UNIVERSITE CLAUDE BERNARD-LYON 1

N° ISSN - 0076-1656

# Publications du Département de Mathématiques

**nouvelle série**

1/A - 1989



bât. n° 101, 43 boulevard du 11 novembre 1918

69622 VILLEURBANNE Cedex

UNIVERSITE CLAUDE BERNARD - LYON 1

N° ISSN - 0076-1656

# Publications du Département de Mathématiques

**nouvelle série**

1/A - 1989



bât. n° 101, 43 boulevard du 11 novembre 1918

69622 VILLEURBANNE Cedex

PUBLICATIONS DU DEPARTEMENT DE MATHEMATIQUES

FASCICULE 1/A - 1989

SOMMAIRE

# LIE ALGEBROID OF A PRINCIPAL FIBRE BUNDLE

Jan KUBARSKI

## INTRODUCTION

The notion of a Lie algebroid, introduced by J.Pradines [22], [23], was invented in connection with studying differential groupoids. Lie algebroids of differential groupoids correspond to Lie algebras of Lie groups. They consist of vector bundles equipped with some algebraic structures ($\mathbb{R}$-Lie algebras in moduli of sections). Since each principal fibre bundle (pfb for short) P determines a differential groupoid ( the so-called Lie groupoid $PP^{-1}$ of Ehresmann), therefore each pfb P defines - in an indirect manner - a Lie algebroid A(P). P.Libermann noticed [12] that the vector bundle of this Lie algebroid A(P), P= P(M,G), is canonically isomorphic to the vector bundle $TP/_G$ (investigated earlier by M.Atiyah [2] in the context of the problem of the existence of a connection in a complex pfb). The problem:

- How to define the structure of the Lie algebroid in $TP/_G$ without using Pradines' construction,

is systematically elaborated in this work (chapt. 1).

The Lie algebroid of a pfb can also be obtained in the third manner as an associated vector bundle with some pfb.

To sum up, three natural constructions of the Lie algebroid A(P) for a given pfb P= P(M,G) are made (chapters 1 and 2):

(1) $A(P)= TP/_G$, the idea of this construction could be found in M.Atiyah [2] and P.Libermann [12], see also [16], [17], [19], [20].

(2) $A(P)= A(PP^{-1})$:= the Lie algebroid of the Ehresmann Lie groupoid $PP^{-1}$, see [3], [9], [22], [23].

(3) $A(P)= W^1(P)\times_{G^1_n}(\mathbb{R}^n\times\mathfrak{g})$ where $\mathfrak{g}=\mathfrak{X}(G)^0$ is the Lie algebra of G defined by right-invariant vector fields, $W^1(P)$ is the 1-st order prolongation of P and $G^1_n$ - the n-dim. 1-st order prolongation of G, n=dimM, [4], [7]; via some left action of $G^1_n$ on $\mathbb{R}^n\times\mathfrak{g}$.

In the theory of Lie groups it is well known that two Lie groups are locally isomorphic if and only if (iff) their Lie algebras are isomorphic. The question:

-What this problem looks like for pfb's?

is answered in this work. A suitable notion of a <u>local</u> <u>homomorphism</u> (and a <u>local</u> <u>isomorphism</u>) between pfb's is found (chapt. 3).

By a <u>local</u> <u>homomorphism</u> [ <u>isomorphism</u> ] $\mathfrak{F}:P(M,G) \longrightarrow P'(M,G')$ we shall mean each family

$$\mathfrak{F}=\{(F_t,\mu_t);\ t\in T\}$$

of "partial homomorphisms" [ isomorphisms ] $(F_t,\mu_t):P\supset D_t \longrightarrow P'$ provided some compatibility axioms are satisfied (def. 3.1).

Every local homomorphism $\mathfrak{F}$ defines an homomorphism of the Lie algebroids $d\mathfrak{F}:A(P)\longrightarrow A(P')$ (prop. 3.2) and, conversely, every homomorphism of the Lie algebroids comes from some local homomorphism of the pfb's (th. 3.4).

Two pfb's are locally isomorphic iff their Lie algebroids are isomorphic (th. 3.5).

Some invariants of isomorphisms of pfb's are invariants of local isomorphisms so they are then de facto some notions of Lie algebroids. For example:

(1) the Ad-associated Lie algebra bundle $P\times_G\mathfrak{g}$,

(2) the flatness (chapt. 4),

(3) the Chern-Weil homomorphism (for some local isomorphisms) (chapt. 5).

One can ask the question:

- How much information about pfb $P$ is carried by the associated Lie algebra bundle $P\times_G \mathfrak{g}$?

It turns out that sometimes none:

- If $G$ is abelian, then $P\times_G\mathfrak{g}$ is trivial (see corollary 1.11),

and sometimes much, and most if $G$ is semisimple:

- Two pfb's with semisimple structural Lie groups are locally isomorphic iff their associated Lie algebra bundles are isomorphic (corollary 7.2.6).

Let $A=(A,[\![\cdot,\cdot]\!],\gamma)$ be an arbitrary Lie algebroid on a manifold $M$. A connection in $A$, ie a splitting of Atiyah sequence

$$0\longrightarrow \mathfrak{g}(A)\hookrightarrow A \underset{\curvearrowleft \lambda}{\xrightarrow{\gamma}} TM \longrightarrow 0 \quad \text{where } \mathfrak{g}(A)=\operatorname{Ker}\gamma,$$

determines a covariant derivative $\nabla$ in the Lie algebra bundle $\mathfrak{g}(A)$ and a tensor $\Omega_M\in\Omega^2(M;\mathfrak{g}(A))$ by the formulae:

(a) $\nabla_X\sigma=[\![\lambda X,\sigma]\!]$,

(b) $\Omega_M(X,Y)=\lambda[X,Y]-[\![\lambda X,\lambda Y]\!]$ (the curvature tensor of $\lambda$).

Now, let $\mathfrak{g}$ be an arbitrary Lie algebra bundle, $\nabla$ - a covariant derivative in $\mathfrak{g}$ and $\Omega_M\in\Omega^2(M;\mathfrak{g})$. The necessary and sufficient conditions for the existence of a Lie algebroid which realizes $(\mathfrak{g},\nabla,\Omega_M)$ via some connection are (see chapt. 6):

(1) $R_{X,Y}\sigma=-[\Omega_M(X,Y),\sigma]$, $R$ being the curvature tensor of $\nabla$,

(2) $\nabla_X[\sigma,\eta]=[\nabla_X\sigma,\eta]+[\sigma,\nabla_X\eta]$,

(3) $\nabla\Omega_M=0$.

The results of chapter 6 are used to give a classification of Lie algebroids in two cases (chapt. 7):

($1^o$) all flat Lie algebroids with abelian isotropy Lie algebras,

($2^o$) all Lie algebroids with semisimple isotropy Lie algebras.

The second looks as follows (th. 7.2.3):

- For any Lie algebra bundle $\mathfrak{g}$ whose fibres are semisimple there exists exactly one (up to an isomorphism) Lie algebroid A for which $\mathfrak{g}(A)=\mathfrak{g}$.

In consequence, two arbitrary pfb's with semisimple structural Lie groups and isomorphic associated Lie algebra bundles have isomorphic Lie algebroids, so they are then locally isomorphic.

- Are they globally isomorphic (in our sense, see p.15) provided their structural Lie groups are, in addition, isomorphic?

It turns out that they are not, even if these Lie groups are assumed to be connected (ex. 8.3).

Some results contained in this work were obtained independently by K.Mackenzie [14], but, in general, using different methods. This concerns some parts of chapters 1, 4 and 6 only (in the text there are more detailed references). The main results of this work /all chap. 2, theorems 3.4, 3.5, 3.6, 5.2, 5.8, 7.1.1, 7.2.3, 8.1 and ex. 8.3 / are included in the remains chapters.

J. K.

## CHAPTER 1

## LIE ALGEBROID A(P) OF A PRINCIPAL FIBRE BUNDLE P(M,G)

All the differential manifolds considered in the present paper are assumed to be smooth (ie $C^\infty$) and Hausdorff.

Take any pfb

$$P = P(M,G)$$

with the projection $\pi: P \longrightarrow M$ and the action $R: P\times G \longrightarrow P$, and define the action

$$R^T: TP\times G \longrightarrow TP, \quad (v,a) \longmapsto (R_a)_* v,$$

$R_a$ being the right action of $a$ on $P$. Denote by

$$A(P)$$

the space of all orbits of $R^T$ with the quotient topology. Let $[v]$ denote the orbit through $v$ and

$$\pi^A: TP \longrightarrow A(P), \quad v \longmapsto [v],$$

the natural projection. In the end, we define the projection

$$p: A(P) \longrightarrow M, \quad [v] \longmapsto \pi z, \quad \text{if} \quad v \in T_z P.$$

For each point $x\in M$, in the fibre $p^{-1}(x)$ there exists exactly one vector space structure (over $\mathbb{R}$) such that

$$[v] + [w] = [v+w] \quad \text{if} \quad \pi_P(v) = \pi_P(w),$$

$\pi_P: TP \longrightarrow P$ being the projection.

$$\pi^A_{|z}: T_z P \longrightarrow A(P)_{|\pi z},$$

is then an isomorphism of vector spaces, $z\in P$.

The pfb $P(M,G)$ determines another pfb

$$TP(TM,TG)$$

with the projection $\pi_*: TP \longrightarrow TM$ and the action

$$R_*: TP\times TG \longrightarrow TP$$

[5]. We can treat $G$ as a closed Lie subgroup of $TG$ ($G \cong \{\theta_a;\ a \in G\}$, $\theta_a$ being the null tangent vector at $a$). The restriction of $R_*$ to $G$ is then equal to $R^T$ [5]. By [6], we see that the structure of a Hausdorff $C^\infty$-manifold, such that $\pi^A$ is a submersion, exists in $A(P)$ (this result is obtained by K.Mackenzie [14,p.282] in another way). We also obtain a pfb $TP(A(P),G)$ with the projection $\pi^A$ and the action $R^T$.

PROPOSITION 1.1. (cf [14,pp.282,283]). For each local trivialization $\varphi: U \times G \longrightarrow P$ of $P(M,G)$, the mapping

(1) $$\varphi^A: TU \times \mathfrak{g} \longrightarrow p^{-1}[U] \subset A(P), \quad (v,w) \longmapsto [\varphi_*(v,w)],$$

is a diffeomorphism, where $\mathfrak{g} = T_eG$.

PROOF. It is easy to see that $\varphi^A$ is a bijection. Besides, the following diagram

$$\begin{array}{ccc} TU \times TG & \xrightarrow{\mathrm{id} \times \theta^R} & TU \times \mathfrak{g} \\ \varphi_* \downarrow & & \downarrow \varphi^A \\ TP & \xrightarrow{\pi^A} & A(P) \end{array}$$

commutes where $\theta^R$ denotes the canonical right-invariant 1-form on $G$. Indeed, if we put

$$\lambda := \varphi(\cdot, e), \quad \text{and} \quad A_z : G \longrightarrow P,\ a \longmapsto za, \quad z \in P,$$

$e$ being the unit of $G$, then we have, for $x \in U$, $v \in T_xU$, $a \in G$ and $w \in T_aG$,

$$\begin{aligned} \pi^A \circ \varphi_*(v,w) &= [\varphi_*(v,w)] = [\varphi(\cdot,a)_*(v) + \varphi(x,\cdot)_*(w)] \\ &= [(R_a \circ \lambda)_*(v) + (A_{\lambda(x)})_*(w)] \\ &= [(R_{a^{-1}})_*((R_a \circ \lambda)_*(v) + (A_{\lambda(x)})_*(w))] \\ &= [\lambda_*(v) + (A_{\lambda(x)})_*(\theta^R(w))] = \varphi^A(v, \theta^R(w)) \\ &= \varphi^A \circ (\mathrm{id} \times \theta^R)(v,w). \end{aligned}$$

Because of the fact that $\pi^A$ and $\mathrm{id} \times \theta^R$ are submersions, we assert that $\varphi^A$ and $(\varphi^A)^{-1}$ are of the $C^\infty$- class. □

REMARK 1.2. Using the bijections $\varphi^A$, we can define the differential structure of $A(P)$ in a more elementary manner than above as the one for which $\varphi^A$ are diffeomorphisms. For this purpose, we must only notice that, for arbitrary local trivializations $\varphi_i : U_i \times G \longrightarrow P$, $i=1,2$, we have:

(a) $(\varphi_1^A)^{-1}[p^{-1}[U_1] \cap p^{-1}[U_2]]$ is open in $TU_1 \times \mathfrak{g}$,

(b) $(\varphi_1^A)^{-1} \circ \varphi_2^A$ is a diffeomorphism.

(a) is trivial. To see (b), we shall calculate that

$$(\varphi_1^A)^{-1} \circ \varphi_2^A(v,w) = (v, \Theta^R(g_*(v)) + \mathrm{Ad}(g(x))(w))$$

for $v \in T_x U$, $x \in U$, $w \in \mathfrak{g}$, where

$$g : U_1 \cap U_2 \longrightarrow G$$

is a transition function, ie $\varphi_2(x,e) = \varphi_1(x,e) \cdot g(x)$, $x \in U_1 \cap U_2$, and Ad denotes the adjoint representation of G. Put

$$\lambda_i := \varphi_i(\cdot, e)$$

and let $l_a$, $r_a$ denote the left and the right translation by a on G. We have

$$\begin{aligned}
(\varphi_1^A)^{-1} \circ \varphi_2^A(v,w) &= (\varphi_1^A)^{-1}([\varphi_{2*}(v,w)]) \\
&= (\varphi_1^A)^{-1}([\lambda_{2*}(v) + (A_{\lambda_2(x)})_*(w)]) \\
&= (\varphi_1^A)^{-1}([(\lambda_1 \cdot g)_*(v) + (A_{\lambda_1(x)\cdot g(x)})_*(w)]) \\
&= (\varphi_1^A)^{-1}([(R_{g^{-1}(x)})_*((R_{g(x)})_*(\lambda_{1*}(v)) \\
&\qquad + (A_{\lambda_1(x)})_*(g_*(v)) + (A_{\lambda_1(x)} \circ l_{g(x)})_*(w))]) \\
&= (\varphi_1^A)^{-1}([\lambda_{1*}(v) + (A_{\lambda_1(x)} \circ r_{g^{-1}(x)})_*(g_*(v)) \\
&\qquad + (A_{\lambda_1(x)})_*((r_{g^{-1}(x)})_*((l_{g(x)})_*(w)))]) \\
&= (\varphi_1^A)^{-1}([\lambda_{1*}(v) + (A_{\lambda_1(x)})_*(\Theta^R(g_*(v)) + \mathrm{Ad}(g(x))(w))]) \\
&= (v, \Theta^R(g_*(v)) + \mathrm{Ad}(g(x))(w)). \quad \Box
\end{aligned}$$

<u>PROPOSITION 1.3</u>. (see [14,p.283]) The system

$$(A(P), p, M) \tag{2}$$

is a vector bundle and (1) is a (strong) isomorphism of the vector bundles (over the manifold $U \subset M$).

<u>PROOF</u>. It is sufficient to notice that

$$\varphi^A_{|x} : T_xU \times \mathfrak{g} \longrightarrow A(P)_{|x}$$

is an isomorphism of vector spaces, $x \in U$. $\square$

EXAMPLE 1.4. (a) For an arbitrary Lie group G (treated as a trivial pfb over a point), we have:

$$A(G) = TG/_G \cong \mathfrak{g}, \quad [w] \longmapsto \theta^R(w).$$

More generally, for $P = M \times G$, we have:

$$A(P) = T(M\times G)/_G \cong TM \times \mathfrak{g}, \quad [(v,w)] \longmapsto (v, \theta^R(w)).$$

(b) $A(L^q(M)) \cong J^q(TM)$, see [11].

Let

$$\operatorname{Sec} A(P)$$

denote the $C^\infty(M)$ - module of all $C^\infty$ global cross-sections of the vector bundle $A(P)$, and

$$\mathfrak{X}^R(P)$$

\- of all $C^\infty$ right-invariant vector fields on P. Each vector field $X \in \mathfrak{X}^R(P)$ determines a cross-section

$$X_o \in \operatorname{Sec} A(P)$$

in such a way that $X_o(x) = [X(z)]$ for $z \in P_{|x}$, $x \in M$. $X_o$ is a $C^\infty$ cross-section because locally $X_o|U = \pi^A \circ X \circ \lambda$ where $\lambda : U \longrightarrow P$ is an arbitrary local cross-section of P. The mapping

(3) $$\mathfrak{X}^R(P) \longrightarrow \operatorname{Sec} A(P), \quad X \longmapsto X_o,$$

is a homomorphism of $C^\infty(M)$ - modules.

PROPOSITION 1.5. (cf [14, pp. 281, 285]) For each cross-section $\eta \in \operatorname{Sec} A(P)$ there exists exactly one $C^\infty$ right-invariant vector field

$$\eta' \in \mathfrak{X}^R(P)$$

such that

(4) $$[\eta'(z)] = \eta(\pi z).$$

The mapping

(5) $$\operatorname{Sec} A(P) \longrightarrow \mathfrak{X}^R(P), \quad \eta \longmapsto \eta',$$

is an isomorphism of $C^{\infty}(M)$-modules, inverse to (3).

<u>PROOF</u>. Formula (4) defines in a unique manner some vector field $\eta'$ on P. $\eta'$ is, of course, right-invariant. To show the smoothness of $\eta'$, we take an arbitrary local trivialization $\varphi: U\times G \longrightarrow P$ and define the mappings $\tilde{\eta}'$ and $\tilde{\eta}$ in such a way that the following diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "\eta'"] & TP \arrow[r, "\pi^A"] & A(P) & M \arrow[l, "\eta"'] \\
U\times G \arrow[u, "\varphi"] \arrow[r, "\tilde{\eta}'"] & TU\times TG \arrow[u, "\varphi_*"'] \arrow[r, "\mathrm{id}\times\Theta^R"] \arrow[dr, "(pr_1)^A"'] & TU\times\mathfrak{g} \arrow[u, "\varphi^A"'] & U \arrow[u, hook] \arrow[l, "\tilde{\eta}"'] \\
 & & T(U\times G)/G \;\; (= A(U\times G)) \arrow[u, "\approx"] &
\end{tikzcd}$$

We read $\tilde{\eta}'$ out as a right-invariant vector field on the trivial pfb $U\times G$, induced by $\tilde{\eta}$:

$$(\mathrm{id}\times\Theta^R)(\tilde{\eta}'(x,a)) = (\varphi^A)^{-1}\circ\pi^A\circ\eta'\circ\varphi(x,a) = (\varphi^A)^{-1}\circ\eta\circ\pi\circ\varphi(x,a) = \tilde{\eta}(x).$$

Therefore, the problem of the smoothness of $\eta'$ reduces to that for the trivial pfb's form $U\times G$. An arbitrary $C^{\infty}$ cross-section $\tilde{\eta}: U \longrightarrow TU\times\mathfrak{g}$ is of the form $\tilde{\eta} = (X,\sigma)$ where $X\in\mathfrak{X}(U)$ and $\sigma: U \longrightarrow \mathfrak{g}$. The right-invariant vector field $\tilde{\eta}'$ on $U\times G$ is then defined by

$$\tilde{\eta}'(x,a) = (X(x),(r_a)_*(\sigma(x))),$$

but this formula asserts the smoothness of $\tilde{\eta}'$.

