathbf{g}$.

The second (last) step is the theorem ([12], [17]) giving the obstruction to the integrability of the so constructed Lie algebroid $TM \oplus \mathbf{g}$.

Example 3. The Lie algebroid $A(\mathfrak{f})$ of a vector bundle $\mathfrak{f}$ [12], [18]. The fibre $A(\mathfrak{f})_x$ of $A(\mathfrak{f})$ is the space of linear homomorphisms $l \colon \mathrm{Sec}\,\mathfrak{f} \to f_x$ such that there exists a vector $u \in T_x M$, the anchor of l, for which

$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu_x,$$
$$f \in C^\infty(M), \quad \nu \in \mathrm{Sec}\,\mathfrak{f}.$$

A cross-section $\xi \in \mathrm{Sec}\,A(\mathfrak{f})$ defines in an obvious manner a covariant differential operator in $\mathfrak{f}$. The bracket of cross-sections of $A(\mathfrak{f})$ is defined in the classical fashion for differential operators. $A(\mathfrak{f})$ is naturally isomorphic to the Lie algebroid of the $GL(V)$-principal bundle (where V is the typical fibre of $\mathfrak{f}$) of all frames of $\mathfrak{f}$. Locally, that is, over some neighbourhood U of a point in M, the Lie algebroid $A(\mathfrak{f})_U$ is isomorphic to the trivial algebroid $TU \times \mathrm{End}(V)$ ([18], 5.4.4). Mackenzie [12] gives an equivalent definition of the Lie algebroid $\mathrm{CDO}(\mathfrak{f})$ of the vector bundle $\mathfrak{f}$.

The concept of Lie algebroid $A(\mathfrak{f})$ is adequate for the definition of a representation of a Lie algebroid on a vector bundle. Namely, by a *representation of a Lie algebroid L* on a *vector bundle* $\mathfrak{f}$ (over the same manifold) [12] (see also [18]) we mean a (strong) homomorphism of Lie algebroids

$$T \colon L \to A(\mathfrak{f}).$$

Equivalently, it can be defined by a flat L-covariant derivative $\nabla_\xi \sigma$, $\xi \in \mathrm{Sec}\,L$, $\sigma \in \mathrm{Sec}\,\mathfrak{f}$, in $\mathfrak{f}$, that is, an operator satisfying the Koszul axioms with one difference: $\nabla_\xi f\sigma = f\nabla_\xi \sigma + \gamma(\xi)(f)\sigma$, where $\gamma(\xi) \in \mathfrak{X}(M)$ is the anchor of ξ, and the curvature tensor R^∇ of $\nabla_\xi \sigma$ is zero: $R^\nabla(\xi_1, \xi_2) = \nabla_{[\![\xi_1, \xi_2]\!]} - [\![\nabla_{\xi_1}, \nabla_{\xi_2}]\!] = 0$.

Each representation $T \colon L \to A(\mathfrak{f})$ induces the *exterior power* $\bigwedge^k T \colon L \to A\big(\bigwedge^k \mathfrak{f}\big)$ defined so that

$$\bigwedge^k T(\xi)(\sigma_1 \wedge \cdots \wedge \sigma_k) = \sum_{i=1}^k \sigma_1 \wedge \cdots \wedge T(\xi)(\sigma_i) \wedge \cdots \wedge \sigma_k,$$

$\xi \in \mathrm{Sec}\,L$, $\sigma_i \in \mathrm{Sec}\,\mathfrak{f}$.

Let $T \colon L \to A(\mathfrak{f})$ be a representation of a Lie algebroid L on a vector bundle $\mathfrak{f}$. A cross-section $\sigma \in \mathrm{Sec}\,\mathfrak{f}$ is said to be *T-invariant* if $T(\xi)(\sigma) = 0$ for all $\xi \in \mathrm{Sec}\,A$.

Let L be a transitive Lie algebroid. Crucial for its properties is the *adjoint representation* $\mathrm{ad}_L \colon L \to A(\mathbf{g})$ of L on its adjoint Lie algebra bundle $\mathbf{g} = \ker \gamma$

defined as follows: $\mathrm{ad}_L(\xi)(\nu) = [\![\xi, \nu]\!]$, $\xi \in \mathrm{Sec}\, L$, $\nu \in \mathrm{Sec}\,\mathbf{g}$. In particular, we obtain the exterior nth-power $\bigwedge^n \mathrm{ad}_L$ of the adjoint representation, which is called the *adjoint representation of the Lie algebroid L on the bundle $\bigwedge^n \mathbf{g}$*.

From the above we see that a cross-section ε of $\bigwedge^n \mathbf{g}$ is $\bigwedge^n \mathrm{ad}_L$-invariant if and only if in each open subset U of M in which ε has the form $\varepsilon_U = (h_1 \wedge \cdots \wedge h_n)_U$, $h_i \in \mathrm{Sec}\,\mathbf{g}$, for all $\xi \in \mathrm{Sec}\, A$ we have

$$\bigwedge^n \mathrm{ad}_A(\xi)(\varepsilon)_U = \sum_{i=1}^n (h_1 \wedge \cdots \wedge [\![\xi, h_i]\!] \wedge \cdots \wedge h_n)_U = 0. \tag{1}$$

Example 4. The Lie algebroid $A(M, \mathcal{F})$ of a tranversally complete foliation $(M, \mathcal{F})$ of a connected Hausdorff paracompact manifold M [5]. We recall that a foliation $(M, \mathcal{F})$ is said to be *transversally complete* (TC for brevity) if at each point $x \in M$ the family $L_c(M, \mathcal{F})$ of complete global $\mathcal{F}$-foliate vector fields (such that the corresponding dynamical system respects the foliation) generates the entire tangent space $T_x M$. We add that

(a) transversally complete foliations perform a crucial role in the theory of Riemannian foliations [6],

(b) among them there exist foliations whose Lie algebroids are non-integrable. This last fact — discovered by Almeida and Molino in 1985 [17] — was one of the most important moments in the theory of Lie algebroids.

The first structure theorem of Molino ([6], Theorem 4.2) states that the closures of leaves of a TC-foliation $\mathcal{F}$ form a simple foliation $\mathcal{F}_b$, said to be *basic*, and are fibres of a locally trivial basic fibration $\pi_b \colon M \to W$ onto a Hausdorff paracompact base manifold W.

Let $Q = TM/E$ $(E = T\mathcal{F})$ be the transverse bundle of $\mathcal{F}$. A cross-section ξ of Q having a foliate vector field as its representative is called a *transversal field*. Its value at a point $x \in M$ uniquely determines the value at each point y of $(L_b)_x$, the leaf through x of the basic foliation $\mathcal{F}_b$. Transverse fields play a similar role to that of right-invariant vector fields on principal bundles. The quotient $A(M, \mathcal{F}) = Q/\equiv$ of Q by the relation identifying the values of transverse fields at the points of a leaf of $\mathcal{F}_b$ (corresponding to right translations of vectors tangent to a principal bundle) has a natural structure of a vector bundle over W. The global cross-sections of $A(M, \mathcal{F})$ correspond to transverse fields. These latter form a Lie algebra, and so does $\mathrm{Sec}\, A(M, \mathcal{F})$. Adding the homomorphism $\gamma \colon A(M, \mathcal{F}) \to W$ defined simply by means of the fibration $\pi_b \colon M \to W$ we obtain a transitive Lie algebroid. A foliation $(M, \mathcal{F})$ on a compact manifold is said to be *transversally parallelizable* (TP for brevity) if the transverse bundle Q is trivial and possesses a basis of transverse fields. Clearly, a TP-foliation is transversally complete.

