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EXPONENTIAL MAPPING FOR LIE GROUPOIDS. APPLICATIONS

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0. Introduction. Let M be a manifold with a covariant derivative. The parallel displacement along any piecewise differentiable curve $\gamma\colon [0, 1] \to M$ defines some isomorphism of the fibre $E_{|\gamma(0)}$ onto the fibre $E_{|\gamma(1)}$. Thus it is appropriate to consider an object consisting of all linear isomorphisms of a fibre onto a fibre. The object has a natural structure of the so-called *Lie groupoid*. The above idea of calling these objects into existence comes from Ehresmann [3]. It turned out later that many problems from differential geometry of higher order are defined, in a natural manner, by means of a Lie groupoid. This gave rise to developing many theories concerning these objects, including the general theory (see, e.g., [1], [2], [6], [8]–[17]). In papers [4] and [5] the author made a uniform approach to the above-mentioned theory. This paper is their continuation.

1. Inducing Lie subgroupoid by Lie subalgebroid.

DEFINITION 1.1. Let $A = (A, [\![\,,]\!], \gamma)$ and $A' = (A', [\![\,,]\!]', \gamma')$ be Lie algebroids (briefly, L.a.'s) ([4], [12]) over any manifold M. We say that A' is a *Lie subalgebroid* (briefly, L.suba.) of A if

(a) A' is a linear subbundle of A,

(b) the inclusion $i\colon A' \hookrightarrow A$ gives a homomorphism of the L.a. (see [4]).

(1.1) *If* $(A, [\![\,,]\!], \gamma)$ *is an L.a. and* A' *is a linear subbundle of* A, *then on* A *there exists a structure of an L.suba. of* A *iff*

(a) $\gamma|A'\colon A' \to TM$ *is an epimorphism,*

(b) $[\![\sigma, \sigma']\!] \in C^\infty(A')$ *for* $\sigma, \sigma' \in C^\infty(A')$.

(1.2) *If a Lie groupoid (briefly, L.g.)* $\boldsymbol{\Phi}'$ *is a Lie subgroupoid (briefly, L.subg.) of some* $\boldsymbol{\Phi}$ *and* $i\colon \Phi' \hookrightarrow \Phi$ *is the inclusion, then* $\tilde{i}_*\colon A' \to A$ *is a monomorphism of their L.a.'s.*

Therefore, one may identify the L.a. of the L.subg. $\boldsymbol{\Phi}'$ with an L.suba. of A.

THEOREM 1.1. *Let* $\boldsymbol{\Phi} = (\Phi, (\alpha, \beta), M, \cdot)$ *be an L.g. and let*

$$A = (i^*(T^\alpha \Phi), [\![\,,]\!], \tilde{\beta}_*)$$

be its L.a. ([4], [12]). Then for every L.suba. A' of A there exists exactly one connected L.subg. of Φ with algebroid equal to A'.

Proof. We take an arbitrary point $x \in M$ and the principal fibre bundle Φ_x (see [4] and [15]). We define a distribution B on the manifold Φ_x by the formula

$$B_h = (\Phi_h)_{*l_{\beta h}}[A'_{|\beta h}] \quad \text{for } h \in \Phi_x.$$

It is a smooth involutive distribution. Let C be a connected maximal integral manifold of B passing through l_x.

(a) *$\beta|C$: $C \to M$ is a surmersion.*

Since $\beta|C$ is coregular, it remains to show that the mapping is "onto". Supposing that $\beta|C$ is not "onto", let $y \in M \setminus \beta[C]$ be any point from the boundary of the set $\beta[C]$ and let $x \xrightarrow{z} y$ be an element of Φ_x with target at y. Consider a connected maximal integral D of the distribution B passing through z. Then for every element $g \in \Phi_{(x,x)}$ the manifold $D_g = R_g[D]$ is also a connected maximal integral of B and it passes through $z \cdot g$ and $\beta[D] = \beta[D_g]$. Let us take a set

$$\Omega = \beta^{-1}[\beta[D]] \cap \Phi_x.$$

Then

$$\Omega = \bigcup_{g \in \Phi_{(x,x)}} D_g.$$

Since the sets $\beta[C]$ and $\beta[D]$ are open, we have $\beta[C] \cap \beta[D] \neq \emptyset$. Let $x \xrightarrow{z'} t$ be an arbitrary element of C such that $t \in \beta[C] \cap \beta[D]$ and let $g \in \Phi_{(x,x)}$ be such that $z' \in D_g$. Hence $C = D_g$ and $y \in \beta[D_g] = \beta[C]$, which contradicts our assumption.

