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Presentation

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Travaux mathématiques

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Special Issue: Proceedings of the 8th Conference on Geometry and Topology of Manifolds

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Editors:

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Proceedings[1] of the 8th Conference on

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(Lie algebroids, dynamical systems and applications)

Luxembourg-Poland-Ukraine conference

Przemyśl (Poland) - L'viv (Ukraine) 30.04.07 - 6.05.2007

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[1]Additionally including two regular papers submitted to "Travaux Mathématiques".

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Foreword

This eighth conference of a cycle, which was initiated in 1998 with a meeting in Konopnica (see `http://im0.p.lodz.pl/konferencje/`), took place in two cities; the first part was held in Przemyśl, Poland, at the State High School of East Europe, the second part in L'viv, Ukraine, at the Ivan Franko National University of L'viv.

The main aim of the conference series is to present and discuss new results on geometry and topology of manifolds with particular attention paid to applications of algebraic methods. The topics that are usually discussed include:

- Dynamical systems on manifolds and applications
- Lie groups (including infinite dimensional ones), Lie algebroids and their generalizations, Lie groupoids
- Characteristic classes, index theory, K-theory, Fredholm operators
- Singular foliations, cohomology theories for foliated manifolds and their quotients
- Symplectic, Poisson, Jacobi and special Riemannian manifolds
- Topology of infinite-dimensional manifolds
- Applications to mathematical physics

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Jet bundles on projective space[1]

by Helge Maakestad

Abstract

In this paper we state and prove some results on the structure of the jet bundles as left and right module over the structure sheaf $\mathcal{O}$ on the projective line and projective space using elementary techniques involving diagonalization of matrices, multilinear algebra and sheaf cohomology.

1 Introduction

In this paper a complete classification of the structure of the jet bundles on the projective line and projective space $\mathbf{P}^N = \mathrm{SL}(V)/P$ as left and right P-module is given. In the first section explicit techniques and known results on the splitting type of the jet bundles as left and right module over the structure sheaf $\mathcal{O}$ are recalled. In the final section the classification of the structure of the jet bundles on projective space as left and right P-module is done using sheaf cohomology, multilinear algebra and representations of $\mathrm{SL}(V)$.

2 On the left and right structure and matrix diagonalization

In this section we recall results obtained in previous papers ([9], [10], [13], [14] and [15]) where the jets are studied as left and right module using explicit calculations involving diagonalization of matrices.

Notation. Let X/S be a separated scheme, and let p, q be the two projection maps $p, q : X \times X \to X$. There is a closed immersion

$$\Delta : X \to X \times X$$

and an exact sequence of sheaves

$$0 \to \mathcal{I}^{k+1} \to \mathcal{O}_{X\times X} \to \mathcal{O}_{\Delta^k} \to 0. \tag{2.1}$$

The sheaf $\mathcal{I}$ is the sheaf defining the diagonal in $X \times X$.

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Definition 2.1. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. We define the *k'th order jets* of $\mathcal{E}$, to be

$$\mathcal{P}^k_X(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^k} \otimes_{X\times X} q^*\mathcal{E}).$$

Let $\mathcal{P}^k_X$ denote the module $\mathcal{P}^k_X(\mathcal{O}_X)$.

There is the following result:

Proposition 2.2. *Let X/S be smooth and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. There exists an exact sequence*

$$0 \to S^k(\Omega^1_X) \otimes \mathcal{E} \to \mathcal{P}^k_X(\mathcal{E}) \to \mathcal{P}^{k-1}_X(\mathcal{E}) \to 0$$

of left $\mathcal{O}_X$-modules, where $k = 1, 2, \dots$.

Proof. For a proof see [11], section 4. □

It follows that for a smooth morphism $X \to S$ of relative dimension n, and $\mathcal{E}$ a locally free sheaf on X of rank e, the jets $\mathcal{P}^k(\mathcal{E})$ is locally free of rank $e\binom{n+k}{n}$.

Given locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ there exist the *sheaf of polynomial differential operators* of order k from $\mathcal{F}$ to $\mathcal{G}$ (following [3] section 16.8), denoted $\mathrm{Diff}^k_X(\mathcal{F}, \mathcal{G})$. There is an isomorphism

$$Hom_X(\mathcal{P}^k_X(\mathcal{F}), \mathcal{G}) \cong \mathrm{Diff}^k_X(\mathcal{F}, \mathcal{G}) \tag{2.2}$$

of sheaves of abelian groups. Hence the sheaf of jets $\mathcal{P}^k_X(\mathcal{F})$ is in a natural way a left and right $\mathcal{O}_X$-module. We write $\mathcal{P}^k(\mathcal{E})^L$ (resp. $\mathcal{P}^k(\mathcal{E})^R$) to indicate we are considering the left (resp. right) structure. Note that for X smooth over S and $\mathcal{E}$ locally free of finite rank, it follows that $\mathcal{P}^k_X(\mathcal{E})$ is locally free of finite rank as left and right $\mathcal{O}_X$-module separately.

By [4] Theorem 2.1 and [5] Theorem 3.1 we know that any finite rank locally free sheaf on $\mathbf{P}^1$ over any field splits into a direct sum of invertible sheaves. The formation of jets commutes with direct sums, hence if we can decompose the jet $\mathcal{P}^k(\mathcal{O}(d))$ into line bundles for any line bundle $\mathcal{O}(d)$ with d an integer it follows we have given an complete classification of the jet $\mathcal{P}^k(\mathcal{F})$ for any locally free finite rank sheaf $\mathcal{F}$ on the projective line. In the paper [10] the decomposition of the sheaf of jets is studied and the following structure theorem is obtained:

Theorem 2.3. *Let $k \geq 1$, and consider $\mathcal{P}^k(\mathcal{O}(d))$ as left $\mathcal{O}_{\mathbf{P}}$-module. If $k \leq d$ or $d < 0$, there is an isomorphism*

$$\mathcal{P}^k(\mathcal{O}(d))^L \cong \oplus^{k+1}\mathcal{O}(d-k) \tag{2.3}$$

of $\mathcal{O}$-modules. If $0 \leq d < k$ there is an isomorphism

$$\mathcal{P}^k(\mathcal{O}(d))^L \cong \mathcal{O}^{d+1} \oplus \mathcal{O}(d-k-1)^k \tag{2.4}$$

as left $\mathcal{O}$-modules. Let $b \in \mathbf{Z}$ and $k \geq 1$. There is an isomorphism

$$\mathcal{P}^k(\mathcal{O}(d))^R \cong \mathcal{O}(d) \oplus \mathcal{O}(d-k-1)^k \tag{2.5}$$

as right $\mathcal{O}$-modules.

Proof. See [10]. □

The proof of this result is done by calculating and diagonalizing the transition matrix of the jet bundle. The result gives a complete classification of the jets of an arbitrary locally free sheaf on the projective line as left and right module over the structure sheaf.

3 On the left and right P-module structure on the projective line

In this section we give a complete classification of the left and right P-module structure of the jets on the projective line over a field of characteristic zero using the same techniques as in [8].

Let in general V be a finite dimensional vector space over a field F of characteristic zero. Any affine algebraic group G is a closed subgroup of $\mathrm{GL}(V)$ for some V, and given any closed subgroup $H \subseteq G$, there exists a quotient map $G \to G/H$ with nice properties. The variety G/H is smooth and quasi projective, and the quotient map is universal with respect to H-invariant morphisms of varieties. F-rational points of the quotient G/H correspond to orbits of H in G (see [7], section I.5). Moreover: any finite-dimensional H-module ρ gives rise to a finite rank G-homogeneous vector bundle $E = E(\rho)$ and by [1], chapter 4 this correspondence sets up an equivalence of categories between the category of linear finite dimensional representations of H and the category of finite rank G-homogeneous vector bundles on G/H. There exists an equivalence of categories between the category of finite rank G-homogeneous vector bundles and the category of finite rank locally free sheaves with a G-linearization, hence we will use these two notions interchangeably.

Fix a line L in V, and let P be the closed subgroup of $\mathrm{SL}(V)$ stabilizing L. The quotient $\mathrm{SL}(V)/P$ is naturally isomorphic to projective space $\mathbf{P}$ parametrizing lines in V, and if we choose a basis $e_0, \cdots, e_n$ for V, the quotient map

$$\pi : \mathrm{SL}(V) \to \mathbf{P}$$

can be chosen to be defined as follows: map any matrix A to its first column-vector. It follows that π is locally trivial in the Zariski topology, in fact it trivializes over the basic open subsets U_i of projective space. One also checks that any $\mathrm{SL}(V)$-homogeneous vectorbundle on $\mathbf{P}$ trivializes over the basic open subsets U_i. Any line bundle $\mathcal{O}(d)$ on projective space is $\mathrm{SL}(V)$-homogeneous coming from a unique character of P since $\mathrm{SL}(V)$ has no characters. The sheaf of jets $\mathcal{P}^k(\mathcal{O}(d))$ is a sheaf of bi-modules, locally free as left and right $\mathcal{O}$-module separately. It has a left and right $\mathrm{SL}(V)$-linearization, hence we may classify the P-module structure of $\mathcal{P}^k(\mathcal{O}(d))$ corresponding to the left and right structure, and that is the aim of this section.

The calculation of the representations corresponding to $\mathcal{P}^k(\mathcal{O}(d))$ as left and right P-module is contained in Theorems 3.4 and 3.6. As a byproduct we obtain the classification of the splitting type of the jet bundles obtained in [10]: this is Corollaries 3.5 and 3.7.

Let, in the following section V be a vector space over F of dimension two. $\mathbf{P} = \mathrm{SL}(V)/P$ is the projective line parametrizing lines in V. There exists two exact sequences of P-modules.

$$0 \to L \to V \to Q \to 0$$

and

$$0 \to m \to V^* \to L^* \to 0,$$

and one easily sees that $m \cong L$ as P-module.

Let $p, q : \mathbf{P} \times \mathbf{P} \to \mathbf{P}$ be the canonical projection maps, and let $I \subseteq \mathcal{O}_{\mathbf{P}\times\mathbf{P}}$ be the ideal of the diagonal. Let $\mathcal{O}_{\Delta^k} = \mathcal{O}_{\mathbf{P}\times\mathbf{P}}/I^{k+1}$ be the $k'th$ *order infinitesimal neighborhood of the diagonal.* Recall the definition of the sheaf of jets:

Definition 3.1. Let $\mathcal{E}$ be an $\mathcal{O}_{\mathbf{P}}$-module. Let $k \geq 1$ and let

$$\mathcal{P}^k(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{E})$$

be the *k'th order sheaf of jets of* $\mathcal{E}$.

If $\mathcal{E}$ is a sheaf with an $\mathrm{SL}(V)$-linearization, it follows that $\mathcal{P}^k(\mathcal{E})$ has a canonical $\mathrm{SL}(V)$-linearization. There is an exact sequence:

$$0 \to I^{k+1} \to \mathcal{O}_{\mathbf{P}\times\mathbf{P}} \to \mathcal{O}_{\Delta^k} \to 0. \tag{3.1}$$

Apply the functor $\mathrm{R}\, p_*(- \otimes q^*\mathcal{O}(d))$ to the sequence 3.1 to obtain a long exact sequence of $\mathrm{SL}(V)$-linearized sheaves

$$0 \to p_*(I^{k+1} \otimes q^*\mathcal{O}(d)) \to p_*q^*\mathcal{O}(d) \to \mathcal{P}^k(\mathcal{O}(d))^L \to \tag{3.2}$$

$$\mathrm{R}^1\, p_*(I^{k+1} \otimes q^*\mathcal{O}(d)) \to \mathrm{R}^1\, p_*q^*\mathcal{O}(d) \to \mathrm{R}^1\, p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{O}(d)) \to \cdots$$

of $\mathcal{O}_{\mathbf{P}}$-modules. We write $\mathcal{P}^k(\mathcal{O}(d))^L$ to indicate that we use the left structure of the jets. We write $\mathcal{P}^k(\mathcal{O}(d))^R$ to indicate right structure. The terms in the sequence 3.2 are locally free since they are coherent and any coherent sheaf with an $\mathrm{SL}(V)$-linearization is locally free.

Proposition 3.2. *Let $\mathcal{E}$ be an $\mathrm{SL}(V)$-linearized sheaf with support in $\Delta \subseteq \mathbf{P}\times\mathbf{P}$. For all $i \geq 1$ the following holds:*

$$\mathrm{R}^i\, p_*(\mathcal{E}) = \mathrm{R}^i\, q_*(\mathcal{E}) = 0.$$

Proof. Let $x \in \mathbf{P}$ be the distinguished point and consider the fiber diagram

$$\begin{tikzcd} \mathbf{P} \arrow[r, "j"] \arrow[d, "\tilde{p},\tilde{q}"] & \mathbf{P} \times \mathbf{P} \arrow[d, "p,q"] \\ \mathrm{Spec}(\kappa(x)) \arrow[r, "i"] & \mathbf{P} \end{tikzcd}$$

Since $\mathrm{R}^i\, p_*(\mathcal{E})$ and $\mathrm{R}^i\, q_*(\mathcal{E})$ have an $\mathrm{SL}(V)$-linearization it is enough to check the statement of the lemma on the fiber at x. We get by [6], Proposition III.9.3 isomorphisms

$$\mathrm{R}^i\, p_*(\mathcal{E})(x) \cong \mathrm{R}^i\, \tilde{p}_*(j^*\mathcal{E})$$

and

$$\mathrm{R}^i\, q_*(\mathcal{E})(x) \cong \mathrm{R}^i\, \tilde{q}_*(j^*\mathcal{E}),$$

and since $j^*\mathcal{E}$ is supported on a zero-dimensional scheme, the lemma follows. □

It follows from the Lemma that $\mathrm{R}^1\, p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{O}(d)) = \mathrm{R}^1\, q_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{O}(d)) = 0$ since $\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{O}(d)$ is supported on the diagonal. Hence we get an exact sequence of P-modules when we pass to the fiber of 3.2 at the distinguished point $x \in \mathbf{P}$. Let $\mathfrak{m} \subseteq \mathcal{O}_{\mathbf{P}}$ be the sheaf of ideals of x. By [6] Theorem III.12.9 and Lemma 3.2 we get the following exact sequence of P-modules:

$$(3.3) \qquad 0 \to \mathrm{H}^0(\mathbf{P}, \mathfrak{m}^{k+1} \otimes \mathcal{O}(d)) \to \mathrm{H}^0(\mathbf{P}, \mathcal{O}(d)) \to \mathcal{P}^k(\mathcal{O}(d))(x) \to$$

$$\mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1} \otimes \mathcal{O}(d)) \to \mathrm{H}^1(\mathbf{P}, \mathcal{O}(d)) \to 0.$$

Proposition 3.3. *Let $k \geq 1$ and $d < k$. Then there is an isomorphism*

$$\mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1} \otimes \mathcal{O}(d)) \cong \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-d-1}(V)$$

of P-modules.

Proof. There is an isomorphism of sheaves $\mathcal{O}(-k-1) \cong \mathfrak{m}^{k+1}$ defined as follows:

$$x_0^{-k-1} \to t^{k+1}$$

on the open set $D(x_0)$ where $t = x_1/x_0$. On the open set $D(x_1)$ it is defined as follows:

$$x_1^{-k-1} \to 1.$$

We get an isomorphism $\mathcal{O}(d-k-1) \cong \mathfrak{m}^{k+1} \otimes \mathcal{O}(d)$ of sheaves, but the corresponding inclusion of sheaves

$$\mathcal{O}(d-k-1) \to \mathcal{O}(d)$$

is not a map of P-linearized sheaves, since it is zero on the fiber at x. Hence we must twist by the character of $\mathfrak{m}^{k+1} = \mathcal{O}(-k-1)$ when we use Serre-duality. We get

$$\mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1} \otimes \mathcal{O}(d)) \cong \mathrm{Sym}^{k+1}(L) \otimes \mathrm{H}^0(\mathbf{P}, \mathcal{O}(k-d-1))^* \cong$$
$$\mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-d-1}(V),$$

and the proposition follows. □

We first give a complete classification of the left P-module structure of the jets. The result is the following.

Theorem 3.4. *Let $k \geq 1$, and consider $\mathcal{P}^k(\mathcal{O}(d))$ as left $\mathcal{O}_{\mathbf{P}}$-module. If $k \leq d$, there exists an isomorphism*

$$\mathcal{P}^k(\mathcal{O}(d))^L(x) \cong \mathrm{Sym}^{k-d}(L^*) \otimes \mathrm{Sym}^k(V^*) \tag{3.4}$$

of P-modules. If $d \geq 0$ there exist an isomorphism

$$\mathcal{P}^k(\mathcal{O}(-d))^L(x) \cong \mathrm{Sym}^{k+d}(L) \otimes \mathrm{Sym}^k(V) \tag{3.5}$$

of P-modules. If $0 \leq d < k$ there exist an isomorphism

$$\mathcal{P}^k(\mathcal{O}(d))^L(x) \cong \mathrm{Sym}^d(V^*) \oplus \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-d-1}(V) \tag{3.6}$$

of P-modules.

Proof. The isomorphism from 3.4 follows from Theorem 2.4 in [8]. We prove the isomorphism 3.5: Since $-d < 0$ we get an exact sequence

$$0 \to \mathcal{P}^k(\mathcal{O}(-d))(x) \to \mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1} \otimes \mathcal{O}(-d)) \to \mathrm{H}^1(\mathbf{P}, \mathcal{O}(-d)) \to 0$$

of P-modules. By proposition 3.3 we get the exact sequence of P-modules

$$0 \to \mathcal{P}^k(\mathcal{O}(-d))^L(x) \to \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{d+k-1}(V) \to \mathrm{Sym}^{d-2}(V) \to 0.$$

The map on the right is described as follows: dualize to get the following:

$$\mathrm{Sym}^{d-2}(V^*) \cong \mathrm{Sym}^{k+1}(L^*) \otimes \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{d-2}(V^*) \cong$$
$$\mathrm{Sym}^{k+1}(L^*) \otimes \mathrm{Sym}^{k+1}(\mathfrak{m}) \otimes \mathrm{Sym}^{d-2}(V^*) \to \mathrm{Sym}^{k+1}(L^*) \otimes \mathrm{Sym}^{d+k-1}(V^*).$$

The map is described explicitly as follows:

$$f(x_0, x_1) \to x_0^{k+1} \otimes x_1^{k+1} f(x_0, x_1).$$

If we dualize this map we get a map

$$\mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{d+k-1}(V) \to \mathrm{Sym}^{d-2}(V),$$

given explicitly as follows:

$$e_0^{k+1} \otimes f(e_0, e_1) \to \phi(f(e_0, e_1)),$$

where ϕ is $k+1$ times partial derivative with respect to the e_1-variable. There exists a natural map

$$\mathrm{Sym}^{k+d}(L) \otimes \mathrm{Sym}^k(V) \to \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{d+k-1}(V)$$

given by

$$e_0^{k+d} \otimes f(e_0, e_1) \to e_0^{k+1} \otimes e_0^{d-1} f(e_0, e_1)$$

and one checks that this gives an exact sequence

$$0 \to \mathrm{Sym}^{k+d}(L) \otimes \mathrm{Sym}^k(V) \to \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{d+k-1}(V) \to$$
$$\mathrm{Sym}^{d-2}(V) \to 0,$$

hence we get an isomorphism

$$\mathcal{P}^k(\mathcal{O}(-d))^L(x) \cong \mathrm{Sym}^{k+d}(L) \otimes \mathrm{Sym}^k(V),$$

and the isomorphism from 3.5 is proved. We next prove isomorphism 3.6: By vanishing of cohomology on the projective line we get the following exact sequence:

$$0 \to \mathrm{H}^0(\mathbf{P}, \mathcal{O}(d)) \to \mathcal{P}^k(\mathcal{O}(d))^L(x) \to \mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1} \otimes \mathcal{O}(d)) \to 0.$$

We get by Proposition 3.3 an exact sequence of P-modules

$$0 \to \mathrm{Sym}^d(V^*) \to \mathcal{P}^k(\mathcal{O}(d))^L(x) \to \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-d-1}(V) \to 0.$$

It splits because of the following:

$$\mathrm{Ext}^1_P(\mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-d-1}(V), \mathrm{Sym}^d(V^*)) =$$
$$\mathrm{Ext}^1_P(\rho, \mathrm{Sym}^{k+1}(L^*) \otimes \mathrm{Sym}^{k-d-1}(V^*) \otimes \mathrm{Sym}^{k-1}(V^*)).$$

Here ρ is the trivial character of P. Since there is an equivalence of categories between the category of P-modules and $\mathrm{SL}(V)$-linearized sheaves we get again by equivariant Serre-duality

$$\mathrm{Ext}^1_P(\rho, \mathrm{Sym}^{k+1}(L^*) \otimes \mathrm{Sym}^{k-d-1}(V^*) \otimes \mathrm{Sym}^{k-1}(V^*)) = \mathrm{H}^1(\mathbf{P}, \mathcal{O}(k+1) \otimes \mathcal{E}_1) =$$
$$\oplus^{r_1} \mathrm{H}^1(\mathbf{P}, \mathcal{O}(k+1)) = \oplus^{r_1} \mathrm{H}^0(\mathbf{P}, \mathcal{O}(-k-3))^* = 0.$$

Here $\mathcal{E}_1$ is the $\mathrm{SL}(V)$-linearized sheaf corresponding to $\mathrm{Sym}^{k-d-1}(V^*) \otimes \mathrm{Sym}^{k-1}(V^*)$ and r_1 is the rank of $\mathcal{E}_1$. Hence we get

$$\mathcal{P}^k(\mathcal{O}(d))^L(x) \cong \mathrm{Sym}^d(V^*) \oplus \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-d-1}(V),$$

and the isomorphism from 3.6 is proved hence the theorem follows. $\square$

As a corollary we get a result on the splitting type of the jets as left module on the projective line.

Corollary 3.5. *The splitting type of $\mathcal{P}^k(\mathcal{O}(d))$ as left $\mathcal{O}_{\mathbf{P}}$-module is as follows: If $k \geq 1$ and $d < 0$ or $d \geq k$ there exist an isomorphism*

$$\mathcal{P}^k(\mathcal{O}(d))^L \cong \oplus^{k+1}\mathcal{O}(d-k) \tag{3.7}$$

of left $\mathcal{O}$-modules. If $0 \leq d < k$ there exist an isomorphism

$$\mathcal{P}^k(\mathcal{O}(d))^L \cong \mathcal{O}^{d+1} \oplus \mathcal{O}(-k-1)^{k-d} \tag{3.8}$$

of left $\mathcal{O}$-modules.

Proof. This follows directly from Theorem 3.4. □

We next give a complete classification of the right P-module structure of the jets. The result is the following.

Theorem 3.6. *Let $k \geq 1$ and consider $\mathcal{P}^k(\mathcal{O}(d))$ as right module. If $d > 0$ there exist an isomorphism*

$$\mathcal{P}^k(\mathcal{O}(-d))^R(x) \cong \mathrm{Sym}^d(L) \oplus \mathrm{Sym}^{k+d+1}(L) \otimes \mathrm{Sym}^{k-1}(V) \tag{3.9}$$

of P-modules. If $d \geq 0$ there exist an isomorphism

$$\mathcal{P}^k(\mathcal{O}(d))^R(x) \cong \mathrm{Sym}^d(L^*) \oplus \mathrm{Sym}^{d-k-1}(L^*) \otimes \mathrm{Sym}^{k-1}(V) \tag{3.10}$$

as P-modules.

Proof. We prove the isomorphism 3.9: Using the functor $\mathrm{R}^i\, q_*(- \otimes q^*\mathcal{O}(-d))$ we get a long exact sequence of $\mathrm{SL}(V)$-linearized sheaves

$$0 \to q_*(I^{k+1}) \otimes \mathcal{O}(-d) \to q_*q^*\mathcal{O}(-d) \to \mathcal{P}^k(\mathcal{O}(-d))^R \to \mathrm{R}^1\, q_*(I^{k+1}) \otimes \mathcal{O}(-d) \to$$

$$\mathrm{R}^1\, q_*(\mathcal{O}_{\mathbf{P}\times\mathbf{P}}) \otimes \mathcal{O}(-d) \to 0.$$

It is exact on the right because of Proposition 3.2. Here we write $\mathcal{P}^k(\mathcal{O}(d))^R$ to indicate we use the right structure of the jets. We take the fiber at $x \in \mathbf{P}$ and using Cech-calculations for coherent sheaves on the projective line, we obtain the following exact sequence of P-modules:

$$0 \to \mathrm{H}^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}) \otimes \mathcal{O}(-d)(x) \to \mathcal{P}^k(\mathcal{O}(-d))^R(x) \to \mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1}) \otimes \mathcal{O}(-d)(x) \to 0$$

Hence we get by Proposition 3.3 the following exact sequence of P-modules:

$$0 \to \mathrm{Sym}^d(L) \to \mathcal{P}^k(\mathcal{O}(-d))^R(x) \to \mathrm{Sym}^d(L) \otimes \mathrm{Sym}^{k+1}(L) \otimes \mathrm{Sym}^{k-1}(V) \to 0$$

It is split exact because of the following argument using Ext's and equivariant Serre-duality:

$$\mathrm{Ext}^1_P(\mathrm{Sym}^{d+k+1}(L)\otimes \mathrm{Sym}^{k-1}(V), \mathrm{Sym}^d(L)) = \mathrm{Ext}^1_P(\rho, \mathrm{Sym}^{k+1}(L^*)\otimes \mathrm{Sym}^k(V^*)),$$

where ρ is the trivial character of P. By equivariant Serre duality we get

$$\mathrm{Ext}^1_P(\rho, \mathrm{Sym}^{k+1}(L^*)\otimes \mathrm{Sym}^k(V^*)) = \mathrm{H}^1(\mathbf{P}, \mathcal{O}(k+1)\otimes \mathcal{E}_2) =$$
$$= \oplus^{r_2}\mathrm{H}^0(\mathbf{P}, \mathcal{O}(-k-3))^* = 0.$$

Here $\mathcal{E}_2$ is the abstract vector bundle corresponding to $\mathrm{Sym}^k(V^*)$ and r_2 is the rank of $\mathcal{E}_2$. Hence we get the desired isomorphism

$$\mathcal{P}^k(\mathcal{O}(-d))^R(x) \cong \mathrm{Sym}^d(L)\oplus \mathrm{Sym}^{d+k+1}(L)\otimes \mathrm{Sym}^{k-1}(V),$$

and isomorphism 3.9 is proved.