Example 5. The Lie algebroid $A(G; H)$ of a non-closed Lie subgroup H of a group G. This is the Lie algebroid of the TC-foliation $\mathcal{F}_{G,H} = \{aH; a \in G\}$ of left cosets of a Lie group G by a connected Lie subgroup H of G. (If H is closed, then $A(G; H)$ is trivial, therefore this Lie algebroid is interesting only if H is not closed.) The algebroid $A(G; H)$ can be constructed without the use of the general theory of TC-foliations [7] since its total space is equal to the space of orbits of the following free right action $\overline{R} \colon Q \times \overline{H} \to Q$ of the closure $\overline{H}$ of H on the transverse

bundle Q: for $t \in \overline{H}$, $\overline{R}_t \colon Q \to Q$ is the automorphism defined by the differential of the right translation $R_t \colon TG \to TG$ with the use of the stability of the tangent space E to the foliation under R_t. A cross-section ξ of Q is a transverse field if and only if ξ is invariant with respect to the action $\overline{R}$. Therefore the relation '$\equiv$' on Q yielding $A(G; H)$ can be defined by the formula $\overline{v} \equiv \overline{w} \Leftrightarrow$ there exists $t \in \overline{H}$ such that $\overline{R}_t(\overline{v}) = \overline{w}$.

We recall that $A(G; H)$ is a transitive Lie algebroid over the homogeneous manifold $G/\overline{H}$ with trivial adjoint bundle of Abelian isotropy Lie algebras $\mathbf{g} \cong G/\overline{H} \times \overline{\mathfrak{h}}/\mathfrak{h}$ [18], [7]. Under this identification we have the Atiyah sequence

$$0 \to G/\overline{H} \times \overline{\mathfrak{h}}/\mathfrak{h} \to A(G, H) \to T(G/\overline{H}) \to 0,$$

and the characteristic flat covariant derivative is equal to the standard derivative of vector-valued functions $\nabla_X = \partial_X$.

Below we present interpretations of the cohomology algebras of the Lie algebroids of principal bundles and TC-foliations.

Examples. (1) If $L = A(P) = TP/G$ for a G-principal fibre bundle $P \to M$, then

$$\Omega_L(M) \cong \Omega^r(P) \hookrightarrow \Omega(P),$$

$\Omega^r(P)$ is the space of G-right invariant differential forms on P and

$$H_L(M) \cong H(\Omega^r(P)) \overset{i}{H}_{dR}(P).$$

The homomorphism i is an isomorphism if G is compact and connected.

(2) If $L = A(M; \mathcal{F}) \to W$ is the Lie algebroid of a TC-foliation $\mathcal{F}$ on M with base manifold W, then

$$\Omega_L(W) \cong \Omega(M; \mathcal{F}),$$

where $\Omega(M; \mathcal{F})$ is the algebra of $\mathcal{F}$-basic differential forms, therefore $H_L(W) \cong H(M; \mathcal{F})$ is the algebra of basic cohomology [19].

1.3. Invariantly oriented Lie algebroids and signature. In what follows we are interested in transitive Lie algebroids with cohomology algebra $H_L(M)$ satisfying the Poincaré duality [19]. TUIO-Lie algebroids [20] (transitive unimodular invariantly oriented) are examples here. To define a TUIO-Lie algebroid consider the Atiyah sequence $0 \longrightarrow \mathbf{g} \hookrightarrow L \overset{\gamma_L}{\longrightarrow} TM \longrightarrow 0$ and assume that

(A1) $m = \dim M$, $n = \operatorname{rank} \mathbf{g} = \dim \mathbf{g}_x$, $\bigwedge^n \mathbf{g}$ is a trivial vector bundle.

Let $\varepsilon \in \operatorname{Sec} \bigwedge^n \mathbf{g}$ be an orientation form of $\mathbf{g}$. A fundamental role is played by the fibre integral [20]

$$\fint_L \colon \Omega_L^*(M) \to \Omega_{dR}^{*-n}(M),$$

which is defined as follows:
if $\deg \Phi < n$, then

$$\fint_L \Phi = 0,$$

while if $k = \deg \Phi \geqslant n$, then

$$\left(\int_L \Phi\right)_x (w_1 \wedge \cdots \wedge w_{k-n}) = (-1)^{n(k-n)} \Phi_x(\varepsilon_x \wedge \widetilde{w}_1 \wedge \cdots \wedge \widetilde{w}_{k-n}),$$

where the $\widetilde{w}_i \in L_x$ are arbitrary vectors such that $\gamma_L(\widetilde{w}_i) = w_i$. In other words

$$\gamma_L^*\left(\int_L \Phi\right) = (-1)^{n(k-n)} \iota_\varepsilon \Phi,$$

where ι_ε is the substitution operator $\iota_\varepsilon : \Omega_L^n(M) \to \Omega_L^{n-k}(M)$ defined by the formula

$$(\iota_\varepsilon \Phi)_x(v_1 \wedge \cdots \wedge v_{n-k}) = \Phi_x(\varepsilon_x \wedge v_1 \wedge \cdots \wedge v_{n-k}), \qquad v_i \in A_x.$$

The operator $\int_L$ commutes with the exterior derivatives d_L and d_M if and only if [20]

   (a) the isotropy Lie algebras $\mathbf{g}_x$ are unimodular, that is, $H^n(\mathbf{g}_x) = \mathbb{R}^1$,
   (b) the cross-section ε is invariant with respect to the adjoint representation of L on $\bigwedge^n \mathbf{g}$, that is, (1) holds.

A Lie algebroid L satisfying (a) and (b) is called a TUIO-Lie algebroid. The fibre integral operator $\int_L : \Omega_L^*(M) \to \Omega_{dR}^{*-n}(M)$ leads then to a homomorphism in cohomology

$$\int_L^\# : H_L^*(M) \to H_{dR}^{*-n}(M).$$

Examples. (1) The Lie algebroid $A(P)$ of a G-principal bundle $P \to M$ is a TUIO-Lie algebroid if G has the following property: $\det(\mathrm{Ad}_G a) = +1$, $a \in G$ [20], [19].

(2) The Lie algebroid $A(M; \mathcal{F})$ of a TP-foliation on a compact simply connected manifold is a TUIO-Lie algebroid.

(3) The Lie algebroid $A(G; H)$ of a non-closed Lie subgroup H of a group G (that is, the Lie algebroid of the TC-foliation of left cosets of H in G) is a TUIO-Lie algebroid. The adjoint Lie algebra bundle of $A(G; H)$ is a trivial bundle of adjoint Lie algebras [7].

Assume that M is compact and oriented. For a compactly supported $(m+n)$-differential form ω on L, $\omega \in \Omega_{L,c}^*(M)$, we set

$$\int_L \omega := \int_M \left(\int_L \omega\right).$$

The operator $\int_L$ leads to an epimorphism [7]

$$\int_L^\# : H_{L,c}(M) \to \mathbb{R}.$$

The Poincaré scalar product

$$H_L^*(M) \times H_{L,c}^{m+n-*}(M) \to \mathbb{R}, \qquad (\alpha, \beta) \longmapsto \int_L^\# \alpha \wedge \beta,$$

is non-degenerate, that is,

$$H_L(M) \cong (H_{L,c}(M))^*.$$

Consequently, we have, for example,

$$H_{L,c}^{m+n}(M) \cong \mathbb{R}.$$

Assume now that

(A2) M is compact, connected, and oriented.

Then we have $H_{L,c}(M) = H_L(M)$, and the scalar Poincaré product

$$\mathcal{P}_L^*: H_L^*(M) \times H_L^{m+n-*}(M) \to \mathbb{R}, \qquad (\alpha, \beta) \longmapsto \int_L^\# \alpha \wedge \beta$$

is non-degenerate. If $m + n = 4k$, then $\mathcal{P}_L^{2k}$ is symmetric and its signature (some-times called the index) is denoted by $\mathrm{Sign}_\varepsilon(L)$ (or briefly $\mathrm{Sign}(L)$) and is called the *signature* of L (the sign $+$ or $-$ of the signature depends on the choice of the invariant orientation ε). In other cases ($m + n \neq 0 \pmod 4$) $\mathrm{Sign}_\varepsilon(L) = 0$ by definition.

Problem 1 (posed in [19]). *Calculate the signature* $\mathrm{Sign}_\varepsilon(L)$ *and give conditions for the equality*

$$\mathrm{Sign}_\varepsilon(L) = 0.$$

Note that this equality holds in the following two classical cases:

(1) $L = \mathfrak{g}$ is a unimodular Lie algebra;
(2) $L = A(P)$ is the Lie algebroid of a principal fibre bundle over a compact orientable manifold with compact connected structure Lie group.