(b) *$G = (\beta|C)^{-1}(\{x\})$ has a structure of a Lie subgroup of $\Phi_{(x,x)}$.*

Since G is an embedding submanifold of C, it has a countable basis. The inclusion i: $G \hookrightarrow \Phi_{(x,x)}$ is smooth. If $z \in C$ and $g \in \Phi_{(x,x)}$, then $z \cdot g \in C$ iff $g \in G$. The mapping

$$G \times G \hookrightarrow \Phi_{(x,x)} \times \Phi_{(x,x)} \xrightarrow{\cdot} \Phi_x$$

is also smooth and its image lies in G. Since it lies also in C, and C is a connected integral of involutive distribution, $\cdot$: $G \times G \to G$ is smooth. Analogously, we can prove the smoothness of $^{-1}$: $G \to G$. Hence G is a Lie subgroup of $\Phi_{(x,x)}$. Of course, the mapping

$$\cdot' = (C \times G \ni (z, g) \mapsto z \cdot g \in C)$$

is also smooth, and the system

$$\mathfrak{C} = (C, \beta|C, M, G, \cdot')$$

is a principal fibre bundle. Moreover, $\mathfrak{C}$ is a subbundle of Φ_x, and the inclusion $C \hookrightarrow \Phi_x$ is an immersive homomorphism which defines an immersion homomorphism of the L.g. $i\colon \mathfrak{C}\mathfrak{C}^{-1} \to \boldsymbol{\Phi}$ (see [3] and [7]). The image $\Psi = i[CC^{-1}]$ is a connected subgroupoid of $\boldsymbol{\Phi}$. On Ψ there exists exactly one differential structure of a manifold such that i is a diffeomorphism. We obtain an L.subg. $\boldsymbol{\Psi}$ of $\boldsymbol{\Phi}$, which is the desired object.

(c) *The L.a. of* $\boldsymbol{\Psi}$ *is* A'.

For $y \in M$ and $z \in \Psi$ with target at y we have

$$T_{l_y}(\Psi_y) = T_{l_y}(\Phi_z[C]) = (\Phi_z)_{*z^{-1}}[T_{z^{-1}}C] = (\Phi_z)_{*z^{-1}}[(\Phi_{z^{-1}})_{*l_y}[A'_{|y}]] = A'_{|y}.$$

(d) *Uniqueness.*

Let $\boldsymbol{H}$ be a connected L.subg. of $\boldsymbol{\Phi}$ with algebroid equal to A'. Then $\boldsymbol{H}$ has the following properties:

(i) H_x is an integral of B passing through l_x;

(ii) the connected component $(H_x)_0$ of l_x is an open submanifold of H_x;

(iii) $\beta[(H_x)_0] = M$, $x \in M$.

To see (iii) observe that the set $W = \beta[(H_x)_0]$ is open in M. Assume that $W \neq M$ and let $y \in M$ be any point from the boundary of W. We take an arbitrary element $x \xrightarrow{h} y$ of H_x, a connected neighbourhood U of y, and a β-section $\sigma\colon U \to H_x$ such that $\sigma(y) = h$.

The mapping

$$\hat{\sigma} = (U \times H_{(x,x)} \ni (s, g) \mapsto \sigma(s)\cdot g \in (\beta|H_x)^{-1}[U])$$

is a diffeomorphism. Hence every connected component of $(\beta|H_x)^{-1}[U]$ is the image under $\hat{\sigma}$ of some connected component of $U \times H_{(x,x)}$. Every such component is equal to $U \times K$, where K is a coset in $H_{(x,x)}$ with respect to the connected component G of l_x in $H_{(x,x)}$. Since y lies in the boundary of W, we have $U \cap W \neq \emptyset$. Let $y_0 \in U \cap W$, $z \in (H_x)_0$, and $\beta(z) = y_0$. There exists a coset K_0 such that $z \in \hat{\sigma}[U \times K_0]$. Hence $\hat{\sigma}[U \times K_0] \subset (H_x)_0$ and, consequently, $y \in U \subset \beta[(H_x)_0] \subset W$, which gives a contradiction to $y \notin W$.

Properties (i) and (ii) imply that $(H_x)_0$ is an open submanifold of C. The set $\Omega = (H_x)_0 (H_x)_0^{-1}$ is open in Ψ and, by (iii), it contains all units. Hence Ω generates Ψ and H. The equality $\boldsymbol{\Psi} = \boldsymbol{H}$ follows from Theorem 1.3 in [5]. Thus the proof is complete.