In the end, we notice that, for the equality

$$(f\cdot\eta)' = f\circ\pi\cdot\eta',$$

(5) is a homomorphism of $C^{\infty}(M)$-modules being inverse to (3). $\square$

Now, we define some $\mathbb{R}$-Lie algebra structure in the $\mathbb{R}$-vector space $\mathrm{Sec}\,A(P)$ by demanding that (5) be an isomorphism of $\mathbb{R}$-Lie algebras.

The bracket in $\mathrm{Sec}\,A(P)$, denoted by $[\![\cdot,\cdot]\!]$, must be defined by

$$[\![\xi,\eta]\!] = ([\xi',\eta'])_0.$$

We also take the mapping

$$\gamma : A(P) \longrightarrow TM, \quad [v] \longmapsto \pi_* v.$$

Of course

$$\gamma_{|x} = \pi_{*z} \circ (\pi^A_{|z})^{-1} \quad \text{for } z \in P_{|x}.$$

<u>DEFINITION 1.6</u>. The object

(6) $$A(P) = (A(P), [\![\cdot,\cdot]\!], \gamma)$$

is called the <u>Lie algebroid</u> of a pfb $P(M,G)$.

The fundamental properties of (6) are described in the following proposition.

<u>PROPOSITION 1.7</u>. (see [14,p.285]).

(a) $(\mathrm{Sec}\, A(P), [\![\cdot,\cdot]\!])$ is an R-Lie algebra,

(b) $\mathrm{Sec}\,\gamma : \mathrm{Sec}\, A(P) \longrightarrow \mathfrak{X}(M)$, $\xi \longmapsto \gamma \circ \xi$, is a homomorphism of Lie algebras,

(c) $\gamma$ is an epimorphism of vector bundles,

(d) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$ for $f \in C^\infty(M)$, $\xi,\eta \in \mathrm{Sec}\, A(P)$,

(e) the vector bundle

$$\mathfrak{g}(P) := \mathrm{Ker}\,\gamma \subset A(P)$$

is a Lie algebra bundle (see [5,p.377]), where the structure of a Lie algebra in a fibre $\mathfrak{g}(P)_{|x}$, $x \in M$, is defined as follows:

(7) $$[v,w] := [\![\xi,\eta]\!](x)$$

where $\xi,\eta \in \mathrm{Sec}\, A(P)$, $\xi(x)=v$, $\eta(x)=w$, $v,w \in \mathfrak{g}(P)_{|x}$.

The mapping

(8) $$\varphi^A_o : U\times\mathfrak{g} \longrightarrow \mathfrak{g}(P)_{|U}, \quad (x,w) \longmapsto \varphi^A(0_x, w),$$

is a local trivialization of the Lie algebra bundle for an arbitrary local trivialization $\varphi$ of P, where $\mathfrak{g} = T_e G$ is the Lie algebra of G defined by right-invariant vector fields.

<u>COROLLARY 1.8</u>. By properties (a) ÷ (d), (6) is a Lie algebroid in the sense of J.Pradines [22], [23].

PROOF OF PROP. 1.7. (a)$\div$(d) see [14,p.285] .

(e) To prove that (7) is a correct definition, we must show that the right-hand side of (7) does not depend on the choice of $\xi$ and $\eta$. For this purpose, we take $\xi_1, \xi_2 \in \mathrm{Sec}\,A(P)$ such that $\xi_1(x) = \xi_2(x)$, $x$ being an arbitrary but fixed point. We prove that

$$[\![\xi_1,\eta]\!](x) = [\![\xi_2,\eta]\!](x)$$

for $\eta \in \mathrm{Sec}\,A(P)$ provided $\eta(x) \in \mathfrak{q}(P)_{|x}$. Put $\nu = \xi_1 - \xi_2$ ; $\nu(x) = 0$. The fact that $A(P)$ is a vector bundle implies the existence of sections $\tilde{\xi}_1, \ldots, \tilde{\xi}_m \in \mathrm{Sec}\,A(P)$, functions $f^1, \ldots, f^m \in C^\infty(M)$ and a nbh $U \subset M$ of $x$, such that $f^i(x) = 0$, $i \leq m$, and $\nu|U = \nu_1|U$ where $\nu_1 = \sum f^i \tilde{\xi}_i$. Making use of (d) and taking a function separating an arbitrary point $y \in U$ in $U$, we see that $[\![\nu,\eta]\!]|U = [\![\nu_1,\eta]\!]|U$. Consequently,

$$\begin{aligned} [\![\xi_1,\eta]\!](x) - [\![\xi_2,\eta]\!](x) &= [\![\nu_1,\eta]\!](x) \\ &= \sum f^i(x)\cdot[\![\tilde{\xi}_i,\eta]\!](x) - \sum (\gamma\circ\eta)(x)(f^i)\cdot\tilde{\xi}_i(x) \\ &= 0. \end{aligned}$$

The correctness now follows from the antisymmetry of $[\![\cdot,\cdot]\!]$.

It remains to show that

$$\varphi^A_{0,x} : \mathfrak{q} \longrightarrow \mathfrak{q}(P)_{|x}$$

is an isomorphism of Lie algebras, $x \in U$. Thanks to the equality

$$\varphi^A_{0,x}(v) = [A_{\lambda(x)*}(v)], \quad v \in \mathfrak{q},$$

we need to show that

$$\hat{z} : \mathfrak{q} \longrightarrow \mathfrak{q}(P)_{|x}, \quad v \longmapsto [A_{z*}(v)], \tag{9}$$

is an isomorphism of Lie algebras, where $z \in P_{|x}$.

Take $v_1, v_2 \in \mathfrak{q}$ and the right-invariant vector fields $X_1, X_2 \in \mathfrak{X}(G)$ determined by $v_1, v_2$, respectively. Let $\xi_1$, $\xi_2$ denote arbitrary but fixed cross-sections of $A(P)$ taking at $x$ the values $\hat{z}(v_1)$, $\hat{z}(v_2)$, respectively. To get the equality

$$\hat{z}([v_1, v_2]) = [\![\xi_1, \xi_2]\!](x)$$

it is sufficient to see that

$$A_{z*}([v_1,v_2]) = [\xi_1',\xi_2'](z).$$

First, we notice that $X_i$ is $A_z$-related to $\xi_i'$:

$$A_{z*}(X_i(a)) = A_{z*}((r_a)_*(v_i)) = (R_a \circ A_z)_*(v_i) = (R_a)_*(\xi_i'(z))$$
$$= \xi_i'(za) = \xi_i'(A_z(a)).$$

Therefore $[X_1,X_2]$ is $A_z$-related to $[\xi_1',\xi_2']$, which implies the assertion. □

EXAMPLE 1.9. ([21]) As the Lie algebroid of a trivial pfb $M\times G$ we take

$$TM \times \mathfrak{g}$$

with the structures

(a) $\gamma = pr_1 : TM\times\mathfrak{g} \longrightarrow TM$,

(b) $[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - \mathcal{L}_Y\sigma + [\sigma,\eta])$, $X,Y\in\mathfrak{X}(M)$, $\sigma,\eta : M \longrightarrow$

(an arbitrary cross-section of $TM\times\mathfrak{g}$ is of the form $(X,\sigma)$ where $X\in\mathfrak{X}(M)$ $\sigma : M \longrightarrow \mathfrak{g}$).

PROPOSITION 1.10. (cf [1] and [14,p.119]). $\mathfrak{g}(P)$ is canonically isomorphic to the Ad-associated Lie algebra bundle $P\times_G\mathfrak{g}$.

PROOF. The mapping

$$\tau : P\times_G\mathfrak{g} \longrightarrow \mathfrak{g}(P), \quad [z,v] \longmapsto [A_{z*}(v)],$$

is an isomorphism of Lie algebra bundles. □

COROLLARY 1.11. If the structural Lie group G is abelian, then $\mathfrak{g}(P)$ is trivial.

PROOF. $\mathfrak{g}(P) \cong P\times_G\mathfrak{g} = (P\times\mathfrak{g})_{/G} \cong P_{/G}\times\mathfrak{g} \cong M\times\mathfrak{g}$. □

— o — o — O — o — o —

DEFINITION 1.12. (cf J.Pradines [23]). By a Lie algebroid (on a manifold M) we shall mean a system

(10) $$A = (A, [\![\cdot,\cdot]\!], \gamma)$$

consisting of a vector bundle A (over M) and mappings

$$[\![\cdot,\cdot]\!] : \mathrm{Sec}\,A \times \mathrm{Sec}\,A \longrightarrow \mathrm{Sec}\,A \quad \text{and} \quad \gamma : A \longrightarrow TM$$

such that

(a) $(\mathrm{Sec}\,A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

(b) $\gamma$, called by K.Mackenzie [14] an anchor, is an epimorphism of vector bundles,

(c) $\mathrm{Sec}\,\gamma : \mathrm{Sec}\,A \longrightarrow \mathfrak{X}(M)$ is a homomorphism of Lie algebras,

(d) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$ for $f \in C^{\infty}(M)$ and $\xi,\eta \in \mathrm{Sec}\,A$.

[J.Pradines [23] does not require for the anchor $\gamma$ to be an epimorphism. The reason is the fact that J.Pradines associates such an object with a differential groupoid, much more general than a Lie groupoid]

With each Lie algebroid (10) we associate a short exact sequence of vector bundles

(11) $$0 \longrightarrow \mathfrak{g}(A) \hookrightarrow A \xrightarrow{\gamma} TM \longrightarrow 0$$

where

$$\mathfrak{g}(A) = \mathrm{Ker}\,\gamma\,,$$

called the Atiyah sequence assigned to (10) (see [14,p.288]).

In each fibre $\mathfrak{g}(A)_{|x}$, some Lie algebra structure is defined by

$$[v,w] := [\![\xi,\eta]\!](x) \quad \text{where } \xi,\eta \in \mathrm{Sec}\,A,\ \xi(x)=v,\ \eta(x)=w, \quad v,w \in \mathfrak{g}(A)_{|x}.$$

$\mathfrak{g}(A)_{|x}$ is called the isotropy Lie algebra of (10) at x.

THEOREM 1.13. (see [14,p.189] and [18,p.50]). For any Lie algebroid (10) on a connected manifold M, the vector bundle $\mathfrak{g}(A)$ is a Lie algebra bundle.

PROOF. Let $[\cdot,\cdot]$ denote here the cross-section of $\mathfrak{g}(A)^{2,1}$ such that $[\cdot,\cdot]_{|x}$ is the Lie algebra structure of $\mathfrak{g}(A)_{|x}$. We must prove that

$$(\mathfrak{g}(A), \{[\cdot,\cdot]\}) \tag{12}$$

is the so-called $\Sigma$-bundle (see [5,p.373]).

Let $\lambda: TM \longrightarrow A$ be any splitting of the Atiyah sequence (11), ie $\gamma\circ\lambda = \mathrm{id}_{TM}$ holds:

$$0 \longrightarrow \mathfrak{g}(A) \hookrightarrow A \underset{\underset{\lambda}{\leftarrow - -}}{\xrightarrow{\gamma}} TM \longrightarrow 0 \tag{13}$$

It is easy to see that the formula

$$\nabla_X \sigma = [\![\lambda X, \sigma]\!], \quad \sigma \in \mathrm{Sec}\,\mathfrak{g}(A), \quad X \in \mathfrak{X}(M),$$

defines some covariant derivative in the vector bundle $\mathfrak{g}(A)$. From the Jacobi identity in $\mathrm{Sec}\,A$ we trivially assert that

$$\nabla_X([\sigma,\eta]) = [\nabla_X\sigma,\eta] + [\sigma, \nabla_X\eta], \quad \text{ie } \nabla([\cdot,\cdot]) = 0.$$

This implies that $\nabla$ is a $\Sigma$-connection in (12), see [5,p.373]. By Theorem II ibidem, the assertion is proved. □

DEFINITION 1.14. ([9,p.273],[14,p.101]). Let $(A, [\![\cdot,\cdot]\!], \gamma)$ and $(A', [\![\cdot,\cdot]\!]', \gamma')$ be two Lie algebroids on the same manifold M. By a homomorphism between them we mean a strong homomorphism

$$H: A \longrightarrow A'$$

of vector bundles, such that
(a) $\gamma'\circ H = \gamma$,
(b) $\mathrm{Sec}\,H: \mathrm{Sec}\,A \longrightarrow \mathrm{Sec}\,A'$ is a homomorphism of Lie algebras.

H determines some homomorphism of the associated Atiyah sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}(A) & \hookrightarrow & A & \xrightarrow{\gamma} & TM & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle H^0} & & \downarrow{\scriptstyle H} & & \| & & \\
0 & \longrightarrow & \mathfrak{g}(A') & \hookrightarrow & A' & \xrightarrow{\gamma'} & TM & \longrightarrow & 0
\end{array}$$

where $H^0 = H|\mathfrak{g}(A)$.

If $H$ is a bijection, then $H^{-1}$ is also a homomorphism of Lie algebroids; then $H$ is called an <u>isomorphism of Lie algebroids</u>.

Each Lie algebroid isomorphic to $TM \times \mathfrak{g}$ (defined in Example 1.9) is called <u>trivial</u>.

<u>REMARK 1.15</u>. A pfb $P$ with a discrete structural Lie group has a trivial Lie algebroid, more exactly, $A(P) \cong TM$. □

<u>REMARK 1.16</u>. (cf [14, p.101]). Let (10) be any Lie algebroid on $M$ and let $U$ be an open submanifold of $M$. Take the restricted vector bundle $A_{|U}$ and $\gamma_{|U} = \gamma|(A_{|U}): A_{|U} \longrightarrow TU$. In the space $\mathrm{Sec}(A_{|U})$ there exists exactly one Lie algebra structure $[\![\cdot,\cdot]\!]_U$ such that $[\![\xi|U, \eta|U]\!]_U = [\![\xi,\eta]\!]|U$, $\xi, \eta \in \mathrm{Sec}\, A$, and the system

$$(A_{|U}, [\![\cdot,\cdot]\!]_U, \gamma_{|U})$$

is a Lie algebroid called <u>restricted</u> to $U$.

Let $\lambda: U \longrightarrow P$ be any cross-section of $P$, then

$$(\varphi_\lambda)^A : TU \times \mathfrak{g} \longrightarrow A(P)_{|U},$$

where $\varphi_\lambda : U \times G \longrightarrow P_{|U}$, $(x,a) \longmapsto \lambda(x)\cdot a$, is an isomorphism of Lie algebroids; therefore $A(P)_{|U}$ is trivial.

Besides, if $H: A \longrightarrow A'$ is any homomorphism of Lie algebroids, then

$$H_{|U} : A_{|U} \longrightarrow A'_{|U}$$

is such a homomorphism, too. □

— o — • — O — o — o —

Each (strong) <u>homomorphism</u> [ <u>isomorphism</u> ]

$$(F,\mu): P(M,G) \longrightarrow P'(M,G')$$

of pfb's / $F: P \longrightarrow P'$, $\mu: G \longrightarrow G'$ such that $\pi' \circ F = \pi$, $\mu$ is a homomorphism

[ isomorphism ] of Lie groups, and $F(za) = F(z)\cdot\mu(a)$ / determines a mapping (see [14,p.289])

$$dF : A(P) \longrightarrow A(P'), \quad [v] \longmapsto [F_*(v)].$$

<u>PROPOSITION 1.17</u>.([14,p.289]). dF is a homomorphism [ isomorphism ] of Lie algebroids. □

The covariant functor

$$P(M,G) \longmapsto A(P), \quad (F,\mu) \longmapsto dF$$

defined above is called the <u>Lie functor for pfb's</u>.

As we have said in the Introduction, the Lie algebroid of a pfb P can also be defined as the Lie algebroid $A(PP^{-1})$ of the Ehresmann Lie groupoid $PP^{-1}$, via the construction of J.Pradines (see [3],[23]).

We recall these constructions.

(a) Let $\Phi$ be any Lie groupoid [20]. We define

$$A(\Phi) = u^* T^\alpha \Phi$$

where $T^\alpha\Phi = \operatorname{Ker}\alpha_*$ ($\alpha : \Phi \longrightarrow M$ - the source, $u : M \longrightarrow \Phi$, $x \longmapsto u_x$, $u_x$ - the unit over x). The right-invariant vector fields on $\Phi$ correspond 1-1 to the cross-sections of $A(\Phi)$. The bracket $[\![\xi,\eta]\!]$ of $\xi,\eta \in \operatorname{Sec} A(\Phi)$ is defined in such a way that the right-invariant vector field corresponding to $[\![\xi,\eta]\!]$ equals the Lie bracket of the corresponding right-invariant vector fields. The mapping $\tilde\gamma : A(\Phi) \longrightarrow TM$ is defined by $\tilde\gamma(v) = \beta_*(v)$ ($\beta$ - the target). The system obtained

$$(A(\Phi), [\![\cdot,\cdot]\!], \tilde\gamma)$$

is a Lie algebroid (for details see for example [9],[14]).

(b) The Ehresmann Lie groupoid $PP^{-1}$ is defined as follows:

Its space equals the space of orbits of the action

$$(P\times P)\times G \longrightarrow P\times P, \quad ((z_1,z_2),a) \longmapsto (z_1a, z_2a),$$

the source and the target are defined by:

$$\alpha([z_1,z_2]) = \pi z_1, \quad \beta([z_1,z_2]) = \pi z_2$$

($[z_1,z_2]$ being the orbit through $(z_1,z_2)$), the partial multiplication by:

$$[z_2,z_3]\cdot[z_1,z_2] = [z_1,z_3].$$

<u>THEOREM 1.18</u>. (cf [12,p.63] and [14,p.119]). $A(P) \cong A(PP^{-1})$.