To investigate the signature of L we use the technique of spectral sequences for the Čech–de Rham complex $K^{*,*}$ of L-differential forms, which is defined similarly to the complex for a manifold, and use the celebrated methods and results in [21]. To this end we recall the main theorem of that paper.

Theorem 2. *Let $E \to B$ be a fibre bundle with typical fibre F satisfying the following conditions*:

(1) *E, B, and F are compact connected oriented manifolds;*
(2) *the fundamental group $\pi_1(B)$ acts trivially on the cohomology ring $H^*(F)$ of F.*

If E, B, and F are coherently oriented so that the orientation of E is induced by those of F and B, then the index of E is the product of the indices of F and B, that is,

$$\mathrm{Sign}(E) = \mathrm{Sign}(F) \cdot \mathrm{Sign}(B).$$

The authors of [21] consider the cohomology spectral sequence $E_s^{p,q}$ of the bundle $E \to B$ with the reals as the coefficient field. The term E_2 by assumption (2) is the bigraded algebra

$$E_2^{j,i} \cong H^j(B; H^i(F)) \cong H^j(B) \otimes H^i(F),$$

therefore

$$E_2^{j,i} = 0 \quad \text{for } j > m \text{ or } i > n.$$

Clearly, E_2 is a Poincaré algebra by assumption (1). Using a spectral-sequences argument the authors observe that

$$(E_s, d_s, \cdot), \qquad s \geqslant 2,$$

are Poincaré algebras with Poincaré differentiation. Hence the infinite term $(E_\infty, \cdot)$ is also a Poincaré algebra and

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = \cdots = \operatorname{Sign} E_\infty.$$

The last step is to prove that $\operatorname{Sign} E_\infty = \operatorname{Sign} H(E)$. Note that this is not so trivial since, in general, the algebras E_∞ and $H(E)$ are not isomorphic (although $E_\infty \cong H(E)$ as bigraded spaces).

We use the exposition in [22] or [23] of the version of the spectral sequence of a fibred manifold in the language of the Čech–de Rham complex on a so-called good covering and generalize it to the case of transitive Lie algebroids.

The double complex $K^{*,*}$ constructed below for arbitrary transitive Lie algebroids coincides in the case of the Lie algebroid $A = A(M, \mathcal{F})$ over the base manifold W of a TP-foliation $(M, \mathcal{F})$ with the double complex in [24] (thanks to the natural DG-isomorphism of algebras $\Omega_L(W) \cong \Omega(M/\mathcal{F})$).

§ 2. Čech–de Rham complex of transitive Lie algebroids

Let L be an arbitrary transitive Lie algebroid on a manifold M with isotropy Lie algebras $\mathbf{g}_x$ isomorphic to a given Lie algebra $\mathfrak{g}$. If U is an open subset of M diffeomorphic to $\mathbb{R}^m$, then the restriction L_U is a Lie algebroid isomorphic to the trivial one $TU \times \mathfrak{g}$. Such an isomorphism is not natural and depends on the choice of a flat connection in L_U. Denote (for brevity) the cohomology algebra $H_{L_U}(U)$ by $H_L(U)$. By Künneth's formula [19],

$$H_L(U) \cong H(U) \otimes H(\mathfrak{g}) \cong H(\mathfrak{g}).$$

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ be a good covering of M, where J is a countable ordered index set. This means that all the U_α and all finite intersections $\bigcap_i U_{\alpha_i}$ are diffeomorphic to the Euclidean space $\mathbb{R}^m$.

We can form the double complex (of Čech–de Rham type)

$$K^{p,q} = C^p(\mathfrak{U}, \Omega_L^q) := \prod_{\alpha_0 < \cdots < \alpha_p} \Omega_L^q(U_{\alpha_0 \ldots \alpha_p}),$$

$p, q \geqslant 0$, with the product structure

$$\cup \colon K^{p,q} \times K^{r,s} \to K^{p+r,q+s}$$

defined by the formula

$$(\omega \cup \eta)_{\alpha_0 \ldots \alpha_{p+r}} = (-1)^{qr} \omega_{\alpha_0 \ldots \alpha_p}|U_{\alpha_0 \ldots \alpha_{p+r}} \wedge \eta_{\alpha_p \ldots \alpha_{p+r}}|U_{\alpha_0 \ldots \alpha_{p+r}}.$$

This complex has two boundary homomorphisms, δ and d.

The vertical homomorphism

$$d \colon C^p(\mathfrak{U}, \Omega_L^q) \to C^p(\mathfrak{U}, \Omega_L^{q+1})$$

acts as an external differential of L-forms:

$$d = (-1)^p d_L.$$

The horizontal homomorphism

$$\delta \colon C^p(\mathfrak{U}, \Omega_L^q) \to C^{p+1}(\mathfrak{U}, \Omega_L^q)$$

acts as a coboundary homomorphism

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_{p+1}}|U_{\alpha_0 \ldots \alpha_{p+1}}.$$

Both horizontal and vertical homomorphisms d and δ are antiderivations of degree $+1$ in the total gradation:

$$d, \delta \colon K^{(r)} \to K^{(r+1)}, \qquad K^{(r)} = \bigoplus_{p+q=r} K^{p,q},$$

$$d(\omega \cup \eta) = d\omega \cup \eta + (-1)^{\deg \omega} \omega \cup d\eta,$$

and the same holds for δ. Therefore,

$$(K, K^{p,q}, \cup, d, \delta)$$

is a double complex of the first quadrant with product structure. It is convenient to represent this complex as a diagram:

$$C^*(\mathfrak{U}, \Omega_L^*) = \left\{ \begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \\
\uparrow d & & \uparrow d & & & & \uparrow d & & \\
C^0(\mathfrak{U}, \Omega_L^q) & \xrightarrow{\delta} & C^1(\mathfrak{U}, \Omega_L^q) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & C^p(\mathfrak{U}, \Omega_L^q) & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & & & \uparrow d & & \\
\vdots & & \vdots & & & & \vdots & & \\
\uparrow d & & \uparrow d & & & & \uparrow d & & \\
C^0(\mathfrak{U}, \Omega_L^1) & \xrightarrow{\delta} & C^1(\mathfrak{U}, \Omega_L^1) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & C^p(\mathfrak{U}, \Omega_L^1) & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & & & \uparrow d & & \\
C^0(\mathfrak{U}, \Omega_L^0) & \xrightarrow{\delta} & C^1(\mathfrak{U}, \Omega_L^0) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & C^p(\mathfrak{U}, \Omega_L^0) & \xrightarrow{\delta} & \cdots
\end{array} \right.$$

The total differential operator $D = d + \delta$ is also an antiderivation. The inclusion

$$r \colon \Omega_L^* \to K^{0,*} \subset K^{(*)}, \qquad r(\omega)_\alpha = \omega|U_\alpha,$$

of Ω_L in the total complex produces a homomorphism of DG-algebras. Indeed, the operator d commutes with r and $\delta \circ r = 0$. Hence $r \circ D = D \circ r$. Consequently, we have a graded homomorphism in cohomology:

$$r^\# \colon H_L^*(M) \to H^{(*)}(K, D).$$

Proposition 3. *The Mayer–Vietoris sequence of a transitive Lie algebroid, that is, the augmented row in the double complex*

$$0 \longrightarrow \Omega_L^q(M) \xrightarrow{r} K^{0,q} \xrightarrow{\delta} K^{1,q} \xrightarrow{\delta} \cdots,$$

is exact.

The proof is standard and can be obtained with the use of a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\mathfrak{U} = \{U_\alpha\}$ and the homotopy cochain operator

$$H \colon K^{p,q} \to K^{p-1,q}, \qquad (H\omega)_{\alpha_0 \ldots \alpha_{p-1}} = \sum_\alpha \rho_\alpha \cdot \omega_{\alpha \alpha_0 \ldots \alpha_{p-1}},$$

for which one has

$$H\delta + \delta H = \mathrm{Id} \colon K^{p,q} \to K^{p,q}, \qquad p \geqslant 1.$$

The following result is standard (see, for example, [22]).

Proposition 4. *If all the rows of an augmented double complex are exact, then the D-cohomology of the complex is isomorphic to the cohomology of the initial column.*

One consequence of it is as follows.