2. Inducing a local homomorphism of Lie groupoids by a homomorphism of Lie algebroids. The problem of the existence of a local homomorphism of L.g.'s with a given homomorphism of L.a.'s was considered by means of other methods in [16].

Let $\boldsymbol{\Phi} = (\Phi, (\alpha, \beta), M, \cdot)$ and $\boldsymbol{\Phi}' = (\Phi', (\alpha', \beta'), M, \cdot')$ be any L.g.'s with the same monifold of units and with algebroids A and A'.

THEOREM 2.1. *For every homomorphism* $\gamma\colon A \to A'$ *there exists a local homomorphism* F *from* $\boldsymbol{\Phi}$ *into* $\boldsymbol{\Phi}'$ *such that* $\tilde{F}_* = \gamma$. *Any two such local homomorphisms coincide in some neighbourhood of all units. If* $\boldsymbol{\Phi}$ *is connected and there exists a global homomorphism* F, *then* F *is uniquely determined.*

Remark. By the *Whitney product* $A \underline{\times} A'$ of the L.a.'s A and A' we mean the L.a. $(A \underline{\times} A', [\![\,,]\!]'', \gamma'')$ in which

(1) $A \underline{\times} A' = \{(v, v') \in A \oplus A' \colon \gamma(v) = \gamma'(v')\}$;

(2) $\sigma,\ \tau \in C^\infty(A'')$, $\sigma = (\mu, \mu')$ and $\tau = (\delta, \delta')$, where $\mu, \delta \in C^\infty(A)$ and $\mu', \delta' \in C^\infty(A')$ imply $[\![\sigma, \tau]\!] = ([\![\mu, \delta]\!], [\![\mu', \delta']\!])$;

(3) $\gamma''(v, v') = \gamma(v)$ for $(v, v') \in A \underline{\times} A'$.

If $\boldsymbol{\Phi} \underline{\times} \boldsymbol{\Phi}'$ is a Whitney product of the L.g. (see [15]) and $i\colon M \to \Phi$, $i'\colon M \to \Phi'$, $i''\colon M \to \Phi \underline{\times} \Phi'$ are natural embeddings, then

$$j = \big(i''(T^\alpha(\Phi \underline{\times} \Phi')) \ni w \mapsto (\pi_{1*} w, \pi_{2*} w) \in i^*(T^\alpha \Phi) \underline{\times} i'^*(T^\alpha \Phi')\big),$$

where $\pi_1\colon \boldsymbol{\Phi} \underline{\times} \boldsymbol{\Phi}' \to \boldsymbol{\Phi}$ and $\pi_2\colon \boldsymbol{\Phi} \underline{\times} \boldsymbol{\Phi}' \to \boldsymbol{\Phi}'$ are projections, is an isomorphism of the L.a.

Proof of Theorem 2.1. We take the subset $\mathfrak{c}$ of the vector bundle

$$C = i^*(T^\alpha \Phi) \underline{\times} i'^*(T^\alpha \Phi')$$

consisting of all elements of the form $(v, \gamma(v))$, $v \in i^*(T^\alpha \Phi')$. The set $\mathfrak{c}$ has a natural structure of an L.suba. of $A \underline{\times} A'$. Let $\mathscr{E}$ be a connected L.subg. of $\boldsymbol{\Phi} \underline{\times} \boldsymbol{\Phi}'$ with algebroid $\mathfrak{c}$. We take a homomorphism π_1' such that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\ \pi_1'\ } & \boldsymbol{\Phi} \\ & \searrow & \uparrow{\scriptstyle \pi_1} \\ & & \boldsymbol{\Phi} \underline{\times} \boldsymbol{\Phi}' \end{array}$$

If v is an α-vertical tangent vector at l_x, then $v \in i^*(T^\alpha \Phi)_{|x}$, $(v, \gamma(v)) \in i^*(T^\alpha \mathscr{E})$, and $\pi_{1*}'(v, \gamma(v)) = v$. Hence

$$(\pi_1' | \mathscr{E}_x)_{*l_x}\colon T_{l_x}(\mathscr{E}_x) \to T_{l_x}(\Phi_x), \qquad x \in M,$$

is a linear isomorphism and such is also $(\pi_1')_{*l_x}$. Consequently, the mapping π_1' is a diffeomorphism in some neighbourhood of each unit. After complicated calculations we shall find a neighbourhood $\Theta \subset \Phi$ which contains all units and the mapping $H\colon \Theta \to \mathscr{E}$ which is a diffeomorphism onto an open set, being inverse to π_1'. Then $F = \pi_2 \circ H$ is the desired local homomorphism.