We next prove the isomorphism 3.10: Using the functor $\mathrm{R}^i\, q_*(-\otimes q^*\mathcal{O}(d))$ we get a long exact sequence of $\mathrm{SL}(V)$-linearized sheaves

$$0 \to q_*(I^{k+1})\otimes \mathcal{O}(d) \to q_*q^*\mathcal{O}(d) \to \mathcal{P}^k(\mathcal{O}(d))^R \to \mathrm{R}^1\, q_*(I^{k+1})\otimes \mathcal{O}(d) \to$$
$$\mathrm{R}^1\, q_*(\mathcal{O}_{\mathbf{P}\times\mathbf{P}})\otimes \mathcal{O}(d) \to 0.$$

We take the fiber at $x\in\mathbf{P}$ and using Cech-calculations for coherent sheaves on the projective line, we obtain the following exact sequence of P-modules:

$$0 \to \mathrm{H}^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}})\otimes \mathcal{O}(d)(x) \to \mathcal{P}^k(\mathcal{O}(d))^R(x) \to \mathrm{H}^1(\mathbf{P}, \mathfrak{m}^{k+1})\otimes \mathcal{O}(d)(x) \to 0.$$

Proposition 3.3 gives the following sequence of P-modules

$$0 \to \mathrm{Sym}^d(L^*) \to \mathcal{P}^k(\mathcal{O}(d))^R(e) \to \mathrm{Sym}^d(L^*)\otimes \mathrm{Sym}^{k+1}(L)\otimes \mathrm{Sym}^{k-1}(V) \to 0$$

It splits because of the following Ext and equivariant Serre-duality argument:

$$\mathrm{Ext}^1_P(\mathrm{Sym}^d(L^*)\otimes \mathrm{Sym}^{k+1}(L)\otimes \mathrm{Sym}^{k-1}(V), \mathrm{Sym}^d(L^*)) =$$
$$\mathrm{Ext}^1_P(\rho, \mathrm{Sym}^{k+1}(L^*)\otimes \mathrm{Sym}^{k-1}(V^*)).$$

Let $\mathcal{E}_3$ be the vector bundle corresponding to $\mathrm{Sym}^{k-1}(V^*)$ and let r_3 be its rank. We get

$$\mathrm{Ext}^1_P(\rho, \mathrm{Sym}^{k+1}(L^*)\otimes \mathrm{Sym}^{k-1}(V^*)) = \mathrm{H}^1(\mathbf{P}, \mathcal{O}(k+1)\otimes \mathcal{E}_3) =$$
$$= \oplus^{r_3}\mathrm{H}^0(\mathbf{P}, \mathcal{O}(-k-3))^* = 0,$$

hence we get the isomorphism

$$\mathcal{P}^k(\mathcal{O}(d))^R(x) \cong \mathrm{Sym}^d(L^*)\oplus \mathrm{Sym}^{d-k-1}(L^*)\otimes \mathrm{Sym}^{k-1}(V),$$

and isomorphism 3.10 follows. $\square$

As a corollary we get a result on the splitting type of the jets as right module.

Corollary 3.7. *Let $k \geq 1$ and $d \in \mathbf{Z}$. There exist an isomorphism*

$$\mathcal{P}^k(\mathcal{O}(d))^R \cong \mathcal{O}(d) \oplus \mathcal{O}(d-k-1)^k \tag{3.11}$$

of right $\mathcal{O}_{\mathbf{P}}$-modules.

Proof. This follows directly from Theorem 3.6. □

Note that Corollary 3.5 and 3.7 recover Theorem 3.4, hence we have used elementary properties of representations of $\mathrm{SL}(V)$ to classify sheaves of left and right modules on the projective line over any field of characteristic zero.

On projective space of higher dimension there is the following result: Let $\mathbf{P}^N = \mathrm{SL}(V)/P$ where P is the subgroup fixing a line, and let $\mathcal{O}(d)$ be the line bundle with $d \in \mathbf{Z}$. It follows $\mathcal{O}(d)$ has a canonical $\mathrm{SL}(V)$-linearization.

Theorem 3.8. *For all $1 \leq k < d$, the representation corresponding to $\mathcal{P}^k(\mathcal{O}(d))^L$ is $\mathrm{Sym}^{d-k}(L^*) \otimes \mathrm{Sym}^k(V^*)$.*

Proof. See [8]. □

Note that the result in Theorem 3.8 is true over any field F if $char(F) > n$.

Corollary 3.9. *For all $1 \leq k < d$, $\mathcal{P}^k(\mathcal{O}(d))$ decompose as left $\mathcal{O}$-module as $\oplus^{\binom{N+k}{N}}\mathcal{O}(d-k)$.*

Proof. See [8]. □

Note. Theorem 3.4 and 3.6 give a complete classification of the structure of the jets as left and right P-module for any line bundle $\mathcal{O}(d)$.

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Around Birkhoff Theorem[1]

by O.Petrenko and I.V.Protasov

Abstract

Let X be a topological space, $f : X \to X$ be a mapping (not necessarily continuous). A point $x \in X$ is recurrent if x is a limit point of the orbit $(f^n(x))_{n\in\mathbb{N}}$. We prove that, for a Hausdorff space X, every bijection has a recurrent point if and only if X is either finite or a one-point compactification of an infinite discrete space.

1 Introduction

Let X be a topological space, $f : X \to X$ be an arbitrary mapping. A point $x \in X$ is said to be *recurrent* if for every neighbourhood U of x and every $n \in \omega$, there exists $m > n$ such that $f^m(x) \in U$ (in other words, x is a limit point of the orbit $(f^n(x))_{n\in\mathbb{N}}$). By Birkhoff Theorem ([1],[2]), every continuous mapping $f : X \to X$ of a compact space X has a recurrent point. We are going to prove the following "discontinuous" version of Birkhoff Theorem.

Theorem 1.1. *For a Hausdorff space X, the following statements are equivalent:*
(i) every mapping $f : X \to X$ has a recurrent point;
(ii) every bijection $f : X \to X$ has a recurrent point;
(iii) X is either finite or a one-point compactification of an infinite discrete space.

2 Auxiliary lemmas

For proof of Theorem, we need two lemmas. Remind that a subspace Y of a topological space X is *discrete in itself* if, for every $y \in Y$ there exists a neighbourhood U of y, such that $U \cap Y = \{y\}$.

Lemma 2.1. *For every infinite Hausdorff space X, there exists a disjoint family of countable discrete in itself subspaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ such that either $X \setminus \bigcup_{\alpha\in\mathcal{A}} X_\alpha = \varnothing$ or $X \setminus \bigcup_{\alpha\in\mathcal{A}} X_\alpha$ is a singleton $\{x\}$ and x is a limit point of every subspace X_α.*

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Proof. We use the following auxiliary statement: every infinite Hausdorff space S has a countable subspace D discrete in itself. Indeed, if S is discrete, it is clear. Otherwise, we fix some non-isolated point $s \in S$, choose an arbitrary element $d_0 \in S$, $d_0 \neq s$ and the disjoint open neighbourhoods U_0, V_0 of d_0 and s. Then we pick an arbitrary element $d_1 \in V_0$ and disjoint open neighbourhoods U_1, V_1 of d_1 and s such that $U_1 \subseteq V_0$, $V_1 \subseteq V_0$. After $\mathbb{N}$ steps we get the sequence $(d_n)_{n\in\mathbb{N}}$ of elements of S and the sequence $(U_n)_{n\in\mathbb{N}}$ of its neighbourhoods such that $U_i \cap U_j = \varnothing$ for all distinct $i, j \in \mathbb{N}$. Then $D = \{d_n : n \in \mathbb{N}\}$ is discrete in itself.

By Zorn Lemma, there exists a maximal disjoint family $\{X_\alpha : \alpha \in \mathcal{A}\}$ of countable discrete in itself subspaces of X. By the auxiliary statement, applying to $S = X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha$, we conclude that $X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha$ is finite.

We assume that $X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha \neq \varnothing$ and take an arbitrary element $y \in X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha$. If y is not a limit point of some subspace X_β, we put $X'_\beta = X_\beta \cup \{y\}$ and $X'_\alpha = X_\alpha$ for all $\alpha \neq \beta$. Then $\{X'_\alpha : \alpha \in \mathcal{A}\}$ is a disjoint family of countable discrete in itself subspaces of X and $|X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha| > |X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X'_\alpha|$. Repeating this arguments, we may suppose that every element $y \in X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha$ is a limit point of each subspace X_α.

If $|X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha| > 1$, we fix two arbitrary points $y, z \in X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha$ and its disjoint neighbourhoods U and V. For one fixed $\alpha_0 \in \mathcal{A}$ we put

$$X'_{\alpha_0} = (X_{\alpha_0} \cap U) \cup \{z\}, \;\; X''_{\alpha_0} = (X_{\alpha_0} \setminus U) \cup \{y\}$$

For every $\alpha \in \mathcal{A} \setminus \{\alpha_0\}$, we put

$$X'_\alpha = (X_\alpha \cap U), \;\; X''_\alpha = (X_\alpha \setminus U)$$

Then $\{X'_\alpha, X''_\alpha : \alpha \in \mathcal{A}\}$ is a disjoint family of countable subspaces discrete in itself, and $|X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} (X'_\alpha \cup X''_\alpha)| < |X \setminus \bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha|$. Repeating the arguments of above and this paragraphs, after finite number of steps we get a desired family of subspaces of X. □

Lemma 2.2. *An infinite Hausdorff space X is a one-point compactification of a discrete space if and only if, for every partition $X = \bigsqcup\limits_{\alpha\in\mathcal{A}} X_\alpha$ of X to countable subspaces, at least one subspace of the partition is not discrete in itself.*

Proof. Let X be a one-point compactification of discrete subspace D, $\{x\} = X \setminus D$. Given any partition $X = \bigsqcup\limits_{\alpha\in\mathcal{A}} X_\alpha$ of X to countable subspaces, we take $\beta \in \mathcal{A}$ such that $x \in X_\beta$. Then x is non-isolated point of X_β, so X_β is not discrete in itself.

On the other hand, let X satisfy the partition condition of lemma. We take the family $\{X_\alpha : \alpha \in \mathcal{A}\}$ given by Lemma 1. If $X \setminus \bigcup_{\alpha\in\mathcal{A}} X_\alpha = \varnothing$, we get a contradiction to the partition condition, so $X \setminus \bigcup_{\alpha\in\mathcal{A}} X_\alpha = \{x\}$. We assume that $X \setminus V$ is infinite for some neighbourhood V of x, and choose a countable discrete in itself subspace D of $X \setminus V$, put $Y = D \cup \{x\}$ and $Y_\alpha = X_\alpha \setminus D$. Since x is a limit point of each X_α, every subspace Y_α is countable. Then $X = Y \cup \bigcup_{\alpha\in\mathcal{A}} Y_\alpha$, each subspace $Y, Y_\alpha, \alpha \in \mathcal{A}$ is countable and discrete in itself, so we again arrive on the contradiction with the partition condition. Hence, $X \setminus V$ is finite for every neighbourhood V of x, so X is a one-point compactification of discrete space $X \setminus \{x\}$. $\square$

3 Proof of main result

Proof of Theorem 1. The implication (i)$\Rightarrow$(ii) is trivial.

To show (ii)$\Rightarrow$(iii), we assume that X is infinite, but X is not a one-point compactification of a discrete space. By Lemma 2, there is a partition $X = \bigsqcup_{\alpha\in\mathcal{A}} X_\alpha$ such that each cell X_α is countable and discrete in itself. For every $\alpha \in \mathcal{A}$, we fix some bijection $f_\alpha : X_\alpha \to X_\alpha$ without periodic points. Put $f = \bigsqcup_{\alpha\in\mathcal{A}} f_\alpha$. Then $f : X \to X$ is a bijection without recurrent points.

If X is finite then every mapping $f : X \to X$ has a periodic point which is recurrent. Let X be a one-point compactification of an infinite discrete space D, $\{x\} = X \setminus D$, $f : X \to X$. If x is not a limit point of the orbit $\{f^n(x) : n \in \mathbb{N}\}$, then there exists a neighbourhood U of x and $n \in \mathbb{N}$ such that $f^m(x) \notin U$ for every $m > n$. Since $X \setminus U$ is finite, at least one point from $X \setminus U$ is periodic. Hence, (iii)$\Rightarrow$(i).

$\square$

Problem 3.1. Detect all Hausdorff spaces such that every continuous mapping $f : X \to X$ has a recurrent point. Does there exist non-countably compact space with this property?

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Zero-emission surfaces of a moving electron[1]

by Yu. A. Aminov

Abstract

The motion of an electron in a constant magnetic field and its electromagnetic emission are presented. We consider a zero-emission ruled surface of the electron. The theorem on existence and uniqueness of a second electron with the same zero-emission surface is proved. The notion of "conjugate" electron is introduced and the formula for distance between two "conjugate" electrons is given.

Mathematics Subject Classification. Primary 53A05, 78A35, 78A40; Secondary 53A25.

Keywords. Ruled surface, helix, electron, electromagnetic field.

"...Maxwell has developed a complete mathematical theory to describe electromagnetism and showed that charges moving with acceleration emit..."
Abdus Salam "Unification of forces"

1 Introduction

The classical problem about a motion of a charge in a constant magnetic field is well investigated. It seems that impossible to discover something new here. But geometrical point of view allows to add interesting knowledge to this old problem. It is well known, that the motion of a charge with acceleration be accompanied by emission of electromagnetic field. This emission is going from every point of the trajectory of the charge at moment, when the charge passes this point. If the charge is an electron, so this emission is considerable. For example, the motion of the stream of electrons in a synchrotron gives the strong synchrotron emission. For proton, which mass is at 2000 times larger than electron one, his emission is not such important. Therefore we speak further about the electron motion.

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There exists physical theory to describe this emission, which we use in present article. The formula for emission, represented in [3], gives two directions, starting from the point of trajectory, for which the emission is equal to zero. We consider the ruled surface in E^3 with the trajectory as a directrix and a generatrix, going in the direction with emission equal to zero. More precisely, we must take the ray in this direction, but complete surface is more comfortable for consideration. That's the way " zero- emission surface" arises.

As there exist two directions with emission equal to zero, so there are two zero-emission surfaces.

This surface has some interesting geometrical properties. In the section 1 we show that the trajectory is an asymptotic curve on it. It is natural to put the following question: does determined zero-emission surface the electron, which its generate. It is purely mathematical question about uniqueness. If to speak more precisely so the talk is going about the trajectory and the law of motion along this trajectory. We show that here the uniqueness does not have place. In the section 2 the following theorem is been proving

Theorem 1.1. *For an electron in a constant magnetic field every zero-emission surface contains a unique second electron such that these electrons lie throughout their motion in the common zero-emission line.*

Denote the first electron Q_1 and the second Q_2. We indicate the place for Q_2 and calculate the distance between Q_1 and Q_2, which is a constant.

In the section 2 the notion of geometrically consistent (coherent) motion of the third electron Q_3 with respect to the electrons Q_1, Q_2 is given. It is motion, when Q_3 lies on the straight -line Q_1Q_2 and distance q between Q_1 and Q_3 is constant. We use the word "geometrically" for emphasize that here we ignore influence by Coulomb fields and emissions of every electrons each other. It have place

Theorem 1.2. *There exists a motion of electron Q_3 along helix $q = const$ on the zero-emission surface geometrically consistent (coherent) with the motions of Q_1 and Q_2.*

2 The trajectory of an electron as the asymptotic line on the zero-emission surface

Let us consider the motion of a point charge (for example, an electron) in a constant magnetic field $H = H_0 = const$ within the bounds of classical electrodynamics. If the motion is with acceleration then there exists an emission of electromagnetic field. In the classical electrodynamics the emission is described by two vector fields - electric E_{emit} and magnetic B_{emit}. The fields E_{emit} and B_{emit} at different points have different intensities and vectors. We have found some correlation between emission and some ruled surfaces (see [1], [2]) and present it here with some new details.

If $x(t)$ is a vector position of a point of the electron trajectory, then the equation of motion is the following

$$\frac{d^2x}{dt^2} = -\frac{e}{mc}[\frac{dx}{dt}H_0], \tag{1}$$

where t is the time, e is the electric charge, m is its mass and c is the light velocity. It is well known that the charge moves along a straight line or a circle or a helix. Denote

$$\mu_0 = -\frac{e}{mc}.$$

By integration of (1) we obtain

$$\frac{dx}{dt} = \mu_0[xH_0] + q, \tag{2}$$

where $q = \{q_i\}$ is the constant vector. Let e_1, e_2, e_3 be an orthogonal frame in E^3 and $H_0 = he_3$. We can rewrite Equation (2) in the form of a system of equations

$$\begin{aligned} \frac{dx_1}{dt} &= \mu_0 h x_2 + q_1, \\ \frac{dx_2}{dt} &= -\mu_0 h x_1 + q_2, \\ \frac{dx_3}{dt} &= q_3. \end{aligned} \tag{3}$$

Let the initial data for $t = 0$ be

$$x_1(0) = x_{10}, x_2(0) = x_{20}, x_3(0) = 0, \frac{dx_i(0)}{dt} = a_i, \quad i = 1, 2, 3.$$

Introduce new coordinates

$$\bar{x}_1 = x_1 - \frac{q_2}{\mu_0 h}, \quad \bar{x}_2 = x_2 + \frac{q_1}{\mu_0 h}, \quad \bar{x}_3 = x_3.$$

Then the Equations (3) will be as follows

$$\frac{d\bar{x}_1}{dt} = \mu_0 h \bar{x}_2,$$

$$\frac{d\bar{x}_2}{dt} = -\mu_0 h \bar{x}_1,$$

$$\frac{d\bar{x}_3}{dt} = q_3,$$

from which we obtain $\bar{x}_1^2 + \bar{x}_2^2 = const = R^2$.

Besides, we can write down

$$\frac{d^2\bar{x}_i}{dt^2} = -(\mu_0 h)^2 \bar{x}_i, \quad i = 1, 2,$$

$$\frac{d^2\bar{x}_3}{dt^2} = 0.$$

So the equations of electron trajectory have the forms

$$\bar{x}_1 = R\cos\mu_0 ht,$$

$$\bar{x}_2 = -R\sin\mu_0 ht,$$

$$\bar{x}_3 = q_3 t.$$

The radius R of a cylinder on which the electron trajectory lies is calculated by the following formula

$$R = \frac{\sqrt{a_1^2 + a_2^2}}{|\mu_0 h|}.$$

The electron motion along trajectory in the constant magnetic field has a constant module of velocity. Let s be the arc length of trajectory with the initial point corresponding to $t = 0$.Then

$$s = \mu t, \quad \mu = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Let ξ_1, ξ_2, ξ_3 be the natural frame of the trajectory and k and κ are its curvature and torsion. The velocity vector $v = \frac{dx}{dt}$ and the acceleration vector $\frac{dv}{dt}$ can be given as follows

$$v = \mu\xi_1, \quad \frac{dv}{dt} = \mu^2 k\xi_2. \tag{4}$$

Using the ordinary expressions for k and κ we can obtain

$$k = \frac{\sqrt{a_1^2 + a_2^2}|\mu_0 h|}{a_1^2 + a_2^2 + a_3^2}. \tag{5}$$

$$\kappa = -\frac{a_3\mu_0 h}{a_1^2 + a_2^2 + a_3^2}. \tag{6}$$

Hence

$$k^2 + \kappa^2 = (\frac{eh}{mc\mu})^2,$$

where μ is the velocity of the particle along its trajectory.

Now we consider the emission of an electron moving along a helix. Since the electron has a curvilinear trajectory, so its acceleration is non-zero and therefore it radiates an electromagnetic field. Assume that the origin of coordinates coincides with the current position of the electron. We denote this point by Q_1.We suppose that electron lies at this point at time t.

Let r be the vector position of a point in space. Then the electric and magnetic emission at a point with the vector position r can be given by the following expressions (see [3](19.17), [4](14.35))

$$E_{emit}(r,t') = \frac{e}{4\pi\epsilon_0 c^2 l^3}[r,[r - \frac{v|r|}{c}, \frac{dv}{dt}]], \tag{7}$$

$$B_{emit}(r,t') = \frac{[r, E_{emit}]}{|r|c},$$

where the right sides are taken for t, $t' = t + \frac{|r|}{c}$, $l = |r| - (vr)/c$ and ϵ_0 is some physical constant.

Two natural ruled surfaces appear which are constructed using the directions with zero emission.

According to Equation (7) for the direction r with zero-emission the vector product $[r - \frac{v|r|}{c}, \xi_2]$ is proportional to r. Let ν be the unit vector along r.Taking into consideration the formula (4), we obtain

$$[\nu - \frac{\mu}{c}\xi_1, \xi_2] = \lambda\nu, \tag{9}$$

where λ is an unknown coefficient. Let us suppose that $\lambda \neq 0$. Then we conclude that ν lies in the plane of ξ_1, ξ_3. We have

$$\nu = \nu^1\xi_1 + \nu^3\xi_3, \quad (\nu^1)^2 + (\nu^3)^2 = 1.$$

Equation (9) can be rewritten as follows

$$\nu^1\xi_3 - \nu^3\xi_1 - \frac{\mu}{c}\xi_3 = \lambda(\nu^1\xi_1 + \nu^3\xi_3).$$

Therefore we have two equations

$$-\nu^3 = \lambda\nu^1,$$

$$\nu^1 - \frac{\mu}{c} = \lambda\nu^3,$$

from which we obtain

$$(\nu^1)^2 + (\nu^3)^2 - \frac{\mu}{c}\nu^1 = 0,$$

and $1 = \frac{\mu}{c}\nu^1$. But the velocity of an electron is less than c. The latter equality is impossible. Hence $\lambda = 0$. The equation for zero-emission direction is of the following form

$$[\nu - \frac{\mu}{c}\xi_1, \xi_2] = 0. \tag{10}$$

This equation is given in [3], [4] . From (10) we can conclude that the zero-emission direction lies in the osculation plane of trajectory

$$\nu = \frac{\mu}{c}\xi_1 + \bar{\lambda}\xi_2,$$

where $\bar{\lambda}$ is the coefficient. As ν is the unit vector, $\bar{\lambda}$ can be given by two expressions

$$\bar{\lambda} = \pm\sqrt{1 - (\frac{\mu}{c})^2}.$$

So, there are two zero-emission directions ν_1 and ν_2, which lie symmetrically with respect to ξ_1. Let $\phi > 0$ be the angle between ν_1 and ξ_1. Evidently $\cos\phi = \frac{\mu}{c}$. We call ϕ the angle of zero emission. We have

$$\nu_1 = \cos\phi\xi_1 + \sin\phi\xi_2,$$

$$\nu_2 = \cos\phi\xi_1 - \sin\phi\xi_2$$

Drawing through every point of the helix a straight line in the direction ν_1 *we obtain a ruled surface* $\Psi_1(Q_1)$*, see Fig.1. It is natural to call it a zero-emission surface.*

Therefore, **zero-emission surface** of a moving electron is the regular ruled surface with the trajectory of the electron as a directrix and a generatrix, going in the direction with the electron emission from the point of trajectory equal to zero.

By analogy, we obtain a ruled surface $\Psi_2(Q_1)$.

Since the principal normal to the trajectory lies in the tangent plane to surface $\Psi_i(Q_1)$, this curve is an asymptotic line.

Remark. One can also construct a ruled zero-emission surface for the general case of a radiating electron (not necessarily in a constant magnetic field). Then the trajectory of the particle will be an asymptotic line on this surface.

Consider now the following question:

Does there exist another electron Q_2 in the zero-emission straight line of Q_1 such that its zero-emission straight line at every moment t coincides with that of the first electron Q_1?