<u>PROOF</u>. For an arbitrary point $x\in M$, we define an isomorphism

$$\rho_x : A(P)_{|x} \longrightarrow A(PP^{-1})_{|x}, \quad [v] \longmapsto \omega_{z*z}(v), \quad v\in T_zP, \quad z\in P_{|x},$$

where

$$\omega_z : P \longrightarrow (PP^{-1})_x, \quad z' \longmapsto [z,z'].$$

The definition of $\rho_x$ is correct which follows from the commutativity of the diagram

$$\begin{tikzcd}
 & T_{za}P \arrow[dl, "(\omega_{za})_{*za}"'] \arrow[dr, "\pi^A_{|za}"] & \\
A(PP^{-1})_{|x} & & A(P)_{|x} \\
 & T_zP \arrow[ul, "\omega_{z*z}"] \arrow[uu, "(R_a)_{*z}"] \arrow[ur, "\pi^A_{|z}"'] &
\end{tikzcd}$$

Now, we establish the smoothness of the mapping

$$\rho : A(P) \longrightarrow A(PP^{-1})$$

defined by $\rho | A(P)_{|x} = \rho_x$. What we need to prove is the smoothness of

$$\rho\circ\pi^A : TP \longrightarrow A(PP^{-1}) \hookrightarrow T((P\times P)/G),$$

but $\rho\circ\pi^A = r_*\circ c$ where $r : P\times P \longrightarrow (P\times P)/G$ is the canonical projection and

$c: TP \longrightarrow T(P\times P)$, $v \longmapsto (0_z, v)$ if $v \in T_zP$, and $r_*$ and $c$ are, of course, smooth.

It remains to show that $\rho$ is an isomorphism of Lie algebroids. The equality $\tilde{\gamma}\circ\rho = \gamma$ is easy to see. The fact that $\mathrm{Sec}\,\rho$ is a homomorphism of Lie algebras is the last thing to consider. Take any $X \in \mathfrak{X}^R(P)$. $X$ is $\omega_z$-related to the right-invariant vector field $(\rho\circ X_0)'$ on $PP^{-1}$. Indeed, for the right translation by $[z,z']$

$$D_{[z,z']}: (PP^{-1})_{\pi z'} \longrightarrow (PP^{-1})_{\pi z}, \qquad [z',z''] \longmapsto [z,z''],$$

we have

$$\omega_z = D_{[z,z']}\circ\omega_{z'}.$$

Thus, for $x' := \pi z'$, we have

$$(\omega_z)_{*z'}(X_{z'}) = (D_{[z,z']}\circ\omega_{z'})_{*z'}(X_{z'}) = (D_{[z,z']})_{*[z,z']}(\rho_{x'}[X_{z'}])$$
$$= (D_{[z,z']})_*(\rho\circ X_0(x')) = (\rho\circ X_0)'(\omega_z(z')) .$$

Although $\omega_z: P \longrightarrow PP^{-1}$ is not a surjective mapping, each right-invariant vector field on $P$ is $\omega_z$-related to exactly one right-invariant vector field on $PP^{-1}$. By this remark and the fact that, for $\xi_1, \xi_2 \in \mathrm{Sec}\,A(P)$, the vector field $[\xi_1', \xi_2']$ $(= [\![\xi_1, \xi_2]\!]')$ is $\omega_z$-related to $[\![\rho\circ\xi_1, \rho\circ\xi_2]\!]'$ and to $(\rho\circ[\![\xi_1,\xi_2]\!])'$ simultaneously, we obtain the equality $\rho\circ[\![\xi_1,\xi_2]\!] = [\![\rho\circ\xi_1, \rho\circ\xi_2]\!]$. $\square$

CHAPTER 2

## $A(P) \cong W^1(P)\times_{G_n^1}(\mathbb{R}^n\times\mathfrak{g})$

Now, we give the third manner of a natural construction of the Lie algebroid for a pfb $P(M,G)$, in the form of the associated vector bundle

(14) $$\tilde{A}(P):=W^1(P)\times_{G_n^1}(\mathbb{R}^n\times\mathfrak{g})$$

with some suitable structures.

We recall [4], [7] that $W^1(P)$ is the smooth fibre bundle of all 1-jets with source $(0,e)$ of the so-called allowable charts on $P(M,G)$, ie of pfb isomorphisms

(15) $$\psi : V\times G \longrightarrow P_{|U}$$

of a trivial pfb $V\times G$ onto $P_{|U}$, where $V$ is open in $\mathbb{R}^n$ and such that $0\in V$ and $U$ is open in $M$, $n=\dim M$.

$W^1(P)$ is a pfb over $M$ with structural Lie group

$$G_n^1 := W_0^1(\mathbb{R}^n\times G) \quad (= \text{the fiber over } 0),$$

provided that both the multiplication in $G_n^1$ and the right action of $G_n^1$ on $W^1(P)$ are defined by means of the composition of jets, ie if

$$u=j^1_{(0,e)}\psi\in W^1(P) \quad \text{and} \quad h=j^1_{(0,e)}Z\in G_n^1, \quad \text{then} \quad uh=j^1_{(0,e)}(\psi\circ Z)\in W^1(P).$$

Each allowable chart (15) is uniquely determined by a couple $(\varkappa,\lambda)$ of a chart $\varkappa :U \xrightarrow{\approx} V\subset\mathbb{R}^n$ $(0\in V)$ on $M$ and a cross-section $\lambda:U \longrightarrow P_{|U}$ such that

$$\psi(x,a)=\lambda(\varkappa^{-1}(x))\cdot a, \quad x\in V, \quad a\in G.$$

From the identification

(16) $$\psi = (\varkappa, \lambda)$$

we deduce that any element $j^1_{(0,e)}\psi \in W^1(P)$ can be identified with a couple $(j^1_0(\varkappa^{-1}), j^1_x\lambda)$, $x := \varkappa^{-1}(0)$, thus with a couple of linear mappings

$$(\hat{\varkappa}_{|x}, \lambda_{*x}) \in \mathrm{Iso}(\mathbb{R}^n; T_xM) \times \widetilde{\mathrm{Hom}}(T_xM; T_{\lambda(x)}P),$$

where $\hat{\varkappa}_{|x} : \mathbb{R}^n \longrightarrow T_xM$, $t \longmapsto \sum t^i \frac{\partial}{\partial x^i}_{|x}$, and, for arbitrary $x \in M$ and $z \in P_{|x}$, by $\widetilde{\mathrm{Hom}}(T_xM; T_zP)$ we mean the set of all linear homomorphisms

$$\lambda_z : T_xM \longrightarrow T_zP \quad \text{such that} \quad \pi_{*z} \circ \lambda_z = \mathrm{id}_{T_xM}.$$

Therefore, we can identify

(17) $$W^1(P)_{|x} = \bigcup_{z \in P_{|x}} \mathrm{Iso}(\mathbb{R}^n; T_xM) \times \widetilde{\mathrm{Hom}}(T_xM; T_zP).$$

According to [8], the group $G^1_n$ can be naturally written as

$$G^1_n = GL(n, \mathbb{R}) \times G \times \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}),$$

and the explicit formula for the multiplication in $G^1_n$ is then of the form

$$(X_1, a_1, \sigma_1) \cdot (X_2, a_2, \sigma_2) = (X_1 \cdot X_2, a_1 \cdot a_2, \mathrm{Ad}(a_2^{-1}) \circ \sigma_1 \circ X_2 + \sigma_2),$$

$X_i \in GL(n, \mathbb{R})$, $a_i \in G$, $\sigma_i \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$.

The action

$$W^1(P) \times G^1_n \longrightarrow W^1(P)$$

can be written as follows:

$$\text{for} \quad (\varkappa_x, \lambda_z) \in \mathrm{Iso}(\mathbb{R}^n; T_xM) \times \widetilde{\mathrm{Hom}}(T_xM; T_zP)$$

$$\text{and} \quad (X, a, \sigma) \in GL(n, \mathbb{R}) \times G \times \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$$

(18) $\quad (\varkappa_x,\lambda_z)\cdot(X,a,\sigma) = (\varkappa_x\circ X,(R_a)_{*z}\circ\lambda_z + (A_{za})_{*e}\circ\sigma\circ X^{-1}\circ\varkappa_x^{-1})$

$$\in \mathrm{Iso}(\mathbb{R}^n;T_xM)\times\tilde{\mathrm{H}}\mathrm{om}(T_xM;T_{za}P).$$

Via identification (17), any allowable chart (16) determines a local cross-section of $W^1(P)$:

$$\psi^W: U \longrightarrow W^1(P), \quad x \longmapsto (\hat{\varkappa}_{|x},\lambda_{*x}).$$

Let $\psi_i = (\varkappa_i,\lambda_i)$, $i=1,2$, be two allowable charts on P, $\varkappa_i$ being with domain $U_i$. Let

$$g: U_1\cap U_2 \longrightarrow G$$

denote the transition function for $\lambda_1$ and $\lambda_2$, ie $\lambda_2(x)=\lambda_1(x)\cdot g(x)$.

The transition function for $\psi_1^W$ and $\psi_2^W$ is equal to

$$\begin{aligned} g^W: U_1\cap U_2 &\longrightarrow \mathrm{Gl}(n,\mathbb{R})\times G\times \mathrm{Hom}(\mathbb{R}^n,\mathfrak{q}) \\ x &\longmapsto (\hat{\varkappa}_{1|x}^{-1}\circ\hat{\varkappa}_{2|x}, g(x), (1_{g^{-1}(x)})_*\circ g_{*x}\circ\hat{\varkappa}_{2|x}). \end{aligned}$$

Really, by (18) for $z:=\lambda_1(x)$ we get

$$\begin{aligned} \psi_1^W(x)\cdot g^W(x) &= (\hat{\varkappa}_{1|x},\lambda_{1*x})\cdot(\hat{\varkappa}_{1|x}^{-1}\circ\hat{\varkappa}_{2|x}, g(x),(1_{g^{-1}(x)})_* g_{*x}\circ\hat{\varkappa}_{2|x}) \\ &= (\hat{\varkappa}_{2|x},(R_{g(x)})_{*z}\circ\lambda_{1*x} + (A_{zg(x)})_{*e}(1_{g^{-1}(x)})_*\circ g_{*x}) \\ &= (\hat{\varkappa}_{2|x},(R_{g(x)})_{*z}\circ\lambda_{1*x} + (A_z)_{*g(x)}\circ g_{*x}) \\ &= (\hat{\varkappa}_{2|x},\lambda_{2*x}) \\ &= \psi_2^W(x). \end{aligned}$$

Now, we see that we can define the pfb $W^1(P)$, independently of the above, as the $G_n^1$-pfb for which $g^W$ are transition functions.

To finish with, what we need to notice is that (2) is a $G_n^1$-vector bundle via some linear action $G_n^1$ on $\mathbb{R}^n\times\mathfrak{q}$. By Prop.1.3, we see that any allowable chart (16), $\varkappa$ being with a domain U, determines a local trivialization of (2) by

$$\hat{\psi} = (\varphi_\lambda)^A \circ (\hat{\varkappa} \times \mathrm{id}) : U \times \mathbb{R}^n \times \mathfrak{g} \longrightarrow A_{|U}$$

where $\varphi_\lambda : U \times G \longrightarrow P$, $(x,a) \longmapsto \lambda(x)\cdot a$, and $\hat{\varkappa} : U \times \mathbb{R}^n \longrightarrow TU$, $(x,t) \longmapsto \hat{\varkappa}_{|x}(t)$. According to remark 1.2, for two allowable charts $\psi_i = (\varkappa_i, \lambda_i)$, $i=1,2$, ($\varkappa_i$ with a domain $U_i$), we have

$$\hat{\psi}_1^{-1} \circ \hat{\psi}_2 : U_1 \cap U_2 \times \mathbb{R}^n \times \mathfrak{g} \longrightarrow U_1 \cap U_2 \times \mathbb{R}^n \times \mathfrak{g},$$

$$(x,t,w) \longmapsto (x, \hat{\varkappa}_{1|x}^{-1} \circ \hat{\varkappa}_{2|x}(t), \Theta^R(g_*(\hat{\varkappa}_{2|x}(t))) + \mathrm{Ad}(g(x))(w)).$$

Take

$$T : (GL(n,\mathbb{R}) \times G \times \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})) \times (\mathbb{R}^n \times \mathfrak{g}) \longrightarrow \mathbb{R}^n \times \mathfrak{g}$$

$$((X,a,\sigma),(t,w)) \longmapsto (X(t), \mathrm{Ad}(a)(w + \sigma(t))).$$

It is easy to see that $T$ is a left smooth action. It remains to notice that

$$\begin{aligned} T(g^W(x),(t,w)) &= T(\hat{\varkappa}_{1|x}^{-1} \circ \hat{\varkappa}_{2|x}, g(x), (l_{g^{-1}(x)})_* \circ g_{*x} \circ \hat{\varkappa}_{2|x}, (t,w)) \\ &= (\hat{\varkappa}_{1|x}^{-1} \circ \hat{\varkappa}_{2|x}(t), \mathrm{Ad}(g(x))(w + (l_{g^{-1}(x)})_* \circ g_{*x} \circ \hat{\varkappa}_{2|x}(t))) \\ &= (\hat{\varkappa}_{1|x}^{-1} \circ \hat{\varkappa}_{2|x}(t), \mathrm{Ad}(g(x))(w) + \Theta^R(g_{*x} \circ \hat{\varkappa}_{2|x}(t))) \\ &= (\hat{\psi}_1^{-1} \circ \hat{\psi}_2)_{|x}(t,w). \end{aligned}$$

From the general theory we obtain an isomorphism correctly defined by the local formula

$$W^1(P) \times_{G_n^1} (\mathbb{R}^n \times \mathfrak{g}) \longrightarrow A(P)$$

$$[(\hat{\varkappa}_{|x}, \lambda_{*x}),(t,w)] \longmapsto [(\varphi_\lambda)_*(\hat{\varkappa}_{|x}(t))]$$

(ie the independence of the choice of an allowable chart (16) holds). One can easily show that it is globally defined by (see (14))

$$(19)\qquad \begin{aligned} H : \tilde{A}(P) &\longrightarrow A(P) \\ [(\varkappa_x, \lambda_z),(t,w)] &\longmapsto [\lambda_z \circ \varkappa_x(t) + (A_z)_*(w)]. \end{aligned}$$

Via (19) we introduce on $\tilde{A}(P)$ some structure of a Lie algebroid

$$(\tilde{A}(P), [\![\cdot,\cdot]\!]^{\sim}, \tilde{\gamma}).$$

Now, we describe this structure without the help of (19):

(a)
$$\begin{aligned} \tilde{\gamma}([(\varkappa_x, \lambda_z),(t,w)]) &= \gamma \circ H([(\varkappa_x, \lambda_z),(t,w)]) \\ &= \gamma([\lambda_z \circ \varkappa_x(t) + (A_z)_*(w)]) \\ &= \varkappa_x(t). \end{aligned}$$

(b) Each allowable chart (16) ($\varkappa$ with a domain U) defines some linear isomorphism of vector bundles

$$\tilde{\psi} : TU \times \mathfrak{g} \longrightarrow \tilde{A}(P)_{|U}, \quad (v,w) \longmapsto [(\hat{\varkappa}_{|x}, \lambda_{*x}),(\hat{\varkappa}_{|x}^{-1}(v), w)], \qquad v \in T_xU,$$

and we have the commuting diagram

$$\begin{tikzcd} TU \times \mathfrak{g} \arrow[rr, "\tilde{\psi}"] \arrow[dr, "(\varphi_\lambda)^A"'] & & \tilde{A}(P)_{|U} \arrow[dl, "H_{|U}"] \\ & A(P)_{|U} & \end{tikzcd}$$

in which $(\varphi_\lambda)^A$ and $H_{|U}$ are isomorphisms of Lie algebroids (see remark 1.16). So $\tilde{\psi}$ must also be an isomorphism of Lie algebroids. Each cross-section $\tilde{\sigma}$ of $A(P)_{|U}$ is of the form

$$(20)\qquad \tilde{\sigma} = [\psi^W, (\tilde{t}, \sigma)]$$

for some (uniquely determined) mappings $\tilde{t} : U \longrightarrow \mathbb{R}^n$ and $\sigma : U \longrightarrow \mathfrak{g}$. (20) determines a vector field X on U by the formula

$$X(x) = \hat{\varkappa}_{|x}(\tilde{t}(x)) \quad \left(= \sum t^i(x) \cdot \frac{\partial}{\partial \varkappa^i}_{|x}\right).$$

Thereby,

$$\tilde{\psi}\circ(X,\sigma)=\tilde{\sigma}.$$

Let $\tilde{\sigma}_i=[\psi^W,(\tilde{t}_i,\sigma_i)]$, $i=1,2$, be two cross-sections of $\tilde{A}(P)_{|U}$ and let $X_i$ be the vector field on U determined by $\tilde{\sigma}_i$. Then we calculate

$$\begin{aligned}
[\![\tilde{\sigma}_1,\tilde{\sigma}_2]\!] &= \psi\,([\![(X_1,\sigma_1),(X_2,\sigma_2)]\!])\\
&= \tilde{\psi}\,([X_1,X_2],\mathcal{L}_{X_1}\sigma_2-\mathcal{L}_{X_2}\sigma_1+[\sigma_1,\sigma_2])\\
&= [\psi^W,(\hat{x}^{-1}\circ[X_1,X_2],\mathcal{L}_{X_1}\sigma_2-\mathcal{L}_{X_2}\sigma_1+[\sigma_1,\sigma_2])].
\end{aligned}$$

## CHAPTER 3

## THE NOTION OF A LOCAL HOMOMORPHISM BETWEEN PFB'S

In the theory of Lie groups the following theorems hold:

THEOREM A. If $G_1$ and $G_2$ are two Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, then, for each homomorphism

$$h:\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$$

of Lie algebras, there exists a local homomorphism

$$H:G_1 \supset \Omega \longrightarrow G_2$$

($\Omega$ is open in $G_1$ and contains the unit of $G_1$) of Lie groups such that

$$dH = h. \quad \square$$

THEOREM B. Two Lie groups $G_1$ and $G_2$ are locally isomorphic iff $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic. $\square$

What does this look like for pfb's ?

First of all, we know [10], [21] that the theorems similar to the above ones hold for Lie groupoids and algebroids, as well. Thus, we have only to discover how to define a suitable notion of a local homomorphism between pfb's in order that it correspond to the notion of a local homomorphism between Lie groupoids.

Here is an answer to this problem.