Corollary 5. *The homomorphism $r^\# \colon H_L^*(M) \to H^{(*)}(K, D)$ is an isomorphism of DG-algebras in cohomology.*

Consider now the 'horizontal' decreasing filtration

$$K_j = \bigoplus_{\substack{p \geqslant j \\ q \geqslant 0}} K^{p,q}.$$

In view of the general construction of spectral sequences (see, for example, [25], 1.4.2; see also [26] and [22]), for the above filtration one can construct, in accordance with the multiplicative structure of the DG-algebra $(K, K^{(r)}, \cup, D, K_j)$, the spectral sequence of the graded differential groups $(E_s^{p,q}, d_s)$:

$$d_s \colon E_s^{p,q} \longrightarrow E_s^{p+s,q-s+1},$$

such that

$$E_{s+1}^{p,q} = H(E_s^{p,q}, d_s),$$

and the group $E_\infty^{p,q}$ is adjoint to the cohomology ring $H(K, D)$ with respect to the filtration induced by the filtration $\{K_j\}$. The filtration K_j is regular, $K_0 = K$, therefore the spectral sequence $(E_s^{p,q}, d_s)$ converges to $H(K, D)$ (see [26]). Moreover, the following results hold:

(1) the E_s, $0 \leqslant s \leqslant \infty$, are graded algebras;

(2) the d_s are antiderivatives with respect to the total gradation of bidegree $(s, 1 - s)$;

(3) the natural identifications

$$\sigma_s \colon E_{s+1} \to H(E_s, d_s), \qquad \sigma_\infty \colon E_\infty \to E_0(H(K, D))$$

are isomorphisms of bigraded algebras. In general, the algebras $H(K, D)$ and E_∞ are not isomorphic, although $H(K, D) \cong E_\infty$ are non-canonically isomorphic as bigraded vector spaces.

The zero term $E_0^{p,q}$ of the spectral sequence is defined by the formula

$$E_0^p = K_p/K_{p+1}, \qquad E_0^{p,q} = K^{p,q}.$$

Hence the differential

$$d_0 \colon E_0^{p,q} \longrightarrow E_0^{p,q+1},$$
$$d_0 \colon K^{p,q} = C^p(\mathfrak{U}, \Omega_L^q) \longrightarrow K^{p,q+1} = C^p(\mathfrak{U}, \Omega_L^{q+1})$$

coincides with d.

The first term of the spectral sequence (E_s, d_s) is as follows:

$$E_1^{p,q} = H^{p,q}(K, d) = C^p(\mathfrak{U}, \mathcal{H}_L^q), \qquad d_1 = \delta^\# \colon E_1^{p,q} \longrightarrow E_1^{p+1,q},$$

where

$$\mathcal{H}_L^* = (U \longmapsto H_L^*(U))$$

is the Leray-type presheaf of cohomology, locally constant on the good covering, with values in the group (more precisely, the algebra) $H^*(\mathfrak{g})$.

Theorem 6. *The second term of the spectral sequence $E_s^{p,q}$ can be calculated by the formula*

$$E_2^{p,q} = H^{p,q}(H(K, D), \delta^\#) = H_{\delta^\#}^p(\mathfrak{U}, \mathcal{H}_L^q).$$

Thus, from Theorem 6 and Corollary 5 one obtains the following consequence.

Theorem 7 (Leray theorem for transitive Lie algebroids). *There exists a spectral sequence $E_s^{j,i}$ converging to the cohomology $H_L(M)$ of the Lie algebroid L,*

$$E_s^{j,i} \Longrightarrow H_L(M)$$

such that $E_2^{j,i} = H_{\delta^\#}^j(\mathfrak{U}, \mathcal{H}_L^i)$.

Remark 1. We make several observations concerning the sheaf $\widetilde{\mathcal{H}}_L^*$ associated with the presheaf $\mathcal{H}_L^*$:

$$(\widetilde{\mathcal{H}}_L^*)(x) = \varinjlim_{U \ni x} H_L^*(U).$$

Note that the bundle $H(\mathbf{g})$ of cohomology of isotropy Lie algebras with typical fibre $H(\mathfrak{g})$ is flat, that is, possesses local trivializations with locally constant transition functions. Indeed, for a neighbourhood $M \supset U \cong \mathbb{R}^m$ and each isomorphism of Lie algebroids

$$\varphi_U \colon L_U \to TU \times \mathfrak{g}$$

we have a local trivialization

$$\widetilde{\varphi}_U \colon H(\mathbf{g})_U \to U \times H(\mathfrak{g}),$$

defined by the formula

$$(\widetilde{\varphi}_U)_x = \big((\varphi_{Ux}^+)^{-1}\big)^{\#} \colon H(\mathbf{g}_x) \to H(\mathfrak{g}),$$

where $\varphi_{Ux}^+ \colon \mathbf{g}_x \to \mathfrak{g}$ is the isomorphism of Lie algebras that is the restriction of φ_U to the kernel $\mathbf{g}_x = \ker(\gamma_L)_x$ of the anchor γ_L.

Denote by $\mathcal{A}_L$ the set of so-defined local trivializations $\{\widetilde{\varphi}_U\}$ of the bundle $H(\mathbf{g})$.

Two local trivializations φ_U and φ_V on neighbourhoods U and V, respectively, give a smooth family of isomorphisms of Lie algebras

$$\lambda_x \colon \mathfrak{g} \to \mathfrak{g}, \qquad \lambda_x = \varphi_{Ux}^+ \circ (\varphi_{Vx}^+)^{-1}, \quad x \in U \cap V.$$

Clearly, the transition function

$$x \longmapsto (\lambda_x)^{\#} \colon H(\mathfrak{g}) \to H(\mathfrak{g})$$

is locally constant.

Indeed, for two arbitrary points x and y belonging to the same connected component of $U \cap V$ and for an arbitrary smooth arc $\alpha \colon \mathbb{R} \to U \cap V$ joining x to y we can consider the composite homomorphism of Lie algebroids

$$T\mathbb{R} \times \mathfrak{g} \xrightarrow{\alpha_* \times \mathrm{id}} T(U \cap V) \times \mathfrak{g} \xrightarrow{\varphi_U \circ (\varphi_V)^{-1}} T(U \cap V) \times \mathfrak{g} \xrightarrow{\mathrm{pr}_2} \mathfrak{g}.$$

The non-strong homomorphism of Lie algebroids over $\mathbb{R} \times \{*\} \to \{*\}$ obtained in this way is a homotopy joining λ_x to λ_y. Consequently, $\lambda_x^{\#} = \lambda_y^{\#}$ (see § 1.1.5).

The above enables us to replace the topology of the bundle $H(\mathbf{g})$ by a topology $H_d(\mathbf{g})$ such that the fibres become discrete and for each $v \in H(\mathfrak{g})$ and each local trivialization φ_U on $U \cong \mathbb{R}^m$ the cross-section $x \mapsto (\varphi_{Ux}^+)^{\#}(v)$ is a homeomorphism onto an open subset of $H_d(\mathbf{g})$.

Clearly, $H_d(\mathbf{g})$ gives rise to a sheaf. In other words, the transition functions are continuous functions with values in the group $\mathrm{Aut}(\mathfrak{g})$ with topology different from the classical topology and defined as follows. Let $\varphi \in \mathrm{Aut}(\mathfrak{g})$ be an automorphism of the algebra $\mathfrak{g}$ and let $\varphi^{\#} \colon H(\mathfrak{g}) \to H(\mathfrak{g})$ be the induced automorphism of the cohomology group. Denote by $\mathrm{Aut}_h(g)$ the stationary subgroup with respect to the action in cohomology, that is, the subgroup of all automorphisms $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi^{\#} = \mathrm{Id}$. We change the topology on the group $\mathrm{Aut}(\mathfrak{g})$ by adding an additional open subset, the subgroup $\mathrm{Aut}_h(\mathfrak{g})$. Let $\mathrm{Aut}^s(\mathfrak{g})$ be the group $\mathrm{Aut}(\mathfrak{g})$

with the new topology. Clearly, the quotient group $\mathrm{Aut}^s(\mathfrak{g})/\mathrm{Aut}_h(\mathfrak{g})$ is discrete and is the structure group of the bundle $H_d(\mathbf{g})$ corresponding to the atlas $\mathcal{A}_L$.