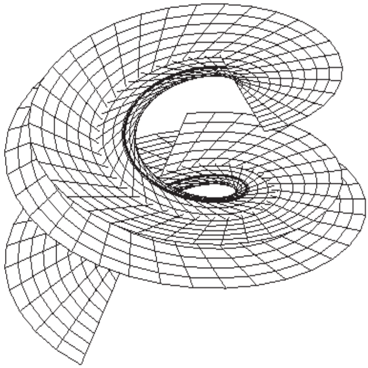

Fig. 1

Now we can prove Theorem 1.1.

Proof. By the existence of a second electron we mean the existence of a trajectory and the existence of a motion of the electron along this trajectory.

We shall say that electrons Q_1 and Q_2 are *"conjugate"* on the surface $\Psi_1(Q_1)$.

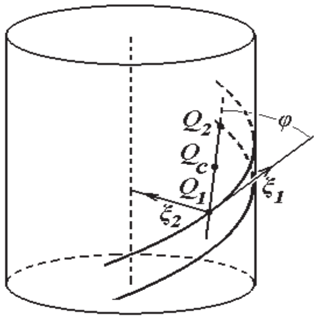

Fig. 2

The place of electron Q_2 can be indicated in a simple way. *The point Q_2 is the intersection point of the ray from Q_1 in the direction ν_1 with the cylinder carrying the trajectory of Q_1.* It is obvious that this intersection point describes a helix parallel to the helix of the point Q_1. The uniqueness of this second electron is a consequence of the following lemmas. Let us write down the equation of surface $\Psi_1(Q_1)$

$$r(p, q) = x(p) + q(\cos\phi\xi_1 + \sin\phi\xi_2),$$

where $x(p)$ is the vector position of the trajectory of Q_1 and the parameter p is its arc length. Denote

$$T = 2k \sin\phi - q(k^2 + \kappa^2 \sin^2\phi).$$

Lemma 2.1. *Asymptotic lines distinct from straight line generators have the the following form except when $q = 0$ or $T = 0$*

$$dp + \frac{2\sin^2\phi}{qT\cos\phi} dq = 0.$$

Lemma 2.2. *An asymptotic line on the surface in question has a constant geodesic curvature only when $q = 0$ or $q = 2k\sin\phi/(k^2 + \kappa^2 \sin^2\phi)$.*

The proofs of these lemmas are given in [1]. From geometrical consideration we have found that $\Psi_1(Q_1) = \Psi_2(Q_2)$. The zero-emission ray of Q_2 with the origin at Q_2 lies on the zero-emission ray of Q_1. The trajectory of the second electron is given by the Equation

$$q = \frac{2k\sin\phi}{k^2 + \kappa^2\sin^2\phi}.$$

The expression in the right side is a constant. It is equal to **the distance between two "conjugate" electrons Q_1 and Q_2.** With the help of expressions (5) and (6) we can calculate this distance in terms of the initial data and the value of magnetic field, h. □

Note the following interesting property of the zero-emission surface: its striction curve takes a medial position between the trajectories of the electrons Q_1 and Q_2. The striction curve consists of central points Q_c (see Fig.2) and lies on the cylinder co-axial with the one carrying the first two helices. But the radius of this cylinder is less than R. At points of the striction line the zero-emission surface $\Psi_1(Q_1)$ has a common tangent plane with this interior cylinder. The striction line is a helix too. The principal normal of the striction line is orthogonal to the tangent plane of $\Psi_1(Q_1)$. *Therefore the striction line is a geodesic on the zero -emission surface.*

3 Geometrically consistent motions.

We consider now the question of stability of configuration consisting of three electrons Q_1, Q_c and Q_2, without regard for their mutual influence. More precisely:

will Q_c lie during their motions at the straight line Q_1, Q_2 and will the distance between Q_3 and Q_1 be a constant ?

More general question: is it possible that electron Q_3 moves along helix $q = const$ by such way, that at all time Q_3 lies on the straight line Q_1, Q_2 ? Such a motion of Q_3 could be named as geometrically consistent (coherent) with the motions of Q_1 and Q_2.

Let us consider the proof of Theorem 1.2.

Proof. We reformulate the question in the following manner: what kind of a particle can move along given helix under the influence of constant magnetic field H_0, if its motion is geometrically consistent with the motions of Q_1 and Q_2 ?

Let $\bar{k}$ and $\bar{\kappa}$ be the curvature and the torsion of the helix $q = const$, which is the trajectory of the particle Q_3, $\bar{e}$ and $\bar{m}$ be its electric charge and mass. We put $\bar{\mu}_0 = -\frac{\bar{e}}{\bar{m}c}$ and denote the module of the velocity of Q_3 by $\bar{\mu}$.

From (5) and (6) we obtain

$$k^2 + \kappa^2 = (\frac{\mu_0 h}{\mu})^2, \quad \bar{k}^2 + \bar{\kappa}^2 = (\frac{\bar{\mu}_0 h}{\bar{\mu}})^2. \tag{11}$$

The trajectory of Q_3 is given in the following form

$$r(p, q) = x(p) + q(\cos\phi\xi_1 + \sin\phi\xi_2),$$

where $q = const$ and the parameter p is the arc length of the basic curve $q = 0$. Denote derivatives with respect to p by $'$. By using the Frenet formulas we obtain

$$r' = (1 - qk\sin\phi)\xi_1 + qk\cos\phi\xi_2 + q\kappa\sin\phi\xi_3,$$

$$r'' = -qk^2\cos\phi\xi_1 + (k - q(k^2+\kappa^2)\sin\phi)\xi_2 + qk\kappa\cos\phi\xi_3,$$

$$r^{(3)} = -k(k - q(k^2+\kappa^2)\sin\phi)\xi_1 - qk(k^2+\kappa^2)\cos\phi\xi_2 + \kappa(k - q(k^2+\kappa^2)\sin\phi)\xi_3.$$

Let $\bar{s}$ be the arc length of the trajectory $q = const$. Then

$$|r'|^2 = (\frac{d\bar{s}}{dp})^2 = 1 - 2qk\sin\phi + q^2(k^2 + \kappa^2\sin^2\phi). \tag{12}$$

As $(\frac{d\bar{s}}{dp})^2$ is a constant, $(r'r'') = 0$. Therefore we have the following expressions for $\bar{k}^2$ and $\bar{\kappa}$

$$\bar{k}^2 = |r''|^2(\frac{dp}{d\bar{s}})^4, \quad \kappa = \frac{(r'r''r^{(3)})}{\bar{k}^2}(\frac{dp}{d\bar{s}})^6. \tag{13}$$

By simple calculations we obtain

$$\bar{k}^2 = \frac{k^2 - 2qk(k^2+\kappa^2)\sin\phi + q^2(k^2+\kappa^2)(k^2+\kappa^2\sin^2\phi)}{[1 - 2qk\sin\phi + q^2(k^2+\kappa^2\sin^2\phi)]^2}. \tag{14}$$

It is not difficult to calculate

$$(r'r''r^{(3)}) = \kappa[k^2 - 2qk(k^2+\kappa^2)\sin\phi + q^2(k^2+\kappa^2)(k^2+\kappa^2\sin^2\phi)]. \tag{15}$$

Hence from (13) and (15) we obtain

$$\bar{\kappa} = \frac{\kappa}{1 - 2qk\sin\phi + q^2(k^2+\kappa^2\sin^2\phi)}. \tag{16}$$

With the help of (14) and (16) we obtain

$$\frac{k^2+\kappa^2}{\bar{k}^2+\bar{\kappa}^2} = 1 - 2qk\sin\phi + q^2(k^2+\kappa^2\sin^2\phi). \tag{17}$$

Taking into account the relations (11) and (17), we can write down the second expression as follows

$$\frac{k^2+\kappa^2}{\bar{k}^2+\bar{\kappa}^2} = (\frac{\mu_0\bar{\mu}}{\bar{\mu}_0\mu})^2. \tag{18}$$

For the velocities of Q_1 and Q_3 along their trajectories we have

$$\mu = \frac{dp}{dt}, \quad \bar{\mu} = \frac{d\bar{s}}{dt} = \mu\frac{d\bar{s}}{dp}.$$

Hence

$$(\frac{\bar{\mu}}{\mu})^2 = (\frac{d\bar{s}}{dp})^2 = 1 - 2qk\sin\phi + q^2(k^2+\kappa^2\sin^2\phi). \tag{19}$$

Substitute this expression into (18) and compare the result with (17). We obtain $\bar{\mu}_0 = \mu_0$.

Hence, if the motion of charge Q_3 is geometrically consistent with motions of Q_1 and Q_2, the ratio $\frac{\bar{e}}{\bar{m}}$ is the same as for Q_1 and Q_2.

Hence the particle Q_3 can be an electron. If at the initial moment its velocity vector is tangent to helix $q = const$ and

$$\bar{\mu} = \mu(1 - 2qk\sin\phi + q^2(k^2+\kappa^2\sin^2\phi))^{\frac{1}{2}},$$

then the motions of Q_1, Q_2 and Q_3 will be geometrically consistent (coherent). □

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Homotopy dimension of orbits of Morse functions on surfaces[1]

by Sergiy Maksymenko

Abstract

Let M be a compact surface, P be either the real line $\mathbb{R}$ or the circle S^1, and $f : M \to P$ be a C^∞ Morse map. The identity component $\mathcal{D}_{\mathrm{id}}(M)$ of the group of diffeomorphisms of M acts on the space $C^\infty(M, P)$ by the following formula: $h \cdot f = f \circ h^{-1}$ for $h \in \mathcal{D}_{\mathrm{id}}(M)$ and $f \in C^\infty(M, P)$. Let $\mathcal{O}(f)$ be the orbit of f with respect to this action and n be the total number of critical points of f. In this note we show that $\mathcal{O}(f)$ is homotopy equivalent to a certain covering space of the n-th configuration space of the interior $\mathrm{Int}M$. This in particular implies that the (co-)homology of $\mathcal{O}(f)$ vanish in dimensions greater than $2n-1$, and the fundamental group $\pi_1\mathcal{O}(f)$ is a subgroup of the n-th braid group $\mathcal{B}_n(M)$.

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Keywords. Morse function, orbits, classifying spaces, homotopy dimension, geometric dimension.

1 Introduction

Let M be a compact surface, P be either the real line $\mathbb{R}$ or the circle S^1. Then the group $\mathcal{D}(M)$ of C^∞ diffeomorphisms of M acts on the space $C^\infty(M, P)$ by the following formula:

$$h \cdot f = f \circ h^{-1} \tag{1.1}$$

for $h \in \mathcal{D}(M)$ and $f \in C^\infty(M, P)$.

We say that a smooth (C^∞) map $f : M \to P$ is *Morse* if

(i) critical points of f are non-degenerate and belong to the interior of M;

(ii) f is constant on every connected component of ∂M.

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Let $f \in C^\infty(M,P)$, Σ_f be the set of critical points of f, and $\mathcal{D}(f,\Sigma_f)$ be the subgroup of $\mathcal{D}(M)$ consisting of diffeomorphisms h such that $h(\Sigma_f) = (\Sigma_f)$.

Then we can define the stabilizers $\mathcal{S}(f)$ and $\mathcal{S}(f,\Sigma_f)$, and orbits $\mathcal{O}(f)$ and $\mathcal{O}(f,\Sigma_f)$ with respect to the actions of the groups $\mathcal{D}(M)$ and $\mathcal{D}(f,\Sigma_f)$. Thus

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \ : \ f \circ h^{-1} = f\}, \qquad \mathcal{O}(f) = \{f \circ h^{-1} \ : \ h \in \mathcal{D}(M)\},$$

$$\mathcal{S}(f,\Sigma_f) = \mathcal{S}(f) \cap \mathcal{D}(f,\Sigma_f).$$

We endow the spaces $\mathcal{D}(M)$ and $C^\infty(M,P)$ with the corresponding C^∞ Whitney topologies. They induce certain topologies on the stabilizers and orbits.

Let $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{D}_{\mathrm{id}}(f,\Sigma_f)$ be the identity path components of the groups $\mathcal{D}(M)$ and $\mathcal{D}(f,\Sigma_f)$, $\mathcal{S}_{\mathrm{id}}(f)$ and $\mathcal{S}_{\mathrm{id}}(f,\Sigma_f)$ be the identity path components of the corresponding stabilizers, and $\mathcal{O}_f(f)$ and $\mathcal{O}_f(f,\Sigma_f)$ be the path-components of f in the corresponding orbits with respect to the induced topologies.

Lemma 1. *If Σ_f is discrete set, e.g. when f is Morse, then $\mathcal{S}_{\mathrm{id}}(f,\Sigma_f) = \mathcal{S}_{\mathrm{id}}(f)$.*

Proof. Since $\mathcal{S}(f,\Sigma_f) \subset \mathcal{S}(f)$, we have that $\mathcal{S}_{\mathrm{id}}(f,\Sigma_f) \subset \mathcal{S}_{\mathrm{id}}(f)$. Conversely, let $h_t : M \to M$ be an isotopy such that $h_0 = \mathrm{id}_M$ and $h_t \in \mathcal{S}(f)$ for all $t \in I$., i.e. $f \circ h_t = f$. We have to show that $h_t \in \mathcal{S}(f,\Sigma_f)$ for all $t \in I$. Notice that $d(f \circ h_t) = h_t^* df = df$, whence $h_t(\Sigma_f) = \Sigma_f$. Since Σ_f is discrete and $h_0 = \mathrm{id}_M$ fixes Σ_f, we see that so does every h_t, i.e. $h_t \in \mathcal{S}(f,\Sigma_f)$. $\square$

Let $f : M \to P$ be a Morse map. Denote by c_i, $(i = 0,1,2)$, the total numbers of critical points of f of index i and let $n = c_0 + c_1 + c_2$ be the total number of critical points of f.

Notice that for every Morse map f its orbits $\mathcal{O}(f)$ and $\mathcal{O}(f,\Sigma_f)$ are Fréchet submanifolds of $C^\infty(M,P)$ of finite codimension, see [4, 5]. Therefore, e.g. [3], these orbits have the homotopy types of CW-complexes. But in general these complexes may have infinite dimensions.

Let X be a topological space which is homotopy equivalent to some CW-complex. Then a *homotopy dimension* $\mathrm{h.d.}\,X$ of X is the minimal dimension of a CW-complex homotopy equivalent to X. In particular $\mathrm{h.d.}\,X$ can be equal to ∞. It is also evident that if $\mathrm{h.d.}\,X < \infty$, then (co-)homology of X vanish in dimensions greater that $\mathrm{h.d.}\,X$.

If π is a finitely presented group π, then the *geometric dimension* of π, denoted $\mathrm{g.d.}\,\pi$, is the homotopy dimension of its Eilenberg-Mac Lane space $K(\pi,1)$:

$$\mathrm{g.d.}\,\pi := \mathrm{h.d.}\,K(\pi,1).$$

In [2, Theorems 1.3, 1.5, 1.9] the author described the homotopy types of $\mathcal{S}_{\mathrm{id}}(f)$, $\mathcal{O}_f(f)$, and $\mathcal{O}_f(f,\Sigma_f)$. It follows from these results that

$$\mathrm{h.d.}\,\mathcal{S}_{\mathrm{id}}(f), \quad \mathrm{h.d.}\,\mathcal{O}_f(f,\Sigma_f) \ \leq \ 1.$$

In fact, $\mathcal{S}_{\mathrm{id}}(f)$ is contractible provided either f has at least one critical point of index 1, i.e., $c_1 \geq 1$ or M is non-orientable. Otherwise $\mathcal{S}_{\mathrm{id}}(f) \simeq S^1$.

Also, $\mathcal{O}_f(f) \simeq S^1$ for Morse mappings $T^2 \to S^1$ and $K^2 \to S^1$ without critical points, and $\mathcal{O}_f(f)$ is contractible in all other cases, where K stands for the Klein bottle.

For $\mathcal{O}_f(f)$ the description is not so complete. But if f is *generic*, i.e., it takes distinct values at distinct critical points, then

$$\mathrm{h.d.}\,\mathcal{O}_f(f) \quad \leq \quad \max\{c_0 + c_2 + 1,\, c_1 + 2\} \quad < \quad \infty.$$

Actually, in this case $\mathcal{O}_f(f)$ is either contractible or homotopy equivalent to T^k or to $\mathbb{R}P^3 \times T^k$ for some $k \geq 0$, where T^k is a k-dimensional torus.

Thus the upper bound for $\mathrm{h.d.}\,\mathcal{O}_f(f)$ (at least in generic case) depends only on the number of critical points of f at each index.

In this note we will show that $\mathrm{h.d.}\,\mathcal{O}_f(f) \leq 2n-1$ for arbitrary Morse mapping $f : M \to P$ having exactly $n \geq 1$ critical points. Notice that if $n = 0$, then f is generic, and in fact $\mathrm{h.d.}\,\mathcal{O}_f(f) \leq 1$, see [2, Table 1.10].

Theorem 2. *Let $f : M \to P$ be a Morse map and n be the total number of critical points of f. Assume that $n \geq 1$. Denote by $\mathcal{F}_n(\mathrm{Int}M)$ the configuration space of n points of the interior $\mathrm{Int}M$ of M. Then $\mathcal{O}_f(f)$ is homotopy equivalent to a certain covering space $\mathcal{F}(f)$ of $\mathcal{F}_n(\mathrm{Int}M)$.*

Corollary 3. $\mathrm{h.d.}\mathcal{O}_f(f) \leq 2n - 1$, *whence (co-)homology of $\mathcal{O}_f(f)$ vanish in dimensions $\geq 2n$.*

Proof. Since $\mathcal{F}_n(\mathrm{Int}M)$ and its connected covering spaces are *open* manifolds of dimension $2n$, they are homotopy equivalent to CW-complexes of dimensions not greater than $2n - 1$. □

For simplicity denote $\pi = \pi_1\mathcal{O}_f(f)$. Since the covering map $\mathcal{F}(f) \to \mathcal{F}_n(\mathrm{Int}M)$ yields a monomorphisms of fundamental groups, we obtain the following:

Corollary 4. *The fundamental group π of $\mathcal{O}_f(f)$ is a subgroup of the n-th braid group $\mathcal{B}_n(M) = \pi_1(\mathcal{F}_n(\mathrm{Int}M))$ of M.*

Corollary 5. *Suppose that M is aspherical, i.e., $M \neq S^2, \mathbb{R}P^2$. Then $\mathcal{O}_f(f)$ is aspherical as well, i.e., $K(\pi,1)$-space, whence* $\mathrm{g.d.}\,\pi \leq 2n - 1$.

Proof. Actually the aspherity of $\mathcal{O}_f(f)$ for the case $M \neq S^2, \mathbb{R}P^2$ is proved in [2, Theorems 1.5, 1.9].

But it can be shown by another arguments. It is well known and can easily be deduced from [1] that for an aspherical surface M every of its configuration spaces $\mathcal{F}_n(\mathrm{Int}M)$ and thus every covering space of $\mathcal{F}_n(\mathrm{Int}M)$ are aspherical as well. Hence so is $\mathcal{F}(f)$ and thus $\mathcal{O}_f(f)$ itself. □

A presentation for π will be given in another paper.

2 Orbits of the actions of $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{D}_{\mathrm{id}}(f, \Sigma_f)$

Proposition 6. *Let $f : M \to P$ be a Morse map and*

$$p : \mathcal{D}(M) \mapsto \mathcal{O}(f), \qquad p(h) = f \circ h^{-1} \tag{2.1}$$

be the natural projection. Then $\mathcal{O}_f(f)$ is the orbit of f with respect to $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{O}_f(f, \Sigma_f)$ is the orbit of f with respect to $\mathcal{D}_{\mathrm{id}}(f, \Sigma_f)$. In other words,

$$p(\mathcal{D}_{\mathrm{id}}(M)) = \mathcal{O}_f(f) \qquad \textit{and} \qquad p(\mathcal{D}_{\mathrm{id}}(f, \Sigma_f)) = \mathcal{O}_f(f, \Sigma_f).$$

Proof. The proof is based on the following general statement. Let G be a topological group transitively acting on a topological space O and $f \in O$. Denote by G_e the *path-component* of the unit e in G and let O_f be the *path-component* of f in O.

Lemma 7. *Suppose that the mapping $p : G \to O$ defined by*

$$p(\gamma) = \gamma \cdot f, \qquad \forall \gamma \in G$$

satisfies a covering path axiom (in particular, this holds when p is a locally trivial fibration). Then O_f is the orbit of f with respect to the induced action of G_e on O, i.e., $p(G_e) = O_f$.

Proof. Evidently, $p(G_e) \subset O_f$. Conversely, let $g \in O_f$. Then there exists a path $\omega : I \to O_f$ between f and g, i.e., $\omega(0) = f$ and $\omega(1) = g$. Since p satisfies the covering path axiom, ω lifts to the path $\tilde{\omega} : I \to G$ such that $\tilde{\omega}(0) = e$ and $\omega = p \circ \tilde{\omega}$. Then $g = \omega(1) = p \circ \tilde{\omega}(1) \in p(G_e)$. Thus $p(G_e) = O_f$. □

It remains to note that the mapping (2.1) is a locally trivial fibration, see e.g. [4, 5], and $\mathcal{D}(M)$ (resp. $\mathcal{D}(f, \Sigma_f)$) transitively acts on the orbit $\mathcal{O}(f)$ (resp. $\mathcal{O}(f, \Sigma_f)$). Therefore the conditions of Lemma 7 are satisfied. □

3 Proof of Theorem 2

Let $\mathcal{F}_n(\mathrm{Int}M)$ be the configuration space of n points of the interior $\mathrm{Int}M$ of M. Thus

$$\mathcal{F}_n(\mathrm{Int}M) = \mathcal{P}_n(\mathrm{Int}M)/\mathbb{S}_n, \tag{3.1}$$

where

$$\mathcal{P}_n(\mathrm{Int}M) = \{(x_1, \dots x_n) \mid x_i \in \mathrm{Int}M \text{ and } x_i \neq x_j \text{ for } i \neq j\}$$

is called the *pure* n-th configuration space of $\mathrm{Int}M$, and $\mathbb{S}_n$ is the symmetric group of n symbols freely acting on $\mathcal{P}_n(\mathrm{Int}M)$ by permutations of coordinates.

We can regard $\mathcal{F}_n(\mathrm{Int}M)$ as the space of n-tuples of mutually distinct points of $\mathrm{Int}M$.

Denote by $\Sigma_f = \{x_1, \dots, x_n\}$ the set of critical points of f. Then for every $g \in \mathcal{O}_f(f)$ the set Σ_g of its critical points is a point in $\mathcal{F}_n(\mathrm{Int}M)$. Hence the correspondence $g \mapsto \Sigma_g$ is a well-defined mapping

$$k : \mathcal{O}_f(f) \to \mathcal{F}_n(\mathrm{Int}M), \qquad k(g) = \Sigma_g.$$

Lemma 8. (i) *The mapping k is a locally trivial fibration. The connected component of the fiber containing f is homeomorphic to $\mathcal{O}_f(f, \Sigma_f)$.*

(ii) *Let $k_i : \pi_i(\mathcal{O}_f(f), f) \to \pi_i(\mathcal{F}_n(\mathrm{Int}M), \Sigma_f)$, $(i \geq 1)$, be the corresponding homomorphism of homotopy groups induced by k. Then k_1 is a* ***monomorphism*** *and all other k_i for $i \geq 2$ are isomorphisms.*

Assuming that Lemma 8 is proved we will now complete our theorem. Let $\mathcal{F}(f)$ be the covering space of $\mathcal{F}_n(\mathrm{Int}M)$ corresponding to the subgroup

$$\pi_1\mathcal{O}_f(f) \approx k_1(\pi_1\mathcal{O}_f(f)) \subset \pi_1\mathcal{F}_n(\mathrm{Int}M).$$

Then k lifts to the mapping $\hat{k} : \mathcal{O}_f(f) \to \mathcal{F}(f)$ which induces isomorphism of all homotopy groups. Since $\mathcal{O}_f(f)$ and $\mathcal{F}(f)$ are connected, we obtain from (2) that $\hat{k}$ is a desired homotopy equivalence. Theorem 2 is proved modulo Lemma 8.

Proof of Lemma 8. (i) Recall, [1], that the following *evaluation* map

$$e : \mathcal{D}_{\mathrm{id}}(M) \to \mathcal{F}_n(\mathrm{Int}M), \qquad e(h) = h(\Sigma_f)$$

is a locally trivial principal fibration with fiber

$$\hat{\mathcal{D}}(f) = \mathcal{D}_{\mathrm{id}}(M) \cap \mathcal{D}(f, \Sigma_f).$$

Let $p : \mathcal{D}_{\mathrm{id}}(M) \to \mathcal{O}_f(f)$ be the projection defined by $p(h) = f \circ h^{-1}$. Then the set of critical points of the function $f \circ h^{-1} \in \mathcal{O}_f(f)$ is $h(\Sigma_f)$. Therefore e coincides with the following composition:

$$e = k \circ p : \mathcal{D}_{\mathrm{id}}(M) \xrightarrow{\ p\ } \mathcal{O}_f(f) \xrightarrow{\ k\ } \mathcal{F}_n(\mathrm{Int}M).$$

Since e and (by Proposition 6) the mapping p are principal locally trivial fibrations, we obtain that k is also a locally trivial fibration with fiber $\hat{\mathcal{O}}(f)$ being the orbit of f with respect to the group $\hat{\mathcal{D}}(f)$.

