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EXPONENTIAL MAPPING FOR LIE GROUPOIDS

BY

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0. Introduction. Lie groupoids have appeared in many problems of differential geometry of higher order, e.g., in the theories of connections of higher order, G-structures of higher order, pseudogroups, and in some theory of differential equations. Originally (see [5]), Lie groupoids were associated with principal fibre bundles. Ngo Van Que [27] formulated in 1967 a precise abstract definition of these objects (see also [7] and [18]). The development of the general theory of Lie groupoids can be found in the papers of Pradines [22]-[26] and Kumpera [11], [12], and also in [3], [4], [8], [15]-[17], [21], [31]-[34].

In [11] an exponential mapping was defined. This paper is devoted to a closer examination of this notion.

A *Lie groupoid* (see [27]) is a collection

$$\Phi = \big(\Phi, (\alpha, \beta), M, \cdot\big) \tag{0.1}$$

in which

(i) Φ and M are smooth (i.e., of class C^∞) manifolds with countable bases, M being a connected manifold;

(ii) $\alpha\colon \Phi \to M$ and $\beta\colon \Phi \to M$ are surmersions (i.e., submersions plus onto);

(iii) $\cdot\colon D \to \Phi$, where $D = \{(z, z') \in \Phi \times \Phi : \alpha(z) = \beta(z')\}$, is a partial multiplication which satisfies

(a) $\beta(z\cdot z') = \beta(z)$ and $\alpha(z\cdot z') = \alpha(z')$ if $(z, z') \in D$ and $\cdot(z, z') = z\cdot z'$,

(b) for every point $x \in M$ there exists an element $l_x \in \Phi$ such that $\alpha(l_x) = \beta(l_x) = x$ and $z\cdot l_x = z$ if $\alpha(z) = x$ and $l_x\cdot z = z$ if $\beta(z) = x$,

(c) for every element $z \in \Phi$ there exists an element $z^{-1} \in \Phi$ such that $z\cdot z^{-1} = l_y$, where $y = \beta(z)$, and $z^{-1}\cdot z = l_x$, where $x = \alpha(z)$;

(iv) the *condition of transitivity* holds, which means that (α, β): $\Phi \to M \times M$ is surjective;

(v) the mapping ${}^{-1} = (\Phi \ni z \mapsto z^{-1} \in \Phi)$ is smooth;

(vi) for every smooth manifold W and for every two smooth mappings $f, g\colon W \to \Phi$ such that $\alpha \circ f = \beta \circ g$, the mapping

$$f \cdot g = \big(W \ni z \mapsto f(z) \cdot g(z) \in \Phi\big)$$

is smooth.

If $\boldsymbol{\Phi} = \big(\Phi, (\alpha, \beta), M, \cdot\big)$ is a Lie groupoid (shortly, L.g.), then Φ is called a *space of the groupoid* $\boldsymbol{\Phi}$, M is said to be a *manifold of units*, α and β are called mappings "*source*" and "*target*".

The set $\Phi_{(x,x)}$ of those elements h, belonging to the space of a Lie groupoid $\boldsymbol{\Phi}$, for which $\alpha(h) = \beta(h) = x$ is called the *isotropy group* of $\boldsymbol{\Phi}$ over x. It is a Lie group [27]. For every Lie groupoid $\boldsymbol{\Phi}$ and every x belonging to M (i.e., to a manifold of the units) a principal fibre bundle

$$\boldsymbol{\Phi}_x = (\Phi_x, M, \gamma, \Phi_{(x,x)}, \cdot)$$

is determined in the following way [27]: The set Φ_x consists of all elements $h \in \Phi$ such that $\alpha h = x$ (i.e., $\alpha(h) = x$). Φ_x is a submanifold of Φ. The projection $\gamma\colon \Phi_x \to M$ is equal to $\beta \,|\, \Phi_x$. The action of the Lie group $\Phi_{(x,x)}$ on Φ_x is determined by the formula $\cdot(h, g) = h \cdot g$, where $h \in \Phi_x$, $g \in \Phi_{(x,x)}$.

A Lie group is a Lie groupoid with a one-element manifold of the units. A typical example is the Lie groupoid $\boldsymbol{\pi}^k(M)$ of all invertible jets of the k-th order of a manifold M, where $\alpha(j^k_x f) = x$, $\beta(j^k_x f) = f(x)$, and $j^k_{g(x)} f \cdot j^k_x g = j^k_x (f \circ g)$ (see [6], [13], and [14]).

An element $h \in \Phi$ such that $\alpha h = x$ and $\beta h = y$ will be denoted by

$$x \xrightarrow{h} y.$$

1. Lie algebroid of a Lie groupoid. Let $\boldsymbol{\Phi} = \big(\Phi, (\alpha, \beta), M, \cdot\big)$ be an L.g. The diffeomorphism

$$\Phi_h = (\Phi_{\beta h} \ni g \mapsto g \cdot h \in \Phi_{\alpha h})$$

is called a *right translation by the element* $h \in \Phi$ (see [11]). The vector field ξ on an open set $\Omega \subset \Phi$ is called *right-invariant* (shortly, r-i) if ξ is α-vertical (i.e., $\alpha_* \xi_g = 0$, $g \in \Omega$) and if ξ is invariant with respect to all right translations by elements $h \in \Phi$ (i.e., $\xi_{gh} = (\Phi_h)_{*g}(\xi_g)$, $g \in \Omega$ and $gh \in \Omega$). See also [19].

Example. Let X be a vector field on an open subset U of a manifold N and let X be generated by a local one-parameter group of diffeomorphisms f_t. Then the family f^k_t determined on the open set $\beta^{-1}[U] \subset \pi^k(N)$ by the formula

$$j^k_t Y = j^k_{\beta Y} f_t \cdot Y, \qquad Y \in \beta^{-1}[U],$$

is a local one-parameter group of diffeomorphisms. It generates a vector field X^k which is r-i (see [30]).