We consider two local homomorphisms $F_1, F_2\colon \Phi|\Omega \to \Phi'$ such that $\tilde{F}_{1*} = \tilde{F}_{2*}$. For some open set $U_m \subset \boldsymbol{R}^m$ star-shaped with respect to $0 \in \boldsymbol{R}^m$ the mapping

$$\overline{\mathrm{Exp}}_\Phi(x_0) = \big(U_m \ni (a^1, \ldots, a^m) \mapsto (\mathrm{Exp}_\Phi \sum_{i=1}^m a^i \xi_i)(x_0) \in \Phi_{x_0}\big),$$

where cross-sections $\xi_1, \ldots, \xi_m \in C_0^\infty(i^*(T^\alpha \Phi))$ are a basis of $i^*(T^\alpha \Phi)$ over an open set $U \ni x_0$, is a diffeomorphism onto the open set $U_{l_{x_0}} \subset \Phi_{x_0}$ (see [4]). The inverse mapping is denoted by Log and called an *exponential coordinate* *system* on Φ_{x_0}. If $(a^1, \ldots, a^m) \in U_m$, then there exists $\varepsilon > 0$ such that, for $|t| < 1+\varepsilon$, we have

$$(\operatorname{Exp} t \sum_{i=1}^{m} a^i \xi_i)(x_0) \in U_{\iota_{x_0}}$$

and

$$F\big((\operatorname{Exp} \sum_{i=1}^{m} a^i \xi_i)(x_0)\big) = \operatorname{Exp}\big(\tilde{F}_* \circ (\sum_{i=1}^{m} a^i \xi_i)\big)(x_0)$$

(see [4]). Hence, for

$$j = \big(\boldsymbol{R}^m \ni (a^1, \ldots, a^m) \mapsto \sum_{i=1}^{m} a^i \xi_i \in C_0^\infty(i^*(T^\alpha \Phi))\big)$$

and for $g \in U_{\iota_{x_0}} \cap \Omega$, we obtain

$$F_1(g) = \big(\operatorname{Exp}(\tilde{F}_{1*} \circ j(\operatorname{Log}(g)))\big)(x_0) = \big(\operatorname{Exp}(\tilde{F}_{2*} \circ j(\operatorname{Log}(g)))\big)(x_0) = F_2(g).$$

Now, it is easy to see that F_1 and F_2 coincide in some neighbourhood containing all units.

Finally, we assume that $F_1, F_2 \colon \boldsymbol{\Phi} \to \boldsymbol{\Phi}'$ are global homomorphisms such that $\tilde{F}_{1*} = \tilde{F}_{2*}$. Clearly, $F_1|\Omega = F_2|\Omega$ for some open set Ω containing all units. Since Ω generates Φ, an arbitrary point $z \in \Phi$ is equal to $z_1 \cdot \ldots \cdot z_n$ for some $n \in \boldsymbol{N}$, $z_1, \ldots, z_n \in \Omega$. As a consequence we obtain

$$F_1(z) = F_1(z_1 \cdot \ldots \cdot z_n) = F_1(z_1) \cdot \ldots \cdot F_1(z_n) = F_2(z_1) \cdot \ldots \cdot F_2(z_n)$$
$$= F_2(z_1 \cdot \ldots \cdot z_n) = F_2(z).$$

COROLLARY. *Two L.g.'s are locally isomorphic if and only if their L.a.'s are isomorphic.*

3. Some characterization of subalgebroid. It is easy to see that if $\boldsymbol{\Psi}$ is an L.subg. of $\boldsymbol{\Phi}$ (see [5]), then the set $C_0^\infty(i^*(T^\alpha \Psi))$ coincides with the set of those $\xi \in C_0^\infty(i^*(T^\alpha \Phi))$ for which the mapping

$$E = \big(M \times \boldsymbol{R} \ni (x, t) \mapsto (\operatorname{Exp} t\xi)(x) \in \Phi\big)$$

has the values in Ψ and, while regarded as the mapping $E\colon M \times \boldsymbol{R} \to \Psi$, it is continuous.