It is easy to see that the identity component of the group $\hat{\mathcal{D}}(f)$ coincides with $\mathcal{D}_{\mathrm{id}}(f, \Sigma_f)$, whence by Proposition 6, the connected component of $\hat{\mathcal{O}}(f)$ containing f is $\mathcal{O}_f(f, \Sigma_f)$.

(ii) As noted above since $n \geq 1$, it follows from [2, Theorems 1.5(i), 1.9] that $\mathcal{O}_f(f, \Sigma_f)$ is contractible. Then from the exact sequence of homotopy groups of the fibration k we obtain that for $i \geq 2$ every k_i is an isomorphism, and k_1 is a monomorphism. Lemma 8 is proved. □

Remark 9. In general the covering map $\mathcal{F}(f) \to \mathcal{F}_n(\mathrm{Int}M)$ is not *regular*, i.e., $\pi_1\mathcal{O}_f(f) \approx \pi_1\mathcal{F}(f)$ is not a normal subgroup of $\mathcal{B}_n(M) = \pi_1\mathcal{F}_n(\mathrm{Int}M)$.

Remark 10. Theorem 2 does not answer the question whether $\mathcal{O}_f(f)$ has the homotopy type of a *finite* CW-complex. Indeed, since M is compact, it follows from (3.1) that $\mathcal{B}_n(M)$ can be regarded as an open cellular (i.e. consisting of full cells) subset of a finite CW-complex $\prod_n M/\mathbb{S}_n$. Therefore if the covering map $\mathcal{F}(f) \to \mathcal{F}_n(\mathrm{Int}M)$ is an infinite sheet covering, i.e., $\pi_1\mathcal{O}_f(f)$ has an infinite index in $\mathcal{B}_n(M)$, then we obtain a priori an infinite cellular subdivision of $\mathcal{F}(f)$. On the other hand, as noted above, for a generic Morse map $f : M \to P$ a finiteness of the homotopy type of $\mathcal{O}_f(f)$ follows from [2].

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On surgery inside a manifold [1]

by Yuri V. Muranov and Rolando Jimenez

Abstract

To study surgery on a submanifold inside an ambient manifold Wall constructed the theory of splitting of a simple homotopy equivalence along a submanifold. The results and methods of splitting theory are very efficient in the classification of manifolds, in the investigation of group actions, and in many others geometric problems. In this paper we compare the methods and results of the abstract surgery with the corresponding methods and results of the surgery inside an ambient manifold. We consider only higher dimensions. We describe some relations between abstract surgery and surgery inside the ambient manifold for a filtered manifold. We obtain new relations between various structure sets and obstruction groups for filtered manifolds and describe some applications of the obtained results to the problem of realizing surgery and splitting obstructions by maps of closed manifolds.

1 Introduction

In surgery theory, one sometimes looks at submanifolds to get additional information about surgery obstruction groups and natural maps (see [2], [5], [7], [9], [13], [14], [15], [16], [17], [18], [19], [24], and [27])). To study surgery on manifold pairs, in [27] Wall introduced the concept of *splitting of a simple homotopy equivalence along a submanifold* in the case of piecewise linear and smooth manifolds and applied this approach to various geometric problems. For topological manifolds this approach was developed by Ranicki in [23], [24], and [25]. The advantage of topological category is exhibited by the possibility of realizing various obstruction groups, structure sets, and natural maps on the spectra level (see [1], [22], [23], [24], [27], and [28]).

In the present paper, we compare the abstract surgery with the surgery inside an ambient manifold. We consider only topological manifolds and topological normal maps. All manifolds pairs $X \subset Y$ will be topological manifold pairs in the sense of Ranicki [24]; in particular, X will be a locally flat submanifold. We consider the case of higher dimensions, that is the dimensions of all closed

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manifolds will be ≥ 5, and the dimensions of all manifolds with boundary will be ≥ 6.

In section 2 we recall necessary definitions of surgery theory for manifold pairs. Afterwards, we describe various natural maps between exact sequences containing surgery obstruction groups and structure sets. Then we give an example of exact computations.

In section 3 we consider manifolds with filtration. At first, we describe the surgery and splitting problem in this case and give a short summary of results on this subject. Afterwards, we obtain new relations between various obstruction groups and describe some geometric applications of the obtained results.

2 Pairs of manifolds

Let X^n be a connected closed n-dimensional manifold. We suppose that the fundamental group $\pi = \pi_1(X)$ of the manifold is equipped with an orientation homomorphism $w\colon \pi \to \{\pm 1\}$ which coincides with the first Stiefel-Whitney class. In what follow, we do not indicate this homomorphism if this doesn't lead to confusion, and we suppose that all homomorphisms of groups preserve orientation.

Consider a degree-one *topological normal map (t-triangulation)*

$$(f, b)\colon (M^n, \nu_M) \to (X^n, \nu_X),$$

of closed topological manifolds, where $b\colon \nu_M \to \nu_X$ is a map of topological bundles covering f, and ν_M is the stable normal bundle of M in an Euclidean space. Two normal maps (f_i, b_i), $(i = 0, 1)$ are said to *be normally bordant (concordant) if* there exists a topological normal map

$$(F, B)\colon (W^{n+1}; \partial_0 W, \partial_1 W) \to (X \times I; X \times \{0\}, X \times \{1\})$$

of manifolds with boundary, whose restrictions to the bottom boundary $\partial_0 W = M_0$ and to the top boundary $\partial_1 W = M_1$ coincide with the normal maps (f_0, b_0) and (f_1, b_1), respectively (see [23] and [24]). The set of concordance classes of normal maps to the manifold X is denoted $\mathcal{T}^{TOP}(X)$ and coincides with the set $[X, G/TOP]$, where TOP is the group of stable homeomorphisms of n-dimensional Euclidean spaces with a base point, and G is the monoid of stable homotopy equivalences of spheres.

A simple homotopy equivalence $f\colon M \to X$ is called a *homotopy triangulation (s-triangulation)* of the manifold X. Two such maps $f_i\colon M_i \to X (i = 0, 1)$ are said to be equivalent *(concordant)* if there exists a topological manifold W^{n+1} with boundary $\partial W = M_0 \cup M_1$ and a simple homotopy equivalence of triads

$$(F; f_0, f_1)\colon (W; M_0, M_1) \to (X \times I; X \times \{0\}, X \times \{1\}).$$

The *topological manifold structure set* $\mathcal{S}^{TOP}(X)$ consists of the concordance classes of s-triangulations of the manifold X.

The structure sets $\mathcal{T}^{TOP}(X)$ and $\mathcal{S}^{TOP}(X)$ fit into the surgery exact sequence

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}^{TOP}(X) \to \mathcal{T}^{TOP}(X) \xrightarrow{\sigma} L_n(\pi) \tag{2.1}$$

where $L_n(\pi)$ is the surgery obstruction group and the map σ provides a surgery obstruction, that is an obstruction for the existence of a simple homotopy equivalence in the class of the normal bordisms (see [4], [24], [25], and [27]). Note that any element of the group $L_n(\pi)$ is represented by a normal map of closed manifolds with boundary [27], and there is a very small number of elements $x \in L_n(\pi)$ which are obstructions to surgery on closed manifolds. Denote by $C_n(\pi) \subset L_n(\pi)$ a subgroup generated by elements which can be realized as obstructions to surgery of normal maps of closed manifolds.

The surgery exact sequence (2.1) is realized on the spectra level [23]. For every group π, the 4-periodic Ω-spectrum $\mathbf{L}(\pi)$ is defined (see [23], [24], and [25]) with

$$\pi_n(\mathbf{L}(\pi)) = L_n(\pi).$$

For any topological space X equipped with an orientation homomorphism

$$w\colon \pi_1(X) \to \{\pm 1\},$$

there exists a cofibration

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbf{L}(\pi_1(X)) \tag{2.2}$$

of spectra, where $\mathbf{L}_\bullet$ is the 1-connected cover of the spectrum $\mathbf{L}(1)$ [23]. The algebraic surgery exact sequence of Ranicki [23]

$$\cdots \to L_{n+1}(\pi_1(X)) \to \mathcal{S}_{n+1}(X) \to H_n(B\pi; \mathbf{L}_\bullet) \to L_n(\pi_1(X)) \to \cdots \tag{2.3}$$

is the homotopy long exact sequence of the cofibration (2.2). Denote by $\mathbf{S}(X)$ the homotopical cofiber of the map (2.2). Then $\pi_i(\mathbf{S}(X)) = \mathcal{S}_i(X)$ and $\mathcal{S}_{n+1}(X) = \mathcal{S}^{TOP}(X)$.

For a topological manifold X, the surgery exact sequence (2.1) is isomorphic to the corresponding part of exact sequence (2.3). For $X = B\pi$ we obtain the algebraic surgery exact sequence containing the Assembly map

$$A\colon H_n(B\pi; \mathbf{L}_\bullet) \to L_n(\pi) \tag{2.4}$$

for the group π. The image of the Assembly map coincides with the group $C_n(\pi)$ [27].

Now let $X^n \subset Y^{n+q}$ be a closed topological manifold pair of codimension q [24]. An s-triangulation $f : N \to Y$ *splits along the submanifold* X if it is homotopic to a map g, transversal to X with $M = g^{-1}(X)$, and the restrictions

$$g|_M\colon M \to X \quad \text{and} \quad g|_{(N\setminus M)}\colon N \setminus M \to Y \setminus X \tag{2.5}$$

are simple homotopy equivalences (see [1], [13], [20], [24], and [27]).

An s-triangulation $g\colon N \to Y$, which satisfies conditions (2.5), is called an *s-triangulation of the manifold pair* (Y, X). Denote by ξ the topological normal block bundle of the submanifold X in Y [24]. Following [24], denote by $\mathcal{S}^{TOP}(Y, X, \xi)$ the set of the concordance classes of s-triangulations of the manifold pair (Y, X).

For a simple homotopy equivalence $f\colon M \to Y$, a splitting obstruction lies in the *splitting obstruction group* $LS_n(F)$, which depends only on n mod 4 and on the square

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(Y \setminus X) \\ \downarrow & & \downarrow \\ \pi_1(U) & \to & \pi_1(Y) \end{pmatrix} \tag{2.6}$$

of fundamental groups, where U is a tubular neighborhood of the submanifold X in Y. In fact, to find a map with properties (2.5) in the homotopy class of the simple homotopy equivalence f means to do surgery on the transversal preimage of the submanifold X inside the manifold N (see [24] and [27]). We have the following exact sequence [24]

$$\cdots \to LS_{n+1}(F) \to \mathcal{S}^{TOP}(Y, X, \xi) \to \mathcal{S}^{TOP}(Y) \to LS_n(F). \tag{2.7}$$

Surgery exact sequence (2.1) algebraically describes the situation when we would like to do abstract surgery starting from a normal map $(f, b) \in \mathcal{T}(Y)$. Exact sequence (2.7) is similar to surgery exact sequence (2.1) and algebraically describes the situation when we start from a simple homotopy equivalence $f : N \to Y$ and would like to do surgery of the normal map

$$f|_{f^{-1}(X)}\colon f^{-1}(X) = M \to X$$

inside an ambient manifold N. For a manifold pair (Y^{n+q}, X^n), the exact sequences (2.1) and (2.7) fit into the commutative diagram

(2.8)
$$\begin{array}{ccccccccc} \cdots \to & LS_{n+1}(F) & \to & \mathcal{S}^{TOP}(Y, X, \xi) & \to & \mathcal{S}^{TOP}(Y) & \to & LS_n(F) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & L_{n+1}(\pi_1(X)) & \to & \mathcal{S}^{TOP}(X) & \to & \mathcal{T}^{TOP}(X) & \to & L_n(\pi_1(X)), \end{array}$$

where vertical maps correspond to pass from surgery inside the ambient manifold Y to abstract surgery.

We have a cofibration (see [1], [2], [11], [23], and [24])

$$\mathbf{S}(Y) \to \Sigma^{q+1}\mathbf{LS}(F), \tag{2.9}$$

where Σ^q is the q-iterated suspension functor [26], and $\mathbf{LS}(F)$ is the 4-periodic Ω-spectrum realizing the groups $LS_*(F)$. Denote by $\mathbf{S}(Y, X, \xi)$ the homotopical fiber of the map in (2.9) with homotopy groups

$$\pi_i(\mathbf{S}(Y, X, \xi)) = \mathcal{S}_i(Y, X, \xi).$$

Note that exact sequence (2.7) is isomorphic to the corresponding part of the homotopy long exact sequence

$$\cdots \to LS_{n+1}(F) \to \mathcal{S}_{n+q+1}(Y,X,\xi) \to \mathcal{S}_{n+q+1}(Y) \to LS_n(F) \to \dots \tag{2.10}$$

of cofibration (2.9) (see [1], [24], and [25]). In particular, $\mathcal{S}_{n+q+1}(Y,X,\xi) = \mathcal{S}^{TOP}(Y,X,\xi)$. Commutative diagram (2.8) also is realized on the spectra level.

Exact sequences (2.1) and (2.10) fits in many various diagrams and braids of exact sequences (see [1], [2], [3], [9], [11], [22], [23], and [24]).

For a manifold pair $X^n \subset Y^{n+q}$, we can consider an opposite situation [2]. Consider a normal map $f\colon N \to Y$ which is transversal to the submanifold X with $M = f^{-1}(X)$, and for which the restriction $f|_M\colon M \to X$ is a simple homotopy equivalence. We define the set $\mathcal{NS}^{TOP}(Y,X)$ as the set of equivalence classes of such maps under the following equivalence relation [2].

Two maps $f_i\colon N_i \to Y$ $(i = 0,1)$, with $M_i = f_i^{-1}(X)$, are *equivalent* if there exists a normal bordism $F\colon W \to X$ such that:

i) $\partial W = N_0 \cup N_1$ and $F|_{N_i} = f_i$ $(i = 0,1)$,

ii) F is transversal to X with $F^{-1}(X) = V$ and $\partial V = M_0 \cup M_1$,

iii) the restriction $F|_V$ is an s-cobordism between $F|_{M_i} = f_i$ $(i = 0,1)$.

Evidently, we have the following natural forgetful maps [2]

$$\mathcal{NS}^{TOP}(Y,X) \to \mathcal{T}^{TOP}(Y) \ \text{ and } \ \mathcal{S}^{TOP}(Y,X,\xi) \to \mathcal{NS}^{TOP}(Y,X). \tag{2.11}$$

The maps in (2.11) are realized on the spectra level by means of the following cofibrations of spectra [2]

$$\mathbf{NS}(Y,X) \to Y_+ \wedge \mathbf{L}_\bullet \ \text{ and } \ \Omega\mathbf{S}(Y,X,\xi) \to \mathbf{NS}(Y,X). \tag{2.12}$$

Denote $\pi_i(\mathbf{NS}(Y,X)) = \mathcal{NS}_i(Y,X)$. Then we have $\mathcal{NS}_{n+q}(Y,X) = \mathcal{NS}^{TOP}(Y,X)$. The following theorem can be found in [2].

Theorem 2.1. *The homotopy long exact sequences of cofibrations in (2.12) fit in the following commutative diagram of exact sequences*

(2.13)

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{S}_{n+q+1}(Y,X,\xi) & \to & \mathcal{NS}_{n+q}(Y,X) & \to & L_{n+q}(\pi_1(Y\setminus X)) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{S}_{n+q+1}(Y) & \to & H_{n+q}(Y;\mathbf{L}_\bullet) & \to & L_{n+q}(\pi_1(Y)) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & LS_n(F) & \to & L_n(\pi_1(X)) & \to & L_{n+q}(\pi_1(Y\setminus X)\to\pi_1(Y)) & \to \cdots, \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

in which all rows and columns are exact, and which is realized on the spectra level.

Consider the following maps

$$\begin{aligned}
&\phi_1\colon \mathcal{S}^{TOP}(Y) = \mathcal{S}_{n+q+1}(Y) \to L_n(\pi_1(X)),\\
(2.14)\qquad &\phi_2\colon \mathcal{NS}^{TOP}(Y,X) = \mathcal{NS}_{n+q}(Y,X) \to L_{n+q}(\pi_1(Y)),\\
&\phi_3\colon L_{n+q+1}(\pi_1(Y\setminus X) \to \pi_1(Y)) \to \mathcal{S}_{n+q}(Y,X,\xi),
\end{aligned}$$

which are obtained as compositions of the maps from diagram (2.13) (see also (2.8)). The maps in (2.14) have very clear geometrical description. For example, for any homotopy triangulation $(f\colon N \to Y) \in \mathcal{S}^{TOP}(Y)$, the element $\phi_1(f) \in L_n(\pi_1(X))$ is the obstruction to surgery of the normal map

$$f|_{f^{-1}(X)}\colon f^{-1}(X) \to X.$$

We describe now the relations between the maps in (2.14).

Theorem 2.2. *The maps in (2.14) are realized on the spectra level by the following maps of spectra*

$$\begin{aligned}
&\Omega^{n+q+1}\mathbf{S}(Y) \to \mathbf{L}(\pi_1(X)),\\
(2.15)\qquad &\Omega^{n+q}\mathbf{NS}(Y,X) \to \Omega^{n+q}\mathbf{L}(\pi_1(Y)),\\
&\Omega^{n+q+1}\mathbf{L}(\pi_1(Y\setminus X) \to \pi_1(Y)) \to \Omega^{n+q}\mathbf{S}(Y,X,\xi).
\end{aligned}$$

The homotopy cofibers of the maps in (2.15) are naturally homotopy equivalent.

Proof. The realization of the maps on the spectra level follows from the realization of diagram (2.13) on spectra level. Now the equivalence of cofibers follows from [20, Lemma 2]. □

Consider a pair of real projective spaces $\mathbb{R}P^{2k} \subset \mathbb{R}P^{2k+1}$ $(k \geq 3)$. The group $\pi_1(\mathbb{R}P^{2k})$ is isomorphic to $\mathbb{Z}/2$ and has a nontrivial orientation.

Proposition 2.3. *For k odd, the nontrivial element of the group $L_{2k}(\mathbb{Z}/2^-) = \mathbb{Z}/2$ does not lie in the image of the map*

$$\phi_1\colon \mathcal{S}^{TOP}(\mathbb{R}P^{2k+1}) \to L_{2k}(\mathbb{Z}/2^-).$$

For k even, the nontrivial element of the group $L_{2k}(\mathbb{Z}/2^-)$ lies in the image of the map ϕ_1.

Proof. In the considered case, the group $LS_n(F)$ is the Browder-Livesay group $LN_n(1 \to \mathbb{Z}/2^+)$ (see [1], [5], [6], and [13]). The map $LS_n(F) \to L_n(\mathbb{Z}/2^-)$ is an isomorphism for $n = 4k$ and is trivial for $n = 4k+2$ (see, for example, [10], [17], [18], [19], and [27]). Now the result follows easily from diagram (2.8) and from [19] (see also [17]). □

In fact, Proposition 2.3 states that for any simple homotopy equivalence

$$f\colon N^{4n+3} \to \mathbb{R}P^{4n+3}$$

the surgery obstruction of the restriction

$$f|_{f^{-1}(\mathbb{R}P^{4n+2})}\colon f^{-1}(\mathbb{R}P^{4n+2}) \to \mathbb{R}P^{4n+2}$$

is trivial. There exists a simple homotopy equivalence

$$f\colon N^{4n+1} \to \mathbb{R}P^{4n+1},$$

for which the surgery obstruction of the restriction

$$f|_{f^{-1}(\mathbb{R}P^{4n})}\colon f^{-1}(\mathbb{R}P^{4n}) \to \mathbb{R}P^{4n}$$

is nontrivial. In a similar way, it is possible to compute the maps in (2.14) for various manifold pairs using the results of [10], [14], [17], and [18].

Note that we exclude from our consideration the case of bordered manifolds. There are a lot of results about surgery on manifold pairs with boundary in papers [9], [11], [24], and [27]. We would like to point out only that the consideration of the structure sets $\mathcal{NS}(Y,X)$ is similar to the consideration of the mixed structures on a manifold with boundary (the submanifold X plays the role of the boundary ∂Y) (see [2], [9], and [27, page 116]).

3 Filtered manifolds

In this section we describe surgery on closed filtered manifolds. At first, we give necessary known results (see [3], [6], [10], [16], [17], [22], and [28]). Then we obtain new relations between various obstruction groups and structure sets for filtered manifolds. We describe also applications of the obtained results to the problem of realizing elements of various surgery and splitting obstruction groups by normal maps of closed manifolds.

Denote by $\mathcal{X}$ a filtration

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{3.1}$$

of a closed manifold X by closed submanifolds. We assume that every pair of manifolds of $\mathcal{X}$ is a closed manifold pair [24, page 570]. Let n_j ($n_0 = n$) be the dimension of the manifold X_j, and $q_j = n_{j-1} - n_j$ ($1 \le j \le k$) be the codimension of X_j in X_{j-1}. Let $s_i = q_i + q_{i-1}$ be the codimension X_i in X_{i-2}. Recall that we are working in higher dimensions, and $n_k \ge 5$.

Let F_i ($0 \le i \le k-1$) be the square of fundamental groups in the splitting problem for the manifold pair (X_i, X_{i+1}).

Denote by ξ_i $(1 \leq i \leq k)$ the normal block bundle of X_i in X_{i-1} and by ν_i $(2 \leq i \leq k)$ the normal bundle of X_i in X_{i-2}. We have the associated spherical bundles [24]

$$(D^{q_i}, S^{q_i-1}) \to (E(\xi_i), S(\xi_i)) \to X_i$$

and

$$(D^{s_i}, S^{s_i-1}) \to (E(\nu_i), S(\nu_i)) \to X_i.$$

For $2 \leq i \leq k$, we assume that the space $E(\nu_i)$ of the normal bundle ν_i is identified with the space $E(\xi_{i-1}|_{E(\xi_i)})$ of the restriction of the bundle ξ_{i-1} to the space $E(\xi_i)$ in such a way that (see [3], [6], [10], [22], and [28])

$$S(\nu_i) = S(\xi_{i-1}|_{E(\xi_i)}) \cup E(\xi_{i-1}|_{S(\xi_i)}). \tag{3.2}$$

We assume also that every triple of manifolds from the filtration $\mathcal{X}$ satisfies the conditions on normal bundles that are similar to conditions in (3.2). Thus the filtration $\mathcal{X}$ is a stratified manifold in the sense of Browder-Quinn (see [6] and [28]).

Let $\mathcal{X}_j^i$ $(0 \leq i \leq j \leq k)$ be a subfiltration

$$X_j \subset X_{j-1} \subset \cdots \subset X_i \tag{3.3}$$

of the filtration $\mathcal{X}$. Denote $\mathcal{X}_k^i$ by $\mathcal{X}^i$ and $\mathcal{X}_j^0$ by $\mathcal{X}_j$. In particular, we have

$$\mathcal{X}^1 = \mathcal{X}_k^1 \quad \text{and} \quad \mathcal{X}_j^j = X_j. \tag{3.4}$$

A topological normal map $f\colon M \to X$ defines a *t-triangulation of the filtration* $\mathcal{X}$ (see [6], [22], [24], and [28]). Thus we can suppose that the map f is topologically transversal to the submanifolds X_i with $f^{-1}(X_i) = M_i$, and we have the filtration $\mathcal{M}$:

$$M_k \subset M_{k-1} \subset \cdots \subset M_1 \subset M_0 = M. \tag{3.5}$$

Denote by $\mathcal{T}^{TOP}(\mathcal{X})$ the set of *concordance classes* (classes of normal bordisms) of such maps. Note that the natural forgetful map $\mathcal{T}^{TOP}(\mathcal{X}) \to \mathcal{T}^{TOP}(X)$ is an isomorphism (see [22] and [24]).

A t-triangulation $(f, b) : M \to X$ is called an *s-triangulation of the filtration* $\mathcal{X}$ if all constituent normal maps of pairs

$$(M_j, M_l) \to (X_j, X_l), 0 \leq j < l \leq k$$

are s-triangulations. Denote by $\mathcal{S}^{TOP}(\mathcal{X})$ the set of concordance classes of s-triangulations of the filtration $\mathcal{X}$ (see [3], [6], [22], and [28]).

The Browder-Quinn group $L_{n_k}^{BQ}(\mathcal{X})$ is the group of obstructions to find an s-triangulation of the filtration $\mathcal{X}$ in the concordance class of a t-triangulation of this filtration (see [3], [6], [22], and [28]). The surgery theory for stratified manifolds is similar to the classical surgery theory. The following result (see [6] and [28]) provides an exact sequence that is similar to surgery exact sequence (2.1).