The mappings $\sigma\colon U \to \Phi$, where U is an open set in M such that $\beta\circ\sigma = \mathrm{id}_U$ and $\alpha\circ\sigma\colon U \to M$ is a diffeomorphism onto an open subset of M, are called *α-admissible β-sections.*

The mapping $\varphi\colon \alpha^{-1}[U] \to \alpha^{-1}[U']$ defined by the formula

$$\varphi(g) = g\cdot\sigma(\alpha g), \quad g \in \alpha^{-1}[U],$$

where $\sigma\colon U \to \Phi$ is an α-admissible β-section such that $\alpha\circ\sigma[U] = U'$, is called a *right translation by the section* σ (see [11]).

THEOREM 1.1. *If ξ is an r-i vector field on an open set $\Omega \subset \Phi$, then there exists exactly one r-i vector field ξ' on $\Omega' = \beta^{-1}[\beta[\Omega]]$ such that the restriction of ξ' to Ω is equal to ξ. If ξ is smooth, then so is ξ'.*

Proof. The existence and uniqueness of ξ' are evident. Assume that ξ is smooth. We take an arbitrary element $x' \xrightarrow{h} y$ belonging to $\beta^{-1}[\beta[\Omega]]$ and an element $x \xrightarrow{g} y$ belonging to Ω $(\beta h = \beta g = y)$. Let $\varphi\colon \alpha^{-1}[U] \to \alpha^{-1}[U']$ be a right translation by an α-admissible β-section $\sigma\colon U \to \Phi$ for which $x \in U$ and $\sigma(x) = g^{-1}\cdot h$. We take $\Theta = \Omega \cap \alpha^{-1}[U]$ and $\Theta' = \varphi[\Theta]$. Then $h \in \Theta'$, $g \in \Theta$ and, clearly, $\xi'|\Theta' = (\varphi|\Theta)_*(\xi|\Theta)$, which completes the proof.

Let us consider the vector bundle (see [24]) $i^*(T^\alpha\Phi)$, i.e., first we take the vector subbundle $T^\alpha\Phi \subset T\Phi$ of the tangent bundle $T\Phi$, consisting of all α-vertical vectors, and next we pull it back by the imbedding $i = (M \ni x \mapsto l_x \in \Phi)$.

The r-i field ξ on an open set $\Omega \subset \Phi$ determines a cross-section ξ_0 of $i^*(T^\alpha\Phi)$ over $\beta[\Omega]$ by the formula

$$(\xi_0)_x = (\Phi_{h^{-1}})_{*h}(\xi_h), \quad x \in \beta[\Omega],\ h \in \Omega, \text{ and } \beta h = x.$$

The correctness of that formula follows from $(\xi_0)_x = \xi'(l_x)$ (for ξ' see Theorem 1.1). If ξ is smooth, then so is ξ_0.

THEOREM 1.2. *Every cross-section η of $i^*(T^\alpha\Phi)$ over an open set $U \subset M$ can be extended uniquely to an r-i vector field η' on $\beta^{-1}[U]$. If η is smooth, then so is η'.*

Proof. We take a cross-section η of $i^*(T^\alpha\Phi)$ over an open set $U \subset M$. It determines a vector field η' on $\beta^{-1}[U] \subset \Phi$ by the formula

$$\eta'(h) = (\Phi_h)_{*l_{\beta h}}(\eta(\beta h)), \quad h \in \beta^{-1}[U].$$

Clearly, η' is r-i. Let η be a smooth cross-section. To prove the theorem it suffices to show that η' is smooth in a neighbourhood of every unit l_x, $x \in U$.

Fix the unit l_{x_0}, $x_0 \in U$, and take a coordinate system $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ of the manifold M in the domain $D_{\bar{x}} \ni x_0$ and a coordinate system

$$\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{x}_{n+1}, \ldots, \hat{x}_{n+m}, \hat{x}_{n+m+1}, \ldots, \hat{x}_{2n+m})$$

of the manifold Φ in the domain $D_{\hat{x}}$ such that

(a) $l_{x_0} \in D_{\hat{x}}$,

(b) $\alpha[D_{\hat{x}}] \subset D_{\bar{x}}$, $\beta[D_{\hat{x}}] \subset D_{\bar{x}}$,

(c) $(\hat{x}_1, \ldots, \hat{x}_n) = \bar{x} \circ \beta \mid D_{\hat{x}}$,

(d) $(\hat{x}_{n+m+1}, \ldots, \hat{x}_{2n+m}) = \bar{x} \circ \alpha \mid D_{\hat{x}}$.

Clearly, there exists a neighbourhood $V \subset D_{\hat{x}}$ of l_{x_0} such that $\tau, \sigma \in V$ implies $\tau \cdot \sigma \in D_{\hat{x}}$. If $\sigma \in V$, then for $i = 1, \ldots, 2n+m$

$$(\eta')_\sigma(\hat{x}_i) = (\Phi_\sigma)_{*\beta h}(\eta_{\beta h})(\hat{x}_i)$$
$$= \eta_{\beta\sigma}(\hat{x}_i \circ \Phi_\sigma) = \eta_{\beta\sigma}\big(\Phi_{\beta\sigma} \ni \tau \mapsto \hat{x}_i(\tau \cdot \sigma) \in R\big).$$

We take the imbedding

$$l = (R^n \times R^m \times R^n \times R^m \times R^n \to R^{2n+m} \times R^{2n+m}),$$
$$(a, b, c, d, e) \mapsto \big((a, b, c), (c, d, e)\big),$$

and we put

$$W = l^{-1}\big[\hat{x}[V] \times \hat{x}[V]\big] \subset R^{3n+2m}.$$

W is an open set containing the point

$$\big(\hat{x}_1(l_{x_0}), \ldots, \hat{x}_{2n+m}(l_{x_0}), \hat{x}_{n+1}(l_{x_0}), \ldots, \hat{x}_{2n+m}(l_{x_0})\big).$$