DEFINITION 3.1. By a local homomorphism from a pfb $P(M,G)$ into a second one $P'(M,G')$ we shall mean a family

$$\mathcal{F} = \{(F_t, \mu_t);\ t \in T\}$$

such that

$$F_t : P \supset D_t \longrightarrow P', \quad D_t \text{ open in } P,$$
$$\mu_t : G \supset U_t \longrightarrow G', \quad U_t \text{ open in } G,\ e \in U_t;$$

provided the following properties hold:

(1) $\mu_t$ is a local homomorphism of Lie groups,

(2) $\bigcup_t \pi[D_t] = M$,

(3) $\pi' \circ F_t = \pi | D_t$ ($\pi$ and $\pi'$ denote the projections),

(4) $F_t(z \cdot a) = F_t(z) \cdot \mu_t(a)$ for $z \in D_t$ and $a \in U_t$ such that $z \cdot a \in D_t$,

(5) If $t, t' \in T$, $z \in D_t$, $a \in G$, $z \cdot a \in D_{t'}$, $a' \in G'$, $z' \in P'$ and

$$F_t(z) = z', \quad F_{t'}(z \cdot a) = z' \cdot a', \text{ then}$$

(a) $F_{t'} = R_{a'} \circ F_t \circ R_{a^{-1}}$ in some nbh of $z \cdot a$,

(b) $\mu_{t'} = \tau_{a'^{-1}} \circ \mu_t \circ \tau_a$ in some nbh of $e \in G$ ($\tau_a(x) = a \cdot x \cdot a^{-1}$, $x \in G$)

If $F_t$ and $\mu_t$ are diffeomorphisms, then

$$\mathcal{F}^{-1} := \{(F_t^{-1}, \mu_t^{-1});\ t \in T\}$$

is a local homomorphism, and $\mathcal{F}$ is then called a <u>local isomorphism</u>.

<u>PROPOSITION 3.2</u>. Let

$$\mathcal{F} = \{(F_t, \mu_t);\ t \in T\} : P(M,G) \longrightarrow P'(M,G')$$

be a local homomorphism between pfb's. Then

$$d\mathcal{F} : A(P) \longrightarrow A(P'), \quad [v] \longmapsto [F_{t*}(v)], \quad v \in T_z P,\ z \in D_t,\ t \in T,$$

is a correctly defined homomorphism of Lie algebroids.

<u>PROOF</u>. We start with proving the correctness of the definition of the linear mapping

$$(d\mathcal{F})_{|x} : A(P)_{|x} \longrightarrow A(P')_{|x}, \quad [v] \longmapsto [F_{t*}(v)],$$

ie its independence of the choice of $z$ and $t$. Let $t' \in T$ and $a \in G$ be arbitrary elements such that $z \cdot a \in D_{t'}$. The independence follows easily from the commutativity of the diagram

$$
\begin{tikzcd}
 & T_zP \arrow[dl, "\pi^A_{|z}"'] \arrow[r, "(F_t)_{*z}"] \arrow[d, "(R_a)_{*z}"'] & T_{z'}P' \arrow[d, "(R_{a'})_{*z'}"] \arrow[dr, "\pi'^A_{|z'}"] & \\
A(P)_{|x} \arrow[rrr, bend right=50, "(d\mathcal{F})_{|x}"'] & T_{za}P \arrow[l, "\pi^A_{|za}"] \arrow[r, "(F_{t'})_{*za}"] & T_{z'a'}P' \arrow[r, "\pi'^A_{|z'a'}"'] & A(P')_{|x}
\end{tikzcd}
$$

where $z' = F_t(z)$, $F_{t'}(za) = z'a'$.

Now, we prove the sought-for properties of $d\mathcal{F}$.

(a) $d\mathcal{F}$ is a $C^\infty$-homomorphism of vector bundles. Indeed, for a point $x\in M$, take an arbitrary $t\in T$ such that $x\in \pi[D_t]$. The smoothness of $d\mathcal{F}$ in some nbh of $x$ follows from the commutativity of the diagram

$$
\begin{tikzcd}
TP \arrow[d, "\pi^A"'] & \supset & \pi_P^{-1}[D_t] \arrow[d, "\pi^A"] \arrow[r, "(F_t)_*"] & TP' \arrow[d, "\pi'^A"] \\
A(P) & \supset & p^{-1}[\pi[D_t]] \arrow[r, "d\mathcal{F}"] & A(P')
\end{tikzcd}
$$

where $\pi_P : TP \longrightarrow M$ is the projection.

(b) $\gamma' \circ d\mathcal{F} = \gamma$ is evident,

(c) $\mathrm{Sec}(d\mathcal{F}) : \mathrm{Sec}\,A(P) \longrightarrow \mathrm{Sec}\,A(P')$ is a homomorphism of Lie algebras. Indeed, for $X\in \mathfrak{X}^R(P)$, the cross-section $d\mathcal{F}\circ X_o$ of $A(P')$ induces the right-invariant vector field $Y := (d\mathcal{F}\circ X_o)'$ on $P'$. It turns out that, for an arbitrary index $t\in T$, the field $X|D_t$ is $F_t$-related to $Y$:

$$
\begin{aligned}
(F_t)_{*z}(X_z) &= (\pi'^A_{|F_t(z)})^{-1}\,(d\mathcal{F})_{|\pi z}(\pi^A(X_z)) \\
&= (\pi'^A_{|F_t(z)})^{-1}(d\mathcal{F}\circ X_o(\pi z)) \\
&= (d\mathcal{F}\circ X_o)'(F_t(z)) \\
&= Y(F_t(z)).
\end{aligned}
$$

The above remark yields (by a standard calculation) that

$$
(d\mathcal{F}\circ [\![\xi_1, \xi_2]\!])\,|\pi[D_t] = [\![d\mathcal{F}\circ \xi_1, d\mathcal{F}\circ \xi_2]\!]\,|\pi[D_t].
$$

The free choice of $t\in T$ ends the proof. $\square$

REMARK 3.3. (1) It is easily seen that $d\mathcal{F}$ is an isomorphism if $\mathcal{F}$ is a local isomorphism. (2) We have

$$d\mathcal{F}[\mathfrak{q}(P)] \subset \mathfrak{q}(P')$$

and we get the commuting diagram

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\hat{z}} & \mathfrak{q}(P)_{|x} \\ {\scriptstyle (\mu_t)_{*e}}\downarrow & & \downarrow{\scriptstyle (d\mathcal{F})_x|\mathfrak{q}(P)_x} \\ \mathfrak{L}' & \xrightarrow{F_t(z)^{\wedge}} & \mathfrak{q}(P')_{|x} \end{array}$$

for $t \in T$, $z \in D_t$ (see (9)).

THEOREM 3.4. Let

$$h: A(P) \longrightarrow A(P')$$

be any homomorphism of Lie algebroids. Then there exists a local homomorphism $\mathcal{F}: P(M,G) \to P'(M,G')$ such that $d\mathcal{F} = h$.

PROOF. Take the Ehresmann Lie groupoids

$$\Phi := PP^{-1} \quad \text{and} \quad \Phi' := P'P'^{-1}$$

corresponding to the pfb's $P(M,G)$ and $P'(M,G')$, respectively. Let

$$\tilde{h}: A(\Phi) \longrightarrow A(\Phi')$$

be the homomorphism of Lie algebroids for which the diagram

$$\begin{array}{ccc} A(P) & \xrightarrow{h} & A(P') \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho'} \\ A(\Phi) & \xrightarrow{\tilde{h}} & A(\Phi') \end{array}$$

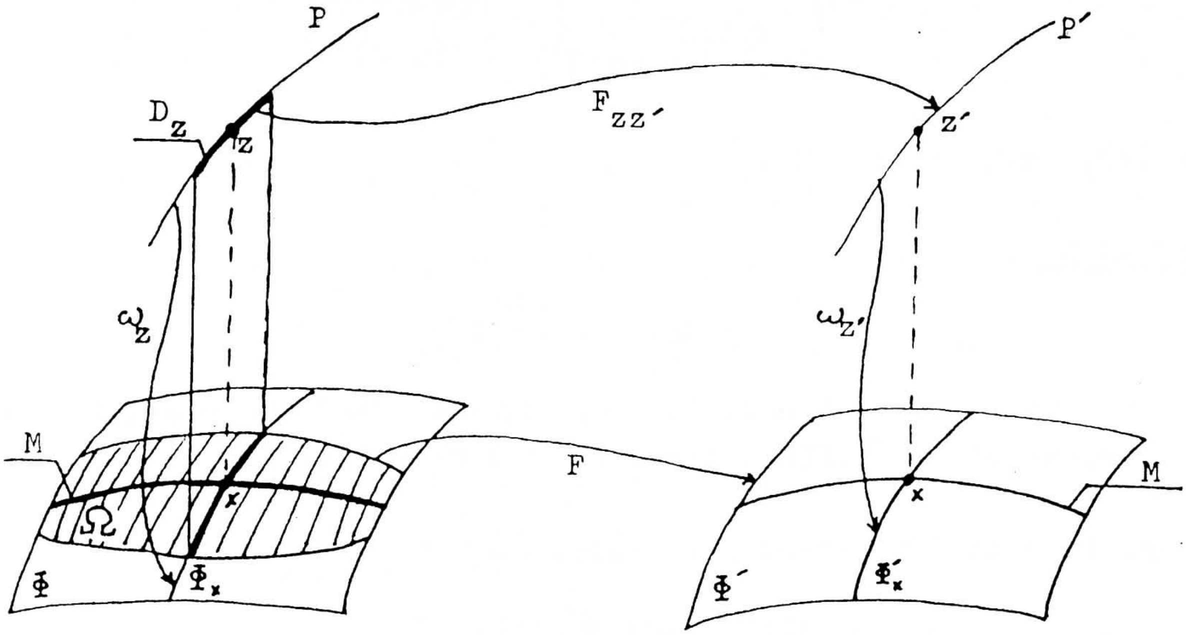
commutes, where $\rho$ and $\rho'$ are natural isomorphisms described in the proof of theorem 1.18. By theorem A, for Lie groupoids, there exists some local homomorphism

(21) $$F: \Phi \supset \Omega \longrightarrow \Phi',$$

$\Omega$ being open in $\Phi$ and covering all units, of Lie groupoids such that $dF = \tilde{h}$. Now, we are able to construct some local homomorphism of pfb's. It will be the family

$$\mathcal{F} := \{(F_{zz'}, \mu_{zz'}); (z,z') \in P \oplus P'\}$$

($P \oplus P' = \{(z,z') \in P \times P'; \pi z = \pi' z'\}$) where $F_{zz'} = \omega_{z'}^{-1} \circ F \circ \omega_z | D_z$, $D_z = \omega_z^{-1}[\Omega \cap \Phi_{\pi z}]$ and $\mu_{zz'} = \mu_{z'}^{-1} \circ F \circ \mu_z | U_z$, $U_z = \mu_z^{-1}[\Omega \cap G_{\pi z}]$, and $\omega_z : P \to \Phi$, $z' \mapsto [z,z']$, $G_{\pi z}$ is the isotropy Lie group at x, $\mu_z : G \to G_{\pi z}$, $a \mapsto [z, za]$, ($\omega_{z'}$, $\mu_{z'}$ are defined in a similar manner), see the figure:




We have to prove that $\mathcal{F}$ is a local homomorphism and $d\mathcal{F} = h$. Properties (1) and (2) of a local homomorphism (see definition 3.1) are evident.

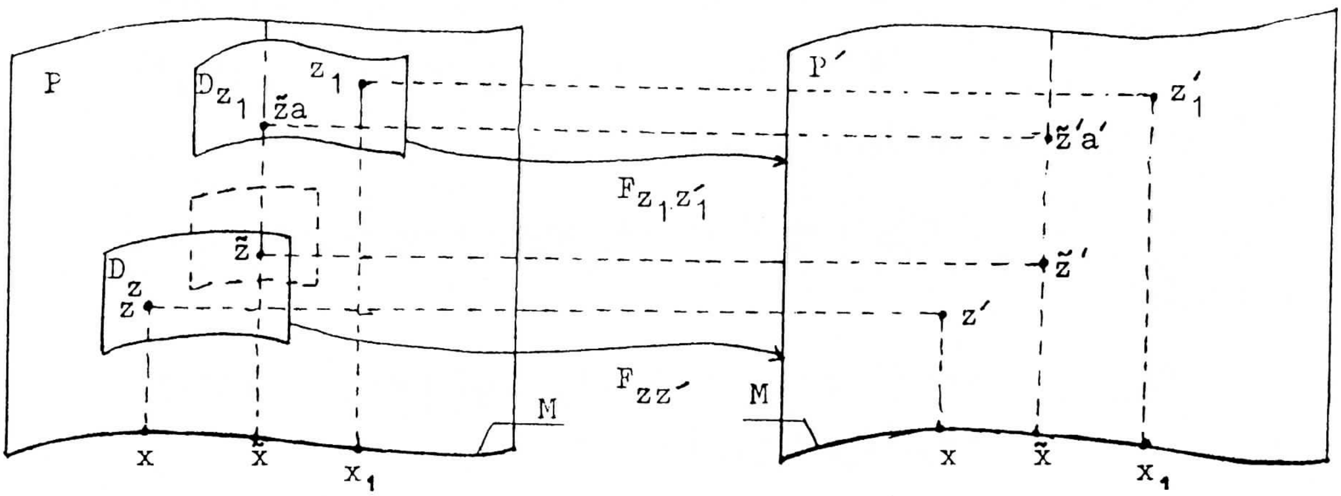
(3): $\pi' \circ F_{zz'}(\tilde{z}) = \pi'(\omega_{z'}^{-1}(F(\omega_z(\tilde{z})))) = \beta'(F([z,\tilde{z}]))$

$$= \beta[z,\tilde{z}] = \pi \tilde{z}.$$

(4): Take $\tilde{z} \in D_z$, $a \in U_z$ such that $\tilde{z} \cdot a \in D_z$. For $z' \in P'$, we have

$$\begin{aligned} F_{zz'}(\tilde{z} \cdot a) &= \omega_{z'}^{-1}(F(\omega_z(\tilde{z} \cdot a))) = \omega_{z'}^{-1}(F([z, \tilde{z} \cdot a])) \\ &= \omega_{z'}^{-1}(F([za, \tilde{z}a] \cdot [z, za])) \\ &= \omega_{z'}^{-1}(F([z,\tilde{z}]) \cdot \mu_{z'}(\mu_{zz'}(a))) \\ &= F_{zz'}(\tilde{z}) \cdot \mu_{zz'}(a). \end{aligned}$$

To prove (5), take $(z,z')$, $(z_1,z_1')\in P\oplus P'$, $a\in G$ and $\tilde z\in D_z$ such that $\tilde z a\in D_{z_1}$. Let $\tilde z' = F_{zz'}(\tilde z)$ and $F_{z_1,z_1'}(\tilde z\cdot a)=\tilde z'\cdot a'$, see the figure:




First of all, we prove that

(i) $F_{zz'}=F_{\tilde z\tilde z'}$ in some nbh of $\tilde z$,

(ii) $\mu_{zz'}=\mu_{\tilde z\tilde z'}$ in some nbh of the unit of G.

We see that, for $\check z\in D_z\cap D_{\tilde z}$,

$$\begin{aligned}F_{zz'}(\check z) &= \omega_{z'}^{-1}(F(\omega_z(\check z)))=\omega_{z'}^{-1}(F([z,\check z]))\\ &=\omega_{z'}^{-1}(F([\tilde z,\check z]\cdot[z,\tilde z]))=\omega_{z'}^{-1}(F([\tilde z,\check z])\cdot F(\omega_z(\tilde z)))\\ &=\omega_{z'}^{-1}(F([\tilde z,\check z])\cdot[z',\tilde z'])=\omega_{z'}^{-1}(D_{[z',\tilde z']}(F([\tilde z,\check z])))\\ &=\omega_{\tilde z'}^{-1}(F(\omega_{\tilde z}(\check z)))=F_{\tilde z\tilde z'}(\check z).\end{aligned}$$

Whereas, for $a\in\mu_{\tilde z}^{-1}[\Omega_{\tilde x}\cap D_{[\tilde z,z]}[\Omega_x]]$ $(\Omega_y:=\Phi_y\cap\Omega,\ \tilde x:=\pi\tilde z)$, we have $\tilde z\cdot a\in D_z\cap D_{\tilde z}$ and

$$F_{zz'}(\tilde z\cdot a)=F_{zz'}(\tilde z)\cdot\mu_{zz'}(a)=\tilde z'\cdot\mu_{zz'}(a),$$

$$\begin{aligned}F_{zz'}(\tilde z\cdot a) &= F_{\tilde z\tilde z'}(\tilde z\cdot a)=F_{\tilde z\tilde z'}(\tilde z)\cdot\mu_{\tilde z\tilde z'}(a)\\ &=\omega_{\tilde z'}^{-1}(F(\omega_{\tilde z}(\tilde z)))\cdot\mu_{\tilde z\tilde z'}(a)=\tilde z'\cdot\mu_{\tilde z\tilde z'}(a).\end{aligned}$$

This yields the equality $\mu_{zz'}(a)=\mu_{\tilde z\tilde z'}(a)$.

Analogously, we prove

(iii) $F_{z_1,z_1'} = F_{\tilde{z}a,\tilde{z}'a'}$ in some nbh of $\tilde{z}a$,

(iv) $\mu_{z_1,z_1'} = \mu_{\tilde{z}a,\tilde{z}'a'}$ in some nbh of the unit of G.

From (i)÷(iv) it follows that it is sufficient to show that

(v) $F_{\tilde{z}a,\tilde{z}'a'} = R_{a'} \circ F_{\tilde{z}\tilde{z}'} \circ R_{a^{-1}}$ (on the set $D_{\tilde{z}a} = R_{a^{-1}}[D_{\tilde{z}}]$),

(vi) $\mu_{\tilde{z}a,\tilde{z}'a'} = \tau_{a'^{-1}} \circ \mu_{\tilde{z}\tilde{z}'} \circ \tau_a$ (on the set $U_{\tilde{z}a} = \tau_{a^{-1}}[U_{\tilde{z}}]$).