Theorem 3.1. *For a filtration $\mathcal{X}$, there exists an exact sequence*

$$\cdots \to L^{BQ}_{n_k+1}(\mathcal{X}) \to \mathcal{S}^{TOP}(\mathcal{X}) \to \mathcal{T}^{TOP}(X) \to L^{BQ}_{n_k}(\mathcal{X}), \tag{3.6}$$

which is realized by a cofibration of spectra

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^{n-n_k}\mathbf{L}^{BQ}(\mathcal{X}) \to \mathbf{S}(\mathcal{X}). \tag{3.7}$$

The corresponding part of the homotopy long exact sequence of cofibration (3.7) coincides with exact sequence (3.6), and we have isomorphisms

$$\pi_i(\mathbf{L}^{BQ}(\mathcal{X})) = L^{BQ}_i(\mathcal{X}) \quad \textit{and} \quad \mathcal{S}^{TOP}(\mathcal{X}) = \mathcal{S}_{n+1}(\mathcal{X}),$$

where $\mathcal{S}_i(\mathcal{X}) = \pi_i(\mathbf{S}(\mathcal{X}))$.

Note that for a filtration $\mathcal{X}$ that is given by a manifold pair $X_1 \subset X_0$, the groups $L^{BQ}_*(\mathcal{X})$ coincide with the surgery obstruction groups $LP_*(F_0)$ (see [3], [10], [22], [24] and [27]).

Let $\mathcal{X}$ be a filtration (3.1). A simple homotopy equivalence $f\colon M \to X$ *splits along the subfiltration* $\mathcal{X}^1$ (see (3.4)) if it is homotopic to a map g for which all restrictions to all pairs of submanifolds, fitting into the filtration $\mathcal{M}$, are s-triangulations (see [3], [10], and [28]). The obstructions groups $LSF_{n_k}(\mathcal{X})$ to splitting a simple homotopy equivalence

$$f\colon M \to X$$

along the subfiltration $\mathcal{X}^1$ is defined in [3] (see, also, [10]). The groups $LSF_*(\mathcal{X})$ are realized by the spectrum $\mathbf{LSF}(\mathcal{X})$.

For a filtration $\mathcal{X}$, that is given by a manifold pair $X_1 \subset X_0$, the groups $LSF_*(\mathcal{X})$ coincide with the splitting obstruction groups $LS_*(F_0)$ (see [3], [10], [24] and [27]).

For a filtration $\mathcal{X}$, we have the following braid of exact sequences [3]

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & \mathcal{S}_{n+1}(\mathcal{X}) \arrow[rr] \arrow[dr] & & H_n(X,\mathbf{L}_\bullet) \arrow[rr] \arrow[dr] & & L_n(\pi_1(X)) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & \mathcal{S}_{n+1}(X) \arrow[ur] \arrow[dr, "\Theta_k"] & & L^{BQ}_{n_k}(\mathcal{X}) \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & L_{n+1}(\pi_1(X)) \arrow[rr, "\delta"] \arrow[ur] & & LSF_{n_k}(\mathcal{X}) \arrow[rr] \arrow[ur] & & \mathcal{S}_n(\mathcal{X}) \arrow[r] \arrow[ur] & {} .
\end{tikzcd} \tag{3.8}$$

Note that for a manifold pair $X_1 \subset X_0$, the braid of exact sequences (3.8) coincides with the braid of exact sequences of [24, Proposition 7.2.6, iv].

Recall that a codimension-one manifold pair $X^n \subset Y^{n+1}$ is called a *Browder-Livesay pair* if X is a one-sided submanifold, and the horizontal maps in the square (2.6) are isomorphisms (see [1], [5], [7], [13], and [24]). Filtration $\mathcal{X}$ is called *a Browder-Livesay filtration* if every pair $X_{i+1} \subset X_i$ of submanifolds of the filtration is a Browder-Livesay pair. An application of $LSF_*(\mathcal{X})$-groups to study the Assembly map is given by the following theorem (see [3], [10], [12], and [22]).

Theorem 3.2. *Let $x \in L_n(\pi)$ be an element for which $\delta(x) \neq 0$ for some Browder-Livesay filtration $\mathcal{X}$ of a manifold X with $\pi_1(X) = \pi$. Then x does not belong to the image of the Assembly map A in (2.4).*

Note that for projective surgery obstruction groups L^p_*, the image of the Assembly map is described by means the invariant δ for filtrations that contains only two or three manifolds (see [12], [13], and [21]). Others applications of surgery on filtered manifolds are given in papers [3], [10], and [17].

The following theorem [3] gives a lot of relations between spectra for surgery and splitting obstruction groups of a filtration $\mathcal{X}$.

Theorem 3.3. *Let $\mathcal{X}$ be a filtration (3.1). Denote by c_i the codimension of the submanifold X_i in the manifold X $(1 \leq i \leq k)$. We have the homotopy commutative diagram of spectra*

(3.9)

$$\begin{array}{ccccccccc} \Omega\mathbf{S}(X) & \to & \Sigma^{c_k}\mathbf{LSF}(\mathcal{X}) & \to & \Sigma^{c_{k-1}}\mathbf{LSF}(\mathcal{X}_{k-1}) & \to \cdots \to & \Sigma^{c_1}\mathbf{LS}(F_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_+ \wedge \mathbf{L}_\bullet & \to & \Sigma^{c_k}\mathbf{L}^{BQ}(\mathcal{X}) & \to & \Sigma^{c_{k-1}}\mathbf{L}^{BQ}(\mathcal{X}_{k-1}) & \to \cdots \to & \Sigma^{c_1}\mathbf{LP}(F_0), \end{array}$$

in which all squares are pullback, and the cofibers of all vertical maps are naturally homotopy equivalent to the spectrum $\mathbf{L}(\pi_1(X))$.

The case of subfiltrations $\mathcal{X}^i_j$ $(0 \leq i \leq k)$ of the filtration $\mathcal{X}$ was considered in papers [3], [10], and [22]. In this case, there are a lot of relations between obstruction groups and structure sets. These results are given by commutative diagrams and braids of exact sequences (see [3], [10], and [22]).

Consider the filtration in (3.1). Let $G_i = \pi_1(X_i)$ be the fundamental group of the manifold X_i, and $\rho_i = \pi_1(X_i \setminus X_{i+1})$. For every pair of manifolds $X_{i+1} \subset X_i$, we have the following pullback square of spectra (see [1], [2], [8], [16], [24], and [27])

$$\begin{array}{ccc} \Sigma^{q_{i+1}}\mathbf{LP}(F_i) & \to & \mathbf{L}(G_i) \\ \downarrow & & \downarrow \\ \Sigma^{q_{i+1}}\mathbf{L}(G_{i+1}) & \to & \mathbf{L}(\rho_i \to G_i). \end{array} \tag{3.10}$$

Note that the homotopy long exact sequences of the maps from this pullback square generate the braid of exact sequences [27, page 264], which is very helpful for computing surgery obstruction groups and natural maps (see [1], [7], [8], [13], [14], [18], and [27]).

Recall that $\mathbf{LP}(F_i) = \mathbf{L}^{BQ}(\mathcal{X}^i_{i+1})$ and c_i is the codimension of X_i in X_0. Using the suspension functor, we can join together the squares (3.10) for various manifold pairs $X_{i+1} \subset X_i$ fitting into the filtration $\mathcal{X}$. We obtain the following homotopy commutative diagram of spectra

$$
(3.11)\qquad
\begin{array}{ccccc}
 & & \mathbf{L}(G_0) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_1}\mathbf{LP}(F_0) & & & & \mathbf{L}(\rho_0 \to G_0) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_1}\mathbf{L}(G_1) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_2}\mathbf{LP}(F_1) & & * & & \Sigma^{c_1}\mathbf{L}(\rho_1 \to G_1) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_2}\mathbf{L}(G_2) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_3}\mathbf{LP}(F_2) & & & & \Sigma^{c_2}\mathbf{L}(\rho_2 \to G_2) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_3}\mathbf{L}(G_3) & & \\
 & & \vdots & & \\
 & & \Sigma^{c_{k-1}}\mathbf{L}(G_{k-1}) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_k}\mathbf{LP}(F_{k-1}) & & & & \Sigma^{c_{k-1}}\mathbf{L}(\rho_{k-1} \to G_{k-1}). \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_k}\mathbf{L}(G_k) & &
\end{array}
$$

We can extend diagram (3.11) to the left direction by means of the pullback construction (see [1], [10], [16], and [22]). Thus we obtain the diagram consisting of the pullback squares of spectra for all Browder-Quinn groups $L_*^{BQ}(\mathcal{X}_j^i)$ $(0 \leq i \leq j \leq k)$ (see [3], [10], and [22]). More precisely, the left part of the extended diagram consists of the suspensions of the pullback squares

$$
(3.12)\qquad
\begin{array}{ccc}
\mathbf{L}^{BQ}(\mathcal{X}_j^i) & \longrightarrow & \Omega^{q_j}\mathbf{L}^{BQ}(\mathcal{X}_{j-1}^i) \\
\downarrow & & \downarrow \\
\mathbf{L}^{BQ}(\mathcal{X}_j^{i+1}) & \longrightarrow & \Omega^{q_j}\mathbf{L}^{BQ}(\mathcal{X}_{j-1}^{i+1}),
\end{array}
$$

where $0 \leq i < i+1 < j \leq k$. Note that the homotopical cofiber of horizontal maps in (3.12) is $\Omega^{-1}\mathbf{LS}(F_{j-1})$, and the homotopical cofiber of the vertical maps is $\Omega^{s_j-1}\mathbf{L}(\rho_i)$ [22]. Every diagram (3.12) generates the braid of exact sequences containing the groups $LS_*(F_j)$, $L_*(\rho_i)$, and the corresponding Browder-Quinn surgery obstruction groups (see [1], [2], [3], [8], [10], and [22]).

The map from the spectrum $X_+ \wedge \mathbf{L}_\bullet$ to the spectrum of the extended diagram (3.11) is defined. The obtained diagram has a form of a pyramid and it is homotopy commutative. The constituent maps of spectra

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^{c_j}\mathbf{L}^{BQ}(\mathcal{X}_j^i)$$

realize the maps

$$\sigma_j^i\colon H_n(X;\mathbf{L}_\bullet)\to L^{BQ}_{n-c_j}(\mathcal{X}_j^i)$$

which have very clear geometric sense. For a normal map $(f,b)\colon M\to X$, the last map gives an obstruction

$$\sigma_j^i(f,b)\in L^{BQ}_{n-c_j}(\mathcal{X}_j^i)$$

to surgery of the normal map of filtered manifolds

$$f|_{f^{-1}(X_i)}\colon \mathcal{M}_j^i\to\mathcal{X}_j^i.$$

The map in Proposition 2.3 is a particular case of this one.

For the case of a Browder-Livesay filtration, the extended diagram (3.11) provides a filtration for the surgery exact sequence of Hambleton and Kharshiladze (see [1], [10], [16], and [22]).

To understand similarity and difference between "abstract surgery theory" and "surgery theory inside an ambient manifold" we need to study more exotic subfiltrations of the filtration $\mathcal{X}$. Now we describe relations between various obstruction groups and structure sets in this case.

Let $A\subset\{0,1,\dots,k\}$ be a nonempty subset. Denote by $\mathcal{Z}_A$ the subfiltration of the filtration $\mathcal{X}$ consisting only of the manifolds that are indexed by A. For example, $\mathcal{Z}_i=\mathcal{X}_i^i=X_i$,

$$\mathcal{Z}_{0,i}\text{ is }X_i\subset X_0,$$

and

$$\mathcal{Z}_{0,i,i+1}\text{ is }X_{i+1}\subset X_i\subset X_0.$$

Note that in [12] the spectrum $\mathbf{LSF}(\mathcal{Z}_{i,j,m})$ is denoted by $\mathbf{LSP}(X_i,X_j,X_m)$. Denote by Ψ_i, $(1\le i\le k)$ the square of fundamental groups in the splitting problem for the manifold pair $X_i\subset X_0$. Note that $\Psi_1=F_0$.

The triple of manifolds $X_2\subset X_1\subset X$ defines a commutative diagram of inclusions

$$\begin{array}{ccc}(X_1\setminus X_2) & \subset & (X\setminus X_2)\\ \cap & & \cap\\ X_1 & \subset & X.\end{array}\tag{3.13}$$

Denote by F_0' the square of fundamental groups in the splitting problem for the pair $(X_1\setminus X_2)\subset(X\setminus X_2)$ (see [23] and [24]).

The horizontal inclusions in (3.13) are inclusions of codimensions q_1, and diagram (3.13) generates the homotopy commutative diagram of spectra

$$\begin{array}{ccc}\mathbf{L}(\pi_1(X_1\setminus X_2)) & \to & \Omega^{q_1}\mathbf{L}(\pi_1(X\setminus X_1)\to\pi_1(X\setminus X_2))\\ \downarrow & & \downarrow\\ \mathbf{L}(\pi_1(X_1)) & \to & \Omega^{q_1}\mathbf{L}(\pi_1(X\setminus X_1)\to\pi_1(X))\\ \downarrow & & \downarrow\\ \mathbf{L}(\pi_1(X_1\setminus X_2)\to(\pi_1(X_1)) & \xrightarrow{tr^{rel}} & \Omega^{q_1}\mathbf{L}(\pi_1(X\setminus X_2)\to\pi_1(X)),\end{array}\tag{3.14}$$

in which all horizontal maps are the transfer maps, and the upper vertical maps are induced by the vertical maps from (3.13). The cofiber of the bottom horizontal map is the spectrum $\Omega^{-1-q_2}\mathbf{LNS}(X, X_1, X_2)$ of the relative transfer map tr^{rel} (see [21]) with homotopy groups

$$LNS_n(X, X_1, X_2) = \pi_n(\mathbf{LNS}(X, X_1, X_2)).$$

The spectrum $\mathbf{LNS}(X, X_1, X_2)$ closely relates with the spectrum

$$\mathbf{LSP}(X, X_1, X_2) = \mathbf{LSF}(\mathcal{Z}_{0,1,2})$$

(see [12] and [21]). In particular, we have the homotopy pullback square of spectra

$$\begin{array}{ccc} \mathbf{LSP}(X, X_1, X_2) & \rightarrow & \Omega^{q_2}\mathbf{LS}(F_0) \\ \downarrow & & \downarrow \\ \mathbf{LS}(\Psi_2) & \rightarrow & \mathbf{LNS}(X, X_1, X_2), \end{array} \tag{3.15}$$

where F_0 is the square in the splitting problem for the pair (X, X_1), and Ψ_2 is the similar square for the pair (X, X_2) (see [12]). The homotopy cofiber of the horizontal maps in (3.15) is $\Omega^{-1}\mathbf{LS}(F_1)$, where F_1 is the square of fundamental groups in the splitting problem for the manifold pair (X_1, X_2). The homotopy cofiber of the vertical maps in (3.15) is $\Omega^{q_2-1}\mathbf{LS}(F_0')$ (see [12] and [21]).

Proposition 3.4. *There exists a homotopy commutative diagram of spectra*

$$\begin{array}{ccccc}
 & & \Sigma^{c_1}\mathbf{LS}(\Psi_1) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_2}\mathbf{LSF}(\mathcal{Z}_{0,1,2}) & & * & & \Sigma^{c_2}\mathbf{LNS}(\mathcal{Z}_{0,1,2}) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_2}\mathbf{LS}(\Psi_2) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_3}\mathbf{LSF}(\mathcal{Z}_{0,2,3}) & & & & \Sigma^{c_3}\mathbf{LNS}(\mathcal{Z}_{0,2,3}) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_3}\mathbf{LS}(\Psi_3) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_4}\mathbf{LSF}(\mathcal{Z}_{0,3,4}) & & & & \Sigma^{c_4}\mathbf{LNS}(\mathcal{Z}_{0,3,4}) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_4}\mathbf{LS}(\Psi_4) & & \\
 & & \vdots & & \\
 & & \Sigma^{c_{k-1}}\mathbf{LS}(\Psi_{k-1}) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{c_k}\mathbf{LSF}(\mathcal{Z}_{0,k-1,k}) & & & & \Sigma^{c_k}\mathbf{LNS}(\mathcal{Z}_{0,k-1,k}), \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{c_k}\mathbf{LS}(\Psi_k) & &
\end{array} \tag{3.16}$$

in which all squares are pullback. We have a map from the spectrum $\Omega\mathbf{S}(X)$ to diagram (3.16) such that the obtained diagram, in the form of a pyramid, is homotopy commutative. The constituent maps of spectra

$$\Omega\mathbf{S}(X) \to \Sigma^{c_i}\mathbf{LS}(\Psi_i),\ 1 \le i \le k$$

and

$$\Omega\mathbf{S}(X) \to \Sigma^{c_i}\mathbf{LSF}(\mathcal{Z}_{0,i-1,i}),\ 2 \le i \le k$$

have very clear geometrical sense. The first map is the realization of the splitting obstruction map

$$\mathcal{S}^{TOP}(X) \to LS_{n-c_i}(\Psi_i)$$

for the manifold pair (X, X_i). The second map is the realization of the map

$$\mathcal{S}^{TOP}(X) \to LSP_{n-c_i}(X, X_{i-1}, X_i)$$

which takes an obstruction to splitting along the submanifold pair $(X_i, X_{i-1}) \subset X$.

Proof. All squares in diagram (3.16) are similar to the square in (3.15), and hence they are pullback. The upper horizontal row in diagram (3.9) is defined for any filtration $\mathcal{X}$ of the manifold X. Hence this row provides the existence of the map from $\Omega\mathbf{S}(X)$ to diagram (3.16). □

Remark. For a filtration $\mathcal{X}$ of the manifold X, diagram (3.16) describes relations between spectra of various splitting obstruction groups. These groups are the groups of obstruction to *"surgery inside the ambient manifold X"*. Diagram (3.16) corresponds to diagram (3.11), which describes relations between abstract surgery obstruction groups for a filtered manifold.

Remark 3.5. If the codimension $c_i \ge 3$, we have homotopy equivalences

$$\mathbf{LS}(\Psi_i) \simeq \mathbf{L}(\pi_1(X_i)),\ \mathbf{LSF}(\mathcal{Z}_{0,i,i+1}) = \mathbf{LP}(F_i),\ \mathbf{LNS}(\mathcal{Z}_{0,i,i+1}) \simeq \Omega^{q_{i+1}}\mathbf{L}(\rho_i \to G_i),$$

where F_i is the square of fundamental groups in the splitting problem for the manifold pair (X_i, X_{i+1}) (see [12] and [24]).

Theorem 3.6. *For $1 \le i < i+1 < j \le k$, we have the homotopy pullback squares of spectra*

$$(3.17)\qquad \begin{array}{ccc} \Sigma^{c_j}\mathbf{LSF}(\mathcal{Z}_{0,i,\dots,j}) & \to & \Sigma^{c_j-1}\mathbf{LSF}(\mathcal{Z}_{0,i,\dots,j-1}) \\ \downarrow & & \downarrow \\ \Sigma^{c_j}\mathbf{LSF}(\mathcal{Z}_{0,i+1,\dots,j}) & \to & \Sigma^{c_j-1}\mathbf{LSF}(\mathcal{Z}_{0,i+1,\dots,j-1}). \end{array}$$

The homotopy cofiber of the horizontal maps in (3.17) is $\Sigma^{c_j+1}\mathbf{LS}(F_{j-1})$. The homotopy cofiber of the vertical maps in (3.17) is $\Sigma^{c_i+1}\mathbf{LS}(\Psi_i')$, where Ψ_i' is the square of fundamental groups for the codimension c_i manifold pair $(X \setminus X_{i+1}, X_i \setminus X_{i+1})$.

Proof. Similarly to diagram (3.11) we can extend diagram (3.16) to the left direction by using the pullback construction. In particular, after the first step we obtain the additional column of spectra $\mathbf{X}_{0,i-2,i-1,i}$ $(3 \leq i \leq k)$ fitting into the pullback squares

$$
\begin{array}{ccc}
\mathbf{X}_{0,i-2,i-1,i} & \to & \Sigma^{c_{i-1}}\mathbf{LSF}(\mathcal{Z}_{0,i-2,i-1}) \\
\downarrow & & \downarrow \\
\Sigma^{c_i}\mathbf{LSF}(\mathcal{Z}_{0,i-1,i}) & \to & \Sigma^{c_{i-1}}\mathbf{LS}(\Psi_{i-1}),
\end{array}
\tag{3.18}
$$

where the cofiber of horizontal maps is $\Sigma^{c_j+1}\mathbf{LS}(F_{j-1})$. Consider diagram (3.9) and squares (3.12) for the filtration $\mathcal{Z}_{0,i-2,i-1,i}$. It follows now that the map

$$
\Sigma^{c_i}\mathbf{LSF}(\mathcal{Z}_{0,i-2,i-1,i}) \to \Sigma^{c_{i-1}}\mathbf{LSF}(\mathcal{Z}_{0,i-2,i-1}),
\tag{3.19}
$$

is defined, and the cofiber of the map in (3.19) is $\Sigma^{c_j+1}\mathbf{LS}(F_{j-1})$. By the unique property of the pullback square, the spectrum $\mathbf{X}_{0,i-2,i-1,i}$ coincides with the spectrum $\Sigma^{c_i}\mathbf{LSF}(\mathcal{Z}_{0,i-2,i-1,i})$ $(3 \leq i \leq k)$. The diagram similar to (3.9) takes place for any subfiltration $\mathcal{Z}_{0,i,i+1,\ldots,j-1,j}$ of the filtration $\mathcal{X}$. Hence, we can iterate our construction. The description of the cofiber of the vertical maps in (3.17) follows from diagram (3.15). □

The extended diagram (3.16) describes natural relations between spectra for the splitting problems along various filtered submanifolds of the given filtration $\mathcal{X}$. The left upper slanting row in this diagram coincides with the part of upper row in diagram (3.9).

Consider the diagram of maps of filtrations

$$
\begin{array}{ccc}
\mathcal{Z}_{0,i,i+1,\ldots,j-1,j} & \to & \mathcal{Z}_{0,i,i+1,\ldots,j-1} \\
\downarrow & & \downarrow \\
\mathcal{Z}_{0,i+1,\ldots,j-1,j} & \to & \mathcal{Z}_{0,i+1,\ldots,j-1},
\end{array}
\tag{3.20}
$$

in which all maps are the "forgetful" maps. For example, the left vertical map is the map of forgetting of the submanifold X_i. The maps in homotopy commutative square (3.17) are induced by maps from diagram (3.20).

The following result follows from Theorem 3.6.

Corollary 3.7. *The homotopy long exact sequences of the pullback square (3.17) provide the braid of exact sequences, which contains the following sequences*

$$
\cdots \to LSF_n(\mathcal{Z}_{0,i,i+1,\ldots,j-1,j}) \to LSF_{n+q_j}(\mathcal{Z}_{0,i,i+1,\ldots,j-1}) \to LS_{n-1}(F_{j-1}) \to \ldots
$$

and

$$
\cdots \to LSF_n(\mathcal{Z}_{0,i,i+1,\ldots,j-1,j}) \to LSF_n(\mathcal{Z}_{0,i+1,\ldots,j-1,j}) \to LS_{n+n_i-n_j-1}(\Psi_i') \to \ldots\ .
$$

Corollary 3.8. *Diagram (3.20) induces the pullback square of spectra*

$$\begin{array}{ccc} \mathbf{S}(\mathcal{Z}_{0,i,i+1,\dots,j-1,j}) & \to & \mathbf{S}(\mathcal{Z}_{0,i,i+1,\dots,j-1}) \\ \downarrow & & \downarrow \\ \mathbf{S}(\mathcal{Z}_{0,i+1,\dots,j-1,j}) & \to & \mathbf{S}(\mathcal{Z}_{0,i+1,\dots,j-1}). \end{array}$$

The homotopy cofiber of the horizontal maps is $\Sigma^{c_j+1}\mathbf{LS}(F_{j-1})$*, and the homotopy cofiber of the vertical maps is* $\Sigma^{c_i+1}\mathbf{LS}(\Psi_i')$*.*

Proof. The pullback construction for spectra $\mathbf{LSF}(\mathcal{Z}_{0,i,\dots,j})$ and Proposition 3.4 provide the maps of the spectra $\Omega\mathbf{S}(\mathcal{X})$ to all spectra of the pullback square (3.17) in such a way, that we obtain a homotopy commutative diagram in the form of a pyramid. The homotopy cofibers of these maps give the pullback squares of spectra of structure sets. Now the result follows from Theorem 3.6 and the properties of the maps between pullback squares. □

Consider the triple $X_2 \subset X_1 \subset X_0$ of manifolds, which is the subfiltration $\mathcal{Z}_{0,1,2}$ of the filtration $\mathcal{X}$. This triple generates the upper square of diagram (3.16). Forgetting the ambient manifold X_0 of the triple $X_2 \subset X_1 \subset X_0$, we obtain a natural map of this square to the pullback square marked by "star" in diagram (3.11) (see [3], [10], [12], and [28]). By a similar way, we can construct the natural forgetful map Υ of diagram (3.16) to the subdiagram of diagram (3.11). The map Υ is given by the maps of spectra

$$\begin{array}{l} \Sigma^{c_i}\mathbf{LS}(\Psi_i) \to \Sigma^{c_i}\mathbf{L}(G_i)\ (1 \le i \le k), \\ \Sigma^{c_i}\mathbf{LSF}(\mathcal{Z}_{0,i-1,i}) \to \Sigma^{c_i}\mathbf{LP}(F_{i-1})\ (2 \le i \le k), \\ \Sigma^{c_i}\mathbf{LNS}(\mathcal{Z}_{0,i-1,i}) \to \Sigma^{c_i-1}\mathbf{L}(\rho_{i-1} \to G_{i-1})\ (2 \le i \le k). \end{array} \tag{3.21}$$

Theorem 3.9. *For a subfiltration* $\mathcal{Z}_{0,i,i+1,\dots,j-1,j}$ *of the filtration* $\mathcal{X}$*, the natural forgetful map*

$$\mathbf{LSF}(\mathcal{Z}_{0,i,i+1,\dots,j-1,j}) \to \mathbf{L}^{BQ}(\mathcal{Z}_{i,i+1,\dots,j-1,j}) \tag{3.22}$$

is defined. The homotopy cofiber of the map in (3.22) is $\Omega^{c_i}\mathbf{L}(\pi_1(X_0 \setminus X_i) \to G_0)$*.*

Proof. See [3]. The pullback construction provides a map Υ^{left} of the extended to the left direction diagram (3.16) to the corresponding subdiagram of the extended diagram (3.11). This map extends the map Υ. From now the result follows using the natural properties of the maps of pullback squares. □

The compositions of the maps from extended diagram (3.16) (see also diagram (3.9)) provide a map

$$\Sigma^{c_k}\mathbf{LSF}(\mathcal{X}_k) \to \Sigma^{c_1}\mathbf{LS}(F_0) \tag{3.23}$$

of spectra, which induces the map

$$\gamma\colon LSF_{n_k}(\mathcal{X}) \to LS_{n_1}(F_0)$$

of splitting obstruction groups.