There exist smooth functions $f_i\colon W \to R$, $i = 1, \ldots, 2n+m$, for which

$$\hat{x}_i(\tau \cdot \sigma) = f_i\big(\hat{x}_1(\tau), \ldots, \hat{x}_{n+m}(\tau), \hat{x}_1(\sigma), \ldots, \hat{x}_{2n+m}(\sigma)\big), \qquad i = 1, \ldots, 2n+m,$$

where $\tau, \sigma \in V$ and $\tau \cdot \sigma$ is defined. Hence

$$(\eta')_\sigma(\hat{x}_i) = \eta_{\beta\sigma}\Big(f_i\big(\hat{x}_1(\cdot), \ldots, \hat{x}_{n+m}(\cdot), \hat{x}_1(\sigma), \ldots, \hat{x}_{2n+m}(\sigma)\big)\Big)$$
$$= \sum_{j=1}^{n+m} \eta_{\beta\sigma}(\hat{x}_j) \cdot f_{i|j}\big(\hat{x}_1(l_{\beta\sigma}), \ldots, \hat{x}_{n+m}(l_{\beta\sigma}), \hat{x}_1(\sigma), \ldots, \hat{x}_{2n+m}(\sigma)\big).$$

Finally, to complete the proof it suffices to see that the function

$$V \ni \sigma \mapsto \eta_{\beta\sigma}(\hat{x}_j), \qquad j = 1, \ldots, n+m,$$
$$V \ni \sigma \mapsto f_{i|j}\big(\hat{x}_1(l_{\beta\sigma}), \ldots, \hat{x}_{n+m}(l_{\beta\sigma}), \hat{x}_1(\sigma), \ldots, \hat{x}_{2n+m}(\sigma)\big),$$
$$i = 1, \ldots, 2n+m, \; j = 1, \ldots, n+m,$$

are smooth.

From Theorem 1.2 it follows that if ξ is an r-i vector field on an open set $\Omega \subset \Phi$, then for any point $g \in \Omega$ there exists a globally defined r-i vector field η on Φ such that $\xi|\Theta = \eta|\Theta$ for an open set $\Theta \subset \Omega$, where $g \in \Theta$.

We shall continue to assume that the r-i vector fields under consideration are smooth.

Let ξ and η be some r-i vector fields on an open set $\beta^{-1}[U]$, $U \subset M$. Then the following statements are true:

(i) *The Poisson bracket $[\xi, \eta]$ is also an r-i vector field.*

(ii) *If f belongs to $C^\infty(M \mid U)$, then the vector field $(f\circ\beta)\cdot\xi$ is also r-i and we have*

$$(f\circ\beta)\cdot\xi = (f\cdot\xi_0)', \quad [\xi, (f\circ\beta)\cdot\eta] = (f\circ\beta)[\xi, \eta] + (\beta_*\xi)(f)\cdot\eta,$$

where ξ_0 is a cross-section of $i^*(T^a\Phi)$ over U determined by the formula $(\xi_0)_x = \xi(l_x)$, $x \in U$ (for $(f\cdot\xi_0)'$ see Theorem 1.2).

(iii) *ξ is β-related to exactly one vector field X on U. If we denote by $\tilde{\beta}_*$ the morphism*

$$i^*(T^a\Phi) \ni v \mapsto \beta_*(v) \in TM,$$

then X is equal to $\tilde{\beta}_* \circ \xi_0$ and it is denoted by $\beta_*\xi$.

(iv) *The vector space of all smooth global cross-sections of $i^*(T^a\Phi)$, namely $C^\infty(i^*(T^a\Phi))$ with bracket $[\![\ ,\ ]\!]$ defined by $[\![\xi, \eta]\!] = [\xi', \eta']_0$, is an R-Lie algebra. This bracket has the property*

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi, \eta]\!] + (\beta_*\xi)(f)\cdot\eta.$$

(v) *The morphism $\tilde{\beta}_*$ has the following properties:*

(a) *$\tilde{\beta}_*$ is an epimorphism,*

(b) *$C^\infty(\tilde{\beta}_*)$: $C^\infty(i^*(T^a\Phi)) \to C^\infty(TM)$ is an R-Lie algebra homomorphism.*

A *Lie algebroid* (shortly, L.a.) is a collection

$$\boldsymbol{A} = (A, [\![\ ,\ ]\!], \gamma)$$

in which

(1) A is a vector bundle over any manifold M;

(2) $[\![\ ,\ ]\!]$: $C^\infty(A) \times C^\infty(A) \to C^\infty(A)$ is a mapping such that $(C^\infty(A), [\![\ ,\ ]\!])$ is a Lie algebra (over R);

(3) γ: $A \to TM$ is an epimorphism of the vector bundles;

(4) if η, $\mu \in C^\infty(A)$ and $f \in C^\infty(M)$, then

$$[\![\eta, f\cdot\mu]\!] = f\cdot[\![\eta, \mu]\!] + (\gamma\circ\eta)(f)\cdot\mu;$$

(5) $C^\infty(\gamma)$: $C^\infty(A) \to C^\infty(TM)$ is an R-Lie algebra homomorphism.

Thus an arbitrary L.g. $\boldsymbol{\Phi} = (\Phi, (\alpha, \beta), M, \cdot)$ determines any object

$$(i^*(T^a\Phi), [\![\ ,\ ]\!], \tilde{\beta}_*) \tag{1.1}$$

which is a Lie algebroid (see [24]).

This definition is almost identical with that of Pradines [24]. It differs in that Pradines does not require for the morphism γ to be an epimorphism. The reason is the fact that Pradines associates such an object with an object more general than L.g., namely with the differential one (see also [1], [2], [9], [10], [20]).