Let us show that there exists an isomorphism of sheaves

$$\rho\colon \widetilde{\mathcal{H}}_L \cong H_d(\mathbf{g})$$

defined as follows: consider the inclusion of bundles $\iota\colon \mathbf{g} \hookrightarrow L$, which can be regarded as a homomorphism of Lie algebroids, where $\mathbf{g}$ is a totally non-transitive Lie algebroid with anchor zero.

Then for each open set U one has the homomorphism of cohomology

$$\iota^{\#}\colon H_L(U) \longrightarrow H_{\mathbf{g}}(U).$$

The space $H_L^*(U)$ is calculated as follows. If one considers an open subset from the good covering, that is, $U \cong \mathbb{R}^m$, then the algebroid L_U is isomorphic to the trivial algebroid $TM \oplus \mathfrak{g}$.

Hence [19] the algebra of L_U-differential forms is isomorphic to the anticommutative tensor product

$$\Omega_L(L_U) = \Omega(TM_U) \otimes \wedge(\mathfrak{g}^*),$$

with differential $d_U \otimes \mathrm{id} + \omega \otimes \delta_{\mathfrak{g}}$, where $\omega(\varphi) = (-1)^{\deg \varphi}\varphi$ and $\delta_{\mathfrak{g}}$ is the Eilenberg–Chevalley differential.

Hence the cohomology ring is

$$H_L(U) = H(U) \otimes H(\mathfrak{g}) \cong H(\mathfrak{g})$$

since

$$H(U) \cong H^0(U) = \mathbb{R}.$$

Moreover, the elements of $H^0(U)$ can be represented by constant functions on U. Hence each element $s \in H_L^*(U)$ is represented by a constant section in the bundle $U \oplus H^*(\mathfrak{g})$. Thus, the image of the homomorphism $\iota^{\#}$ consists of constant sections of the bundle $H_{\mathbf{g}}(U)$, that is, continuous sections of the sheaf $H_d(\mathbf{g})$.

In other words the homomorphism of sheaves ρ defined by the formula

$$\widetilde{\mathcal{H}}_L(x) = \varinjlim_{U \ni x} H_L(U) \xrightarrow{\rho_x^{\#}} H(\mathbf{g}_x), \qquad [s] \longmapsto [\rho_x^{\#} s],$$

where $\rho_x\colon \mathbf{g}_x \hookrightarrow L$ is a homomorphism of Lie algebroids over the inclusion $\iota\colon \{x\} \hookrightarrow M$, is an isomorphism.

We now give an example in which the LAB $\mathbf{g}$ is trivial with respect to the structure group $\mathrm{Aut}^s(\mathfrak{g})$. We assume that the isotropy Lie algebras $\mathbf{g}_x$ are Abelian and the LAB $\mathbf{g}$ is trivial in the usual sense: $\mathbf{g} \cong M \times \mathfrak{g}$. Then $\mathbf{g}$ is also trivial with respect to $\mathrm{Aut}^s(\mathfrak{g})$, or equivalently, the sheaf $\widetilde{\mathcal{H}}_L \cong H_d(\mathbf{g})$ is globally constant; in particular, $\widetilde{\mathcal{H}}_L$ has trivial monodromy if the characteristic covariant derivative in $M \times \mathfrak{g}$ is the standard derivative of vector-valued functions $\nabla_X = \partial_X$. The Lie algebroid $A(G; H)$ of a non-closed Lie subgroup H of a group G is an example here [8].

Indeed, the Atiyah sequence of A is as follows: $0 \longrightarrow M \times \mathfrak{g} \hookrightarrow L \xrightarrow{\gamma_L} TM \longrightarrow 0$. Hence choosing locally defined flat connections $\lambda\colon TU \to A_U$ we can define local trivializations $\varphi_U\colon L_U \to TU \times \mathfrak{g}$ by the formula $(\varphi_U)^{-1}(X, \sigma) = \lambda(X) + \sigma$, therefore $(\varphi_U)^+ = \mathrm{id}\colon \mathfrak{g} \to \mathfrak{g}$, which implies that the subatlas of A_L defined above gives one local trivializations of $H_d(\mathbf{g})$ with trivial structure group $\{\mathrm{id}\}$. Therefore, $\widetilde{\mathcal{H}}_L \cong M \times H(\mathfrak{g})_d$, where $H(\mathfrak{g})_d$ is the vector space with discrete topology.

§3. Signature of transitive Lie algebroids with trivial monodromy

We assume that

(A3) the presheaf $\mathcal{H}_L^*$ is constant on some good covering $\mathfrak{U}$, that is, for an arbitrary open set $U \in \operatorname{Open}_{\mathfrak{U}}(M)$ (the subcategory of finite intersections $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$ of open sets in $\mathfrak{U}$) there exists an isomorphism of algebras

$$\varphi_U \colon H_L(U) \cong H(\mathfrak{g})$$

such that all composites of the form $\varphi_V \rho_V^U \varphi_U^{-1}$ are equal to the identity of the algebra $H(\mathfrak{g})$, where the $\rho_V^U \colon H_L(U) \to H_L(V)$ ($V, U \in \operatorname{Open}_{\mathfrak{U}}(M)$, $V \subset U$) are restriction operators in the presheaf. In other words, the diagram

$$
\begin{array}{ccc}
H_L(U) & \xrightarrow{\ \varphi_U\ } & H(\mathfrak{g}) \\
\downarrow{\scriptstyle \rho_V^U} & & \downarrow{\scriptstyle \mathrm{Id}} \\
H_L(V) & \xrightarrow{\ \varphi_V\ } & H(\mathfrak{g})
\end{array}
$$

is commutative.

Condition (A3) is equivalent to the following one:

(A3′) the monodromy representation of the presheaf $\mathcal{H}_L^*$ of the fundamental group

$$\rho \colon \pi_1(M) = \pi_1(N(\mathfrak{U})) \to \operatorname{Aut}(H(\mathfrak{g}))$$

is trivial.

It is clear that both conditions (A3) and (A3′) are equivalent to the triviality of the monodromy representation of the fundamental group of the manifold M for the adjoint bundle of isotropy Lie algebras, that is, the kernel of the anchor $\mathbf{g} := \ker \gamma_L$.

Examples. The condition of the triviality of the monodromy holds in the following cases:

(1) the manifold M is simply connected ($\pi_1(N(\mathfrak{U})) = \pi_1(M) = 0$, therefore $\rho = 0$);

(2) the adjoint bundle of isotropy Lie algebras $\mathbf{g}$ is trivial with respect to the canonical atlas of local trivializations $\mathcal{A}_L$ and the structure group $\operatorname{Aut}^s(\mathbf{g})$ (see Remark 1);

(3) $\operatorname{Aut}(G) = \operatorname{Int}(G)$, where G is a simply connected Lie group with Lie algebra $\mathbf{g}_x$.

Leray's formula now yields

$$E_2^{j,i} = H_{\delta\#}^j(\mathfrak{U}, \mathcal{H}_L^i) \cong H^j(\mathfrak{U}, H^i(\mathfrak{g})) \cong H^j(\mathfrak{U}, \mathbb{R}) \otimes H^i(\mathfrak{g}) \cong H_{dR}^j(M) \otimes H^i(\mathfrak{g}).$$

All isomorphisms here are canonical isomorphisms of bigraded algebras. This means that E_2 lives in the rectangle $j \leqslant m$, $i \leqslant n$ and

$$E_2^{(m+n)} = E_2^{m,n} = H_{dR}^m(M) \otimes H^n(\mathfrak{g}).$$

By (A1), (A2), and the additional condition

(A4) the isotropy Lie algebra $\mathbf{g}$ is unimodular, that is, $\dim H^n(\mathfrak{g}) = 1$,

we obtain the following results:

(1) $\dim E_2^{(m+n)} = \dim H^m(M) \cdot \dim H^n(\mathfrak{g}) = 1$;
(2) $E_2 = H_{dR}(M) \otimes H(\mathfrak{g})$ is a Poincaré algebra with respect to the total gradation, with top group $E_2^{(m+n)} = E_2^{m,n}$.