COROLLARY. *If two connected L.subg.'s $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ of $\boldsymbol{\Phi}$ coincide as topological spaces, then they coincide as L.g.'s.*

This corollary will be considerably strengthened by using the following

THEOREM 3.1. *Let $\boldsymbol{\Psi}$ be an L.subg. of $\boldsymbol{\Phi}$. Then the set $C_0^\infty(i^*(T^\alpha \Psi))$ is equal to the set of those $\xi \in C_0^\infty(i^*(T^\alpha \Phi))$ for which $(\operatorname{Exp} t\xi)(x) \in \Psi$ for $(x, t) \in M \times \boldsymbol{R}$.*

Proof. Take any vector subbundle $\mathfrak{m}$ of $i^*(T^\alpha \Phi)$ such that $i^*(T^\alpha \Phi) = i^*(T^\alpha \Psi) \oplus \mathfrak{m}$. Let $\xi_1, \ldots, \xi_n \in C_0^\infty(i^*(T^\alpha \Psi))$ and $\xi_{n+1}, \ldots, \xi_{n+m} \in C_0^\infty(\mathfrak{m})$ be cross-sections which are bases of $i^*(T^\alpha \Psi)$ and $\mathfrak{m}$, respectively, over an open subset $U \ni x$. Then the system of cross-sections $\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m}$ is a basis of $i^*(T^\alpha \Phi)$ over U. Let $\overline{\mathrm{Exp}}_\Phi$ and $\overline{\mathrm{Exp}}_\Psi$ be defined for the above cross-sections. There exist neighbourhoods $U' \subset U$ of x, $U_m \subset \boldsymbol{R}^m$ of 0, and $U_n \subset \boldsymbol{R}^n$ of 0 such that the mapping $\lambda\colon U_m \times U_n \times U' \to \Phi$ defined by

$$\lambda(a, b, y) = \overline{\mathrm{Exp}}_\Phi\big((a, 0), \beta \circ \overline{\mathrm{Exp}}_\Phi((0, b), y)\big) \cdot \overline{\mathrm{Exp}}_\Phi((0, b), y)$$

for $a \in U_m$, $b \in U_n$ and $y \in U'$ is a diffeomorphism onto its open image Θ (see [4]) and $\overline{\mathrm{Exp}}_\Psi\colon U_n \times U \to \Psi$ is also a diffeomorphism onto an open set Ω such that its topology is induced from Φ. Since the mapping

$$\gamma = \big(U_m \times U \ni (b, y) \mapsto \beta \circ \overline{\mathrm{Exp}}_\Phi((0, b), y) \in M\big)$$

is smooth and $\gamma(0, x) = x$, there exist some open neighbourhoods $U'_m \subset U_m$ of 0 and $U' \subset U$ of x such that $\gamma[U'_m \times U'] \subset U$. For $y \in U'$ we put

$$\mathfrak{a}_y = \{b \in U'_m;\ \overline{\mathrm{Exp}}_\Phi((0, b), y) \in \Psi\} \quad \text{and} \quad \Theta' = \lambda[U_n \times U'_m \times U'].$$

Hence

$$\Theta' \cap \Psi_y = \bigcup_{b \in \mathfrak{a}_y} \overline{\mathrm{Exp}}_\Phi[U_n \times \{0\} \times \{\beta \circ \overline{\mathrm{Exp}}_\Phi((0, b), y)\}] \cdot \overline{\mathrm{Exp}}_\Phi((0, b), y)$$

and the set

$$\overline{\mathrm{Exp}}_\Phi[U_n \times \{0\} \times \{d\}] = \overline{\mathrm{Exp}}_\Psi[U_n \times \{d\}]$$

is open in Ψ_d, where $d = \beta \circ \overline{\mathrm{Exp}}_\Phi((0, b), y)$. Consequently, the set

$$\overline{\mathrm{Exp}}_\Phi[U_n \times \{0\} \times \{d\}] \cdot \overline{\mathrm{Exp}}_\Phi((0, b), y)$$

is open in Ψ_y for $b \in \mathfrak{a}_y$, $y \in U'$. For different $b \in \mathfrak{a}_y$ these sets are disjoint, which follows from the injectivity of λ. The set $\mathfrak{a}_y$ is then at most countable. We put

$$(\pi\colon \Theta'_y \to U'_m) = (\mathrm{pr}_2 \circ \lambda_y^{-1}),$$

where $\underline{\Theta'_y}$ denotes the topological space $\Phi_y | \Theta'_y$ ($\underline{A}$ denotes the set of points of the space A). Then π is continuous and

$$\pi\big(\overline{\mathrm{Exp}}_\Phi((a, 0), \beta \circ \overline{\mathrm{Exp}}_\Phi((0, b), y)) \cdot \overline{\mathrm{Exp}}_\Phi((0, b), y)\big) = b$$

for $(a, b) \in U_n \times U'_m$. Thus

$$\pi | \underline{\Theta'_y} \cap \underline{\Psi_y}\colon \Theta'_y | \underline{\Theta'_y} \cap \underline{\Psi_y} \to U'_m$$

is also continuous and $\pi[\underline{\Theta'_y} \cap \underline{\Psi_y}] \subset \mathfrak{a}_y$. Therefore, it induces the continuous mapping

$$\tilde{\pi}\colon \Theta'_y | \underline{\Theta'_y} \cap \underline{\Psi_y} \to \mathfrak{a}_y.$$