Example. The L.a. of the L.g. $\pi^k(M)$ is isomorphic to

$$\left(J^k(TM), [\![\ ,\ ]\!], \tilde{\beta}\right),$$

where

(i) $J^k(TM)$ is the vector bundle of the k-th order jets of vector fields of M,

(ii) $\tilde{\beta} = \left(J^k(TM) \ni j^k_x\Theta \mapsto \Theta(x) \in TM\right)$,

(iii) $[\![\ ,\ ]\!]$: $C^\infty\left(J^k(TM)\right) \times C^\infty\left(J^k(TM)\right) \to C^\infty\left(J^k(TM)\right)$ is the only mapping for which $\left(C^\infty\left(J^k(TM)\right), [\![\ ,\ ]\!]\right)$ is a Lie algebra and

$$[\![\sigma, f\cdot\eta]\!] = f[\![\sigma, \eta]\!] + (\tilde{\beta}\sigma)(f)\cdot\eta$$

for $\sigma, \eta \in C^\infty\left(J^k(TM)\right)$ and $f \in C^\infty(M)$.

Investigations of such objects were carried out by Libermann [13], [14].

Example. The L.a. of the trivial L.g. $M \times G \times M$ is a collection $(TM \times \mathfrak{g}, [\![\ ,\ ]\!], \tilde{\beta})$, where

(i) $\mathfrak{g}$ is the Lie algebra of the Lie group G;

(ii) $TM \times \mathfrak{g}$ is the vector bundle over M in which a fibre over $x \in M$ is equal to $T_xM \times \mathfrak{g}$;

(iii) $\tilde{\beta} = \left(TM \times \mathfrak{g} \ni (v, u) \mapsto v \in TM\right)$;

(iv) if X, X' are two vector fields on M and h, h': $M \to \mathfrak{g}$ are two smooth mappings, then

$$[\![(X, h), (X', h')]\!] = ([X, X'], \mathcal{L}_X h' - \mathcal{L}_{X'} h + [h, h']).$$

A smooth mapping F: $\Phi|\,\Omega \to \Phi'$, where Φ and Φ' are spaces of the L.g.'s

(1.2) $$\boldsymbol{\Phi} = \left(\Phi, (\alpha, \beta), M, \cdot\right), \quad \boldsymbol{\Phi}' = \left(\Phi', (\alpha', \beta'), M, \cdot'\right),$$

and Ω contains all units 1_x, $x \in M$, is called a *local homomorphism* from $\boldsymbol{\Phi}$ into $\boldsymbol{\Phi}'$ if

(i) $\alpha' \circ F = \alpha|\,\Omega$, $\beta' \circ F = \beta|\,\Omega$,

(ii) $z, z', z\cdot z' \in \Omega$ implies $F(z\cdot z') = F(z)\cdot F(z')$.

If $\Omega = \Phi$, then the local homomorphism is called a *homomorphism* from $\boldsymbol{\Phi}$ into $\boldsymbol{\Phi}'$ (see [27]).

Let F: $\Phi|\,\Omega \to \Phi'$ be a local homomorphism from $\boldsymbol{\Phi}$ into $\boldsymbol{\Phi}'$. It is easy to see that the mapping

$$\tilde{F}_* = \left(i^*(T^\alpha\Phi) \ni v \mapsto F_*(v) \in i'^*(T^\alpha\Phi')\right)$$

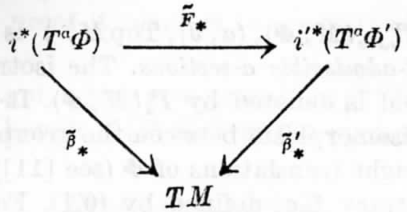
has the following properties:

(i) $\tilde{F}_*$ is a morphism of the vector bundles;

(ii) if ξ is a cross-section of $i^*(T^a\Phi)$, then ξ' and $(\tilde{F}_* \circ \xi)'$ are F-related (see Theorem 1.2);

(iii) $C^\infty(\tilde{F}_*)$ is a Lie algebra homomorphism;

(iv) the diagram



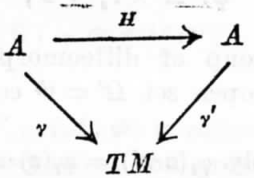
is commutative.

From these properties we infer that the following definition is justified [11], [24]:

Definition 1.1. Let

$$\boldsymbol{A} = (A, [\![\,,\,]\!], \gamma) \quad \text{and} \quad \boldsymbol{A}' = (A', [\![\,,\,]\!]', \gamma')$$

be arbitrary L.a.'s over a manifold M. A morphism $H\colon A \to A'$ of the vector bundles is called an L.a. morphism if $C^\infty(H)\colon C^\infty(A) \to C^\infty(A')$ is a Lie algebra homomorphism and the following diagram is commutative:



The assignment of the L.a. (1.1) to an L.g. (0.1) and of the L.a. homomorphism $\tilde{F}_*$ to an L.g. homomorphism F is a covariant functor from the category of L.g. into the one of L.a. It is called the *Lie functor for L.g.*

2. Groupoid of β-admissible α-sections $\Gamma_{a,\mathrm{loc}}(M, \Phi)$. The L.g. Φ defined by (0.1) determines another very important object, namely the groupoid of β-admissible α-sections (see [11] and [12]) $\Gamma_{a,\mathrm{loc}}(M, \Phi)$. It consists of such local sections $\sigma\colon M\,|\,U \to \Phi$ of the surmersion $\alpha\colon \Phi \to M$ for which U and $U' = \beta\circ\sigma[U] \subset M$ are open sets and $\beta\circ\sigma\colon M|\,U \to M\,|\,U'$ is a diffeomorphism. The element $\sigma(x)$ will often be denoted by σ_x, and the topology of M by Top$M$. We define the mappings

$$a, b\colon \Gamma_{a,\mathrm{loc}}(M, \Phi) \to \mathrm{Top}M$$

by the formulas

$$a(\sigma) = D_\sigma, \quad b(\sigma) = \beta\circ\sigma[D_\sigma],$$

where $\sigma\colon M|D_\sigma \to \Phi$ (i.e., D_σ is the domain of the σ) for σ belonging to $\Gamma_{a,\mathrm{loc}}(M, \Phi)$. The multiplication $\tau\cdot\sigma$ for $\sigma\colon M|U \to \Phi$ and $\tau\colon M|U' \to \Phi$ belonging to $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ is defined if $\beta\circ\sigma[U] = U'$, and we have

$$\tau\cdot\sigma = (U \ni x \mapsto \tau_{\beta(\sigma_x)}\cdot\sigma_x \in \Phi).$$

The collection $\bigl(\Gamma_{a,\mathrm{loc}}(M, \Phi), (a, b), \mathrm{Top}M, \cdot\bigr)$ is a groupoid. It is called a *groupoid of β-admissible a-sections*. The isotropy group over the unit M of this groupoid is denoted by $\Gamma_a(M, \Phi)$. It is easy to find that there exists a natural isomorphism between the groupoid $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ and the groupoid of local right translations of Φ (see [11]).