We recall (see [21]) that a graded finite-dimensional algebra

$$\left(A = \bigoplus_{0 \leqslant r < \infty} A^r, \cup \right)$$

is called a *Poincaré algebra* if the following properties hold:

(1) there exists a superscript n_0 such that $A^r = 0$ for $r > n_0$ and $\dim A^{n_0} = 1$;
(2) $x \cup y = (-1)^{ij} y \cup x$ for $x \in A^i$ and $y \in A^j$, that is, $(A, \cup)$ is an anticommutative algebra;
(3) let $\xi \in A^{n_0}$, $\xi \neq 0$, be a basis element of A^{n_0}. Then the bilinear form

$$\langle \cdot , \cdot \rangle \colon A^r \times A^{n_0 - r} \to \mathbb{R}$$

relative to ξ (that is, the form defined by the formula $\langle x, y \rangle \xi = x \cup y$) is non-degenerate.

Then $A^r \cong (A^{n_0 - r})^*$ and $\dim A^r = \dim A^{n_0 - r}$. The key to the further investigation is the concept of *Poincaré differentiation*, that is, a linear homomorphism $d \colon A \to A$ satisfying the following conditions:

(1) $d^2 = 0$;
(2) $d[A^r] \subset A^{r+1}$;
(3) d is an antiderivation;
(4) $d[A^{n_0 - 1}] = 0$.

By analogy with the signature of an oriented manifold we have the *signature of the Poincaré algebra* $\left(A = \bigoplus A^r, \cup \right)$ relative to $\xi \in A^{n_0}$, $\xi \neq 0$. We set it equal to zero if $n_0 \neq 0 \pmod{4}$, while if $n_0 = 4k$, then $\operatorname{Sign} A$ is the signature of the non-degenerate symmetric bilinear form $\langle \cdot , \cdot \rangle \colon A^{2k} \times A^{2k} \to \mathbb{R}$ defined relative to ξ.

The next lemma is very useful for what follows.

Lemma 8 [21]. *If $(A^*, \cup, d)$ is a Poincaré algebra with Poincaré differentiation, then the graded cohomology algebra $(H^*(A), \cup)$ is a Poincaré algebra relative to the same element $\xi \in A^{n_0} = H^{n_0}(A, d)$, $\xi \neq 0$, and one has the equality*

$$\operatorname{Sign} A = \operatorname{Sign} H(A).$$

Examples. (1) Let E be an arbitrary finite-dimensional vector space. Then the exterior algebra $(\wedge E, \wedge)$ is a Poincaré algebra of signature zero.

(2) Let $\mathfrak{g}$ be an arbitrary real Lie algebra. Then the system

$$(\wedge \mathfrak{g}^*, \wedge, \delta_{\mathfrak{g}})$$

with Chevalley–Eilenberg differential $\delta_{\mathfrak{g}}$ is a Poincaré algebra with Poincaré differentiation if and only if $\mathfrak{g}$ is unimodular. The above lemma shows that if $\mathfrak{g}$ is unimodular, then the cohomology algebra $H(\mathfrak{g})$ is also a Poincaré algebra and

$$\operatorname{Sign} H(\mathfrak{g}) = \operatorname{Sign} \wedge \mathfrak{g}^* = 0.$$

If L is a transitive Lie algebroid over a compact oriented connected manifold M with unimodular isotropy Lie algebra and trivial monodromy representation in the cohomology of the adjoint bundle of isotropy Lie algebras, then for a good covering the second term E_2 of the spectral sequence is a Poincaré algebra living in the rectangle $j \leqslant m$, $i \leqslant n$ with top group $E_2^{(m+n)} = E_2^{m,n}$.

Our aim is to compare the signature of E_2 with the signature of L in the case when L is a TUIO-Lie algebroid. The Chern–Hirzebruch–Serre arguments [21] are purely algebraic and lead to the following general result.

Theorem 9. *Let $(A, A^r, \cup, D, A_j)$ be a DG-algebra with decreasing filtration A_j, and $(E_s^{j,i}, d_s)$ its spectral sequence. Assume that there exist positive integers m and n such that*

(1) *$E_2^{j,i} = 0$ for $j > m$ and $i > n$;*

(2) *E_2 is a Poincaré algebra with respect to the total gradation, with top group $E_2^{(m+n)} = E_2^{m,n}$.*

Then each term $(E_s^{(*)}, \cup, d_s)$, $2 \leqslant s < \infty$, is a Poincaré algebra with Poincaré differentiation. The infinite term $(E_\infty^{(*)}, \cup)$ is also a Poincaré algebra and

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = \cdots = \operatorname{Sign} E_\infty.$$

Proof. Clearly, all terms $E_3, E_4, \ldots, E_\infty$ live in the same rectangle $j \leqslant m$, $i \leqslant n$. We can use the following property of spectral sequences [26]: if $E_s^{j,i} = 0$ for some s, j, i, then $0 = E_s^{j,i} = E_{s+1}^{j,i} = \cdots = E_\infty^{j,i}$. Now, the bidegree argument $(d_2 \colon E_2^{j,i} \to E_2^{j+2,i-1})$ shows that

$$d_2[E_2^{(m+n-1)}] = 0$$

(see Fig. 1). Therefore, $E_2^{(m+n)} = E_3^{(m+n)} = \cdots = E_\infty^{(m+n)}$, that is, we have $E_2^{m,n} = E_3^{m,n} = \cdots = E_\infty^{m,n}$.

Thus, we start with the assumption that $(E_2, \cup, d_2)$ is a Poincaré algebra with Poincaré differentiation. By Lemma 8 we obtain that $E_3 = H(E_2, d_2)$ is a Poincaré algebra, and relative to the same element $0 \neq \xi \in E_3^{m,n} = E_2^{m,n}$ we have

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3.$$

The same argument shows that d_3 is Poincaré differentiation and $\operatorname{Sign} E_3 = \operatorname{Sign} E_4$, and so on. As a result,

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = \operatorname{Sign} E_4 = \cdots.$$
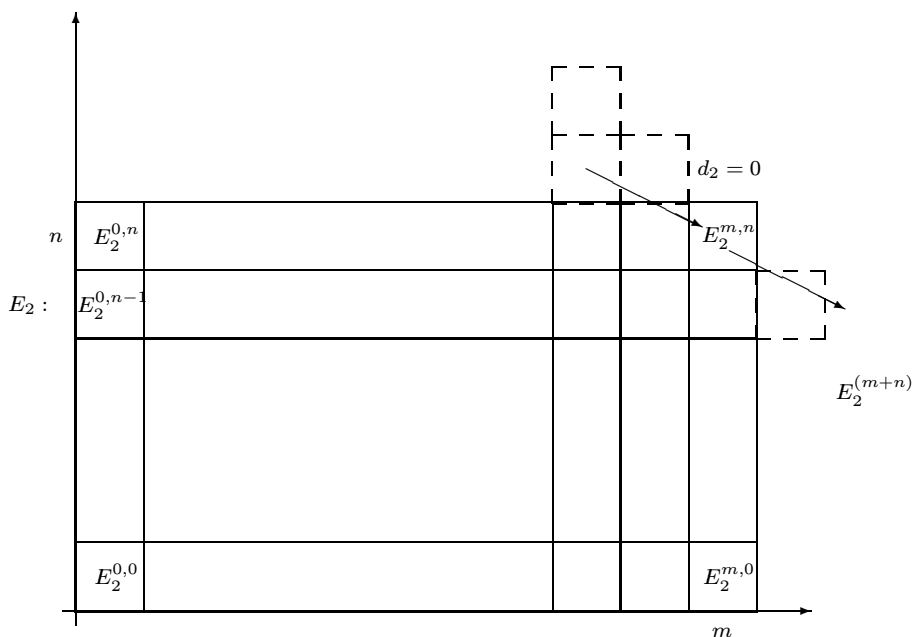
Figure 1

Since the term E_2 lies in a finite rectangle, the spectral sequence E_s collapses in some term E_{s_0}. Hence $E_s^{(r)} \cong E_\infty^{(r)}$ for $s > \max\{s_0, n\}$, which means that $(E_\infty, \cup)$ is a Poincaré algebra and $\operatorname{Sign} E_{s_0} = \operatorname{Sign} E_\infty$. As a result we obtain

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = \cdots = \operatorname{Sign} E_\infty.$$

Conclusion 10. *If L is a transitive Lie algebroid on a compact oriented connected manifold M with unimodular isotropy Lie algebras $\mathbf{g}_x \cong \mathfrak{g}$ and monodromy in the cohomology of the adjoint bundle $\mathbf{g}$ of isotropy Lie algebras is trivial, then the terms $E_2, \ldots, E_\infty$ are Poincaré algebras and*

$$0 = \operatorname{Sign} M \cdot \operatorname{Sign} \mathfrak{g} = \operatorname{Sign} E_2 = \cdots = \operatorname{Sign} E_\infty.$$

It remains to prove the equality $\operatorname{Sign} E_\infty = \operatorname{Sign} H_L(M)$. The same arguments as in the above-mentioned original paper by Chern, Hirzebruch, and Serre lead to the following general result.