From the fact that every connected subset of a countable set in $\boldsymbol{R}^n$ is one-point it follows that the image of the connected component of l_y in

$$\Theta'_y|\underline{\Theta'_y} \cap \underline{\Psi_y} = \Phi_y|\Theta'_y \cap \Psi_y$$

under $\tilde{\pi}$ is one-point, of course 0. Since the set

$$\tilde{\pi}^{-1}[\{0\}] = \overline{\mathrm{Exp}_\Psi}[U_n \times \{y\}]$$

is connected, it is a component of l_y in $\Phi_y|\underline{\Theta'_y} \cap \underline{\Psi_y}$.

We consider again the mapping E, an open neighbourhood $U'' \subset U$ of x, and $\varepsilon > 0$ such that $E[U'' \times I_\varepsilon] \subset \Theta'$. Since the mapping

$$E_y = (I_\varepsilon \ni t \mapsto E(y, t) \in \Phi_y)$$

is continuous for $y \in U''$ and $E_y[I_\varepsilon] \subset \Theta' \cap \Psi_y$, we infer that

$$E_y\colon\ I_\varepsilon \to \Phi_y|\Theta'_y \cap \underline{\Psi_y}$$

is also continuous. The set $E_y[I_\varepsilon]$ is connected and $E_y(0) = l_y$, so it is contained in the connected component of l_y in $\Phi_y|\underline{\Theta'_y} \cap \underline{\Psi_y}$, i.e., in $\overline{\mathrm{Exp}_\Psi}[U_n \times \{y\}]$. This proves that $E[U'' \times I_\varepsilon]$ is contained in $\overline{\mathrm{Exp}_\Psi}[U_n \times U'']$ and, consequently, in $\Omega \subset \Psi$. Therefore,

$$E|U'' \times I_\varepsilon\colon\ U'' \times I_\varepsilon \to \Psi$$

is continuous and, of course, smooth. Hence $E|U'' \times I_\varepsilon$ generates a cross-section of $i^*(T^\alpha \Psi)$ over U'' (see [4] and [6]), of course, $\xi|U''$. Since $x \in M$ is arbitrary, $\xi \in C_0^\infty(i^*(T^\alpha \Psi))$.

COROLLARY. *If $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ are two connected L.subg.'s of $\boldsymbol{\Phi}$ whose sets of points are equal, then $\boldsymbol{\Psi}_1$ is equal to $\boldsymbol{\Psi}_2$ as an L.g.*

COROLLARY. *Let $\boldsymbol{K}$ and $\boldsymbol{H}$ be L.subg.'s of $\boldsymbol{\Phi}$ and let $\boldsymbol{K}$ be connected. If the set of points of $\boldsymbol{K}$ is contained in the same one of the $\boldsymbol{H}$, then $\boldsymbol{K}$ is an L.subg. of $\boldsymbol{H}$.*

4. Images and pre-images.

THEOREM 4.1. *Let $F\colon \boldsymbol{\Phi} \to \boldsymbol{\Psi}$ be an L.g. homomorphism and let $\mathfrak{k} = \operatorname{im} \tilde{F}_*$ be a vector subbundle of $i^*(T^\alpha \Psi)$. Then*

(a) *$\mathfrak{k}$ determines a subalgebroid of $i^*(T^\alpha \Psi)$, say $\boldsymbol{k}$;*

(b) *on $\operatorname{im} F$ there exists a structure of the L.subg. of $\boldsymbol{\Psi}$ with algebroid $\boldsymbol{k}$.*

Proof. It is easy to find that $\mathfrak{k}$ determines an L.suba. Let $\boldsymbol{H}$ be a connected L.subg. of $\boldsymbol{\Psi}$ with algebroid $\boldsymbol{k}$. For $x \in M$ and $\xi \in C_0^\infty(i^*(T^\alpha \Phi))$, the elements $(\mathrm{Exp}_\Phi \xi)(x)$ generate Φ, so the elements

$$F((\mathrm{Exp}_\Phi \xi)(x)) = \mathrm{Exp}_\Psi(\tilde{F}_* \circ \xi)(x)$$

generate $\operatorname{im} F$. Hence $\operatorname{im} F$ is generated by $(\mathrm{Exp}_\Psi \eta)(x)$ for $x \in M$ and

$\eta \in C_0^\infty(\mathfrak{f})$. Since these elements generate also $\boldsymbol{H}$, the equality $\underline{H} = \operatorname{im} F$ holds. This completes the proof.