Let Φ be an arbitrary L.g. defined by (0.1). For each cross-section ξ of $i^*(T^a\Phi)$ the r-i vector field generated by ξ is denoted by ξ' (see Theorem 1.2). Every integral curve γ of ξ' lies in Φ_x for some point $x \in M$, namely if γ passes through h, then γ lies in Φ_{ah}.

Let ξ be an arbitrary fixed global cross-section of $i^*(T^a\Phi)$. It is easy to see that the following statements are true:

(i) If z, z' belong to Φ, $z\cdot z'$ is defined, and γ is an integral curve of ξ' passing through z, then $\gamma' = \Phi_{z'}\circ\gamma$ is also an integral curve of ξ' and it passes through $z\cdot z'$.

(ii) We take a certain point $x_0 \in M$ and

$$\varphi\colon \Omega'\times I_\varepsilon \to \Phi,$$

a local one-parameter group of diffeomorphisms (shortly, l.o-p.g.d.), which generates ξ' on an open set $\Omega' \subset \Phi$ containing l_{x_0}, $I_\varepsilon = (-\varepsilon, \varepsilon)$. Then

(a) $z, z', z\cdot z' \in \Omega'$ imply $\varphi_t(z\cdot z') = \varphi_t(z)\cdot z'$, $t \in I_\varepsilon$;

(b) $z \in \Omega'$ and $l_{\beta z} \in \Omega'$ imply $\varphi_t(z) = \varphi_t(l_{\beta z})\cdot z$, $t \in I_\varepsilon$;

(c) if $l_x \in \Omega'$, $s \in I_\varepsilon$, $l_{\beta(\varphi_s(l_x))} \in \Omega'$, then for every $t \in I_\varepsilon$ such that $t+s \in I_\varepsilon$ we have

$$\varphi_{t+s}(l_x) = \varphi_t(l_{\beta(\varphi_s(l_x))})\cdot\varphi_s(l_x).$$

THEOREM 2.1. *Let ξ belong to $C^\infty\bigl(i^*(T^a\Phi)\bigr)$. For every point $x \in M$ there exist a neighbourhood $U \subset M$ of x, a number $\varepsilon > 0$, and an l.o-p.g.d.*

$$\varphi'\colon \beta^{-1}[U]\times I_\varepsilon \to \Phi \tag{2.1}$$

which generates ξ' on $\beta^{-1}[U]$.

Proof. Let us take an arbitrary fixed point $x \in M$ and an l.o-p.g.d. $\varphi\colon \Omega\times I_\varepsilon \to \Phi$ which generates ξ' on the open set $\Omega \subset \Phi$ containing the unit l_x. Put $U = i^{-1}[\Omega]$. By (ii) the mapping φ' must be defined by

$$\varphi'(z, t) = \varphi(l_{\beta z}, t)\cdot z,\quad z \in \beta^{-1}[U],\ t \in I_\varepsilon.$$

It suffices to show that φ' is an l.o-p.g.d. which generates $\xi'|\beta^{-1}[U]$. By the above it is easy to find that

(a) $\varphi'(z, 0) = z$, $z \in \beta^{-1}[U]$;

(b) $\varphi'_{t+s}(z) = \varphi'_t(\varphi'_s(z))$, $z, \varphi'_s(z) \in \beta^{-1}[U]$, $s, t, s+t \in I_\varepsilon$;

(c) φ' generates $\xi'|\beta^{-1}[U]$.

To complete the proof it remains to see that $\varphi'(\cdot, t)$, $t \in I_\varepsilon$, is a diffeomorphism.

Let φ' be an l.o-p.g.d. which generates ξ' on $\beta^{-1}[U]$, where ξ is a cross-section of the vector bundle $i^*(T^\alpha\Phi)$ over the open set $U \subset M$.

Put

$$\mathrm{Exp}(t, \xi) = \varphi'_t \circ i \,|\, U.$$

The mapping $\mathrm{Exp}(t, \xi)\colon M|U \to \Phi$ is an α-section. It is easy to find that

(i) the mapping

$$\psi = \big(U \times I_\varepsilon \ni (y, t) \mapsto (\beta \circ \mathrm{Exp}(t, \xi))(y) \in M\big)$$

is an l.o-p.g.d. which generates $\beta_*\xi'$;

(ii) $\mathrm{Exp}(t, \xi)$ is a β-admissible α-section, i.e.

$$\mathrm{Exp}(t, \xi) \in \Gamma_{a,\mathrm{loc}}(M, \Phi);$$

(iii) if $U' = \psi_t[U]$, then

$$\varphi'_t\colon \Phi|\beta^{-1}[U] \to \Phi|\beta^{-1}[U'], \quad t \in I_\varepsilon,$$

are left translations.