The following result is a natural generalization of results from [3] and [22], and it is similar to Theorem 3.2.

Theorem 3.10. *Let F be a square of fundamental groups in a splitting problem. Let $\mathcal{X}$ be a filtration of a closed manifold X such that $F_0 = F$, and the subfiltration $\mathcal{X}^1$ of $\mathcal{X}$ is a Browder-Livesay filtration. If an element $x \in LS_{n_1}(F)$ does not belong to the image of the map*

$$\gamma\colon LSF_{n_k}(\mathcal{X}) \to LS_{n_1}(F),$$

then the element x cannot be realized as an obstruction to splitting of a simple homotopy equivalence of closed manifolds.

Proof. The proof is similar to the one given in [3] (see also [10] and [22]). □

In fact, in higher codimensions (≥ 3) the bottom part of diagram (3.16) coincides with the corresponding part of diagram (3.11), as follows easily from Remark 3.5, Theorem 3.8 and properties of maps in (3.21). Recall that q_1 is the codimension of the submanifold X_1 in the manifold X. If $q_1 \geq 3$, then the map Υ provides an isomorphism of diagram (3.16) into the subdiagram of diagram (3.11).

If $q_1 = 2$, then the upper square of diagram (3.16) has the following form

$$(3.24)\qquad \begin{array}{ccccc} & & \Sigma^{c_1}\mathbf{LS}(\Psi_1) & & \\ & \nearrow & & \searrow & \\ \Sigma^{c_2}\mathbf{LSF}(\mathcal{Z}_{0,1,2}) & & * & & \Sigma^{c_2}\mathbf{LNS}(\mathcal{Z}_{0,1,2}), \\ & \searrow & & \nearrow & \\ & & \Sigma^{c_2}\mathbf{L}(G_2) & & \end{array}$$

and the other bottom part of the diagram coincides with the corresponding part of diagram (3.11).

The most interesting case is $q_1 = q_2 = 1$. In this case, the upper part of diagram (3.16) has the following form

$$(3.25)\qquad \begin{array}{ccccc} & & \Sigma^{c_1}\mathbf{LS}(\Psi_1) & & \\ & \nearrow & & \searrow & \\ \Sigma^{c_2}\mathbf{LSF}(\mathcal{Z}_{0,1,2}) & & * & & \Sigma^{c_2}\mathbf{LNS}(\mathcal{Z}_{0,1,2}) \\ & \searrow & & \nearrow & \\ & & \Sigma^{c_2}\mathbf{LS}(\Psi_2) & & \\ & \nearrow & & \searrow & \\ \Sigma^{c_3}\mathbf{LSF}(\mathcal{Z}_{0,2,3}) & & & & \Sigma^{c_3}\mathbf{LNS}(\mathcal{Z}_{0,2,3}), \\ & \searrow & & \nearrow & \\ & & \Sigma^{c_3}\mathbf{L}(G_3) & & \end{array}$$

and the other bottom part of the diagram coincides with the corresponding part of diagram (3.11).

In the cases $q_1 = 2$ and $q_1 = q_2 = 1$, diagram (3.16) can be extended to the left, as well, using the pullback construction. The obtained diagrams contain **L**-, **LSF**-, **LS**-, and $\mathbf{L}^{BQ}$-spectra of various obstruction groups for the filtration $\mathcal{X}$. Similarly to Corollary 3.8 we can construct pullback squares of spectra of the corresponding structure sets. Note that braids of homotopy long exact sequences of such pullback squares are very effective for computing obstruction groups, structure sets, and natural maps (see [8], [9], [10], [13], [14], and [17]). A lot of geometric applications of Browder-Livesay filtrations are given in [3], [10], [17], and [22].

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Inönü–Wigner contraction to the truncated current Lie algebras[1]

by Paolo Casati and Giovanni Ortenzi

Abstract

We show how the truncated current Lie algebras can be constructed by performing a generalized Inönü–Wigner contraction from a direct sum of copies of the same Lie algebra $\mathfrak{g}$. Further we explain how the sometime existing ad–invariant non degenerated bilinear form may be obtained using the same procedure.

1 Introduction

The idea of Lie algebra contraction [7] has born in order to provide a rigorous and appealing argument to justify the claim that the theory of special relativity coincides with the classical mechanics if the light travels infinite velocity. Since then it has found fertile ground in many fields of mathematics. In the applications, this idea is manly used to deduce informations on one Lie algebra by regarding it as contraction of an other better known one (see e.g. [1, 2]).
This will be also the case of the present paper. We shall use the tool of Lie algebra contraction to study a class of in general non semisimple Lie algebras widely investigated in the literature under different names, among which that of truncated current Lie algebra [12, 11], which we shall keep thorough the whole paper.

A truncated current Lie algebra can be associated to any Lie algebra and any positive integer number as follows. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and let $n \in \mathbb{N}$ be a positive integer number, then the Lie algebra

$$\mathfrak{g}^{(n)} = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[\lambda] / \mathbb{K}\lambda^{n+1}. \tag{1.1}$$

with Lie bracket

$$\left[X \otimes \lambda^i, Y \otimes \lambda^j\right] = [X, Y] \otimes \lambda^{i+j} \quad X, Y \in \mathfrak{g}, \quad i, j \in \mathbb{N} \tag{1.2}$$

is called a *truncated current Lie algebra* .

These Lie algebras are naturally graded by non–negative degrees in λ:

$$\mathfrak{g}_k = \{x \otimes \lambda^k \in \mathfrak{g}^{(n)} | \; 0 \le k \le n\} \tag{1.3}$$

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and if $n > 0$ are not semisimple being $\mathfrak{g}_n$ in the radical of its Killing form. What makes these the algebra interesting is that they appear in many different contest like the theory of integrable systems [8], in algebraic geometry [9, 6] or finally more recently in the theory of Lie algebra and their representations [4, 5, 10, 12].

Surprising enough, however, one of their maybe most important property, namely the presence of a bilinear symmetric ad–invariant non degenerated form even in many case when these Lie algebras are non reductive, has been discovered only very recently [3]. The aim of this paper is to show how a generalized Inönü–Wigner contraction could be useful to proceed in the investigations of such Lie algebras. The starting point is the observation that, for a fixed $n \in \mathbb{N}$, $\mathfrak{g}^{(n)}$ is, as vector space, isomorph to the direct sum $\mathfrak{g}^n = \oplus_{i=0}^n \mathfrak{g}$. This suggests to obtain the Lie algebra $\mathfrak{g}^{(n)}$ by contraction of the direct sum of $n+1$ copies of $\mathfrak{g}$. This will be indeed the case, as will be shown in the next section. Using this contraction moreover we shall carry any bilinear ad–invariant form of $\mathfrak{g}^{n+1}$ in an interesting bilinear ad–invariant form of $\mathfrak{g}^{(n)}$, which turns out to be non degenerated if that of $\mathfrak{g}$ has this latter property. Further it seems fairly natural to hope that such technique could be useful to study other aspects of the truncated current Lie algebra as their cohomology groups or their Poisson cohomology just to mention a few.

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2 Truncated current Lie algebras

The aim of this first section is to present the truncated current Lie algebras already mentioned in the introduction.
Despite to the fact that our constructions can be straightforward generalized to the infinite dimensional Lie algebras we restrict ourself to the maybe simpler finite dimensional case.

Definition 2.1. For any Lie algebra $\mathfrak{g}$ and any positive integer number $n \in \mathbb{N}$ let us denote by $\mathfrak{g}^{(n)}$ the Lie algebra given by the tensor product

$$\mathfrak{g} \otimes \mathbb{C}^{(n)} \tag{2.1}$$

where $\mathbb{C}^{(n)} = \mathbb{C}[\lambda]/(\lambda)^{n+1}$, $\mathbb{C}[\lambda]$ is the ring of polynomials in λ and $(\lambda)^{n+1}$ is the principal ideal generated by λ^{n+1}.

This algebra may be identified with the Lie algebra of polynomial maps from $\mathbb{C}[\lambda]/(\lambda^{n+1})$ in $\mathfrak{g}$, hence an element $X(\lambda)$ in $\mathfrak{g}^{(n)}$ can be viewed as the mapping $X : \mathbb{C} \to \mathfrak{g}$, $X(\lambda) = \sum_{k=0}^{n} X_k\lambda^k$ where $X_k \in \mathfrak{g}$.

In this setting the Lie bracket of two elements in $\mathfrak{g}^{(n)}$, $X(\lambda) = \sum_{k=0}^{n} X_k\lambda^k$ and $Y(\lambda) = \sum_{k=0}^{n} Y_k\lambda^k$ can be written explicitly as

$$[X(\lambda), Y(\lambda)] = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} [X_j, Y_{k-j}]_{\mathfrak{g}} \right) \lambda^k \tag{2.2}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie bracket defined on $\mathfrak{g}$.
If on $\mathfrak{g}$ is defined an ad–invariant bilinear form $\langle\cdot,\cdot\rangle$ (for instance the Killing form) then we can carry it to a bilinear form $\langle\cdot,\cdot\rangle_\lambda$ on $\mathfrak{g}(\lambda)$ by setting on $\mathbb{C}(\lambda)$ the standard inner product

$$(p(\lambda), q(\lambda)) = \left(\sum_{i=0}^{n} p_i\lambda^i, \sum_{j=0}^{n} q_j\lambda^j \right) = \int_{|\lambda|=1} \lambda^{-1} p(\lambda)\overline{q(\lambda)}d\lambda = \sum_{i=0}^{\infty} p_i\overline{q_i} \tag{2.3}$$

(where $\overline{z}$ denotes the complex conjugate of z and one has to keep in mind that all the sums are on a finite set and that on the unit circle both $p(\lambda)$ and $p(\lambda^{-1})$ are well defined). More precisely on $\mathfrak{g}(\lambda)$ will be defined the bilinear form:

$$\begin{aligned} \langle X(\lambda), Y(\lambda)\rangle_\lambda &= \langle \sum_{i=0}^{n} X_i\lambda^i, \sum_{j=0}^{n} Y_j\lambda^j \rangle_\lambda \\ &= \int_{|\lambda|=1} \lambda^{-1}\langle X(\lambda), Y(\lambda^{-1})\rangle d\lambda = \sum_{i=0}^{n} \langle X_i, Y_i\rangle. \end{aligned} \tag{2.4}$$

Unfortunately this bilinear form, while it is not degenerate if the bilinear form on $\mathfrak{g}$ is not degenerate, turns out to be in general not ad–invariant even if the form chosen on $\mathfrak{g}$ (like in the case of the Killing form) is ad–invariant. Suppose indeed $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ endowed with the Killing form and set

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then

$$\langle H, [X\lambda, Y\lambda]\rangle_\lambda = \langle H, H\lambda^2\rangle_\lambda = 0$$

but

$$\langle [H, X\lambda], Y\lambda\rangle_\lambda = \langle 2X\lambda, Y\lambda\rangle_\lambda = 2 \neq 0.$$

On the other hand there exist on $\mathfrak{g}^{(n)}$ non trivial symmetric ad–invariant forms, which are surely more exotic than that previously considered but non–degenerated. As proved in [3] it holds indeed

Theorem 2.2. *Suppose that on $\mathfrak{g}$ is defined symmetric ad–invariant non–degenerated bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ then for any set of complex numbers $\mathcal{A}=\{a_j\}_{j=0,\dots n}$, the bilinear form*

$$
\begin{array}{ll}
\mathfrak{g}^{(n)}\times\mathfrak{g}^{(n)} & \to \mathbb{C} \\
(X(\lambda),Y(\lambda)) & \mapsto \langle X(\lambda),Y(\lambda)\rangle^{(n)}_{\mathcal{A}}=\sum_{j=0}^{n}a_j\sum_{i=0}^{j}\langle X_i,Y_{j-i}\rangle_{\mathfrak{g}}.
\end{array}
\tag{2.5}
$$

is a symmetric bilinear, ad–invariant and, if $a_n\neq 0$, non degenerate form.

3 Lie Algebras Contraction

This section is devote to construct the Lie algebra $\mathfrak{g}^{(n)}$ as contraction of the direct sum $\mathfrak{g}^n=\oplus_{i=0}^{n}\mathfrak{g}$. To begin with let us observe that the formula (2.2) suggests an alternative but obviously equivalent definition of the same truncated current Lie algebras:

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{K}$ and n be a positive integer number, the truncated current Lie algebra $\mathfrak{g}^{(n)}$ associated to the pair $(\mathfrak{g},n)$ is the Lie algebra given as vector space by the direct sum $\mathfrak{g}^n=\oplus_{i=0}^{n}\mathfrak{g}$ of $n+1$ copies of the Lie algebra $\mathfrak{g}$ and with Lie bracket given by:

$$
\begin{array}{rll}
[(X_0,X_1,\dots,X_n),(Y_0,Y_1,\dots Y_n)] & =(Z_0,Z_1,\dots,Z_n) & \\
Z_k & =\sum_{j=0}^{k}[X_j,Y_{k-j}] & k=0,\dots n\ .
\end{array}
$$

On behalf of this definition one can regard to the truncated current Lie algebra $\mathfrak{g}^{(n)}$ as the space $\mathfrak{g}^n=\oplus_{i=0}^{n}\mathfrak{g}_i$ endowed with modified Lie bracket with respect to that corresponding to the direct sum of Lie algebras. It is therefore quite natural to wonder if such modified brackets may be obtained from the usual one by performing a suitable Lie algebra contraction. For example in the first non trivial case namely that corresponding to the truncated current Lie algebra $\mathfrak{g}^{(1)}$, with $\mathfrak{g}$ finite dimensional, is indeed the case. In this particular case we can choose a basis for the direct sum $\mathfrak{g}^1$ of the type $X_j^{(k)}$, $k=0,1$ $j=1,\dots,r=\dim(\mathfrak{g})$ $X_j^{(k)}\in\mathfrak{g}^k$ $k=0,1$, $j=1,\dots,r$, with respect to which the canonical Lie bracket are written in the form

$$
\begin{array}{lll}
\left[X_j^{(k)},X_i^{(k)}\right] & =\sum_{l=1}^{r}c_{ji}^{l}X_l^{(k)} & k=0,1 \quad i,j=1,\dots r \\
\left[X_j^{(0)},X_i^{(1)}\right] & =0 & i,j=1,\dots r
\end{array}
\tag{3.1}
$$

i.e., with the same structure constants for each copy of $\mathfrak{g}$. Then by performing the parameter depending change of basis

$$
\begin{array}{ll}
Y_j^{(0)}=X_i^{(0)}+X_i^{(1)} & i=1,\dots r \\
Y_j^{(1)}=\zeta X_i^{(1)} & i=1,\dots r
\end{array}
$$

the equations (3.1) become

$$\begin{aligned}
\left[Y_j^{(0)}, Y_i^{(0)}\right] &= \textstyle\sum_{l=1}^r c_{ji}^l Y_l^{(0)} && i,j=1,\dots r\\
\left[Y_j^{(0)}, Y_i^{(1)}\right] &= \textstyle\sum_{l=1}^r c_{ji}^l Y_l^{(1)} && i,j=1,\dots r\\
\left[Y_j^{(1)}, Y_i^{(1)}\right] &= \zeta\textstyle\sum_{l=1}^r c_{ji}^l Y_l^{(1)} && i,j=1,\dots r.
\end{aligned}$$

Now computing the limit for $\zeta \to 0$ in these latter equations one obtains exactly the Lie bracket previously defined on $\mathfrak{g}^{(1)}$. Unfortunately this procedure, which coincides with the canonical Inönü–Wigner contraction, cannot be directly and straightforward extended to the general case. In this case one has to move much more carefully choosing an appropriate parameter dependence of the new basis as explained in the

Theorem 3.2. *Let $\mathfrak{g}$ be Lie algebra of dimension r and let us denote by $\mathfrak{g}^n = \oplus_{i=0}^n \mathfrak{g}$ the direct sum of $n+1$ copies of $\mathfrak{g}$. Let us moreover choose $X_i^{(k)}$, $i=1,\dots r$, $k=0,\dots n$ a basis for $\mathfrak{g}^n$ such that $X_i^{(k)} \in \mathfrak{g}_k$ for all $i=1,\dots r$, so that with respect to this basis the canonical bracket of $\mathfrak{g}^n$ will have the form*

(3.2)

$$\begin{aligned}
\left[X_i^{(k)}, X_j^{(k)}\right] &= \textstyle\sum_{l=1}^r c_{ij}^l X_l^{(k)} && k=0,\dots n \quad i,j=1,\dots r\\
\left[X_i^{(k)}, X_j^{(m)}\right] &= 0 && k,m=0,\dots n,\ k\neq m \quad i,j=1,\dots r.
\end{aligned}$$

Then if we write equations (3.2) with respect to the basis

$$Y_i^{(k)} = \sum_{l=0}^{n-k} \zeta^{k2^l} X_i^{(l)} \qquad k=0,\dots,n \quad i=1,\dots r, \tag{3.3}$$

and then perform the limit $\zeta \to 0$ we end up with the Lie bracket for the Lie algebra $\mathfrak{g}^{(n)}$:

$$\begin{aligned}
\left[Y_i^{(l)}, Y_j^{(m)}\right] &= \textstyle\sum_{l=1}^r c_{ij}^l Y_l^{(l+m)} && l+m\le n \qquad i,j=1,\dots r\\
\left[Y_i^{(l)}, Y_j^{(m)}\right] &= 0 && l+m> n \qquad i,j=1,\dots r.
\end{aligned} \tag{3.4}$$

Proof Let us first compute the Lie bracket of two elements of the new basis say $Y_i^{(l)}$ and $Y_j^{(m)}$ (where we omit here and in what follows the explicit dependence from ζ) in terms of the old ones:

$$\left[Y_p^{(k)}, Y_q^{(j)}\right] = \sum_{t=1}^r \sum_{l=0}^{\mathrm{Min}(n-k,n-j)} \zeta^{(k+j)2^l} c_{pq}^t X_t^{(l)}. \tag{3.5}$$

If $k+j<n+1$ we write them as

$$\left[Y_p^{(k)}, Y_q^{(j)}\right] = \sum_{t=1}^{r} \sum_{l=0}^{n-k-j} \zeta^{(k+j)2^l} c_{pq}^t X_t^{(l)} + \sum_{l=n-k-j+1}^{\mathrm{Min}(n-k,n-j)} \zeta^{(k+j)2^l} c_{pq}^t X_t^{(l)}, \tag{3.6}$$

which using the definition of the elements $Y_p^{(k)}$ can be written as

$$\left[Y_p^{(k)}, Y_q^{(j)}\right] = \sum_{t=1}^{r} c_{pq}^t Y_t^{(j+k)} + \sum_{l=n-k-j+1}^{\mathrm{Min}(n-k,n-j)} \zeta^{(k+j)2^l} c_{pq}^t X_t^{(l)}. \tag{3.7}$$

To prove our claim we need to show that if ζ goes to zero then for $k+j>n$ the whole right hand of equation (3.5) and for $k+j \leq n$ the second term in the right hand of equation (3.7) vanish.

From equation (3.3) follows by induction over l that

$$\zeta^{n2^l} X_i^{(l)} = \sum_{k=n-l}^{n} p_{kl}(\zeta) Y_i^{(k)} \tag{3.8}$$

with $p_{kl}(\zeta)$ a polynomial in ζ. Indeed in the case $l=0$ this coincides with the equation (3.3). Now if we set $k=l+1$ in equation (3.3) and multiply both hands of it by $\zeta^{(l+1)2^{l+1}}$ we get

$$\zeta^{(l+1)2^{l+1}} Y_i^{(n-l-1)} = \sum_{k=0}^{l} \zeta^{n2^k+(l+1)(2^{l+1}-2^k)} X_i^{(k)} + \zeta^{n2^{l+1}} X_i^{(l+1)}$$

which using the induction hypothesis gives equation (3.8) for $l+1$. Now substituting this in the expression (3.5) with $k+j>n$, one obtains only positive powers of ζ on the right hand side of $\left[Y_p^{(k)}, Y_q^{(j)}\right]$. Hence taking the limit $\zeta \to 0$ gives that $\left[Y_p^{(k)}, Y_q^{(j)}\right] = 0$.

In case of equations (3.7) (which correspond to the case $k+j \leq n$) we need the

Lemma 3.3. *The elements $\zeta^{2^{k(n-k)}} X^{(k)}$ $k=0,\dots n$ can be written as*

$$\zeta^{2^k(n-k)} X_p^{(k)} = \sum_{j=0}^{k} q_j^k(\zeta^{-1}) Y_p^{(n-j)} \tag{3.9}$$

where q_j^k is a polynomial of degree $deg(q_j^k) = 2^k - 2^j$.

Proof First we rewrite the formula (3.3)

$$Y_i^{(n-k)} = \sum_{l=0}^{k} \zeta^{2^l(n-k)} X_i^{(l)} \qquad k=0,\dots,n \quad i=1,\dots r$$

and we proceed by induction on the index k. For $k = 0$ formula (3.9) is immediately true. For $k = 1$ we have $\zeta^{2(n-1)}X^{(1)} = Y^{(n-1)} - \zeta^{-1}Y^{(n)}$. So let us suppose (3.9) is true for k and let us prove it for $k+1$. We have

$$Y_p^{(n-k-1)} = \sum_{l=0}^{k} \zeta^{2^l(n-k-1)} X_p^{(l)} + \zeta^{2^{k+1}(n-k-1)} X_p^{(k+1)},$$

which can be written in the form

$$\zeta^{2^{k+1}(n-k-1)} X_p^{(k+1)} = Y_p^{(n-k-1)} - \sum_{l=0}^{k} \zeta^{2^l(l-k-1)} \zeta^{2^l(n-l)} X_p^{(l)}$$

then by induction

$$\zeta^{2^{k+1}(n-k-1)} X_p^{(k+1)} = Y_p^{(n-k-1)} - \sum_{l=0}^{k} \sum_{j=0}^{l} \zeta^{2^l(l-k-1)} q_j^l(\zeta^{-1}) Y_p^{(n-j)}.$$

By commuting the two sums in the last term of this expression we have

$$\zeta^{2^{k+1}(n-k-1)} X_p^{(k+1)} = Y_p^{(n-k-1)} - \sum_{j=0}^{k} \sum_{l=j}^{k} \zeta^{2^l(l-k-1)} q_j^l(\zeta^{-1}) Y_p^{(n-j)}.$$

from which we have

$$q_l^{k+1}(\zeta^{-1}) = -\sum_{l=j}^{k} \zeta^{2^l(l-k-1)} q_j^l(\zeta^{-1})$$

Since it is easily checked that the leading terms (in ζ^{-1}) in this sum is that corresponding to the index $l = k$, we have finally

$$\deg(q_j^{k+1}) = 2^k + \deg(q_j^k) \qquad j = 0, \dots k.$$

Then again by induction

$$\deg(q_j^{k+1}) = 2^k + 2^k - 2^j = 2^{k+1} - 2^j$$

which concludes the proof of our lemma.