(v): From the equalities

$$\omega_{\tilde{z}a} = \omega_{\tilde{z}} \circ R_{a^{-1}} \quad \text{and} \quad \omega_{\tilde{z}'a'} = \omega_{\tilde{z}'} \circ R_{a'^{-1}}$$

we obtain

$$F_{\tilde{z}a,\tilde{z}'a'}(\check{z}) = \omega_{\tilde{z}'a'}^{-1}(F(\omega_{\tilde{z}a}(\check{z}))) = R_{a'}(\omega_{\tilde{z}'}^{-1}(F(\omega_{\tilde{z}}(R_{a^{-1}}(\check{z})))))$$
$$= R_{a'} \circ F_{\tilde{z}\tilde{z}'} \circ R_{a^{-1}}(\check{z}).$$

(vi): From the equalities

$$\mu_{\tilde{z}a} = \mu_{\tilde{z}} \circ \tau_a \quad \text{and} \quad \mu_{\tilde{z}'a'} = \mu_{\tilde{z}'} \circ \tau_{a'}$$

we get

$$\mu_{\tilde{z}a,\tilde{z}'a'}(\check{a}) = \mu_{\tilde{z}'a'}^{-1}(F(\mu_{\tilde{z}a}(\check{a}))) = \tau_{a'}^{-1}(\mu_{\tilde{z}'}(\mu_{\tilde{z}}(\tau_a(\check{a}))))$$
$$= \tau_{a'}^{-1}(\mu_{\tilde{z}\tilde{z}'}(\tau_a(\check{a}))).$$

It remains to show that

$$d\mathcal{F} = h.$$

Take arbitrary $x \in M$ and $z \in P_{|x}$. For $v \in T_z P$, we have (see theorem 1.18)

$$(d\mathcal{F})_{|x}[v] = [F_{zz'*z}(v)] = [\omega_{z'*z'}^{-1} \circ F_{*u_x} \circ \omega_{z*z}(v)] = \rho_x'^{-1} \circ F_{*u_x} \circ \rho_x(v)$$
$$= \rho_x'^{-1} \circ \tilde{h}_{|x} \circ \rho_x(v) = h(v). \quad \square$$

As a corollary we obtain

<u>THEOREM 3.5</u>. Two pfb's $P(M,G)$ and $P'(M,G')$ are locally isomorphic iff their Lie algebroids are isomorphic. $\square$

Take now two pfb's

$$P = P(M,G) \quad \text{and} \quad P' = P'(M,G)$$

over M, with the same structural Lie group G. Let

$$\{(U_t,\varphi_t);\ t\in T\}, \quad \{(U_t,\varphi'_t);\ t\in T\}$$

be two families of local trivializations of P and P′, respectively, (over the same covering $\{U_t;\ t\in T\}$ of M) with the transition functions equal to

$$g_{tt'}, g'_{tt'}, \quad t,t'\in T,$$

respectively. Put

$$\psi_t = \varphi'_t\circ\varphi_t^{-1}, \quad t\in T.$$

When is the family

$$\mathcal{F} = \{(\psi_t, \mathrm{id});\ t\in T\}$$

a local homomorphism between pfb's ?

THEOREM 3.6. The following conditions are equivalent:

(1) $\mathcal{F}$ is a local homomorphism,

(2) for any $t,t'\in T$, the transition functions

$$g_{tt'}, g'_{tt'} : U_t\cap U_{t'} \longrightarrow G$$

differ locally by an element from the subgroup

$$\{\bar a\in G:\ \bigwedge_{a\in G}(\tau_a(\bar a)\in Z_{G_o})\}$$

where $Z_{G_o}$ is the centralizer of $G_o$ and $G_o$ is the connected component of the unit of G.

COROLLARY 3.7. Under the assumption of the connectedness of G, condition (2) is equivalent to

(2′) for any $t, t' \in T$, the transition functions $g_{tt'}, g'_{tt'}$ differ locally by an element from the centre $Z_G$ of $G$. □

COROLLARY 3.8. Under the assumption that $G$ is abelian, condition (2) is equivalent to

(2″) for any $t, t' \in T$, the transition functions $g_{tt'}$, $g'_{tt'}$ differ locally by a constant. □

PROOF OF THEOREM 3.6. The family $\mathcal{F}$ always fulfils conditions 1÷4 from definition 3.1. Therefore $\mathcal{F}$ is a local homomorphism (so it is a local isomorphism because $\psi_t$, $t \in T$, are diffeomorphisms) iff it fulfils condition 5.

Take arbitrary $t \in T$, $z_o \in D_t := \pi^{-1}[U_t]$, $a \in G$ and let $z_o a \in D_{t'} := \pi^{-1}[U_{t'}]$. Then $x_o := \pi z_o \in U_t \cap U_{t'}$. Let $\psi_t(z_o) = z'_o$ and $\psi_{t'}(z_o a) = z'_o a'$. We prove that a necessary and sufficient condition for

(a) $\psi_{t'} = R_{a'} \circ \psi_t \circ R_{a^{-1}}$ in some nbh of $z_o a$,
(b) $id = \tau_{a'^{-1}} \circ \tau_a$ in some nbh of $e \in G$

to hold is that the transition functions $g_{tt'}, g'_{tt'}$ should fulfil in some nbh of $x_o$ the condition:

$$g'_{tt'}(x) = g_{tt'}(x) \cdot \bar{a}$$

for some $\bar{a} \in G$ such that $\tau_a(\bar{a}) \in Z_{G_o}$ for all $a \in G$.

Let

$$\lambda_t = \varphi_t(\cdot, e) \quad \text{and} \quad \lambda'_t = \varphi'_t(\cdot, e).$$

for $a_o \in G$ such that $z_o = \lambda_{t'}(x_o) \cdot a_o$, we have

$$(*) \qquad a' = a_o^{-1} \cdot g_{tt'}^{-1}(x_o) \cdot g'_{tt'}(x_o) \cdot a_o \cdot a.$$

Indeed,

$$z'_o = \psi_t(z_o) = \varphi'_t \circ \varphi_t^{-1}(\lambda_{t'}(x_o) \cdot a_o) = \varphi'_t \circ \varphi_t^{-1}(\lambda_t(x_o) \cdot g_{tt'}(x_o) \cdot a_o)$$
$$= \varphi'_t(x_o, g_{tt'}(x_o) \cdot a_o) = \lambda'_t(x_o) \cdot g_{tt'}(x_o) \cdot a_o \;;$$

on the other hand,

$$z_o'\cdot a' = \psi_t(z_o\cdot a) = \varphi_{t'}'(\varphi_{t'}^{-1}(\lambda_{t'}(x_o)\cdot a_o\cdot a)) = \varphi_{t'}'(x_o\cdot a_o\cdot a)$$
$$= \lambda_{t'}'(x_o)\cdot a_o\cdot a = \lambda_t'(x_o)\cdot g_{tt'}'(x_o)\cdot a_o\cdot a,$$

so $\lambda_t'(x_o)\cdot g_{tt'}'(x_o)\cdot a_o\cdot a = \lambda_t'(x_o)\cdot g_{tt'}(x_o)\cdot a_o\cdot a'$, whence

$$g_{tt'}'(x_o)\cdot a_o\cdot a = g_{tt'}(x_o)\cdot a_o\cdot a',$$

which proves ( * ).

What does condition (b) say? It turns out that

$id = \tau_{a'^{-1}}\cdot\tau_a$ in some nbh of the unit of G iff

$id = \tau_{a'^{-1}\cdot a}$ on $G_o$ iff

(b$'$) $a'^{-1}\cdot a \in Z_{G_o}$.

Now, we explain condition (a). Because of the fact that each nbh of $z_o a$ contains the nbh consisting of all points of the form

$$\lambda_{t'}(x)\cdot a_o\cdot a\cdot g$$

for x from some nbh of $x_o$ and g from some nbh of the unit of G, we see that condition (a) is equivalent to

(a$'$) for x and g as above, the equality

$$\psi_{t'}(\lambda_{t'}(x)\cdot a_o\cdot a\cdot g) = R_{a'}\circ\psi_t\circ R_{a^{-1}}(\lambda_{t'}(x)\cdot a_o\cdot a\cdot g)$$

holds.

But its left-hand side is equal to

$$L = \varphi_{t'}'\circ\varphi_{t'}^{-1}(\lambda_{t'}(x)\cdot a_o\cdot a\cdot g) = \varphi_{t'}'(x, a_o\cdot a\cdot g)$$
$$= \lambda_{t'}'(x)\cdot a_o\cdot a\cdot g = \lambda_t'(x)\cdot g_{tt'}'(x)\cdot a_o\cdot a\cdot g,$$

while the right-hand side to

$$R = R_{a'}(\varphi_t'\circ\varphi_t^{-1}(\lambda_{t'}(x)\cdot a_o\cdot a\cdot g\cdot a^{-1}))$$
$$= R_{a'}(\varphi_t'\varphi_t^{-1}(\lambda_t(x)\cdot g_{tt'}(x)\cdot a_o\cdot a\cdot g\cdot a^{-1}))$$

$$= R_a \circ \varphi'_t(x, g_{tt'}(x)\cdot a_0\cdot a\cdot g\cdot a^{-1})$$
$$= \lambda'_t(x)\cdot g_{tt'}(x)\cdot a_0\cdot a\cdot g\cdot a^{-1}\cdot a',$$

therefore (a′) is equivalent to

(a″) for x and g as above, we have

$$g'_{tt'}(x)\cdot a_0\cdot a\cdot g = g_{tt'}(x)\cdot a_0\cdot a\cdot g\cdot a^{-1}\cdot a'.$$

In particular, for $g = e$, we get

$$g^{-1}_{tt'}(x)\cdot g'_{tt'}(x)\cdot a_0\cdot a = a_0\cdot a'.$$

This means that

$$g^{-1}_{tt'}(x)\cdot g'_{tt'}(x) = a_0\cdot a'\cdot a^{-1}\cdot a_0^{-1} \quad (=\text{const}),$$

which proves that the function

$$x \longmapsto g^{-1}_{tt'}(x)\cdot g'_{tt'}(x)$$

is locally constant. Let

(* *) $g'_{tt'}(x) = g_{tt'}(x)\cdot \bar{a}$ for x from some nbh of $x_0$.

Then we can observe that (a) is (by ( * ) and (* *)) equivalent to

(a‴) for $g\in G_0$, we have $\bar{a}\cdot(a_0 a)\cdot g = (a_0 a)\cdot g\cdot a^{-1}\cdot a_0^{-1}\cdot\bar{a}\cdot(a_0 a)$.

But we have the following equivalences:

$$(b') = (a_0^{-1}\cdot\bar{a}\cdot a_0\cdot a)^{-1}\cdot a\in Z_{G_0} \Leftrightarrow \tau_{(a_0 a)^{-1}}(\bar{a})\in Z_{G_0} \Leftrightarrow (a''').$$

Thereby, the system of conditions (a) and (b) is equivalent to the following fact:

— the transition functions $g'_{tt'}$ and $g_{tt'}$ differ locally by a constant $\bar{a}$ such that, for arbitrary $a_0, a$, we have $\tau_{(a_0 a)^{-1}}(\bar{a})\in Z_{G_0}$, which means that, for an arbitrary $a\in G$, we have $\tau_a(\bar{a})\in Z_{G_0}$. □

## CHAPTER 4

## CONNECTIONS IN LIE ALGEBROIDS

DEFINITION 4.1. ([2,p.188], [14,p.140]). By a connection in Lie algebroid (10) we mean a splitting of Atiyah sequence (13), ie a mapping

(22) $$\lambda : TM \longrightarrow A$$

such that $\gamma\circ\lambda = id_{TM}$, or, equivalently, a subbundle $B \subset A$ such that

$$A = q(A) \oplus B.$$

We define its connection form (called by K.Mackenzie [14,p.140] a back connection )

$$\omega^A : A \longrightarrow q(A)$$

as a unique form such that
(a) $\omega^A | q(A) = id$,
(b) $\mathrm{Ker}\,\omega^A = \mathrm{Im}\,\lambda$.

Let (22) be an arbitrary but fixed connection in (10) and let

$$A = A(P)$$

for some pfb $P = P(M,G)$. For each point $z \in P$, we define a subspace

$$H^\lambda_{|z} := \mathrm{Im}[(\pi^A_{|\pi z})^{-1} \circ \lambda_{\pi z}] \subset T_z P.$$

PROPOSITION 4.2. (see [14,p.292]). $z \longmapsto H^\lambda_{|z}$, $z \in P$, is a connection in P.

PROOF. The equality

(23) $$\pi^A_{|za} \circ (R_a)_{*z} = \pi^A_{|z}$$

implies

$$H^\lambda_{|za} = (R_a)_{*z}[H^\lambda_{|z}].$$

On the other hand, $\pi_{*z}|H^{\lambda}_{|z}:H^{\lambda}_{|z} \longrightarrow T_{\pi z}M$ is a linear isomorphism, thus

$$T_zP = H^{\lambda}_{|z} \oplus \operatorname{Ker}(\pi_{*z}).$$

It remains to show the smoothness of the distribution $H^{\lambda}$. Let $X_i$, $i \leq n$, be a local basis of $\mathfrak{X}(M)$ on $U \ni x$, $x$ being an arbitrary point of M. Then $(\lambda \circ X_i)'$, $i \leq n$, forms a local basis of $H^{\lambda}$ on $\pi^{-1}[U]$. □

PROPOSITION 4.3. ([14,p.292]). The correspondence

(24)
$$\lambda \longmapsto H^{\lambda}$$

sets up a bijection between connections in (6) and in P(M,G).

PROOF. Let H be any connection in P(M,G). Put

$$B_{|x} = \pi^{A}_{|z}[H_{|z}]$$

where $z \in P_{|x}$, $x \in M$. By (23), we see that $B_{|x}$ is independent of the choice of $z \in P_{|x}$. Evidently,

$$A(P)_{|x} = B_{|x} \oplus \mathfrak{T}(P)_{|x}$$

because $\gamma_{|x}|B_{|x}:B_{|x} \longrightarrow T_xM$ is an isomorphism as a superposition $\pi_{*z}|H_{|z} \circ (\pi^{A}_{|z}|H_{|z})^{-1}$.

$$B := \bigcup_x B_{|x} \subset A(P)$$

is a vector subbundle. Indeed, take a basis of the distribution H on a set $\pi^{-1}[U]$, $U \ni x$, x being an arbitrary point of M, consisting of right-invariant vector fields $Y_1,\dots,Y_n$ and take a local cross-section $\sigma:U \longrightarrow P$. Then the system of smooth cross-sections $\pi^{A} \circ Y_i \circ \sigma$, $i \leq n$, forms a basis of B on U, which proves that B is a vector subbundle. B defines a connection $\lambda^{H}:TM \longrightarrow A(P)$ by $\lambda^{H}_{|x} = (\gamma_{|x}|B_{|x})^{-1}$. The correspondence $H \longmapsto \lambda^{H}$ is inverse to (24). □

Fix a connection H in a pfb P. It determines the connection form $\omega \in \Omega^1(P;\mathfrak{g})$ and the curvature form $\Omega \in \Omega^2(P;\mathfrak{g})$. $\Omega$ is Ad-equivariant and horizontal at the same time [5,p.257], ie is a basic $\mathfrak{g}$-valued form on P. Via the classical manner (see for example [5,p.406]) the space

$$\Omega_B(P;\mathfrak{g})$$

of all basic $\mathfrak{g}$-valued forms on $P(M,G)$ is naturally isomorphic to the space of all forms on M with values in the associated Lie algebra bundle $P\times_G\mathfrak{g}$:

$$\Omega_B(P;\mathfrak{g}) \xrightarrow{\approx} \Omega(M;P\times_G\mathfrak{g}), \quad \Theta \longmapsto \tilde{\Theta},$$
$$\tilde{\Theta}(x;v_1,\dots,v_q) = [z,\Theta(z;v_1^z,\dots,v_q^z)], \quad v_i\in T_xM,$$

where $z\in P_{|x}$, while $v^z$ denotes a lifting of $v\in T_xM$ to $T_zP$ (for example with respect to some connection in P).

Considering the canonical isomorphism $P\times_G\mathfrak{g}\cong\mathfrak{g}(P)$ (see prop.1.10), we obtain an isomorphism (see (9))

$$\begin{aligned}&\Omega_B(P;\mathfrak{g}) \xrightarrow{\approx} \Omega(M;\mathfrak{g}(P)), \quad \Theta \longmapsto \Theta_M,\\ (25)\qquad &\Theta_M(x;v_1,\dots,v_q) = \hat{z}(\Theta(z;v_1^z,\dots,v_q^z)), \quad z\in P_{|x}.\end{aligned}$$

Via isomorphism (25) we define the so-called <u>curvature base form</u> (or the <u>curvature tensor</u>) $\Omega_M$ of H. Now, let $\lambda: TM \longrightarrow A(P)$ be the connection in (6) corresponding to H with connection form $\omega^A$. Of course, the following diagram commutes

$$\begin{array}{ccc}\mathfrak{g} & \xleftarrow{\ \omega_{|z}\ } & T_zP\\ {\scriptstyle\hat{z}}\downarrow & & \downarrow{\scriptstyle\pi^A_{|z}}\\ \mathfrak{g}(P)_{|x} & \xleftarrow{\ \omega^A_{|x}\ } & A(P)_{|x}\end{array}$$

<u>PROPOSITION 4.4.</u>

(26) $$\Omega_M(X,Y) = -\omega^A([\![\lambda\circ X, \lambda\circ Y]\!]), \quad X,Y\in\mathfrak{X}(M).$$

<u>PROOF</u>. By the equality $\pi^A_{|z}(v^z)=\lambda(v)$, $v\in T_xM$, we see that, for $X\in\mathfrak{X}(M)$, the right-invariant vector field $(\lambda\circ X)'$ on P is equal to the horizontal

lifting $\tilde{X}$ of $X$. By the classical equality

$$\Omega'(\tilde{X},\tilde{Y}) = -\omega([\tilde{X},\tilde{Y}]),$$

we obtain ($z \in P_{|x}$)

$$\begin{aligned}\Omega_M(X,Y)(x) &= \hat{z}(\Omega(z;\tilde{X}(z),\tilde{Y}(z))) = \hat{z}(-\omega(z;[\tilde{X},\tilde{Y}](z)))\\ &= -\omega^A(x;\pi^A_{|z}([(\lambda\circ X)',(\lambda\circ Y)'](z)))\\ &= -\omega^A(x;\pi^A_{|z}([\![\lambda\circ X,\lambda\circ Y]\!]'(z)))\\ &= -\omega^A(x;[\![\lambda\circ X,\lambda\circ Y]\!](x))\\ &= -\omega^A([\![\lambda\circ X,\lambda\circ Y]\!])(x). \quad \square\end{aligned}$$

Prop. 4.4 asserts that the curvature tensor $\Omega_M$ of a connection H in a pfb $P(M,G)$ corresponding to a connection $\lambda$ in the Lie algebroid $A(P)$ depends on $\lambda$ only.