The mapping $S\colon I_\varepsilon \to \Gamma_{a,\mathrm{loc}}(M, \Phi)$ is called a *local smooth one-parameter subgroup* (shortly, l.s.o-p.s.) of the groupoid $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ on th open set $U \subset M$ ([11], [12]) if

(i) $S_t := S(t)$ is defined on the set U;

(ii) the mapping $\hat{S} = \big(U \times I_\varepsilon \ni (x, t) \mapsto S_t(x) \in \Phi\big)$ is smooth;

(iii) $S_0 = i\,|\,U$;

(iv) $s, t, s+t \in I_\varepsilon$ and $x, \beta \circ S_s(x) \in U$ imply

$$S_{s+t}(x) = S_t(\beta \circ S_s(x)) \cdot S_s(x).$$

THEOREM 2.2. (a) *If ξ is a local cross-section of $i^*(T^\alpha\Phi)$ over an open set $U \subset M$, and φ' is an l.o-p.g.d.* (2.1) *which generates ξ', then*

$$S = \big(I_\varepsilon \ni t \mapsto \mathrm{Exp}(t, \xi) \in \Gamma_{a,\mathrm{loc}}(M, \Phi)\big)$$

is an l.s.o-p.s. of $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ over U. (In this case S is said to be *generated* by ξ.)

(b) *Conversely, every l.s.o-p.s. of $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ over an open set $U \subset M$ is generated by exactly one cross-section of $i^*(T^\alpha\Phi)$ over U.*

Proof. (a) is evident.

(b) Let S be an l.s.o-p.s. of $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ over an open set $U \subset M$. The uniqueness of ξ which generates S follows from the equality

$$\xi(x) = \big(S_{(\cdot)}(x)\big)_{*0}\left(\frac{\partial}{\partial t}\Big|_0\right), \quad x \in U. \tag{2.2}$$

We take the cross-section defined by (2.2). It is easy to see that the mapping

$$\varphi' = \big(\beta^{-1}[U] \times I_\varepsilon \ni (z, t) \mapsto S_t(\beta z)\cdot z \in \Phi\big)$$

is an l.o-p.g.d. which generates ξ'. The above considerations prove that ξ' is smooth, and so is ξ. Now we see that

$$\big(\mathrm{Exp}(t, \xi)\big)(x) = \varphi'(l_x, t) = S_t(x), \quad x \in U.$$

A vector field on a manifold is called *complete* if it is globally defined and generated by a global one-parameter group of diffeomorphisms (shortly, g.o-p.g.d.). An r-i vector field η on an L.g. Φ defined by (0.1) is complete if and only if the vector field $\beta_* \eta$ is complete on the manifold M (see [11] and [12]).

A cross-section ξ of $i^*(T^a \Phi)$ is called *complete* if ξ' is complete on Φ.

An l.s.o-p.s. of $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ is called a ***global smooth one-parameter subgroup*** (shortly, g.s.o-p.s.) of the group $\Gamma_a(M, \Phi)$ if it is over M and is defined on R.

A mapping $S\colon R \to \Gamma_a(M, \Phi)$ is a g.s.o-p.s. of $\Gamma_a(M, \Phi)$ if and only if

(i) *the mapping $\hat{S} = \big(M \times R \ni (x, t) \mapsto S(t)(x) \in \Phi\big)$ is smooth;*

(ii) *S is a homomorphism of the additive group R into $\Gamma_a(M, \Phi)$.*

If ξ is a complete cross-section of $i^*(T^a \Phi)$, then

$$S = \big(R \ni t \mapsto \mathrm{Exp}(t, \xi) \in \Gamma_a(M, \Phi)\big)$$

is a g.s.o-p.s. of $\Gamma_a(M, \Phi)$, and

(i) $\big(S(\cdot)(x)\big)_{*0}\big((\partial/\partial t)|_0\big) = \xi(x)$, $x \in M$,

(ii) *$\beta \circ S$ is a g.o-p.g.d. which generates $\beta_* \xi$.*

Every g.s.o-p.s. of $\Gamma_a(M, \Phi)$ is generated by exactly one complete cross-section of $i^*(T^a \Phi)$.

3. Exponential mapping for Lie groupoids. We denote by $C_0^\infty\big(i^*(T^a \Phi)\big)$ the set of all global cross-sections ξ of $i^*(T^a \Phi)$ such that $\beta_* \xi$ has a compact support. They are complete.

The mapping

$$\mathrm{Exp}_\Phi = \big(C_0^\infty\big(i^*(T^a \Phi)\big) \ni \xi \mapsto \mathrm{Exp}(1, \xi) \in \Gamma_a(M, \Phi)\big)$$

is called an *exponential mapping on the L.g. Φ* defined by (0.1) (see [11], [27], and [28]).

For a cross-section $\xi \in C_0^\infty(i^*(T^a\Phi))$ and a number $s \in R$ the equality $\mathrm{Exp}(1, s\xi) = \mathrm{Exp}(s, \xi)$ holds.

THEOREM 3.1. *Let $F\colon \Phi|\Omega \to \Phi'$ be a local homomorphism from Φ into Φ' defined by* (1.2). *Assume that x_0 is an arbitrary point in M, $\xi \in C_0^\infty(i^*(T^a\Phi))$ a cross-section, and $\varepsilon > 0$ a number such that the relation*

$$(\mathrm{Exp}_\Phi t\xi)(x_0) \in \Omega$$

holds for every $t \in I_{1+\varepsilon}$. Then there exist a number ε' $(0 < \varepsilon' \leqslant \varepsilon)$ and a neighbourhood U of x_0 such that

(i) $(\mathrm{Exp}_\Phi t\xi)(x) \in \Omega$ *for* $x \in U$ *and* $|t| < 1+\varepsilon'$,

(ii) $\big(\mathrm{Exp}_{\Phi'}(\tilde{F}_*\circ \xi)\big)(x) = F\big((\mathrm{Exp}_\Phi \xi)(x)\big)$ *for* $x \in U$.