□

Now this lemma allows us to estimate $\zeta^{2^l(k+j)} X_p^l$, we have indeed that

$$\begin{aligned} \zeta^{2^l(k+j)} X_p^l &= \zeta^{2^l(k+j)+2^l(n-l-n+l)} X_p^l = \zeta^{2^l(k+j-n+l)} \zeta^{2^l(n-l)} X_p^l \\ &= \zeta^{2^l(k+j-n+l)} \textstyle\sum_{s=0}^{l} q_j^k(\zeta^{-1}) Y_p^{(n-s)} \\ &= \textstyle\sum_{s=0}^{l} \zeta^{2^l(k+j-n+l)} q_j^k(\zeta^{-1}) Y_p^{(n-s)} \quad . \end{aligned}$$

Therefore being $\deg(q_s^l) = 2^l - 2^s$ and $j + k - n + l > 1$:

$$\lim_{\zeta \to 0} \zeta^{2^l(k+j-n+l)} q_j^k(\zeta^{-1}) = 0 \qquad \forall l \quad n-k-j+1 \le l \le \mathrm{Min}(n-k, n-j).$$

□

4 Bilinear form contraction

In this section we proceed further in the application of the Lie algebra contraction to the truncated current Lie algebras by showing how our generalized Inönü–Wigner contraction allows us to obtain also the wanted ad–invariant symmetric non degenerate bilinear forms on $\mathfrak{g}^{(n)}$ (2.5) as well. This result can be actually achieved but unfortunately one has first to modify the canonical bilinear form defined on the direct sum $\mathfrak{g}^n = \oplus_{k=0}^n \mathfrak{g}$ by multiplying its entries by factors depending in a quite complicate way from the parameter ζ. More precisely it holds the

Theorem 4.1. *Let us consider on the Lie algebra given by the direct sum $\mathfrak{g}^n = \oplus_{k=0}^n \mathfrak{g}$ the canonical bilinear form induced from that defined on $\mathfrak{g}$ $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$:*

$$\begin{aligned} \langle\cdot,\cdot\rangle^n : \mathfrak{g}^n \times \mathfrak{g}^n &\to \mathbb{K} \\ ((X_0,\dots,X_n),(Y_0,\dots,Y_n)) &\mapsto \langle X,Y\rangle^n = \textstyle\sum_{k=0}^n \langle X_k, Y_k\rangle_{\mathfrak{g}} \ . \end{aligned} \tag{4.1}$$

Then there exist ζ–depending factors $d_k(\zeta)$ such that the modified inner product on $\mathfrak{g}^n$ given by

$$\langle X,Y\rangle^n(\zeta) = \sum_{k=0}^n d_k(\zeta)\langle X_k, Y_k\rangle_{\mathfrak{g}} \tag{4.2}$$

has the following property

$$lim_{\zeta\to 0}\langle Y_p^{(j)}(\zeta), Y_q^{(k)}(\zeta)\rangle^n(\zeta) = \langle Y_p^{(j)}, Y_q^{(k)}\rangle_{\mathcal{A}}^{(n)} \tag{4.3}$$

Proof. Let us first consider the relations (4.3) when $j=0$ and k running between 0 and n. This relations form a system of $n+1$ independent linear equations

$$\langle Y_p^{(0)}(\zeta), Y_q^{(k)}(\zeta)\rangle = a_k\omega_{pq} \qquad k=0,\dots n,\ p,q=1,\dots \dim(\mathfrak{g}) \tag{4.4}$$

where $\{\omega_{pq}\}_{p,q=1,\dots,\dim\mathfrak{g}}$ is the matrix form of the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.
We actually should simply verify the weaker assumption

$$\langle Y_p^{(0)}(\zeta), Y_q^{(k)}(\zeta)\rangle \sim a_k\omega_{pq} \quad k=0,\dots n,\ p,q=1,\dots,\dim(\mathfrak{g})$$

but since this does not affect our proof, let us consider the equation (4.4) instead. From this equation one can explicitly compute the coefficients $d_k(\zeta)$, $k=0,\dots,n$, although their expression in terms of ζ turns out to be complicate. Likely enough we need only to know that the $d_k(\zeta)$ satisfy (4.4) and use it to to prove the following technical

Lemma 4.2. *The coefficients $d_k(\zeta)$ have the following asymptotic expansion*

$$\zeta^{(n-k)2^k} d_k(\zeta) \sim (-1)^k a_n \zeta^{-2^k+1}. \tag{4.5}$$

Proof. Let us proceed by induction from formula (4.4) we have immediately

$$\zeta^n d_0(\zeta) = a_n$$

and similarly

$$\zeta^{(n-1)2} d_1(\zeta) \sim -a_n \zeta^{-2+1}.$$

Let us suppose (4.5) is true for k and let us prove it for $k+1$. Using (4.4) we have

$$\zeta^{(n-k-1)2^{k+1}} d_{k+1}(\zeta) = a_{n-k-1} - \sum_{i=0}^{k} \zeta^{(n-k-1)2^i} d_i(\zeta)$$

therefore using the induction hypothesis we obtain

$$\begin{aligned}\zeta^{(n-k-1)2^{k+1}} d_{k+1}(\zeta) &= a_{n-k-1} - \textstyle\sum_{i=0}^{k} \zeta^{(i-k-1)2^i} \zeta^{2^i(n-i)} d_i(\zeta) \\ &\sim a_{n-k-1} - \textstyle\sum_{i=0}^{k} (-1)^i \zeta^{(i-k-1)2^i - 2^i + 1} a_n \\ &\sim -\textstyle\sum_{i=0}^{k} (-1)^i a_n \zeta^{2^i(i-k-2)+1} \sim (-1)^{k+1} a_n \zeta^{-2^{k+1}+1}\end{aligned}$$

which is formula (4.5) for $k+1$ proving the claim.

□

We can complete our theorem. For $j+k > n$ we have indeed

$$\langle Y_p^{(j)}(\zeta), Y_q^{(k)}(\zeta)\rangle(\zeta) = \sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} d_l(\zeta) \zeta^{(j+k)2^l} \langle X_p^{(l)}, X_q^{(l)}\rangle_{\mathfrak{g}}. \tag{4.6}$$

Using lemma 4.2 we can estimate this equation as

$$\begin{aligned}&\textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k-n+l)2^l} \zeta^{(n-l)2^l} d_l(\zeta) \\ &\sim \textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} (-1)^l a_n \zeta^{(j+k-n+l)2^l - 2^l + 1} = \textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} (-1)^l a_n \zeta^{2^l(j+k-n+l-1)+1}\end{aligned}$$

and since $j+k-n+l > 0$ (being $l \geq 0$) we have that

$$\lim_{\zeta\to 0} \langle Y_p^{(k)}, Y_q^{(j)}\rangle = 0 \qquad \text{for } j+k > n$$

as wanted. If vice versa $j+k \leq n$, we have that as

$$\begin{aligned}\langle Y_p^{(k)}, Y_q^{(j)}\rangle &= \textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta)\omega_{pq} \\ &= \textstyle\sum_{l=0}^{n-k-j} \zeta^{(j+k)2^l} d_l(\zeta)\omega_{pq} + \textstyle\sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta)\omega_{pq}\end{aligned}$$

using (4.4) we obtain immediately that

$$\sum_{l=0}^{n-k-j} \zeta^{(j+k)2^l} d_l(\zeta) = a_{k+j},$$

while for the second summand we have, using lemma 4.2

$$\sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta) \sim \sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} a_n \zeta^{(j+k-n+l-1)2^l+1}.$$

But this implies, because $l > n-k-j$ and therefore $j+k-n+l-1 \geq 0$ that the parameter ζ in the addends of the above written equation appears with powers bigger then one, and therefore that

$$\lim_{\zeta\to 0} \left(\sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta) \right) = 0$$

and then finally that

$$\lim_{\zeta\to 0} \langle Y_p^{(k)}, Y_q^{(j)} \rangle = \omega_{pq} \qquad \text{for } j+k \leq n \quad j,k \geq 1 \; p,q = 1,\dots,\dim(\mathfrak{g})$$

which concludes the proof of our theorem.

□

Remark 4.3. Of course the most interesting and more studied case in the literature is when the "source" algebra $\mathfrak{g}$ is simple. In this case indeed the corresponding truncated Lie algebras are metric and one may use such ad–invariant non degenerated bilinear form to construct exactly as in the case of simple Lie algebras their affinization. These latter Lie algebras may be obtained as generalized Inönü–Wigner contraction from the direct sum of the corresponding Kac–Moody Lie algebra as well.

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Reconstruction theorems for two remarkable groups of diffeomorphisms[1]

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Abstract

The notion of modular group of diffeomorphisms is introduced. The group of Hamiltonian diffeomorphisms of a Poisson manifold is considered. It is shown that these groups determine uniquely the underlying manifolds and the related geometric structures. In the proofs a general reconstruction theorem of M. Rubin is applied.

1 Introduction

It is well known ([2], [11]) that the group of all C^r-diffeomorphisms ($1 \leq r \leq \infty$) of a C^r-manifold defines uniquely the topological and smooth structure of the manifold. Analogous results are true for the automorphism groups of some geometric structures. In the present note we show reconstruction theorems in case of some nontransitive geometric structures by making use of a very deep and general reconstruction theorem concerning homeomorphism groups proved by M. Rubin (Theorem 1.3).

Let us first recall some concepts and facts from papers of M. Rubin ([11], [12], [3]). For a topological space X let $H(X)$ denote the group of all homeomorphisms of X. Let G be a subgroup of $H(X)$. For a subset U of X we define

$$G\boxed{U} = \{g \in G| \quad g|_{(X\setminus U)} = \mathrm{id}\}.$$

Definition 1.1. (Rubin) Group G is said to be *factorizable* if for every open cover $\mathcal{U}$ of X, the set $\bigcup_{U\in\mathcal{U}} G\boxed{U}$ generates G.

Observe that we may also express the condition of factorizability in a slightly different way. For $g \in G$ let $\mathrm{supp}(g) := \overline{\{x \in X|g(x) \neq x\}}$ be the support of g. For any open $U \subset X$ let G_U be the totality of g with $\mathrm{supp}(g) \subset U$. Then G is *factorizable (in our sense)* if for every open cover $\mathcal{U}$ of X, the set $\bigcup_{U\in\mathcal{U}} G_U$ generates G. It is immediate that this condition is stronger than that in Definition 1.1.

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Definition 1.2. Group G is said to be *non-fixing* if $G(x) \neq \{x\}$ for every $x \in X$, where $G(x) := \{g(x) | g \in G\}$.

Theorem 1.3. *Let X, Y be regular topological spaces (i.e they satisfy T3 condition) and let G and H be factorizable (in the sense of Definition 1.1), non-fixing homeomorphism groups of X and Y, resp. Suppose that there is an isomorphism $\varphi : G \cong H$. Then there is a homeomorphism $\tau : X \cong Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$.*

In this note we will confine ourselves mainly to the C^∞-smoothness category. The main reason is that the notion of the regularity of a diffeomorphism group (see [8]) can be expressed naturally in this category.

Finally, we would like to emphasize that the reconstruction problem is very geometrical in its spirit. Namely, the fact that the automorphism group of a geometric structure of a manifold determines uniquely the manifold and the structure could be viewed as a modern counterpart of the Erlangen program of F. Klein [7] (see also [2]).

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2 Singular foliations and modular diffeomorphism groups

Recently E. Ben Ami and M. Rubin showed in [3] very interesting reconstruction theorems concerning homeomorphism and diffeomorphism groups related to a regular foliation. Here we will deal with the case of singular foliations.

Let $1 \leq r \leq \infty$ and let L be a subset of a C^r-manifold M endowed with a C^r-differentiable structure which makes it an immersed submanifold. Then L is *weakly imbedded* if for any locally connected topological space N and a continuous map $f : N \to M$ satisfying $f(N) \subset L$, the map $f : N \to L$ is continuous as well. It follows that in this case such a differentiable structure is unique .

Definition 2.1. A *foliation of class C^r* is a partition $\mathcal{F}$ of M into weakly imbedded submanifolds, called leaves, such that the following condition holds. If x belongs to a k-dimensional leaf, then there is a local chart (U, ϕ) with $\phi(x) = 0$, and $\phi(U) = V \times W$, where V is open in $\mathbb{R}^k$, and W is open in $\mathbb{R}^{n-k}$, such that if $L \in \mathcal{F}$ then $\phi(L \cap U) = V \times l$, where $l = \{w \in W | \phi^{-1}(0, w) \in L\}$.

Foliation is called *regular* if all leaves have the same dimension and *singular* if this condition is not satisfied. From now on by foliation we will mean a singular foliation.

Let $(M_i, \mathcal{F}_i)$, $i = 1, 2$, be foliated manifolds. A map $f : M_1 \to M_2$ is called *foliation preserving* if $f(L_x) = L_{f(x)}$ for any $x \in M_1$, where L_x is the leaf passing through x. Next, if $(M_1, \mathcal{F}_1) = (M_2, \mathcal{F}_2)$ then f is said to be *leaf preserving* if $f(L_x) = L_x$ for all $x \in M_1$. Throughout the symbol $\mathrm{Diff}^r(M, \mathcal{F})$ will stand for the group of all leaf preserving C^r-diffeomorphisms of a foliated manifold $(M, \mathcal{F})$.

In view of P. Stefan [15] foliations can be regarded as collections of accessible sets in the following sense.

Definition 2.2. A smooth mapping ϕ of a open subset of $\mathbb{R} \times M$ into M is said to be a C^r*-arrow*, $1 \leq r \leq \infty$, if
(1) $\phi(t, \cdot) = \phi_t$ is a local C^r-diffeomorphism for each t, possibly with empty domain,
(2) $\phi_0 = \mathrm{id}$ on its domain,
(3) $\mathrm{dom}(\phi_t) \subset \mathrm{dom}(\phi_s)$ whenever $0 \leqslant s < t$.

Given an arbitrary set of arrows $\mathcal{A}$, let $\mathcal{A}^*$ be the totality of local diffeomorphisms ψ such that $\psi = \phi(t, \cdot)$ for some $\phi \in \mathcal{A}$, $t \in \mathbb{R}$. Next $\hat{\mathcal{A}}$ denotes the set consisting of all local diffeomorphisms being finite compositions of elements from $\mathcal{A}^*$ or $(\mathcal{A}^*)^{-1} = \{\psi^{-1} | \psi \in \mathcal{A}^*\}$, and of the identity.

Definition 2.3. The orbits of $\hat{\mathcal{A}}$ are called *accessible* sets of $\mathcal{A}$.

For $x \in M$ let $\mathcal{A}(x)$, $\bar{\mathcal{A}}(x)$ be the vector subspaces of $T_x M$ generated by

$$\{\dot{\phi}(t, y) | \phi \in \mathcal{A}, \phi_t(y) = x\}, \quad \{d_y \psi(v) | \psi \in \hat{\mathcal{A}}, \psi(y) = x, v \in \mathcal{A}(y)\},$$

respectively.

Theorem 2.4. *Let $\mathcal{A}$ be an arbitrary set of C^r-arrows on M. Then:*
(i) Every accessible set of $\mathcal{A}$ admits a (unique) C^r -differentiable structure of a connected weakly imbedded submanifold of M.
(ii) The collection of accessible sets defines a foliation $\mathcal{F}$.
(iii) $\{\bar{\mathcal{A}}(x)\}$ is the tangent distribution of $\mathcal{F}$ (cf. [15], [16] for the notion of tangent distribution to a foliation).

Let $G(M) \subset \mathrm{Diff}^\infty(M)$ be any diffeomorphism group. Throught out the symbol $G_c(M)$ will stand for the subgroup of all compactly supported elements. By a smooth path (or isotopy) in $G(M)$ we mean any family $\{f_t\}_{t\in\mathbb{R}}$ with $f_t \in G(M)$ such that the map $(t, x) \mapsto f_t(x)$ is smooth. Next, $G(M)_0$ denotes the subgroup of all $f \in G(M)$ such that there is a smooth path $\{f_t\}_{t\in\mathbb{R}}$ with $f_t = \mathrm{id}$ for $t \leqslant 0$ and $f_t = f$ for $t \geqslant 1$, and such that each f_t stabilizes outside a fixed compact set.

Given $G(M)$ the totality of f_t as above constitutes a set of arrows. This set determines uniquely a foliation. Likewise, the flow of a C^r-vector field is an arrow. Therefore any set of vector fields on M defines a foliation.

As usual, $\mathfrak{X}(M)$ denotes the Lie algebra of vector fields on M, and $\mathfrak{X}(M)_c$ its Lie subalgebra of compactly supported elements.

Definition 2.5. Let $G(M) \subset \mathrm{Diff}^\infty_c(M)$ be a group of compactly supported diffeomorphisms of M. To any smooth path f_t in $G(M)_0$ one can attach a family of vector fields given by

$$\dot{f}_t = \frac{df_t}{dt}(f_t^{-1}).$$

$G(M)$ is called *regular* (cf. [8]) if there exists a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}(M)_c$ such that there is a one-to-one correspondence between isotopies f_t in $G(M)_0$ and smooth paths X_t in $\mathfrak{g}$ given by

$$\frac{df_t}{dt} = X_t \circ f_t \quad \text{with} \quad f_0 = \mathrm{id}, \tag{2.1}$$

that is $\dot{f}_t = X_t$. In particular, f_t is a flow if and only if the corresponding $X_t = X$ is time-independent. $\mathfrak{g}$ is then called the Lie algebra of $G(M)$.

Definition 2.6. A Lie algebra of compactly supported vector fields $\mathfrak{g} \subset \mathfrak{X}(M)_c$ is called *modular* if it is a $C^\infty(M)$-module. A regular group of diffeomorphisms $G(M)$ is said to be *modular* if its Lie algebra $\mathfrak{g}$ is modular.

Example 2.7. Let $\mathcal{F}$ be a C^∞-foliation on a manifold M. Then $\mathrm{Diff}^\infty(M, \mathcal{F})$ is modular with the Lie algebra $\mathfrak{X}(M, \mathcal{F})_c$, the Lie algebra of all compactly supported vector fields tangent to $\mathcal{F}$.

Problem 2.8. The following question seems to have a deep geometrical meaning: given a modular group $G(M)$, under what conditions one has the equality $G(M) = \mathrm{Diff}^\infty(M, \mathcal{F}_{G(M)})$, where $\mathcal{F}_{G(M)}$ is the foliation induced by $G(M)$? Notice that the inclusion $G(M) \subset \mathrm{Diff}^\infty(M, \mathcal{F}_{G(M)})$ is always fulfilled.

Proposition 2.9. *Suppose that $G(M)$ is modular and let $\{U_i\}$ be a finite family of open balls of M. If f_t is an isotopy in $G(M)$ such that $\bigcup_t \mathrm{supp}(f_t) \subset \bigcup U_i$, then there are isotopies f_t^j, supported in $U_{i(j)}$, which satisfy $f_t = f_t^s \circ \cdots \circ f_t^1$. In particular, $G(M)$ satisfies Definition 1.1.*

Proof. Let f_t be as above and let X_t be the corresponding family in $\mathfrak{X}_G(M)$. By considering $f_{(p/m)t} f^{-1}_{(p-1/m)t}$, $p = 1, \ldots, m$, instead of f_t we may assume that f_t is close to the identity.

First we choose a new family of open balls, $\{V_j\}_{j=1}^s$, satisfying $\mathrm{supp}(f_t) \subset V_1 \cup \cdots \cup V_s$ for each t and which is starwise subordinate than $\{U_i\}$, that is

$$(\forall j)(\exists i)\ \mathrm{star}(V_j) \subset U_{i(j)}, \quad \text{where} \quad \mathrm{star}(V_j) = \bigcup_{\overline{V}_j \cap \overline{V}_k \neq \emptyset} V_k.$$

Let $(\lambda_j)_{j=1}^s$ be a partition of unity subordinate to $\{V_j\}$, and let $Y_t^j = \lambda_j X_t$.
We set

$$X_t^j = Y_t^1 + \cdots + Y_t^j, \quad j = 1, \ldots, s,$$

and $X_t^0 = 0$. Each of the smooth families X_t^j integrates to an isotopy g_t^j with support in $V_1 \cup \dots \cup V_j$. We get the partition

$$f_t = g_t^s = f_t^s \circ \dots \circ f_t^1,$$

where $f_t^j = g_t^j \circ (g_t^{j-1})^{-1}$, with the required inclusions

$$\mathrm{supp}(f_t^j) = \mathrm{supp}(g_t^j \circ (g_t^{j-1})^{-1}) \subset \mathrm{star}(V_j) \subset U_{i(j)}$$

which hold if f_t is sufficiently small. □

A Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}(M)$ is called *non-fixing* if $\mathfrak{g}_x = \{X(x)|X \in \mathfrak{g}\} \neq 0$ for all $x \in M$. An important result of I. Amemiya [1] yields the following fact.

Theorem 2.10. *Let $\mathfrak{g}(M)$ and $\mathfrak{g}(N)$ be modular and non-fixing Lie algebras of vector fields on manifolds M and N, resp. If there is a Lie algebra isomorphism $\Psi : \mathfrak{g}(M) \to \mathfrak{g}(N)$ then there is a diffeomorphism $\psi : M \to N$ such that $\psi_* = \Psi$.*

Observe that the theorem of Amemiya is still true for non-compactly supported vector fields. As a consequence of Theorems 1.3 and 2.10 we have

Theorem 2.11. *Let $G(M)$ and $G(N)$ be modular and non-fixing groups of diffeomorphisms of manifolds M and N, resp. If there is an isomorphism $\varphi : G(M)_0 \to G(N)_0$ then there is a unique foliation preserving diffeomorphism $\tau : M \to N$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G(M)$. Here we refer to the foliations $\mathcal{F}_{G(M)}$ and $\mathcal{F}_{G(N)}$.*

Proof. In view of Theorem 1.3 it is clear the existence of a homeomorphism $\tau : M \to N$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G(M)$. Moreover it follows from the definition of the foliations $\mathcal{F}_{G(M)}$ and $\mathcal{F}_{G(N)}$ that τ is foliation preserving.

Next we prove the uniqueness of τ. Let $\bar\tau$ be another bijection such that $\varphi(g) = \bar\tau g \bar\tau^{-1}$ for all $g \in G(M)$. Assume that $\tau \neq \bar\tau$ and put $\chi = \bar\tau^{-1}\tau \neq \mathrm{id}$. We have $\chi g \chi^{-1} = g$ for any $g \in G(M)$. There is an $x \in M$ such that $y = \chi(x) \neq x$. Let $z \in L_x, z \neq x, z \neq y$, and let y, z lie in the same component of $L_x - \{x\}$ whenever $\dim(L_x) = 1$. Then one can find f from the isotropy subgroup of $G(M)$ at x such that $f(y) = z$. Therefore $\chi f \chi^{-1}(y) = \chi f(x) = \chi(x) = y$ and $f(y) = z \neq y$, a contradiction.

It remains to show that τ (and by symmetry τ^{-1}) is smooth. To this end we will apply Theorem 2.10.

We wish to define a Lie algebra isomorphism $\Psi : \mathfrak{g}(M) \to \mathfrak{g}(N)$, where $\mathfrak{g}(M)$ and $\mathfrak{g}(N)$ are the Lie algebra of $G(M)$ and $G(N)$, resp. In fact, Ψ induces a diffeomorphism $\psi : M \to N$ satisfying $\psi_* = \Psi$. On the other hand, the definition of Ψ below implies $\tau_* = \Psi$. Thus $\tau = \psi$ and τ is a diffeomorphism.

Let $X \in \mathfrak{g}(M)$ and let g_t be its one parameter group of diffeomorphisms. Consider $h_t = \tau g_t \tau^{-1}$. Then the h_t's are C^∞-diffeomorphisms and they obviously satisfy $h_{t+s} = h_t h_s$ and $h_0 = \mathrm{id}$. Since the action of $\mathbb{R}$ on N by $(t,x) \mapsto h_t(x)$ is continuous, in view of a well known theorem of Bochner and Montgomery [4], it is C^∞-smooth in both variables. Therefore, it is a one parameter transformation group of an element of $\mathfrak{X}(N)$, say $\Psi(X)$.

Then we have the following equality

$$(\Psi(X)u) \circ \tau \;=\; X(u \circ \tau) \quad \forall u \in C^\infty(N),\, \forall X \in \mathfrak{g}(M). \tag{2.2}$$

This implies also that the right hand side of (2.2) is meaningful for any u and X. In fact, let g_t and $h_t = \tau g_t \tau^{-1}$ be the one-parameter transformation groups of X and $\Psi(X)$, resp. Then $(u \circ \tau \circ g_t)(x) = (u \circ h_t \circ \tau)(x)$ for any $x \in M$ and the derivative $\frac{d}{dt}(u \circ \tau \circ g_t)(x)|_{t=0}$ exists. Therefore

$$X(u \circ \tau) \;=\; \frac{d}{dt}(u \circ \tau \circ g_t)|_{t=0} \;=\; \frac{d}{dt}(u \circ h_t)|_{t=0} \circ \tau \;=\; (\Psi(X)u) \circ \tau.$$

By iterating the above we have also $(\Psi(X_1)\Psi(X_2)u) \circ \tau \;=\; X_1(X_2(u \circ \tau))$.