<u>COROLLARY 4.5.</u>

$$\Omega_M(X,Y) = \lambda\circ[X,Y] - [\![\lambda\circ X,\lambda\circ Y]\!]. \tag{26'}$$

<u>PROOF</u>. $\lambda\circ[X,Y] - [\![\lambda\circ X,\lambda\circ Y]\!] \in \mathrm{Sec}\,\mathfrak{g}(P)$, therefore

$$\begin{aligned}\lambda\circ[X,Y] - [\![\lambda\circ X,\lambda\circ Y]\!] &= -\omega^A([\![\lambda\circ X,\lambda\circ Y]\!])\\ &= \Omega_M(X,Y). \quad \square\end{aligned}$$

Equation (26) or (26') can be taken (see [14,p.295]) as a definition of a <u>curvature tensor</u> of a connection $\lambda$ in Lie algebroid (10).

<u>COROLLARY 4.6.</u> The following properties are equivalent to one another:

(1) H is flat (ie $\Omega = 0$),

(2) $\Omega_M = 0$,

(3) $\mathrm{Sec}\,\lambda : \mathfrak{X}(M) \longrightarrow \mathrm{Sec}\,A(P)$ is a homomorphism of Lie algebras. $\square$

Any connection (22) in Lie algebroid (10) is called <u>flat</u> iff $Sec\lambda$ is a homomorphism of Lie algebras or, equivalently, if its curvature tensor $\Omega_M$ defined by (26) or by (26′) vanishes.

Lie algebroid (10) is called <u>flat</u> iff it possesses a flat connection.

A pfb P(M,G) is flat iff its Lie algebroid (6) is flat.

By theorem 3.5, we obtain (as a corollary)

<u>THEOREM 4.7.</u> If both pfb's P(M,G) and P′(M,G′) are locally isomorphic and one of them is flat, then the second one is flat, too. Consequently, flatness is an invariant of local isomorphisms. □

<u>EXAMPLE 4.8.</u> Every trivial Lie algebroid is flat. The canonical flat connection in the trivial Lie algebroid $TM\times\mathfrak{g}$ is defined by

$$\lambda : TM \longrightarrow TM\times\mathfrak{g}, \quad v \longmapsto (v,0). \quad \Box$$

<u>COROLLARY 4.9.</u> If Lie algebroid (6) of a pfb P(M,G) is trivial, then P(M,G) is flat. □

## THE CHERN-WEIL HOMOMORPHISM

We prove that the Chern-Weil homomorphisms of pfb's (over an arbitrary but fixed connected manifold M) are invariants of some local isomorphisms between them and, in the case of pfb's with connected structural Lie groups, these homomorphisms are invariants of all local isomorphisms.

Let $P = P(M,G)$ be any pfb with a Lie algebroid $A(P)$. Let

$$\overset{k}{\vee}\mathfrak{g}^{*} \text{ and } \overset{k}{\vee}\mathfrak{g}(P)^{*}$$

be the k-symmetric power of the vector space $\mathfrak{g}^{*}$ and the vector bundle $\mathfrak{g}(P)^{*}$, respectively;

$$\vee\mathfrak{g}^{*} = \overset{k}{\bigoplus}(\overset{k}{\vee}\mathfrak{g}^{*}).$$

In the sequel any element of $\overset{k}{\vee}\mathfrak{g}^{*}$ (analogously of $\overset{k}{\vee}(\mathfrak{g}(P)_{|x})^{*}$) is treated as a symmetric k-linear homomorphism $\mathfrak{g}\times\ldots\mathfrak{g} \to \mathbb{R}$ via the isomorphism

$$\overset{k}{\vee}\mathfrak{g}^{*} \cong \mathcal{L}^{k}_{s}(\mathfrak{g};\mathbb{R})$$

$$t_1\vee\ldots\vee t_k \longmapsto ((v_1,\ldots,v_k) \mapsto \frac{1}{k!}\sum_{\sigma} t_{\sigma(1)}(v_1)\cdot\ldots\cdot t_{\sigma(k)}(v_k)).$$

Define the mapping (see (9))

$$\Theta : (\vee\mathfrak{g}^{*})_I \longrightarrow \overset{k}{\bigoplus}(\mathrm{Sec}\ \overset{k}{\vee}\mathfrak{g}(P)^{*})$$

$$\Theta(\bar{\Gamma})_x = \overset{k}{\vee}(\hat{z}^{-1})^{*}(\Gamma)$$

for $\bar{\Gamma} \in (\overset{k}{\vee}\mathfrak{g}^{*})_I$ where $z \in P_{|x}$, $x \in M$. From the Ad-invariance of $\bar{\Gamma} \in (\overset{k}{\vee}\mathfrak{g}^{*})_I$ and the fact that

$$(za)\hat{\ } = \hat{z} \circ \mathrm{Ad}\, a, \quad z \in P, \ a \in G,$$

we see the correctness of this definition, ie the independence of $\Theta(\bar{\Gamma})_x$ of the choice of $z \in P_{|x}$. To prove the smoothness of $\Theta(\Gamma)$, we take a local section $\lambda : U \to P$ of $P$. $\lambda$ determines a local trivialization of $\overset{k}{\vee}\mathfrak{g}(P)^*$ of the form

$$\varphi^{\vee} : U \times \overset{k}{\vee}\mathfrak{g}^* \to \overset{k}{\vee}\mathfrak{g}(P)^*, \quad (x,u) \mapsto \overset{k}{\vee}(\lambda(x)^{\wedge -1})^*(u) ;$$

of course, $\varphi^{\vee -1} \circ \Theta(\bar{\Gamma})|U$ is a constant cross-section $x \mapsto (x, \Gamma)$, thus a smooth one. Denote the image $\operatorname{Im}\Theta^k$ $(\Theta^k := \Theta|(\overset{k}{\vee}\mathfrak{g}^*)_I)$ by

$$(\operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I.$$

Of course,

$$\Theta^k : (\overset{k}{\vee}\mathfrak{g}^*)_I \longrightarrow (\operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I$$

is an isomorphism of vector spaces.

<u>PROPOSITION 5.1.</u> Let $\Gamma \in \operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*$, then $\Gamma \in (\operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I$ iff, for any $z_1, z_2 \in P$, we have

$$\overset{k}{\vee}(\hat{z}_1)^*(\Gamma_{\pi z_1}) = \overset{k}{\vee}(\hat{z}_2)^*(\Gamma_{\pi z_2}). \quad \square$$

<u>THEOREM 5.2.</u> The mapping

$$h^{A(P)} : \overset{k}{\bigoplus}(\operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I \longrightarrow H(M)$$

for which the diagram

$$\begin{tikzcd} \overset{k}{\bigoplus}(\operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I \arrow[r, "h^{A(P)}"] & H(M) \\ (\vee\mathfrak{g}^*)_I \arrow[u, "\Theta", "\approx"'] \arrow[ur, "h_P"'] & \end{tikzcd}$$

commutes is defined by

$$\Gamma \longmapsto [\Gamma_*(\underbrace{\Omega_M, \ldots, \Omega_M}_{k\text{-times}})] \quad \text{for } \Gamma \in (\operatorname{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I$$

where $\Omega_M$ is the curvature base form of any connection in P and

$$\Gamma_*(\Omega_M,\dots,\Omega_M)(x;v_1,\dots,v_{2k})$$
$$= \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma\cdot\Gamma_x(\Omega_M(x;v_{\sigma(1)},v_{\sigma(2)}),\dots,\Omega_M(x;v_{\sigma(2k-1)},\dots,v_{\sigma(2k)})),$$

$v_i\in T_xM$, $x\in M$.

PROOF. We must only prove that

$$\pi^*((\otimes^k\bar\Gamma)_*(\Omega_M,\dots,\Omega_M)) = \bar\Gamma_*(\Omega,\dots,\Omega) \tag{27}$$

where $\Omega_M$ and $\Omega$ are the curvature base form and the curvature form of the same connection in P. Both sides of (27) are horizontal forms, so, to show the theorem, we must notice the equality on the horizontal vectors only. Let $z\in P_{|x}$ and $v_1,\dots,v_{2k}\in T_xM$, then we have (see (25))

$$\pi^*((\otimes^k\bar\Gamma)_*(\Omega_M,\dots,\Omega_M))(z;v_1^z,\dots,v_{2k}^z)$$
$$= (\otimes^k\bar\Gamma)_*(\Omega_M,\dots,\Omega_M)(\pi z;v_1,\dots,v_{2k})$$
$$= \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma(\otimes^k\bar\Gamma)_x(\Omega_M(x;v_{\sigma(1)},v_{\sigma(2)}),\dots,\Omega_M(x;v_{\sigma(2k-1)},v_{\sigma(2k)}))$$
$$= \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma\cdot\bar\Gamma(\hat z^{-1}(\Omega_M(x;v_{\sigma(1)},v_{\sigma(2)})),\dots,\hat z^{-1}(\Omega_M(x;v_{\sigma(2k-1)},v_{\sigma(2k)})))$$
$$= \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma\cdot\bar\Gamma(\Omega(z;v^z_{\sigma(1)},v^z_{\sigma(2)}),\dots,\Omega(z;v^z_{\sigma(2k-1)},v^z_{\sigma(2k)}))$$
$$= \bar\Gamma_*(\Omega,\dots,\Omega)(z;v_1^z,\dots,v_{2k}^z). \quad \square$$

Now, we describe the relationship between the Chern-Weil homomorphisms for local isomorphic pfb's.

Let $\mathcal{F}=\{(F_t,\mu_t);\ t\in T\}:P(M,G)\longrightarrow P'(M,G')$ be a local homomorphism between pfb's $P(M,G)$ and $P'(M,G')$ and let

$$\omega'\in\Omega^1(P',\mathfrak{g}')$$

be a connection form on $P'$ where $\mathfrak{g}':=\mathfrak{gl}(G')^0$.

PROPOSITION 5.3. There exists exactly one connection form

$$\omega \in \Omega^1(P;\mathfrak{g})$$

on P such that for each $t\in T$

$$\omega | \pi_P^{-1}[D_t] = (\mu_t)_{*e}^{-1}(F_t^*\omega').$$

PROOF. Correctness of the definition of $\omega$: Let $z_0 \in D_t \cap D_{t'}$. If $F_{t'}(z_0) = F_t(z_0)\cdot a'$ ($a' \in G'$), then $F_{t'} = R_{a'} \circ F_t$ in some nbh of $z_0$, and $\mu_{t'} = \tau_{a'^{-1}} \circ \mu_t$ in some nbh of the unit of G. That is why, for z from some nbh of $z_0$ and for $v \in T_zP$, we obtain

$$\begin{aligned}
(\mu_{t'})_{*e}^{-1}(F_{t'}^*\omega')(z;v) &= (\tau_{a'^{-1}}\circ\mu_t)_{*e}^{-1}((R_{a'}\circ F_t)^*\omega')(z;v)\\
&= ((\mu_t)_{*e}^{-1}\circ \mathrm{Ad}(a'))(F_t^* R_{a'}^*\omega')(z;v)\\
&= ((\mu_t)_{*e}^{-1}\circ \mathrm{Ad}(a'))(F_t^*(\mathrm{Ad}(a'^{-1})\omega'))(z;v)\\
&= (\mu_t)_{*e}^{-1} F_t^*\omega'(z;v).
\end{aligned}$$

$\omega$ is a connection form: (a) $\omega(z;(A_z)_{*e}(v)) = v$; indeed, let $z\in D_t$, then

$$\begin{aligned}
\omega(z;(A_z)_{*e}(v)) &= (\mu_t)_{*e}^{-1}(F_t^*\omega')(z;(A_z)_{*e}(v))\\
&= (\mu_t)_{*e}^{-1}(\omega'(F_t(z);(F_t)_*(A_{z*}(v))))\\
&= (\mu_t)_{*e}^{-1}(\omega'(F_t(z);(A_{F_t(z)})_*(\mu_t)_*(v)))\\
&= v.
\end{aligned}$$

(b) $R_a^*\omega = (\mathrm{Ad}\, a^{-1})\omega$; indeed, let $z\in D_t$, $z'\in P'$, $a\in G$, $a'\in G'$, $za\in D_{t'}$, $F_t(z) = z'$, $F_{t'}(za) = z'\cdot a'$. Then $F_{t'} = R_{a'}\circ F_t \circ R_{a^{-1}}$ in some nbh of za, and $\mu_{t'} = \tau_{a'^{-1}}\circ\mu_t\circ\tau_a$ in some nbh of the unit of G. So

$$\begin{aligned}
(R_a^*\omega)(z;v) = \omega(za;(R_a)_*(v)) &= (\mu_{t'})_{*e}^{-1}(F_{t'}^*\omega')(za;(R_a)_*(v))\\
&= (\mu_{t'})_{*e}^{-1}\omega'(F_{t'}(za);(F_{t'})_*((R_a)_*(v)))\\
&= (\mu_{t'})_{*e}^{-1}(\omega'(z'\cdot a';(R_{a'*}F_{t*}(v))))
\end{aligned}$$

$$= (\tau_{a'^{-1}} \circ \mu_t \circ \tau_a)^{-1}_{*e}(R^*_{a'}\omega')(z'; F_{t*}(v))$$

$$= (\mathrm{Ad}\, a^{-1}) \circ (\mu_t)^{-1}_{*e}(\omega'(z'; F_{t*}(v)))$$

$$= \mathrm{Ad}(a^{-1})\omega(z; v). \quad \square$$

The connection form $\omega$ obtained in proposition 5.3 is called <u>induced</u> by $\mathcal{F}$ from $\omega'$. $\omega$ and $\omega'$ induce some connections $\lambda$ and $\lambda'$ in $A(P)$ and $A(P')$, respectively, which next determine connection forms $\omega^A$ and $\omega'^A$ in them. The following diagram commutes

$$(28) \qquad \begin{tikzcd} \mathfrak{g}(P) \arrow[d, "(d\mathcal{F})^o"'] & A(P) \arrow[l, "\omega^A"'] \arrow[d, "d\mathcal{F}"] \arrow[dl, phantom, "①"] & TM \arrow[l, "\lambda"'] \arrow[d, equal] \\ \mathfrak{g}(P') & A(P') \arrow[l, "\omega'^A"'] & TM \arrow[l, "\lambda'"'] \end{tikzcd} .$$

Indeed, the commutativity of the left-hand side of (28) follows from the commutativity of all the remaining squares in the diagram

$$\begin{tikzcd} T_zP \arrow[rrr, "(F_t)_{*z}"] \arrow[dr, "\pi^A_{|z}"] \arrow[ddd, "\omega_{|z}"'] & & & T_{F_t(z)}P' \arrow[dl, "\pi'^A_{|F_t(z)}"'] \arrow[ddd, "\omega'_{|F_t(z)}"] \\ & A(P)_{|x} \arrow[r, "(d\mathcal{F})_{|x}"] \arrow[d, "\omega^A_{|x}"'] \arrow[dr, phantom, "①"] & A(P')_{|x} \arrow[d, "\omega'^A_{|x}"] & \\ & \mathfrak{g}(P)_{|x} \arrow[r, "(d\mathcal{F})^o_{|x}"] & \mathfrak{g}(P')_{|x} & \\ \mathfrak{g} \arrow[ur, "\hat{z}"] \arrow[rrr, "(\mu_t)_{*e}"] & & & \mathfrak{g}' \arrow[ul, "F_t(z)^\wedge"'] \end{tikzcd}$$

The commutativity of the right-hand side of (28) follows easily from the above because, for each $v \in T_xM$, the vector $(d\mathcal{F})_{|x}(\lambda(v))$ is horizontal and its projection on $T_xM$ is v. $\square$

<u>PROPOSITION 5.4</u>. The relationship between the curvature base form $\Omega_M$

and $\Omega'_M$ of $\omega^A$ and $\omega'^A$ [ the curvature forms $\Omega$ and $\Omega'$ of $\omega$ and $\omega'$ ], respectively, is described by the equality

$$(d\mathcal{F})^o_* \Omega_M = \Omega'_M \quad [\Omega \,|\, \pi_P^{-1}[D_t] = (\mu_t)^{-1}_{*e} F_t^* \Omega' ].$$

PROOF. For $X, Y \in \mathcal{X}(M)$, we have

$$\begin{aligned}(d\mathcal{F})^o \Omega_M(X,Y) &= (d\mathcal{F})^o(-\omega[\![\lambda X, \lambda Y]\!]) = -\omega(d\mathcal{F}[\![\lambda X, \lambda Y]\!]) \\ &= -\omega[\![d\mathcal{F}(\lambda X), d\mathcal{F}(\lambda Y)]\!] = -\omega[\![\lambda' X, \lambda' Y]\!] \\ &= -\Omega'_M(X,Y).\end{aligned}$$

The equality in the square brackets is classical [5, p.278] but we may obtain it immediately in the following way: by (25), for $z \in D_t$, $v \in T_z P$, we have

$$\begin{aligned}\mu_{t*e} \Omega(z; v_1^z, v_2^z) &= \mu_{t*e} \hat{z}^{-1} \Omega_M(x; v_1, v_2) = F_t(z)^{\wedge -1} (d\mathcal{F})^o_{|x} \Omega_M(x; v_1, v_2) \\ &= F_t(z)^{\wedge -1} \Omega'_M(x; v_1, v_2) = \Omega'(F_t(z); v_1^{F_t(z)}, v_2^{F_t(z)}) \\ &= (F_t^* \Omega')(z; v_1^z, v_2^z). \quad \square\end{aligned}$$

PROPOSITION 5.5. If $M$ is connected, then, for any $t, t' \in T$, there exist $a \in G$ and $a' \in G'$ such that $\mu_{t'} = \tau_{a'^{-1}} \circ \mu_t \circ \tau_a$ in some nbh of $e \in G$.

PROOF. Let $t, t' \in T$. Take arbitrary $x \in \pi[D_t]$, $x' \in \pi[D_{t'}]$ and let $\tau: \langle 0,1 \rangle \longrightarrow M$ be any path such that $\tau(0) = x$, $\tau(1) = x$. We can choose some sequence of indices $t_1, \dots, t_n \in T$ such that $t = t_1$, $t' = t_n$,

$$\bigcup_i \pi[D_{t_i}] \supset \operatorname{Im} \tau \quad \text{and} \quad \pi[D_{t_i}] \cap \pi[D_{t_{i+1}}] \neq \emptyset,$$

and some sequence of elements $z_1, \dots, z_n \in P$ such that

$$z_i \in D_{t_i}, \quad x_i := \pi z_i \in \pi[D_{t_i}] \cap \pi[D_{t_{i+1}}].$$

Let $a_i \in G$ and $a_i' \in G'$ be elements such that

$$z_i \cdot a_i \in D_{t_{i+1}}, \quad F_{t_i}(z_i) = z_i' \quad \text{and} \quad F_{t_{i+1}}(z_i \cdot a_i) = z_i' \cdot a_i'.$$

Then $\mu_{t_{i+1}} = \tau_{a_i'^{-1}} \circ \mu_{t_i} \circ \tau_{a_i}$ in some nbh of $e \in G$. Thereby,

$$\mu_{t'} = \mu_{t_n} = \tau_{(a_n' \cdot \ldots \cdot a_1')^{-1}} \circ \mu_t \circ \tau_{(a_1 \cdot \ldots \cdot a_n)}. \qquad \square$$

DEFINITION 5.6. A local isomorphism $\mathfrak{F}$ is said to have a property Ch-W if for all $t \in T$

$$(29) \qquad \bigvee (\mu_t)_{*e} [(\bigvee \mathfrak{q}'^*)_I] \subset (\bigvee \mathfrak{q}^*)_I$$

[ or, equivalently, if there exists $t \in T$ such that (29) holds (by prop.5.5) provided M is connected ].