Proof. Since $\tilde{\beta}'_*\circ(\tilde{F}_*\circ\xi)(x) = \tilde{\beta}_*\circ\xi(x)$, we have

$$\tilde{F}_*\circ\xi \in C_0^\infty(i'^*(T^a\Phi')).$$

Put

$$S(t) = \mathrm{Exp}_\Phi t\xi.$$

$\hat{S}^{-1}[\Omega]$ is an open subset of $M \times R$ containing $\{x_0\} \times I_{1+\varepsilon}$. It is easily seen that there exist a neighbourhood U of x_0 and a number ε' $(0 < \varepsilon' \leqslant \varepsilon)$ such that

$$U \times I_{1+\varepsilon'} \subset \hat{S}^{-1}[\Omega].$$

For an arbitrary point $(x, t) \in U \times I_{1+\varepsilon'}$ we have

$$(\mathrm{Exp}_\Phi t\xi)(x) = \hat{S}(x, t) \in \Omega.$$

Hence the mapping

$$S' = \Big(I_{1+\varepsilon'} \ni t \mapsto F\circ\big((\mathrm{Exp}_\Phi t\xi)|\,U\big) \in \Gamma_{a,\mathrm{loc}}(M, \Phi)\Big)$$

is an l.s.o-p.s. of $\Gamma_{a,\mathrm{loc}}(M, \Phi)$ over U, which is generated by $(\tilde{F}_*\circ\xi)|\,U$. Therefore, equality (ii) holds.

Let us take the cross-sections $\xi_1, \ldots, \xi_m \in C_0^\infty(i^*(T^a\Phi))$ which are a basis of $i^*(T^a\Phi)$ over $U \subset M$. The r-i vector fields $\xi'_1|\,\Phi_x, \ldots, \xi'_m|\,\Phi_x$ are a basis of $T(\Phi_x)$ over $\Phi_x \cap \beta^{-1}[U]$ for an arbitrary point $x \in M$. A basic property of the exponential mapping is given in the sequel.

THEOREM 3.2. *For each point $x_0 \in U$ there exist open neighbourhoods $U_m \subset R^m$ of 0 and $U' \subset U$ of x_0 such that the mapping*

$$\overline{\mathrm{Exp}_\Phi} = \Big(U_m \times U' \ni (a^1, \ldots, a^m, x) \mapsto \Big(\mathrm{Exp}_\Phi \sum_{i=1}^{m} a^i\,\xi_i\Big)(x) \in \Phi\Big)$$

is a diffeomorphism onto its open image.

For the proof we need the following lemma which is known from theory of differential equations:

LEMMA. *If $Z_1, \ldots, Z_m$ are some vector fields on the manifold M, then for any point $y_0 \in M$ there exist a neighbourhood W of y_0 and a cube*

$$\Omega_K^m = \{(a^1, \ldots, a^m) \in R^m\colon |a^i| < K,\ i = 1, \ldots, m\}, \quad 0 < K < \infty,$$

such that for each point $y \in W$ and $a \in \Omega_K^m$ there exists an integral curve

$$\varphi_{y,a}\colon I_2 \to M, \quad \varphi_{y,a}(0) = y,$$

of the vector field $\sum_{i=1}^{m} a^i Z_i$, and the mapping

$$\varphi = \big(W \times I_2 \times \Omega_K^m \ni (y, t, a) \mapsto \varphi_{y,a}(t) \in M\big)$$

is smooth.

Proof of Theorem 3.2. From the above lemma it follows that for any point $x_0 \in U$ there exist neighbourhoods $U_m \subset R^m$ of 0 and $U' \subset U$ of x_0 such that the mapping $\overline{\mathrm{Exp}}_\Phi$ is smooth. To prove the theorem it suffices to show that the differential $(\overline{\mathrm{Exp}}_\Phi)_{*(0,x_0)}$ is an isomorphism. First, we shall prove that the differential at the point $a = 0$ of the mapping

$$\overline{\mathrm{Exp}}_\Phi(x_0) = \Big(U_m \ni (a^1, \ldots, a^m) \mapsto \Big(\mathrm{Exp}_\Phi \sum_{i=1}^{m} a^i \xi_i\Big)(x_0) \in \Phi_{x_0}\Big)$$

is an isomorphism. Denote this mapping by $\varkappa$ and identify the tangent spaces $T_0(R^m)$ and $T_{l_{x_0}}(\Phi_{x_0})$ with R^m by means of the following isomorphism:

$$T_0(R^m) \ni \sum_{i=1}^{m} c^i e_i \mapsto (c^1, \ldots, c^m) \in R^m,$$

$$T_{l_{x_0}}(\Phi_{x_0}) \ni \sum_{i=1}^{m} c^i \xi_i'(l_{x_0}) \mapsto (c^1, \ldots, c^m) \in R^m.$$

Then $\varkappa_{*0} = \mathrm{id}_{R^m}$. Indeed, if $b = (b^1, \ldots, b^m) \in R^m$ is an arbitrary point and $\lambda_b = (R \ni s \mapsto s \cdot b \in R^m)$, then

$$\varkappa_{*0}(b) = (\varkappa \circ \lambda_b)_{*0}\left(\frac{\partial}{\partial t}\Big|_0\right) = b$$

because $\varkappa \circ \lambda_b$ is an integral curve of the vector field $\sum_{i=1}^{m} b^i \xi_i' | \Phi_{x_0}$ and $\varkappa \circ \lambda_b(0) = x_0$. Now, the theorem is implied by the following fact:

If $p\colon P \to N$, $q\colon P' \to N$ are coregular mappings and $f\colon P \to P'$ is a mapping such that $q \circ f = p$, and $p_0 \in P$, $x_0 = p(p_0)$,

$$f_1 = f | p^{-1}[\{x_0\}]\colon P | p^{-1}[\{x_0\}] \to P' | q^{-1}[\{x_0\}],$$

and $(f_1)_{*p_0}$ is an isomorphism, then f_{*p_0} is also an isomorphism.

The mapping inverse to $\overline{\mathrm{Exp}}_\Phi$ is called the *exponential coordinate system determined by the cross-sections* $\xi_1, \ldots, \xi_m \in C_0^\infty(i^*(T^a\Phi))$ which are a basis in a neighbourhood of $x \in M$.