Let $(A, A^r, \cup, D, A_j)$ be an arbitrary DG-algebra with decreasing filtration with regularity $A_0 = A$:

$$A = A_0 \supset \cdots \supset A_j \supset A_{j+1} \supset \cdots,$$

compatible with the DG-structure, that is,

$$A_i A_j \subset A_{i+j}, \qquad D(A_j) \subset A_j, \qquad A_j = \bigoplus_r (A^r \cap A_j).$$

Let $(E_s^{j,i}, d_s)$ be the spectral sequence associated with this filtration. Assume also that

(1) the infinite term $E_\infty^{p,q}$ lives in the rectangle $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant n$;
(2) $\dim E_\infty^{m,n} = 1$;
(3) E_∞ is a Poincaré algebra with respect to the total gradation.

In particular, $\dim E_\infty$ is finite.

Theorem 11. *Under the above assumptions about the algebra A its cohomology algebra satisfies the following conditions*:

(1) $H^{m+n}(A) \cong E_\infty^{m,n}$, *that is* $\dim H^{m+n}(A) = 1$;
(2) *the algebra* $H(A) = \bigoplus_{r=0}^{m+n} H^r(A)$ *is a Poincaré algebra*;
(3) *the signature of the cohomology $H(A)$ is equal to that of the infinite term of the spectral sequence*

$$\operatorname{Sign} E_\infty = \operatorname{Sign} H(A)$$

under a suitable choice of generators of the top groups.

Proof. (1) This is obvious because $H^{m+n}(A) \cong E_\infty^{(m+n)} \cong E_\infty^{m,n}$ and, by assumption, $\dim E_\infty^{m,n} = 1$.

(2) Let $A^{j,i} = A^{j+i} \cap A_j$. The algebra $H(A)$ possesses a graded filtration $H^{j,i}(A) = \pi[Z \cap A^{j,i}]$, $H^{0,i}(A) = H^i(A)$, where Z is the kernel of the differential D and π is the quotient homomorphism modulo the range of the differential D. The associated space $E_0(H(A))$ has the bigradation

$$E_0^{j,i}(H(A)) \cong H^{j,i}(A)/H^{j+1,i-1}(A).$$

Consider the decreasing filtration

$$H^r(A) = H^{0,r}(A) \supset H^{1,r-1}(A) \supset \cdots \supset H^{r,0} \supset 0, \qquad (2)$$

its part for $0 \leqslant j \leqslant r$

$$H^{j,r-j} \supset H^{j+1,r-j-1} \supset \cdots \supset H^{r-1,1} \supset H^{r,0} \supset 0,$$

and the associated non-canonical isomorphisms

$$H^{j,r-j} \cong E_0^{j,r-j} \oplus E_0^{j+1,r-j-1} \oplus \cdots \oplus E_0^{r,0}, \qquad E_0^{r,0} = H^{r,0}, \qquad (3)$$

which are compatible with one another for each j.

The bigraded spaces $H(A,D)$, $E_0(H(A))$, and E_∞ possess a natural structure of bigraded algebras such that there exists an isomorphism

$$\sigma_\infty \colon E_\infty \to E_0(H(A))$$

of bigraded algebras uniquely defined by the condition of the commutativity of the following diagram:

$$
\begin{array}{ccc}
Z \cap A^{j,i} & \twoheadrightarrow & H^{j,i} \\
\downarrow & & \downarrow \\
E_\infty^{j,i} & \xrightarrow{\sigma_\infty^{j,i}} & E_0^{j,i}(H(A)) \quad = H^{j,i}(A)/H^{j+1,i-1}(A).
\end{array}
\qquad (4)
$$

Hence if $x \in H^{j,i}(A)$, $y \in H^{j',i'}(A)$, and $\sigma_\infty^{j,i}(\overline{x}) = [x]$, $\sigma_\infty^{j',i'}(\overline{y}) = [y]$, then

$$\sigma_\infty(\overline{x} \cup \overline{y}) = [x] \cup [y] = [x \cup y]. \tag{5}$$

Since σ_∞ is a bigraded isomorphism, $E_0^{j,i}(H(A))$ lives in the same rectangle $j \leqslant m$, $i \leqslant n$, that is,

$$E_0^{j,i}(H(A)) = 0 \quad \text{for } j > m \quad \text{or } i > n, \tag{6}$$

and

$$\dim E_0^{m,n}(H(A)) = 1.$$

By (3),

$$H^{j,i}(A) = 0, \qquad j > m. \tag{7}$$

One consequence of (7) is as follows:

$$E_0^{m,n}(H(A)) = H^{m,n}(A)/H^{m+1,n-1} = H^{m,n}(A) = H^{m+n}(A).$$

The last equality follows from (3), the equality $H^{m+n}(A) \cong E_\infty^{m+n} = E_\infty^{m,n} \cong E_0^{m,n}$, and the dimension argument

$$1 = \dim H^{m+n}(A) = \dim E_0^{m,n}(H(A)) = \dim H^{m,n}(A).$$

Hence

$$H^{m+n}(A) = H^{0,m+n}(A) = H^{1,m+n-1}(A) = \cdots = H^{m,n}(A)$$

and

$$H^{m+n-i,i}(A) = 0, \qquad i < n. \tag{8}$$

If $j = m$ and $i = n$, then the right arrow in (4) is an identity:

$$
\begin{array}{ccc}
Z \cap A^{m,n} & \longrightarrow & H^{m,n}(A) \\
\downarrow & & \| \\
E_\infty^{m,n} & \xrightarrow[\cong]{\sigma_\infty^{m,n}} & H^{m,n}(A).
\end{array}
$$

Consequently, by (5) we obtain the following result: if $x \in H^{j,i}(A)$, $y \in H^{j',i'}(A)$, $j + j' = m$, $i + i' = n$, and $\sigma_\infty^{j,i}(\overline{x}) = [x]$, $\sigma_\infty^{j',i'}(\overline{y}) = [y]$, then

$$\sigma_\infty(\overline{x} \cup \overline{y}) = x \cup y. \tag{9}$$

Using the isomorphisms $\sigma_\infty^{m,n} \colon E_\infty^{m,n} \to H^{m,n}(A)$, we fix generators ξ and ξ' distinct from zero such that

$$0 \neq \xi \in E_\infty^{m,n}, \qquad 0 \neq \xi' = \sigma_\infty^{m,n}(\xi) \in H^{m,n}(A) = H^{m+n}(A).$$

Relative to ξ and ξ' we consider the symmetric bilinear forms

$$\langle \cdot, \cdot \rangle_\infty \colon E_\infty^{(j)} \times E_\infty^{(m+n-j)} \to \mathbb{R},$$
$$\langle \cdot, \cdot \rangle_A \colon H^j(A) \times H^{m+n-j}(A) \to \mathbb{R},$$

that is, the forms such that $\langle x, y \rangle_\infty \xi = x \cup y$ for the first and a similar equality holds for the second. The bilinear function $\langle \cdot, \cdot \rangle_\infty$ is non-degenerate by assumption. We claim that $\langle \cdot, \cdot \rangle_A$ is non-degenerate. The dimension $\dim H^r(A)$ is finite because

$$H^r(A) \cong E_0^{(r)}(H(A)) \cong E_\infty^{(r)},$$

and E_∞ is finite-dimensional. Hence we must check that if $x \in H^j(A)$, $x \neq 0$, then there exists an element $y \in H^{m+n-j}(A)$ such that $\langle x, y \rangle_A \neq 0$. The element x belongs to the minimal member of the filtration (2):

$$x \in H^{p,j-p}(A), \qquad x \notin H^{p+1,j-p-1}(A),$$

therefore $[x] \in E_0^{p,j-p}$, $[x] \neq 0$. We take $\overline{x} \in E_\infty^{p,j-p}$ such that $\sigma_\infty^{p,j-p}(\overline{x}) = [x]$. By the non-degeneracy of $\langle \cdot, \cdot \rangle_\infty$ we can find an element $y \in H^{m-p,n-j-p}$ such that $\langle \overline{x}, \overline{y} \rangle_\infty \neq 0$, where $\overline{y} \in E_\infty^{m-p,n-j-p}$ and $\sigma_\infty^{m-p,n-j-p}(\overline{y}) = [y]$. By (9) we obtain $x \cup y = \sigma_\infty(\overline{x} \cup \overline{y})$, therefore $\langle x, y \rangle_A = \langle \overline{x}, \overline{y} \rangle_\infty \neq 0$, which proves the non-degeneracy of $\langle \cdot, \cdot \rangle_A$. Thus, $H(A)$ is a Poincaré algebra.