EXAMPLE 5.7. $\mathfrak{F}$ has the property Ch-W if it satisfies one of the following properties:

(a) G is connected,

(b) there exists $t \in T$ such that $\mu_t$ can be extended to some globally defined homomorphism $G \to G'$ (provided M is connected),

(c) there exists $t \in T$ such that for each $a \in G$, there exists $a' \in G'$ such that $\mu_{t*e} \circ \mathrm{Ad}\, a = \mathrm{Ad}\, a' \circ \mu_{t*e}$ (provided M is connected).

First, we easily show that each local isomorphism fulfilling property (c) has the property Ch-W. Now, we trivially notice that (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c). $\square$

THEOREM 5.8. If $\mathfrak{F}$ has the property Ch-W, then

$$(30) \qquad \overset{k}{\bigvee} (d\mathfrak{F})^\circ [(\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{q}(P')^*)_I] \subset (\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{q}(P)^*)_I$$

and the following diagram commutes:

$$\begin{tikzcd}
(\overset{k}{\bigvee} \mathfrak{q}'^*)_I \arrow[r, "\theta'^k"] \arrow[d, "\overset{k}{\bigvee} \mu_{t*e}^*"'] & (\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{q}(P')^*)_I \arrow[r, "h^{A(P')}"] \arrow[d, "\overset{k}{\bigvee} (d\mathfrak{F})^{\circ *}"] & H(M) \\
(\overset{k}{\bigvee} \mathfrak{q}^*)_I \arrow[r, "\theta^k"] & (\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{q}(P)^*)_I \arrow[ur, "h^{A(P)}"'] &
\end{tikzcd}$$

PROOF. To prove the left-hand side of the above diagram, and the inclusion (30), we need to show the commutativity of

$$\begin{tikzcd}
(\overset{k}{\vee}\mathfrak{g}'^{*})_I \arrow[r, "\Theta'^k"] \arrow[d, "\overset{k}{\vee}\mu^{*}_{t*e}"'] & \mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P')^{*} \arrow[d, "\overset{k}{\vee}(d\mathcal{F})^{o*}"] \\
(\overset{k}{\vee}\mathfrak{g}^{*})_I \arrow[r, "\Theta^k"] & \mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P)^{*}.
\end{tikzcd}$$

$$\begin{aligned}
(\overset{k}{\vee}(d\mathcal{F})^{o*}\circ\Theta'^k(\bar{\Gamma}))_x &= \Theta'^k(\bar{\Gamma})_x\circ(d\mathcal{F}^o\times\ldots\times d\mathcal{F}^o)_x \\
&= \bar{\Gamma}\circ((F_t z)^{\wedge-1}\times\ldots\times(F_t z)^{\wedge-1})\circ(d\mathcal{F}^o_{|x}\times\ldots\times d\mathcal{F}^o_{|x}) \\
&= \bar{\Gamma}\circ(\mu_{t*e}\times\ldots\times\mu_{t*e})\circ(\hat{z}^{-1}\times\ldots\times\hat{z}^{-1}) \\
&= \Theta^k(\bar{\Gamma}\circ(\mu_{t*e}\times\ldots\times\mu_{t*e}))_x \\
&= (\Theta^k\circ\overset{k}{\vee}\mu^{*}_{t*e}(\bar{\Gamma}))_x,
\end{aligned}$$

$z\in P_{|x}\cap D_t$, $\quad x\in\pi[D_t]$.

To end the proof, we notice that (by prop.5.4)

$$\begin{aligned}
h^{A(P)}\circ\overset{k}{\vee}(d\mathcal{F})^{o*}(\Gamma) &= h^{A(P)}(\Gamma\circ(d\mathcal{F}^o\times\ldots\times d\mathcal{F}^o)) \\
&= [\Gamma\circ(d\mathcal{F}^o\times\ldots\times d\mathcal{F}^o)_x(\Omega_M,\ldots,\Omega_M)] \\
&= [\Gamma_*(d\mathcal{F}^o\Omega_M,\ldots,d\mathcal{F}^o\Omega_M)] \\
&= [\Gamma_*(\Omega'_M,\ldots,\Omega'_M)] \\
&= h^{A(P')}. \quad \square
\end{aligned}$$

<u>COROLLARY 5.9</u>. The Chern-Weil homomorphisms of pfb's are invariants of local isomorphisms having the property Ch-W. In the case of pfb's with connected structural Lie groups, the Chern-Weil homomorphisms are invariants of all local isomorphisms. $\square$

# CHAPTER 6

## A STRUCTURAL THEOREM

Here we prove that any Lie algebroid $A$ is uniquely determined (up to an isomorphism) by its Lie algebra bundle $\boldsymbol{g}(A)$, a covariant derivative $\nabla$ in $\boldsymbol{g}(A)$ and a 2-tensor $\Omega\in\Omega^2(M;\boldsymbol{g}(A))$, fulfilling some conditions. Cf [1; chapt. VIII] and [14,p.224].

Let (10) ba any Lie algebroid on a manifold M with the Lie algebra bundle $\boldsymbol{g}$. Let $\lambda: TM \longrightarrow A$ be any connection in this Lie algebroid,

(31)
$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \overset{\gamma}{\underset{\curvearrowleft \lambda}{\longrightarrow}} TM \longrightarrow 0,$$

with the curvature base form

$$\Omega_M \in \Omega^2(M;\boldsymbol{g}).$$

Corollary 4.5 states that

(i)
$$[\![\lambda X,\lambda Y]\!] = \lambda[X,Y] - \Omega_M(X,Y), \quad X,Y\in\mathfrak{X}(M),$$

$\lambda X := \lambda\circ X$. The connection $\lambda$ determines a covariant derivative $\nabla$ in $\boldsymbol{g}$ by the formula

(ii)
$$\nabla_X\sigma = [\![\lambda X,\sigma]\!], \quad X\in\mathfrak{X}(M), \quad \sigma\in \mathrm{Sec}\,\boldsymbol{g},$$

(see the proof of theorem 1.13). $\nabla$ is called <u>corresponding</u> to $\lambda$ or after K.Mackenzie [14,p.295] the <u>adjoint connection</u> of $\lambda$.

We notice that the bracket $[\![\cdot,\cdot]\!]$ in the Lie algebra $\mathrm{Sec}\,A$ is uniquely determined by the system $(\boldsymbol{g},\nabla,\Omega_M)$ and $\lambda$, namely

(iii)
$$[\![\lambda X+\sigma,\lambda Y+\eta]\!] = \lambda[X,Y] - \Omega_M(X,Y) + \nabla_X\eta - \nabla_Y\sigma + [\sigma,\eta],$$

$X,Y\in\mathfrak{X}(M)$, $\sigma,\eta\in\mathrm{Sec}\,\boldsymbol{g}$.

$\nabla$ determines the so-called <u>exterior covariant derivative</u> in $\Omega(M;\boldsymbol{g})$ by the classical formula:

for $\Psi\in\Omega^q(M;\boldsymbol{g})$, we have $\nabla\Psi\in\Omega^{q+1}(M;\boldsymbol{g})$, and

(iv) $\nabla\Psi(X_0,\ldots,X_q) = \sum_{j=0}^{q}(-1)^q\nabla_{X_j}(\Psi(X_0,\ldots,\hat{X}_j,\ldots,X_q))$

$+\sum_{i<j}(-1)^{i+j}\Psi([X_i,X_j],\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_q), \quad X_j\in\mathfrak{X}(M).$

<u>PROPOSITION 6.1</u>. The elements $\nabla$ and $\Omega_M$ fulfil the following assertions

($1^o$) $R_{X,Y}\sigma = -[\Omega_M(X,Y),\sigma]$, $X,Y\in\mathfrak{X}(M)$, $\sigma\in\mathrm{Sec}\,\mathfrak{g}$, where R denotes the curvature tensor of $\nabla$, ie

$$\nabla^2\sigma = -[\Omega_M,\sigma],\quad \sigma\in\mathrm{Sec}\,\mathfrak{g},\quad \text{(the Ricci identity).}$$

($2^o$) $\nabla_X[\sigma,\eta] = [\nabla_X\sigma,\eta]+[\sigma,\nabla_X\eta]$, $X\in\mathfrak{X}(M)$, $\sigma,\eta\in\mathrm{Sec}\,\mathfrak{g}$, ie $\nabla$ is a $\Sigma$-connection in $(\mathfrak{g},\{[\cdot,\cdot]\})$ ( see the proof of theorem 1.13) (called in the sequel a <u>$\Sigma$-connection</u> in $\mathfrak{g}$ or after [14,p.143] a <u>Lie connection</u> in $\mathfrak{g}$ ).

($3^o$) $\nabla\Omega_M = 0$ (the Bianchi identity).

<u>PROOF</u>. Trivial calculations. □

<u>THEOREM 6.2</u>. (cf [1,p.372] and [14,p.223]). (a) Let a system

$$(\mathfrak{g},\nabla,\Omega_M)$$

be given, consisting of

(i) a Lie algebra bundle $\mathfrak{g}$ on a manifold M,

(ii) a covariant derivative $\nabla$ in $\mathfrak{g}$,

(iii) a 2-form $\Omega_M\in\Omega^2(M;\mathfrak{g})$,

fulfilling conditions ($1^o$)$\div$($3^o$) (from proposition 6.1).

Then, for a vector bundle $A\supset\mathfrak{g}$ and mappings $\gamma$, $\lambda$, such that

(*) in the diagram (31) the row is exact and $\gamma\circ\lambda = \mathrm{id}_{TM}$,

there exists in the vector space Sec A exactly one Lie algebra structure $[\![\cdot,\cdot]\!]$ fulfilling conditions:

— $(A,[\![\cdot,\cdot]\!],\gamma)$ is a Lie algebroid with the Lie algebra bundle equal to $\mathfrak{g}$,

— equalities (i) and (ii) hold.

The bracket $[\![\cdot,\cdot]\!]$ is defined by formula (iii).

(b) For another vector bundle $A' \supset \mathfrak{g}$ (on $M$) and mappings $\gamma'$, $\lambda'$, fulfilling the analogous properties, there exists exactly one isomorphism $F: A' \longrightarrow A$ of Lie algebroids such that the diagram

$$\begin{tikzcd} & A' \arrow[dd, "F"] \arrow[dr, "\gamma'"] & \\ \mathfrak{g} \arrow[ur, hook] \arrow[dr, hook] & & TM \\ & A \arrow[ur, "\gamma"'] & \end{tikzcd}$$

commutes. $F$ is defined by the formula $F(\lambda'(v) + w) = \lambda(v) + w$, $v \in TM$, $w \in \mathfrak{g}$.

(c) If $\Omega_M = 0$, then the Lie algebroid constructed in (a) is flat.

PROOF. (a) The uniqueness of $[\![\cdot,\cdot]\!]$ is evident. To prove the existence of the sought-for structure, we need to demonstrate that (iii) defines it. The bilinearity and antisymmetry of $[\![\cdot,\cdot]\!]$ and properties (i) and (ii) are very easy to see.

The Jacobi identity:

$$\begin{aligned} &[\![ [\![\lambda X + \sigma, \lambda Y + \eta]\!], \lambda Z + \delta ]\!] + \text{cycl} \\ &= [\![ \lambda[X,Y] - \Omega_M(X,Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma,\eta], \lambda Z + \delta ]\!] + \text{cycl} \\ &= [\lambda[X,Y], Z] - \Omega_M([X,Y],Z) + \nabla_{[X,Y]}\delta + \nabla_Z(\Omega_M(X,Y)) - \nabla_Z \nabla_X \eta \\ &\qquad + \nabla_Z \nabla_Y \sigma - \nabla_Z[\sigma,\eta] - [\Omega_M(X,Y),\delta] + [\nabla_X \eta, \delta] \\ &\qquad - [\nabla_Y \sigma, \delta] + [[\sigma,\eta],\delta] + \text{cycl} \\ &= 0. \end{aligned}$$

The last equality is obtained from the Jacobi identity in $\mathfrak{X}(M)$ and in $\mathrm{Sec}\,\mathfrak{g}$ and from assumptions $(1^\circ) \div (3^\circ)$.

The equality $[\![\lambda X + \sigma, f\cdot(\lambda Y + \eta)]\!] = f \cdot [\![\lambda X + \sigma, \lambda Y + \eta]\!] + X(f)\cdot(\lambda Y + \eta)$ is easy to obtain.

(b) To prove the second part of our theorem, we notice that

- $\gamma \circ F = \gamma'$ (trivial),

- $\operatorname{Sec} F : \operatorname{Sec} A' \longrightarrow \operatorname{Sec} A$ is a homomorphism of Lie algebras,

indeed:

$$
\begin{aligned}
F(\llbracket \lambda' X + \sigma, \lambda' Y + \eta \rrbracket) &= F(\lambda'[X,Y] - \Omega_M(X,Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma, \eta]) \\
&= \lambda[X,Y] - \Omega_M(X,Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma, \eta] \\
&= \llbracket \lambda X + \sigma, \lambda Y + \eta \rrbracket \\
&= \llbracket F(\lambda' X + \sigma), F(\lambda' Y + \eta) \rrbracket .
\end{aligned}
$$

(c) Trivial because then $\operatorname{Sec} \lambda : \mathfrak{X}(M) \longrightarrow \operatorname{Sec} A$ is a homomorphism of Lie algebras. □

## CHAPTER 7

## CLASSIFICATIONS OF LIE ALGEBROIDS OF SOME TYPES

Let $\lambda$, $\lambda_1 : TM \to A$ be two connections in a Lie algebroid (10). Then

$$c := \lambda_1 - \lambda$$

has its values in the bundle $\mathfrak{g}(A)$ of course.

PROPOSITION 7.1. If $\nabla$, $\nabla_1$ are two covariant derivatives in $\mathfrak{g}(A)$ corresponding to $\lambda$, $\lambda_1$, respectively, then $\nabla = \nabla_1$ iff $c : TM \to \mathfrak{g}(A)$ is a central homomorphism, ie such that $c(v)$ belongs to the centre of the Lie algebra $\mathfrak{g}(A)_{|x}$ for $v \in T_xM$, $x \in M$.

PROOF. By the definition we have: $\nabla_v \sigma = [\![\lambda(v), \sigma]\!]$, $(\nabla_1)_v \sigma = [\![\lambda_1(v), \sigma]\!]$, $v \in TM$, $\sigma \in \mathrm{Sec}\,\mathfrak{g}(A)$. Therefore $\nabla = \nabla_1$ iff, for all $v \in TM$ and $\sigma \in \mathrm{Sec}\,\mathfrak{g}(A)$, $[\![\lambda(v) - \lambda_1(v), \sigma]\!] = 0$, thus iff $[c(v), w] = 0$ for all $(v, w) \in T_xM \times \mathfrak{g}(A)_{|x}$, $x \in M$. $\square$

COROLLARY 7.2. If the isotropy Lie algebras are abelian, then to all connections there corresponds the same covariant derivative. $\square$

COROLLARY 7.3. If the isotropy Lie algebras are without the centre, then to different connections there correspond different covariant derivatives. $\square$

### 7.1. A CLASSIFICATION OF FLAT LIE ALGEBROIDS WITH ABELIAN ISOTROPY LIE ALGEBRAS.

THEOREM 7.1.1. Let $\mathfrak{g}$ be an arbitrary vector bundle on a manifold M, considered as a bundle of abelian Lie algebras. Then there exists a bijection between the set of all classes of isomorphic flat Lie algebroids with the Lie algebra bundle $\mathfrak{g}$ and the set of all equivalent flat covariant derivatives in $\mathfrak{g}$, where by the equivalent covariant derivatives we mean both $\nabla$ and $\nabla^1$ such that there exists a vector bundle isomorphism $f : \mathfrak{g} \to \mathfrak{g}$ for which $\nabla^1_X \sigma = \nabla_X (f \circ \sigma)$,

$X \in \mathfrak{X}(M)$, $\sigma \in \mathrm{Sec}\,\mathfrak{g}$.

PROOF. Fix any vector bundle $A \supset \mathfrak{g}$ and mappings $\gamma, \lambda$, such that the condition ( $*$ ) (see theorem 6.2) holds. With each flat covariant derivative $\nabla$ in $\mathfrak{g}$ we associate the system

$$(\mathfrak{g}, \nabla, 0), \quad 0 \in \Omega^2(M;\mathfrak{g}),$$

and with the latter - according to theorem 6.2 - some flat Lie algebroid $A^{\nabla} = (A, [\![\cdot,\cdot]\!]^{\nabla}, \gamma)$ (for the bundle A taken above). Lie algebroids obtained in this manner are - for different $A, \gamma, \lambda$ - isomorphic (see theorem 6.2). Of course, by prop.6.1 and theorem 6.2, each flat Lie algebroid with the Lie algebra bundle $\mathfrak{g}$ can be obtained (up to an isomorphism) with the help of some flat covariant derivative in $\mathfrak{g}$.