COROLLARY. *There exists an open set* $\Omega \subset \Phi$ *which contains all units and is contained in the set*

$$E = \{(\mathrm{Exp}_\Phi \xi)(x) \colon x \in M,\ \xi \in C_0^\infty(i^*(T^a\Phi))\}.$$

THEOREM 3.3. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be any subbundles of* $i^*(T^a\Phi)$ *such that* $i^*(T^a\Phi) = \mathfrak{m} \oplus \mathfrak{n}$. *Let* $\xi_1, \ldots, \xi_m \in C_0^\infty(\mathfrak{m})$ *and* $\xi_{m+1}, \ldots, \xi_{m+n} \in C_0^\infty(\mathfrak{n})$ *be cross-sections which are a basis of* $\mathfrak{m}$ *and* $\mathfrak{n}$, *respectively, over a non-empty subset* $U \subset M$. *Then the cross-sections* $\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_{m+n}$ *are a basis of* $i^*(T^a\Phi)$ *over* U. *Let* $\overline{\mathrm{Exp}}_\Phi$ *be defined for these sections. Then for each point* $x \in U$ *there exist neighbourhoods* $U' \subset U$ *of* x, $U_m \subset R^m$ *of* 0, *and* $U_n \subset R^n$ *of* 0 *such that the mapping*

$$\lambda \colon U_m \times U_n \times U' \to \Phi$$

defined by

$$\lambda(a, b, y) = \overline{\mathrm{Exp}}_\Phi\big((a, 0), \beta \circ \overline{\mathrm{Exp}}_\Phi((0, b), y)\big) \cdot \overline{\mathrm{Exp}}_\Phi((0, b), y)$$

is a diffeomorphism onto its open image.

Proof. Let $\overline{\mathrm{Exp}}_\Phi \colon W \times U_1 \to \Phi$ be the diffeomorphism onto its open image Ω, where $W \subset R^{m+n} = R^m \times R^n$ and $U_1 \subset U$ are open sets such that $0 \in W$ and $x \in U_1$.

Since λ is smooth, we can take neighbourhoods $U_m \subset R^m$ of 0, $U_n \subset R^n$ of 0, and $U' \subset U_1$ of x such that $\lambda[U_m \times U_n \times U'] \subset \Omega$. We denote by $(x^1(z), \ldots, x^{m+n}(z), a(z))$ the exponential coordinates of $z \in \Omega$. Then

$$x^i\big(\overline{\mathrm{Exp}}_\Phi((a, 0), y)\big) = a^i, \quad i = 1, \ldots, m+n,\ y \in U,\ a \in W.$$

Let us take the mapping

$$t = (\overline{\mathrm{Exp}}_\Phi)^{-1} \circ \lambda \colon U_m \times U_n \times U' \to W \times U_1$$

which is of the form

$$t(a, b, y) = \big(t_y(a, b), y\big).$$

To prove the theorem it suffices to show that the differential $(t_x)_{*(0,0)}$ is an isomorphism. For $i = 1, \ldots, m+n$ we have

$$t_x^i(a^1, \ldots, a^m, b^1, \ldots, b^n)$$
$$= x^i\Big(\Big(\mathrm{Exp}_\Phi \sum_{i=1}^m a^i \xi_i\Big)\Big(\beta \circ \Big(\mathrm{Exp}_\Phi \sum_{i=1}^n b^i \xi_{m+i}\Big)(x)\Big) \cdot \Big(\mathrm{Exp}_\Phi \sum_{i=1}^n b^i \xi_{m+i}\Big)(x)\Big).$$

Put

$$\tau_{i,j} = \big(I_\varepsilon \ni s \mapsto t_x^i(\underbrace{0, \ldots, 0}_{j-1}, s, 0, \ldots, 0) \in R\big) \quad \text{for } i, j = 1, \ldots, m+n,$$

where $\varepsilon > 0$ and $\bigtimes^{m+n} I_\varepsilon \subset U_m \times U_n$. Then $\tau_{i,j}(s) = s \cdot \delta_j^i$. Therefore, $t_{x|j}^i(0, 0) = \delta_j^i$ and

$$\det[t_{x|j}^i(0, 0)\colon i, j \leqslant m+n] = 1 \neq 0,$$

which completes the proof.

THEOREM 3.4. *An injective homomorphism $F\colon \Phi \to \Phi'$ of the L.g.'s (1.2) is an immersion.*

Proof. It is easy to prove that a homomorphism $F = \Phi \to \Phi'$ is an immersion if and only if $\tilde{F}_*\colon i^*(T^a\Phi) \to i'^*(T^a\Phi')$ is a monomorphism of the vector bundles. Let F be injective. We shall prove that $\tilde{F}_*$ is a monomorphism. Let us take a vector $v \in i^*(T^a\Phi)$ such that $\tilde{F}_*(v) = 0$. Assume that $\xi \in C_0^\infty\big(i^*(T^a\Phi)\big)$ is a cross-section for which $\xi(x) = v$. Then (Theorem 3.1)

$$\big(\mathrm{Exp}_{\Phi'} t(\tilde{F}_* \circ \xi)\big)(x) = (F \circ \mathrm{Exp}_\Phi t\xi)(x).$$

Since $\tilde{F}_* \circ \xi(x) = \tilde{F}_*(v) = 0$, we have $(\tilde{F}_* \circ \xi)'(l_x) = 0$. Hence the integral curve of $(\tilde{F}_* \circ \xi)'$ passing through l_x is constant, and since $t \mapsto \big(\mathrm{Exp}_{\Phi'} t(\tilde{F}_* \circ \xi)\big)(x)$ is such a curve, we obtain

$$\big(\mathrm{Exp}_{\Phi'} t(\tilde{F}_* \circ \xi)\big)(x) = l_x.$$

Consequently, $F\big((\mathrm{Exp}_\Phi t\xi)(x)\big) = l_x$, and from the injectivity of F we get $(\mathrm{Exp}_\Phi t\xi)(x) = l_x$. Since $t \mapsto (\mathrm{Exp}_\Phi t\xi)(x)$ is an integral curve of ξ', we have $v = \xi(x) = \xi'(l_x) = 0$.

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