Let $F\colon \boldsymbol{\Phi} \to \boldsymbol{\Psi}$ be an L.g. homomorphism. Let us take an L.subg. $\boldsymbol{\Psi}' \subset \boldsymbol{\Psi}$ having the topology induced from Ψ. Put $H_1 = F^{-1}[\Psi']$ and $\mathfrak{u} = \tilde{F}_*^{-1}[i^*(T^\alpha \Psi')]$. Assume that $\mathfrak{u}$ is a vector subbundle of $i^*(T^\alpha \Phi)$; then $\mathfrak{u}$ determines an L.suba. of $i^*(T^\alpha \Phi)$, say $\boldsymbol{u}$.

THEOREM 4.2. *Let $\boldsymbol{H}$ be a connected L.subg. of $\boldsymbol{\Phi}$ with algebroid $\boldsymbol{u}$. Then on H_1 there exists a structure of L.subg. of $\boldsymbol{\Phi}$, say $\boldsymbol{H}_1$, with topology induced from Φ and such that $\boldsymbol{H}$ is an open L.subg. of $\boldsymbol{H}_1$ with topology induced from H_1.*

Proof. It is enough to prove that

(1) $H \subset H_1$,

(2) the topology of H is induced from H_1,

(3) H is open in H_1.

To verify (2) it suffices to prove (1), (3) and to see that

(4) H is closed in H_1.

(1) For $\xi \in C_0^\infty(\mathfrak{u})$ we have $\tilde{F}_* \circ \xi \in C_0^\infty(i^*(T^\alpha \Psi'))$ and

$$F((\operatorname{Exp}\xi)(x)) = (\operatorname{Exp}(\tilde{F}_* \circ \xi))(x) \in \Psi',$$

whence $(\operatorname{Exp}\xi)(x) \in H_1$. Since the elements of the form $(\operatorname{Exp}\xi)(x)$ generate H (see [4]), we get $H \subset H_1$.

(3) Take an arbitrary point $x_0 \in M$ and some cross-sections

$$\xi_1, \ldots, \xi_s \in C_0^\infty(\mathfrak{u}), \qquad \xi_1, \ldots, \xi_s, \xi_{s+1}, \ldots, \xi_k \in C_0^\infty(i^*(T^\alpha \Phi)),$$

$$\eta_1, \ldots, \eta_m \in C_0^\infty(i^*(T^\alpha \Psi')), \qquad \eta_1, \ldots, \eta_m, \eta_{m+1}, \ldots, \eta_r \in C_0^\infty(i^*(T^\alpha \Psi))$$

such that over an open set $U \ni x_0$ they are bases of the corresponding vector bundles. Let $\overline{\operatorname{Exp}}_\Phi$ and $\overline{\operatorname{Exp}}_\Psi$ be defined for these cross-sections. Choose neighbourhoods $U_s \subset \boldsymbol{R}^s$ of 0, $U_{k-s} \subset \boldsymbol{R}^{k-s}$ of 0, $U_m \subset \boldsymbol{R}^m$ of 0, $U_{r-m} \subset \boldsymbol{R}^{r-m}$ of 0, and $U' \subset U$ of x_0 such that the mappings

$$\overline{\operatorname{Exp}}_\Phi\colon U_s \times U_{k-s} \times U' \to \Phi \quad \text{and} \quad \overline{\operatorname{Exp}}_\Psi\colon U_m \times U_{r-m} \times U' \to \Psi$$

are diffeomorphisms onto their open images, and

$$\overline{\operatorname{Exp}}_\Phi[U_s \times \{0\} \times U'] = \overline{\operatorname{Exp}}_\Phi[U_s \times U_{k-s} \times U'] \cap H,$$

$$\overline{\operatorname{Exp}}_\Psi[U_m \times \{0\} \times U'] = \overline{\operatorname{Exp}}_\Psi[U_m \times U_{r-m} \times U'] \cap \Psi'.$$