Let $\nabla$ and $\nabla^1$ be two covariant derivatives in $\mathfrak{g}$ such that the Lie algebroids $A := A^{\nabla}$ and $A^1 := A^{\nabla^1}$ are isomorphic (via some isomorphism F):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g} & \hookrightarrow & A & \xrightarrow{\gamma} & TM & \longrightarrow & 0 \\
 & & \approx \big\downarrow F^0 & & \approx \big\downarrow F & & \| & & \\
0 & \longrightarrow & \mathfrak{g} & \hookrightarrow & A^1 & \xrightarrow{\gamma^1} & TM & \longrightarrow & 0
\end{array}$$

Let $\lambda : TM \longrightarrow A$ be any connection in A; then $F \circ \lambda$ is a connection in $A^1$. According to corollary 7.2, we have $\nabla_X \sigma = [\![\lambda X, \sigma]\!]$, $\nabla^1_X \sigma = [\![F \circ \lambda(X), \sigma]\!]$, $X \in \mathfrak{X}(M)$, $\sigma \in \mathrm{Sec}\,\mathfrak{g}$. Thereby, since F is an isomorphism of Lie algebroids,

$$\nabla^1_X(F^0 \circ \sigma) = [\![F \circ \lambda(X), F^0 \circ \sigma]\!] = [\![F \circ (\lambda X), F \circ \sigma]\!] = [\![\lambda X, \sigma]\!] = \nabla_X \sigma,$$

which means that $\nabla$ and $\nabla^1$ are equivalent. $\square$

## 7.2. A CLASSIFICATION OF LIE ALGEBROIDS WITH SEMISIMPLE ISOTROPY LIE ALGEBRAS.

Let $\mathfrak{g}$ be any bundle of semisimple Lie algebras on a manifold M.

PROPOSITION 7.2.1. For any $\Sigma$-connection $\nabla$ in $\mathfrak{g}$, there exists exactly one 2-form

$$\Omega_M \in \Omega^2(M;\mathfrak{g})$$

fulfilling condition ($1^o$) from prop.6.1. $\Omega_M$ fulfils the Bianchi identity ($3^o$).

PROOF. It is easy to check that

$$R_{v,w}:\mathfrak{g}_{|x} \longrightarrow \mathfrak{g}_{|x}$$

for $v,w\in T_xM$ is a derivation of the Lie algebra $\mathfrak{g}_{|x}$, $R$ being the curvature tensor of $\nabla$. From the assumption that $\mathfrak{g}_{|x}$ is semisimple we have the existence and the uniqueness of an element

$$\Omega_M(x;v,w)\in\mathfrak{g}_{|x}$$

such that

$$R_{v,w}(u) = -[\Omega_M(x;v,w),u], \quad u\in\mathfrak{g}_{|x} .$$

Of course, we have thus defined a 2-form $\Omega_M\in\Omega^2(M;\mathfrak{g})$.

By a standard calculation and the fact that $\mathfrak{g}_{|x}$, $x\in M$, are without the centre, we obtain the equality $\nabla\Omega_M = 0$:

$$\begin{aligned}
[\nabla\Omega_M(X,Y,Z),\sigma] &= [\nabla_X(\Omega_M(Y,Z)),\sigma] - [\nabla_Y(\Omega_M(X,Z)),\sigma] \\
&\quad + [\nabla_Z(\Omega_M(X,Y)),\sigma] - [\Omega_M([X,Y],Z),\sigma] \\
&\quad + [\Omega_M([X,Z],Y),\sigma] - [\Omega_M([Y,Z],X),\sigma] \\
&= -\nabla_X(R_{Y,Z}\sigma) + R_{Y,Z}(\nabla_X\sigma) + \nabla_Y(R_{X,Z}\sigma) \\
&\quad - R_{X,Z}(\nabla_Y\sigma) - \nabla_Z(R_{X,Y}\sigma) + R_{X,Y}(\nabla_Z\sigma) \\
&\quad + R_{[X,Y],Z}\sigma - R_{[X,Z],Y}\sigma + R_{[Y,Z],X}\sigma \\
&= 0. \quad \square
\end{aligned}$$

By the above, we see that any $\Sigma$-connection in $\mathfrak{g}$ determines exactly one Lie algebroid (see theorem 6.2).

PROPOSITION 7.2.2. If $\mathfrak{g}$ is the Lie algebra bundle assigned to a Lie algebroid A, and a covariant derivative $\nabla$ in $\mathfrak{g}$ corresponds to a connection $\lambda$ in A, then the 2-form $\Omega_M \in \Omega^2(M;\mathfrak{g})$ defined by ($1^o$) is exactly the curvature tensor of $\lambda$.

PROOF. We need to notice that

$$R_{X,Y}\sigma = -[\lambda[X,Y] - [\![\lambda X, \lambda Y]\!], \sigma]$$

knowing that $\nabla_X \sigma = [\![\lambda X, \sigma]\!]$; but this is a standard calculation. □

THEOREM 7.2.3. For a given Lie algebra bundle $\mathfrak{g}$ whose fibres are semi-simple, there exists exactly one (up to an isomorphism) Lie algebroid A for which $\mathfrak{g}(A) = \mathfrak{g}$.

PROOF. The existence: According to [5, p.380], there exists in $\mathfrak{g}$ a $\Sigma$-connection. Let $A, \gamma, \lambda$ be elements as before (see (31) and (*) in theorem 6.2). Give any $\Sigma$-connection $\nabla$ in $\mathfrak{g}$ and the 2-form $\Omega_M \in \Omega^2(M;\mathfrak{g})$ fulfilling ($1^o$). For this homomorphism $\lambda$, we define in A some structure of a Lie algebroid according to theorem 6.2.

The uniqueness. Let A be any Lie algebroid for which $\mathfrak{g}(A) = \mathfrak{g}$. Let $\nabla^\lambda$ denote the covariant derivative in $\mathfrak{g}$ corresponding to a connection $\lambda : TM \longrightarrow A$.

LEMMA 7.2.4. The correspondence

$$\lambda \longmapsto \nabla^\lambda$$

establishes a bijection between the set of all connections in A and the set of all $\Sigma$-connections in $\mathfrak{g}$.

PROOF. By corollary 7.3, this correspondence is an injection.

Let $\nabla$ be an arbitrary $\Sigma$-connection in $\mathfrak{g}$. Of course,

$$T = \nabla - \nabla_0$$

is a tensor

$$T : TM \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

where $\nabla_o$ is a $\Sigma$-connection corresponding to an arbitrary but fixed connection $\lambda_o$.

Besides

$$\nabla_v \sigma = \nabla_{ov} \sigma + T(v, \sigma(x)), \quad v \in T_x M, \; \sigma \in Sec \mathfrak{g}, \quad x \in M.$$

We want to find a homomorphism

$$c : TM \longrightarrow \mathfrak{g}$$

such that

$$\nabla_v \sigma = [\![ (\lambda_o + c)(v), \sigma ]\!]$$

which will mean that

$$\nabla = \nabla^{\lambda_o + c}.$$

First, we notice that

$$T(v, \cdot) : \mathfrak{g}_{|x} \longrightarrow \mathfrak{g}_{|x}, \quad v \in T_x M,$$

is a derivation of the Lie algebra $\mathfrak{g}_{|x}$. Because of the fact that $\mathfrak{g}_{|x}$ is semisimple, we see that the derivation $T(v, \cdot)$ is inner which means that there is an uniquely determined element $c(v)$ such that

$$T(v, \cdot) = [c(v), \cdot].$$

It remains to show that the mapping

$$c : TM \longrightarrow \mathfrak{g}, \quad v \longmapsto c(v),$$

is a $C^\infty$-vector bundle homomorphism. Of course, it is a vector bundle homomorphism, so we must prove the smoothness of $c$ only. Since $\mathfrak{g}$ is a locally trivial Lie algebra bundle, the smoothness of $c$ is obtained locally by the following assertion:

— For a Lie algebra $\mathfrak{h}$ without the centre, a manifold $N$ and a $C^\infty$-linear representation $T : N \times \mathfrak{h} \longrightarrow \mathfrak{h}$, such that $T(v, \cdot) = [c(v), \cdot]$, $v \in N$, for some $c : N \longrightarrow \mathfrak{h}$, we have: $c$ is $C^\infty$.

This assertion is easy to show, see the diagram

$$\begin{tikzcd} N \arrow[r, "c"] \arrow[rr, bend right, "\tilde{T}"'] & \mathfrak{h} \arrow[r, "ad"] & \mathcal{L}(\mathfrak{h},\mathfrak{h}) \end{tikzcd}$$

in which $\tilde{T}(v)(w) = T(v,w)$. $\square$

The continuation of the proof of the theorem: Let $A^1$, $A^2$ be two Lie algebroids for which

$$\mathfrak{g}(A^1) = \mathfrak{g}(A^2) = \mathfrak{g} .$$

Take an arbitrary $\Sigma$-connection $\nabla$ in $\mathfrak{g}$, and denote by $\lambda_1$, $\lambda_2$, the corresponding connections in $A^1$, $A^2$, respectively (according to the lemma above). Then

$$F: A^1 \longrightarrow A^2, \quad (\lambda_1(v)+w \longmapsto \lambda_2(v)+w), \quad v \in TM, \quad w \in \mathfrak{g},$$

is an isomorphism of Lie algebroids. Indeed

$$\begin{aligned} F([\![\lambda_1 X+\sigma, \lambda_1 Y+\eta]\!]) &= F(\lambda_1 [X,Y] - \Omega_M(X,Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma,\eta]) \\ &= \lambda_2 [X,Y] - \Omega_M(X,Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma,\eta] \\ &= [\![\lambda_2 X+\sigma, \lambda_2 Y+\eta]\!]. \quad \square \end{aligned}$$

COROLLARY 7.2.5. Two Lie algebroids with semisimple isotropy Lie algebras are isomorphic iff their Lie algebra bundles are isomorphic. $\square$

Theorem 3.5 and the last corollary give the following

COROLLARY 7.2.6. Two pfb's with semisimple structural Lie groups are locally isomorphic iff their associated Lie algebra bundles are isomorphic. $\square$

CHAPTER 8

## SOME EXAMPLES

A/ We ask two questions:

$1^{\circ}$) Does, for any pfb $P = P(M,G)$ and a Lie group $G'$ locally isomorphic to $G$, there exists a pfb $P' = P'(M,G')$ such that $A(P) \cong A(P')$ ?

$2^{\circ}$) Are pfb's $P = P(M,G)$, $P' = P'(M,G')$ globally isomorphic provided their structural Lie groups $G$ and $G'$ and their Lie algebroids $A(P)$ and $A(P')$ are isomorphic ?

It turns out that the answers for both these questions are negative (even the Lie groups $G$ and $G'$ are assumed to be connected).

$1^{\circ}$: Consider the Hopf bundle

$$\xi = (S^3 \longrightarrow S^2)$$

(being an $S^1$-pfb) and the universal covering $\mathbb{R} \longrightarrow S^1$.

THEOREM 8.1. There exists no $\mathbb{R}$-pfb with the Lie algebroid isomorphic to $A(\xi)$.

PROOF. Suppose $P(S^2,\mathbb{R})$ is such a pfb. According to [6,p.58], this pfb has a global section, thus is trivial. Therefore its Lie algebroid is trivial; consequently, $A(\xi)$ is trivial, so (by corollary 4.9) $\xi$ is flat. But $S^2$ is simply connected, so, by Atiyah-Milnor's theorem [2,prop.14], [15,lemma 1], $\xi$ is trivial, which yields the contradiction because $\xi$ has no global section. $\square$

$2^{\circ}$: Without the assumption of the connectedness of $G$ and $G'$, the negative answer to $2^{\circ}$) is easy to obtain.

EXAMPLE 8.2. Let $\tilde{M} \longrightarrow M$ be the universal covering of $M$ and let $\pi_1(M) \neq 0$.

Then $\pi_1(M)$-pfb's $\tilde{M} \longrightarrow M$ and $M \times \pi_1(M) \longrightarrow M$ are not isomorphic although its Lie algebroids are isomorphic (see remark 1.15).

It turns out that the assumption of connectedness and even, in addition, the semisimplicity of G and G′ are not sufficient for a positive answer.

<u>EXAMPLE 8.3</u>. (The idea of this example was suggested to me by Th.Friedrich). Because of the fact that $H^1(\mathbb{R}P(5);Z_2) = Z_2$, there exist [25] two distinct Spin(3)-structures of the trivial pfb $\mathbb{R}P(5)\times SO(3)$ . One of them, say $P^1$, is of course trivial: $P^1 = RP(5)\times Spin(3)$, but the second one, say $P^2$, according to [24] <u>is not trivial</u>! Thus, between $P^1$ and $P^2$ there exists no global fibre isomorphism (so, no global pfb's isomorphism in any sense). However, Lie algebroids $A(P^1)$ and $A(P^2)$ are isomorphic. Indeed, there exist (by the definition of spin structures) homomorphisms

$$(F^i, \lambda): P^i \longrightarrow \mathbb{R}P(5)\times SO(3), \quad i=1,2,$$

where $\lambda: Spin(3) \longrightarrow SO(3)$ is the standard homomorphism from Spin(3) to SO(3). $\lambda$ being a covering is a local isomorphism, which implies that the homomorphisms of Lie algebroids

$$dF^i : A(P^i) \longrightarrow A(\mathbb{R}P(5)\times SO(3)), \quad i=1,2,$$

are isomorphisms, and then $A(P^1)$ and $A(P^2)$ are isomorphic (and, of course, are trivial).

B/ Both, R.Almeida and P.Molino [17], [18] constructed a Lie algebroid which cannot be realized as the Lie algebroid of any pfb. Now, we give a simple example of a Lie algebroid which cannot be realized as the Lie algebroid of any pfb with abelian structural Lie group.

Namely, we construct a Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \gamma)$ such that the vector bundle $\mathfrak{g}(A)$ is not trivial but all isotropy Lie algebras $\mathfrak{g}(A)_{|x}$

are abelian. Then, according to corollary 1.11, there exists no pfb with an abelian structural Lie group and with the Lie algebroid A.

EXAMPLE 8.4. Let $\mathfrak{g}$ be any vector bundle on a manifold M which is not trivial but admits of a flat covariant derivative $\nabla$. Put

$$A = \mathfrak{g} \oplus TM \quad \text{and} \quad \gamma = pr_2 : \mathfrak{g} \oplus TM \longrightarrow TM.$$

Let $\lambda : TM \longrightarrow A$ be any splitting of the following exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \oplus TM \xrightarrow{\gamma} TM \longrightarrow 0$$

In the $C^{\infty}(M)$-module $Sec(\mathfrak{g} \oplus TM)$ we introduce a structure of a Lie algebra $[\![\cdot,\cdot]\!]$ (see th. 6.2) by the formula:

$$[\![\lambda X + \sigma, \lambda Y + \eta]\!] = \lambda [X, Y] + \nabla_X \eta - \nabla_Y \sigma .$$

We obtain a Lie algebroid $(A, [\![\cdot,\cdot]\!], \gamma)$ in which the isotropy Lie algebras $\mathfrak{g}(A)_{|x}$ are abelian.

## REFERENCES


[1] M.A. ARAGNOL, "Sur la geometrie differentielle des espaces fibres", Ann. Sci. Ec. Norm. Sup., (3) 75 (1958), 257-407.

[2] M.F. ATIYAH, "Complex analytic connections in fibre bundles", Trans. A.M.S. Vol.85, No 1, 181-207, 1957.

[3] C. EHRESMANN, "Les connexions infinitesimales dans un espace fibre differentiable", Colloq. Topologie (Bruxelles 1950), Liege 1951.

[4] C. EHRESMANN, "Les prolongements d'un espace fibre differentiable", C.R.A.S. Paris, A 240 (1955), 1755-1757.

[5] W. GREUB, S. HALPERIN, R. VANSTONE, "Connections, Curvature, and Cohomology", Vol.II, Academic Press, New York and London, 1973.

[6] S. KOBAYASHI, K. NOMIZU, "Foundations of differential geometry", Vol.I, Interscience Publishers, New York, London, 1963.

[7] I. KOLAŘ, "Canonical forms on the prolongations of principal fibre bundles", Rev. Roumaine Math. Pures Appl. 1971, 16, 1091-1106.

[8] I. KOLAŘ, "Generalized G-Structures and G-Structures of Higher Order", Bollettino U.M.I. (4) 12, Supl. fasc. 3, (1975), 235-256.

[9] J. KUBARSKI, "Exponential mapping for Lie groupoids", Coll. Math. Vol. XLVII, 1982, 267-282.

[10] J. KUBARSKI, "Exponential mapping for Lie groupoids. Applications", Coll. Math., Vol. LIV, p.39-48, 1987.

[11] P. LIBERMANN, "Pseudogroupes infinitesimaux", Coll. Int. CNRS, Lille, 1959, Bull. Soc. Math. France, 87, 1959, 409-425.

[12] P. LIBERMANN, "Sur les prolongements des fibres principaux et les groupoides differentiables", Seminaire Analyse Global, Montreal, 1969.

[13] P. LIBERMANN, "Parallelismes", Journal of Diff. Geom. Vol.8, No 4, Dec. 1973, 511-539.

[14] K. MACKENZIE, "Lie Groupoids and Lie Algebroids in Differential Geometry", Cambridge University Press 1987, London Mathematical Society, Lecture Note Series 124.

[15] J. MILNOR, "On the existence of a connection with curvature zero", Comment. Math. Helv. 32 (1958), 215-223.

[16] P. MOLINO, "Champs d'elements sur un espace fibre principal differentiable", Ann. Inst. Fourier, Grenoble, 14, 2, (1964) 163-220.

[17] P. MOLINO, R. ALMEIDA, "Suites d'Atiyah et feuilletages transversalement complets", C.R.A.S. Paris, t.300, 13-15, 1985.

[18] P. MOLINO, R. ALMEIDA, "Suites d'Atiyah, feuilletages et quantification geometrique", Esctrait du Seminaire du Geometrie Differentielle, Montpellier.

[19] H. K. NICKERSON, "On differential operators and connections", Trans A.M.S. Vol.99, 509-539, 1961.

[20] NGO-VAN-QUE, "Du prolongement des espaces fibres et des structures infinitesimales", Ann. Inst. Fourier, Grenoble, 17,1, 1967, 157-223.

[21] NGO-VAN-QUE, "Sur l'espace de prolongement differentiable", J. of Diff. Geom. Vol.2, No 1, 1968, 33-40.

[22] J. PRADINES, "Theorie de Lie pour les groupoides differentiables, Calcul differential dans la categorie des groupoides infinitesimaux", C.R.A.S. Paris, t.264, 245-248, 1967.

[23] J. PRADINES, "Theorie de Lie pour les groupoides differentiables", Atti Conv. Intern. Geom. Diff. Bologna, 1967, Bologna-Amsterdam.

[24] TH. FRIEDRICH, "Zur Abhangigkeit des Dirac-operators Von Der Spin-Struktur", Coll. Math. Vol.XLVIII, Fasc 1, 57-62, 1984.

[25] J. MILNOR, "Spin Structures On Manifolds", L'Enseignement Mathematique (2) 9 (1963), 198-203.

JAN KUBARSKI
Institute of Mathematics
Technical University of Łódź
Al. Politechniki 11
90-924 Łódź
P O L A N D