(3) We assume in what follows that $m+n = 4k$ (otherwise the signatures are zero by definition). Consider the symmetric bilinear forms relative to the generators ξ and ξ',

$$\langle \cdot, \cdot \rangle_\infty \colon E_\infty^{(2k)} \times E_\infty^{(2k)} \to \mathbb{R},$$
$$\langle \cdot, \cdot \rangle_A \colon H^{2k}(A) \times H^{2k}(A) \to \mathbb{R},$$

that is, the forms such that $\langle x, y \rangle_\infty \xi = x \cup y$ in the first case and a similar equality holds in the second. The space $H^{2k}(A)$ has the decreasing filtration (2), and by (3) and (6),

$$H^{2k-n,n}(A) = H^{2k-s,s}(A), \qquad s \geqslant n.$$

We must consider two cases, $m \geqslant n$ and $m \leqslant n$. In the case $m \geqslant n$ we have $2k - n = (n+m)/2 - n = (m-n)/2 \geqslant 0$. Hence

$$H^{2k}(A) = H^{2k-n,n}(A) \supset H^{2k-n+1,n-1} \supset \cdots \supset H^{2k,0} \supset 0,$$

and

$$H^{2k-q,q}(A)/H^{2k-q+1,q-1}(A) \overset{\sigma_\infty}{\cong} E_\infty^{2k-q,q}, \qquad q = 0, \ldots, n.$$

We choose subspaces $V_0, \ldots, V_n$, $V_q \subset H^{2k-q,q}$, such that

$$H^{2k-q,q}(A) = V_q \oplus H^{2k-q+1,q-1}(A).$$

Of course,

$$V_0 = H^{2k,0}, \qquad H^{2k} = V_0 \oplus V_1 \oplus \cdots \oplus V_n, \qquad E_\infty^{2k-q,q} \cong V_q. \qquad (10)$$

If $x \in V_i$ and $y \in V_j$, $i+j < n$, then $x \cup y \in H^{4k-i-j,i+j}(A) = 0$ by (8). If $i+j = n$ and elements $\overline{x} \in E_\infty^{2k-i,i}$ and $\overline{y} \in E_\infty^{2k-j,j}$ correspond to x and y by virtue of σ_∞, then by (9) we obtain

$$\langle \overline{x}, \overline{y} \rangle_\infty = \langle x, y \rangle_A.$$

Since E_∞ is a Poincaré algebra, it follows that

$$\dim E_\infty^{2k-i,i} = \dim E_\infty^{2k-n+i,n-i}$$

and if $\dim E_\infty^{2k-i,i} > 0$, then

$$\langle \cdot, \cdot \rangle_\infty \colon E_\infty^{2k-i,i} \times E_\infty^{2k-n+i,n-i} \to \mathbb{R}$$

is a non-degenerate bilinear form. Hence the matrix of the bilinear symmetric function $\langle \cdot, \cdot \rangle_A \colon H^{2k}(A) \times H^{2k}(A) \to \mathbb{R}$ can be written in blocks as follows:

$$\begin{pmatrix} 0 & \ldots & 0 & L_0 \\ 0 & \ldots & L_1 & * \\ \vdots & \diagup & \vdots & \vdots \\ L_n & \ldots & * & * \end{pmatrix}, \qquad (11)$$

where the L_i are non-singular square matrices and L_i is the transpose of L_{n-i}. Moreover, the matrix

$$\begin{pmatrix} 0 & \ldots & 0 & L_0 \\ 0 & \ldots & L_1 & 0 \\ \vdots & \diagup & \vdots & \vdots \\ L_n & \ldots & 0 & 0 \end{pmatrix}, \qquad (12)$$

which contains the same blocks on the antidiagonal as the first matrix and zeros elsewhere, is the matrix of $\langle \cdot, \cdot \rangle_\infty$ on $E_\infty^{(2k)}$.

If some spaces V_q have dimension zero, then we modify the matrices (11) and (12) by removing the rows and the columns corresponding to zero-dimensional spaces and repeat the arguments.

The case $m \leqslant n$ is a word-for-word repetition of the above-considered case, including the procedure of the removal of the rows and the columns corresponding to zero-dimensional spaces in the decomposition (10). Similar arguments demonstrate that the bilinear form in the cohomology is non-degenerate in all dimensions. Obviously, (see [21], Lemma 1) we have $\operatorname{Sign} E_\infty = \operatorname{Sign} H(A)$.

Using the same Lemma 1 in [21] we can additionally assert that $\operatorname{Sign} H(A)$ is equal to 0 if n is odd and to $\operatorname{Sign} L_{n/2}$ if n is even.

Corollary 12. *Let $(A, A^r, \cup, D, A_j)$ be an arbitrary DG-algebra with decreasing filtration A_j and with regularity condition $A_0 = A$, and let $(E_s^{j,i}, d_s)$ be its spectral sequence. Assume that there exist positive integers m and n such that*

   (1) $E_2^{j,i} = 0$ *for $j > m$ and $i > n$;*
   (2) E_2 *is a Poincaré algebra with respect to the total gradation and the top group $E_2^{(m+n)} = E_2^{m,n}$.*

Then $H(A) = \bigoplus_{r=0}^{m+n} H^r(A)$ is a Poincaré algebra, $\dim H^{m+n}(A) = 1$, and

$$\operatorname{Sign} E_2 = \operatorname{Sign} H(A).$$

As a consequence we obtain the following result.

Theorem 13 (Chern–Hirzebruch–Serre theorem for transitive Lie algebroids).
*Let L be a transitive Lie algebroid on a compact oriented connected manifold with
unimodular isotropy Lie algebras $\mathfrak{g}_x \cong \mathfrak{g}$ and trivial monodromy (that is, the pre-
sheaf $\mathcal{H}_L^*$ of cohomology of $\mathfrak{g}$ is constant on some good covering). Then the
cohomology algebra $H_L(M)$ is a Poincaré algebra and*

$$\operatorname{Sign} L = \operatorname{Sign} E_2 = \operatorname{Sign} M \cdot \operatorname{Sign} \mathfrak{g} = 0.$$

There exist many TUIO-Lie algebroids among examples of Lie algebroids with
unimodular isotropy Lie algebras and trivial monodromy. In particular, Theorem 13
holds if

(1) M is simply connected, for example, if $A = A(M; \mathcal{F})$ is the Lie algebroid of
a TP-foliation on a compact simply connected manifold;

(2) $\operatorname{Aut} G = \operatorname{Int} G$, where G is a simply connected Lie group with Lie algebra $\mathfrak{g}$,
for instance, if $\mathfrak{g}$ is a simple Lie algebra of type B_l, C_l, E_7, E_8, F_4, or G_2
(see [27], Appendix D.8);

(3) the adjoint Lie algebra bundle **g** is trivial with respect to the structure group
$\operatorname{Aut}^s(\mathfrak{g})$, for example, if $A(G; H)$ is the Lie algebroid of the TC-foliation of
left cosets of a non-closed Lie subgroup H of an arbitrary Lie group G (see
the example at the end of §2).

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