We put

$$i = \Big(U_s \times U_{k-s} \times U' \ni (a, b, x) \mapsto \sum_{i=1}^{s} a^i \xi_i(x) + \sum_{i=1}^{k-s} b^i \xi_{s+i}(x) \in \Phi\Big),$$

$$j = \Big(U_m \times U_{r-m} \times U' \ni (c, d, x) \mapsto \sum_{i=1}^{m} c^i \eta_i(x) + \sum_{i=1}^{r-m} d^i \eta_{m+i}(x) \in \Psi\Big).$$

Let $U'_s \subset U_s$ and $U'_{k-s} \subset U_{k-s}$ be neighbourhoods of 0 and let $U'' \subset U$ be a neighbourhood of x_0. Suppose

$$\tilde{F}_*\left[i\left[U'_s \times U_{k-s} \times U''\right]\right] \subset j\left[U_m \times U_{r-m} \times U'\right].$$

Putting $\Omega = \overline{\mathrm{Exp}}_\Phi\left[U'_s \times U'_{k-s} \times U''\right]$, we obtain an open subset of Φ containing l_{x_0}, and $H_1 \cap \Omega = H$. Indeed, for $z \in H_1 \cap \Omega$ we have $F(z) \in \Psi'$, and $z = \overline{\mathrm{Exp}}_\Phi((a_1, a_2), x)$ for $a_1 \in U'_s$, $a_2 \in U'_{k-s}$. If

$$((b_1, b_2), x) = j^{-1}\left(\tilde{F}_*(i(a_1, a_2), x)\right),$$

then $b_1 \in U_m$, $b_2 \in U_{r-m}$ and

$$\begin{aligned} F(z) = \overline{\mathrm{Exp}}_\Psi((b_1, b_2), x) \in \overline{\mathrm{Exp}}_\Psi\left[U_m \times U_{r-m} \times U'\right] \cap \Psi' \\ = \overline{\mathrm{Exp}}_\Psi\left[U_m \times \{0\} \times U'\right]. \end{aligned}$$

Hence $b_2 = 0$ and $F(z) = \overline{\mathrm{Exp}}_\Psi(((b_1, 0), x))$, $(b_1, x) \in U_m \times U'$, and $i(b_1, x) \in i^*(T^\alpha \Psi')$, which means that

$$\tilde{F}_*\left(i((a_1, a_2), x)\right) \in i^*(T^\alpha \Psi').$$

Therefore $i((a_1, a_2), x) \in U$, whence $a_2 = 0$. Finally,

$$z = \overline{\mathrm{Exp}}_\Phi((a_1, a_2), x) = \overline{\mathrm{Exp}}_\Phi((a_1, 0), x) \in H.$$

We now take $z \in H$ such that $\beta(z) = x_0$. Let $\sigma\colon U' \to H$ be an arbitrary α-admissible β-section such that $\sigma(x_0) = z$. Put $W = \alpha\left[\sigma[U']\right]$ and $f = \alpha \circ \sigma$. Let

$$L = (\alpha^{-1}[U'] \ni g \mapsto g \cdot \sigma_{\alpha(g)} \in \alpha^{-1}[W])$$

be a right translation in Φ by σ. Then $L[\Omega]$ is an open set of z. It is easy to see that

$$L[\Omega] \cap H_1 \subset H.$$

Since x_0 is an arbitrary point, H is open in H_1.

(4) We prove first that H_x is closed in $(H_1)_x$ for an arbitrary $x \in M$. Let $x \xrightarrow{z} y$ be an element of $H_1 \setminus H$. Then $\Phi_z\colon (H_1)_y \to (H_1)_x$ is a homeomorphism and $\Phi_z[H_y]$ is open in the space $(H_1)_x$ disjoint with H_x and containing z. Hence $(H_1)_x \setminus H_x$ is open in $(H_1)_x$, and so H_1 is closed in $(H_1)_x$.

We now take a sequence $z_n \to z_0$ as $n \to \infty$, $z_n \in H$. Let $\alpha(z_n) = x_n$ and $\beta(z_n) = y_n$. Then $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$, where $\alpha(z_0) = x_0$ and $\beta(z_0) = y_0$. For an arbitrary sequence $t_n \in H$ such that $\alpha(t_n) = x_0$, $\beta(t_n) = x_n$, and $t_n \to l_{x_0}$, we have $z_n \cdot t_n \to z_0 \cdot l_{x_0} = z_0$ as $n \to \infty$. Since $z_n \cdot t_n \in H_{x_0}$, we have $z_0 \in H_{x_0}$.

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