It follows easily from the above equalities that Ψ is a homomorphism. By uniqueness, φ^{-1} must determine the homeomorphism τ^{-1}. Consequently, the respective mapping from $\mathfrak{g}(N)$ to $\mathfrak{g}(M)$ is the inverse of Ψ. Thus Ψ is a Lie algebra isomorphism. This completes the proof. □

Corollary 2.12. *Let $\mathcal{F}_M$ and $\mathcal{F}_N$ be foliations on manifolds M and N, resp. If there is an isomorphism $\varphi : \mathrm{Diff}^\infty(M, \mathcal{F}_M)_0 \to \mathrm{Diff}^\infty(N, \mathcal{F}_N)_0$ then there is a foliation preserving diffeomorphism $\tau : M \to N$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in \mathrm{Diff}^\infty(M, \mathcal{F}_M)_0$.*

Conjecture 2.13. *By a standard argument using approximation theorems (cf. [6]) one has $\mathrm{Diff}^r(\mathbb{R}^n)_0 = \mathrm{Diff}^r(\mathbb{R}^n)_c$, where $1 \le r \le \infty$. Now, let $0 < k < n$ and let $\mathcal{F}_{n-k}$ be a foliation on $\mathbb{R}^{n-k}$ with $L_0 = \{0\}$. Then $\mathcal{F} = \{\mathbb{R}^k \times L\}_{L \in \mathcal{F}_{n-k}}$ is a foliation on $\mathbb{R}^n$. Note that in light of a splitting theorem by P. Dazord [5] every foliation on a manifold assumes locally this form.*

Now our conjecture is that $\mathrm{Diff}^r(\mathbb{R}^n, \mathcal{F})_0 = \mathrm{Diff}^r(\mathbb{R}^n, \mathcal{F})_c$ in view of a possible "foliated" adaptation of the proof of an approximation theorem (this is beyond the scope of this note). As a consequence we would obtain the factorization property and the reconstruction theorem for $\mathrm{Diff}^r(\mathbb{R}^n, \mathcal{F})_c$; that is, we could detect from the algebraic structure of $\mathrm{Diff}^r(\mathbb{R}^n, \mathcal{F})_c$ the dimension k and the foliation $\mathcal{F}$.

3 Hamiltonian group of a Poisson manifold

The reconstruction problem in case of Poisson structures has been studied by one of us in [14] but here we use a different definition of the Hamiltonian group (by using regularity) and the resulting group is possibly larger and more natural than that in [14].

Poisson structures are nontransitive generalizations of symplectic structures. A Poisson structure on a manifold M can be introduced by a 2-vector Λ on M such that $[\Lambda, \Lambda] = 0$, where $[.,.]$ is the Schouten-Nijenhuis bracket (see [17]). In general, the rank of Λ_x varies but it is even everywhere. The ring of the real smooth functions on M, $C^\infty(M)$, admits a Lie algebra structure by means of the bracket

$$\{u, v\} = \Lambda(\mathrm{d}u, \mathrm{d}v) \quad \text{for any } u, v \in C^\infty(M) \tag{3.1}$$

and every adjoint homomorphism of this bracket is a derivation of $C^\infty(M)$, that is the Leibniz rule is satisfied

$$\{u, vw\} = v\{u, w\} + w\{u, v\}. \tag{3.2}$$

Equivalently, any 2-linear bracket on $C^\infty(M)$ verifying (3.2) and the Jacobi rule defines uniquely a Poisson structure Λ by (3.1).

We have the 'sharp' bundle homomorphism associated with Λ

$$\sharp : \Omega^1(M) \to \mathfrak{X}(M), \quad \sigma(\rho^\sharp) = \Lambda(\rho, \sigma),$$

for any 1-forms $\rho, \sigma \in \Omega^1(M)$, where $\rho^\sharp = \sharp(\rho)$. In the case where Λ is nondegenerate (i.e. $\mathrm{rank}(\Lambda) = \dim(M)$), $\sharp$ is an isomorphism and we get a symplectic structure. In general, the distribution $\sharp(T^*_x M)$, $x \in M$, integrates to a generalized foliation $\mathcal{F}(\Lambda)$ such that Λ restricted to any leaf induces a symplectic structure. The symplectic form living on $L \in \mathcal{F}(\Lambda)$ will be denoted by ω^Λ_L.

A smooth mapping f of (M, Λ) into itself is called a Poisson morphism if $\{u \circ f, v \circ f\} = \{u, v\} \circ f$ for any $u, v \in C^\infty(M)$. Let $\mathrm{Diff}^\infty(M, \Lambda)$ be the group of all Poisson diffeomorphisms of (M, Λ) which are leaf preserving with respect to $\mathcal{F}(\Lambda)$. We will show that $\mathrm{Diff}^\infty(M, \Lambda)_c$ is a regular group.

Recall that a vector field X is an infinitesimal automorphism of (M, Λ) if $[\Lambda, X] = 0$, that is if $\mathcal{L}_X \Lambda = 0$, where $\mathcal{L}$ is the Lie derivative. By $\mathfrak{X}(M, \Lambda)_c$ we denote the Lie algebra of all infinitesimal automorphisms of (M, Λ) with compact support and tangent to $\mathcal{F}(\Lambda)$. Next, let $\mathfrak{X}^*(M, \Lambda)$ be the ideal of $\mathfrak{X}(M, \Lambda)$ of all Hamiltonian vector fields, i.e. $X \in \mathfrak{X}^*(M, \Lambda)$ iff there exists a compactly supported $u \in C^\infty(M)$ such that $X = [\Lambda, u]$ or, equivalently, $X = (\mathrm{d}u)^\sharp$. We have the inclusion $[\mathfrak{X}(M, \Lambda), \mathfrak{X}(M, \Lambda)] \subset \mathfrak{X}^*(M, \Lambda)$ as a consequence of the equality $[X_1, X_2] = [\Lambda, u]$, where u is defined by $u(x) = \iota(X_1(x) \wedge X_2(x))\omega^\Lambda_{L_x}$ (cf. [9]) .

Proposition 3.1. *Let (M, Λ) be an arbitrary Poisson manifold. Suppose that f_t is a compactly supported smooth path with $f_0 = \mathrm{id}$ and X_t is the corresponding path in $\mathfrak{X}(M)_c$ given by (2.1). Then $f_t \in \mathrm{Diff}^\infty(M, \Lambda)_0$ for each t if and only if $X_t \in \mathfrak{X}(M, \Lambda)_c$ for each t.*

Proof. By restricting f_t to a leaf L and using the equality

$$\frac{\partial}{\partial t}(f_t^* \omega_t) = f_t^* \left(\mathcal{L}_{\dot{f}_t} \omega_t + \frac{\partial}{\partial t} \omega_t \right)$$

we have

$$\frac{d}{dt}f_t^*\omega_L^\Lambda = f_t^*(\mathcal{L}_{X_t}\omega_L^\Lambda) = f_t^*(\iota(X_t)\mathrm{d}\omega_L^\Lambda + \mathrm{d}(\iota(X_t)\omega_L^\Lambda)) = f_t^*\mathrm{d}(\iota(X_t)\omega_L^\Lambda).$$

It follows that the claim is true on any leaf, and consequently so is on M. □

A smooth path satisfying Proposition 3.1 is called a *Poisson isotopy.* A Poisson isotopy f_t is *Hamiltonian* if the corresponding $X_t \in \mathfrak{X}^*(M,\Lambda)_c$ for each t. A diffeomorphism f of (M,Λ) is called *Hamiltonian* if there exists a Hamiltonian isotopy f_t such that $f_0 = \mathrm{id}$ and $f_1 = f$. The totality of all Hamiltonian diffeomorphisms is denoted by $\mathcal{H}(M,\Lambda)$. It is obvious that $\mathcal{H}(M,\Lambda)_0 = \mathcal{H}(M,\Lambda)$.

Proposition 3.2. *$\mathcal{H}(M,\Lambda)$ is a normal subgroup of $\mathrm{Diff}^\infty(M,\Lambda)_c$. Moreover, $\mathcal{H}(M,\Lambda)$ is a regular group with Lie algebra $\mathfrak{X}^*(M,\Lambda)_c$.*

Proof. First we show that $\mathcal{H}(M,\Lambda)$ satisfies the group axioms. Let f_t, g_t be Hamiltonian isotopies, that is $\dot{f}_t = (\mathrm{d}u_t)^\sharp$, $\dot{g}_t = (\mathrm{d}v_t)^\sharp$ for some smooth families of C^∞-functions u_t and v_t. Then $f_t \circ g_t$ is still a Hamiltonian isotopy with the corresponding smooth path in $\mathfrak{X}^*(M,\Lambda)$ being $(\mathrm{d}(u_t + v_t \circ f_t^{-1}))^\sharp$, (cf. [8]). Furthermore, we have that f_t^{-1} is Hamiltonian as well with the corresponding smooth path $(\mathrm{d}(-u_t \circ f_t))^\sharp$. Therefore $\mathcal{H}(M,\Lambda)$ is a group.

Next, if f_t is a Hamiltonian isotopy as above and g is a Poisson diffeomorphism then $g^{-1}\circ f_t \circ g$ is also Hamiltonian with the corresponding smooth path $(\mathrm{d}(u_t\circ g))^\sharp$. This means that $\mathcal{H}(M,\Lambda)$ is a normal subgroup of $\mathrm{Diff}^\infty(M,\Lambda)_c$. The second assertion is trivial. □

Lemma 3.3. *Let $X \in \mathfrak{X}^*(M,\Lambda)_c$ with $\mathrm{supp}(X) \subset \bigcup_{i=1}^k U_i$, where the U_i are open. Then there is a decomposition $X = X_1 + \cdots + X_k$ such that $X_i \in \mathfrak{X}^*(M,\Lambda)_c$ and $\mathrm{supp}(X_i) \subset U_i$. The same is true for smooth curves in $\mathfrak{X}^*(M,\Lambda)_c$ instead of elements of $\mathfrak{X}^*(M,\Lambda)_c$.*

The proof is a consequence of the definition of $\mathfrak{X}^*(M,\Lambda)$. Although $\mathfrak{X}^*(M,\Lambda)_c$ is not modular, Lemma 3.3 combined with [10], chapter X, enable us to prove the following analogue of Theorem 2.10.

Theorem 3.4. *Let (M,Λ_M) and (N,Λ_N) be Poisson manifolds with no leaves of dimension zero. If there is a Lie algebra isomorphism $\Psi : \mathfrak{X}^*(M,\Lambda_M)_c \to \mathfrak{X}^*(N,\Lambda_N)_c$ then there is a foliation preserving diffeomorphism $\psi : M \to N$ such that $\psi_* = \Psi$ and $\psi_*\Lambda_M = \mu\Lambda_N$ for some nowhere vanishing function $\mu \in C^\infty(M)$ which is constant on leaves.*

Another consequence of Lemma 3.3 is the fragmentation property for $\mathcal{H}(M,\Lambda)$.

Proposition 3.5. *Let $f \in \mathcal{H}(M,\Lambda)$ and $M = \bigcup_{i=1}^k U_i$, where the U_i are open balls. Then there exist $f_j \in \mathcal{H}(M,\Lambda)$, $j = 1,\ldots,l$, with $f = f_1 \circ \ldots \circ f_l$ such that $\mathrm{supp}(f_j) \subset U_{i(j)}$ for all j. The same is true for isotopies in $\mathcal{H}(M,\Lambda)$.*

The proof exploits the correspondence between isotopies in $\mathcal{H}(M,\Lambda)$ and smooth curves in $\mathfrak{X}^*(M,\Lambda)_c$ given by (2.1) and is similar to that of Proposition 2.9.

Theorem 3.6. *Let (M,Λ_M) and (N,Λ_N) be Poisson manifolds with no leaves of dimension 0. If there is an isomorphism $\varphi : \mathcal{H}(M,\mathcal{F}_M) \to \mathcal{H}(N,\mathcal{F}_N)$ then there is a unique foliation preserving diffeomorphism $\tau : M \to N$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in \mathcal{H}(M,\mathcal{F}_M)$ and $\tau_*\Lambda_M = \mu\Lambda_N$ for some nowhere vanishing function $\mu \in C^\infty(M)$ which is constant on leaves.*

Proof. The existence of τ is an immediate consequence of Theorem 1.3 and Proposition 3.5. The proof of the uniqueness of τ is analogous to that in the proof of Theorem 2.11. In the remaining part of the proof we follow the proof of Theorem 2.11 and we use Theorem 3.4. □

The following fact can be proved in a "traditional" way ([2], [13]), but here is a consequence of the above results and Theorem 1.3 in [3].

Corollary 3.7. *Let (M,ω_M) and (N,ω_N) be symplectic manifolds and let $\mathrm{Diff}^\infty(M,\omega_M)$ and $\mathrm{Diff}^\infty(M,\omega_N)$ be the corresponding symplectomorphism groups. If there is an isomorphism $\varphi : \mathrm{Diff}^\infty(M,\omega_M) \to \mathrm{Diff}^\infty(N,\omega_N)$ then there is a diffeomorphism $\tau : M \to N$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in \mathrm{Diff}^\infty(M,\omega_M)$ and $\tau^*\omega_N = C\omega_M$ for some constant C.*

A possible reconstruction theorem for the group of Poisson diffeomorphisms (even if the Poisson structure is regular) is still an open and probably difficult problem.

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Bernoulli-Taylor formula in the case of Q-umbral Calculus [1]

by Ewa Krot-Sieniawska

Abstract

In this note we derive the Q-difference Bernoulli-Taylor formula with the rest term of the Cauchy form by the Viskov's method [1]. This is an extension of technique presented in [5, 6, 7] by the use of Q-extended Kwaśniewski's $*_\psi$-product [3, 4]. The main theorems of Q-umbral calculus were given by G. Markowsky in 1968 (see [2]) and extended by A.K.Kwaśniewski [3, 4].

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1 Introduction - Q-umbral calculus

We shall denote by $\mathbf{P}$ the algebra of polynomials over the field $\mathbb{F}$ of characteristic zero.

Let us consider a one parameter family $\mathcal{F}$ of sequences. Then a sequence ψ is called admissible ([3, 4]) if $\psi \in \mathcal{F}$. Where

$$\mathcal{F} = \{\Psi : \mathbb{R} \supset [a,b]; q \in [a,b] : \Psi(q) : \mathbb{Z} \to \mathbb{F}; \Psi_0(q) = 1, \Psi_n(q) \neq 0, \\ \Psi_{-n}(q) = 0, n \in \mathbb{N}\}.$$

Now let us to introduce the Ψ-notation [3, 4]:

$$n_\psi = \Psi_{n-1}(q)\Psi_n^{-1}(q);$$

$$n_\psi! = n_\psi(n-1)_\psi \cdots 2_\psi 1_\psi = \Psi_n^{-1}(q),$$

$$n_\psi^{\underline{k}} = n_\psi(n-1)_\psi \cdots (n-k+1)_\psi,$$

$$\binom{n}{k}_\psi = \frac{n_\psi^{\underline{k}}}{k_\psi!},$$

$$exp_\psi\{y\} = \sum_{k=0}^{\infty} \frac{y^k}{k_\psi!}.$$

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Definition 1.1. [2, 3] Let Q be a linear map $Q : \mathbf{P} \to \mathbf{P}$ such that:

$$\forall_{p\in\mathbf{P}} \quad \deg(Qp) = (\deg p) - 1,$$

$\deg p = -1$ means $p = \text{const} = 0$. Then Q is called a generalized differential operator [2].

Definition 1.2. [2] A normal sequence of polynomials $\{q_n(x)\}_{n\geq 0}$ has the following properties:

(a) $\forall_{n\geq 0} \quad \deg q_n(x) = n$;

(b) $\forall_{x\in\mathbb{F}} \quad q_0(x) = 1$;

(c) $\forall_{n\geq 1} \quad q_n(0) = 0$.

Definition 1.3. [2, 3] Let $\{q_n(x)\}_{n\geq 0}$ be the normal sequence of polynomials. Then we call it the ψ-basic sequence of the generalized differential operator Q (or $Q-\psi-$basic sequence) if:

$$\forall_{n\geq 0} \quad Qq_n(x) = n_\psi q_{n-1}(x).$$

In [2] it is shown that once a differential operator Q is given a unique ψ-basic polynomial sequence is determined and the other way round: given a normal sequence $\{q_n(x)\}_{n\geq 0}$ there exists a uniquely determined generalized differential operator Q.

Definition 1.4. [2, 3] The $\hat{x}_Q$-operator (Q-multiplication operator, the operator dual to Q) is the linear map $\hat{x}_Q : \mathbf{P} \to \mathbf{P}$ such that:

$$\forall_{n\geq 0} \quad \hat{x}_Q q_n(x) = \frac{n+1}{(n+1)_\psi} q_{n+1}(x).$$

Note that: $\quad [Q, \hat{x}_Q] = id.$

Definition 1.5. [3] Let $\{q_n(x)\}_{n\geq 0}$ be a $Q-\psi-$basic sequence. Let

$$E^y_q(Q) = E^y(Q) = exp_{Q,\psi}\{yQ\} = \sum_{k=0}^{\infty} \frac{q_k(y)Q^k}{k_\psi!}.$$

$E^y_q(Q) = E^y(Q)$ is called the $Q-\psi-$generalized translation operator.

As was announced in [5, 7], the notion of Kwaśniewski's $*_\psi$ product and its properties presented in [3] can be easily Q-extended as follows.

Definition 1.6. [3]

$$x *_Q q_n(x) = \hat{x}_Q(q_n(x)) = \frac{n+1}{(n+1)_\psi} q_{n+1}(x), \quad n \geq 0$$

$$x^n *_Q q_n(x) = (\hat{x}_Q^n)(q_1(x)) = \frac{(n+1)!}{(n+1)_\psi!} q_{n+1}(x), \quad n \geq 0$$

Therefore

$$x *_Q \alpha 1 = x *_Q \alpha q_0(x) = \hat{x}_Q(\alpha q_0(x)) = \alpha \hat{x}_Q(q_0(x)) = \alpha x *_Q 1$$

and

$$f(x) *_Q q_n(x) = f(\hat{x}_Q) q_n(x)$$

for every formal series f ($f \in \mathbb{F}[[x]]$).

Definition 1.7. According to definition above and [3] we can define Q-powers of x by recurrence relation:

$$x^{0*_Q} = 1 = q_0(x)$$

$$x^{n*_Q} = x *_Q (x^{(n-1)*_Q}) = \hat{x}_Q(x^{(n-1)*_Q}).$$

It is easy to show that:

$$x^{n*_Q} = x *_Q x *_Q \ldots *_Q 1 = \frac{n!}{n_\psi!} q_n(x), \quad n \geq 0.$$

Also note that:

$$x^{n*_Q} *_Q x^{k*_Q} = \frac{n!}{n_\psi!} q_{k+n}(x)$$

and

$$x^{k*_Q} *_Q x^{n*_Q} = \frac{k!}{k_\psi!} q_{k+n}(x)$$

so in general i.e. for arbitrary admissible ψ and for every $\{q_n(x)\}_{n \geq 0}$ it is non-commutative.

Due to definition above one can prove the following Q-extended properties of Kwaśniewski's $*_\psi$ product [3, 4].

Proposition 1.8. Let f, g be formal series, ($f, g \in \mathbb{F}[[x]]$). Then for $*_Q$ defined above holds:

(a) $Qx^{n*_Q} = nx^{(n-1)*_Q}$, $n \geq 0$;

(b) $\exp_{Q,\psi}[\alpha x] = \exp\{\alpha \hat{x}_Q\} 1$, where $\exp_{Q,\psi}\{\alpha x\} = \sum_{k \geq 0} \frac{q_k(x)\alpha^k}{k_\psi!}$;

(c) $Q(x^k *_Q x^{n*_Q}) = (Dx^k) *_Q x^{n*_Q} + x^k *_Q (Qx^{n*_Q})$;

(d) $Q(f *_Q g) = (Df) *_Q g + f *_Q (Qg)$, ($Q$-Leibnitz rule);

(e) $f(\hat{x}_Q)g(\hat{x}_Q)1 = f(x) *_Q \tilde{g}$; $\tilde{g}(x) = g(\hat{x}_Q)1$.

According to [2, 3], let us to define Q-integration operator which is a right inverse operation to generalized differential operator Q i.e.:

$$Q \circ \int d_Q t = id.$$

Definition 1.9. We define Q-integral as a linear operator such that

$$\int q_n(x) d_Q x = \frac{1}{(n+1)_\psi} q_{n+1}(x); \quad n \geq 0.$$

Proposition 1.10.

(a) $Q \circ \int_\alpha^x f(t) d_Q t = f(x)$;

(b) $\int_\alpha^x (Qf)(t) d_Q t = f(x) - f(\alpha)$;

(c) formula for integration "per partes" :

$$\int_\alpha^\beta (f *_Q Qg)(x) d_Q x = [(f *_Q g)(x)]_\alpha^\beta - \int_\alpha^\beta ((Df) *_Q g)(x) d_Q x.$$

2 Q-umbral calculus Bernoulli-Taylor formula

In [1] O.V.Viskov establishes the following identity

$$\hat{p} \sum_{k=0}^{n} \frac{(-\hat{q})^k \hat{p}^k}{k!} = \frac{(-\hat{q})^n \hat{p}^{n+1}}{n!} \tag{2.1}$$

what he calls the Bernoulli identity. Here $\hat{p}, \hat{q}$ stand for linear operators satisfying condition:

$$[\hat{p}, \hat{q}] = \hat{p}\hat{q} - \hat{q}\hat{p} = id. \tag{2.2}$$

Now let $\hat{p}$ and $\hat{q}$ be as below:

$$\hat{p} = Q, \quad \hat{q} = \hat{x}_Q - y, \quad y \in \mathbb{F}.$$

From definition (1.4) we have $[Q, \hat{x}_Q - y] = id$. After substitution into (2.1) we get:

$$Q \sum_{k=0}^{n} \frac{(y - \hat{x}_Q)^k Q^k}{k!} = \frac{(y - \hat{x}_Q)^n Q^{n+1}}{n!}.$$

Now let us apply it to any polynomial (formal series) f:

$$Q\sum_{k=0}^{n}\frac{(y-\hat{x}_Q)^k(Q^kf)(t)}{k!}=\frac{(y-\hat{x}_Q)^n(Q^{n+1}f)(t)}{n!}.$$

Now using definitions (1.6) and (1.7) of $*_Q$-product we get:

$$Q\sum_{k=0}^{n}\frac{(y-x)^{k*_Q}*_Q(Q^kf)(t)}{k!}=\frac{(y-x)^{n*_Q}*_Q(Q^{n+1}f)(t)}{n!}.$$

After integration $\int_y^x d_Qt$ using proposition (1.2) it gives Q-difference calculus Bernoulli-Taylor formula of the form:

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}(x-y)^{k*_Q}*_Q(Q^kf)(y)+R_{n+1}(x) \tag{2.3}$$

where R_{n+1} stands for the rest term of the Cauchy type :

$$R_{n+1}(x)=\frac{1}{n!}\int_y^x(x-t)^{n*_Q}*_Q(Q^{n+1}f)(t)d_\psi t. \tag{2.4}$$

3 Special cases

1. An example of generalized differential operator is $Q\equiv\partial_\psi$. Then $q_n=x^n$ and $\partial_\psi x^n=n_\psi x^{n-1}$ for an admissible ψ. ∂_ψ is called ψ-derivative. Then also $\hat{x}_Q\equiv\hat{x}_\psi$ and $\hat{x}_\psi x^n=\frac{(n+1)}{(n+1)_\psi}x^{n+1}$ and $[\partial_\psi,\hat{x}_\psi]=id$. In this case we get ∂_ψ -difference calculus Bernoulli-Taylor formula presented in [5, 6, 7] of the form
$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}(x-\alpha)^{k*_\psi}*_\psi(\partial_\psi^kf)(\alpha)+R_{n+1}(x)$$
with
$$R_{n+1}(x)=\frac{1}{n!}\int_\alpha^x(x-t)^{n*_\psi}*_\psi(\partial_\psi^{n+1}f)(t)d_\psi t.$$
In [3] there is given a condition for the case $Q=Q(\partial_\psi)$ for some admissible ψ, (see Section 2, Observation 2.1).

2. For $Q=\partial_\psi$, $q_n(x)=x^n$, $n\geq 0$ the choice $\psi_n(q)=\frac{1}{[R(q^n)]!}$, $R(x)=\frac{1-x}{1-q}$ gives the well known q-factorial $n_q!=n_q(n-1)_q\dots 2_q1_q$, for $n_q=1+q+q^2+\dots+q^{n-1}$. Then $\partial_\psi=\partial_q$ becomes the well known Jackson's derivative ∂_q:
$$(\partial_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x}$$
The ∂_q-difference version of the Bernoulli-Taylor formula was given in [5] by the use of $*_q$-product.

3. By the choice $Q \equiv D \equiv \frac{d}{dx}$, $q_n(x) = x^n$ and $\psi_n = \frac{1}{n!}$ after substitution to (2.3), (2.4) we get the classical Bernoulli-Taylor formula of the form:

$$f(x) = \sum_{k=0}^{n} \frac{(x-\alpha)^k}{k!} f^{(k)}(\alpha) + \int_{\alpha}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)dt$$

where $f^{(k)}(\alpha) = (D^k f)(\alpha)$.

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On the pseudoharmonic functions[1]

by Iryna Yurchuk

Abstract

We will obtain the conditions for pseudoharmonic functions defined on a disk to be topologically equivalent in term of their invariant that is a finite connected graph with a strict partial order on vertices. The inverse problem of the realization of a graph as a combinatorial invariant of such functions will be solved.

1 Introduction

In [1, 2, 3, 9] the conditions for a topological equivalence of functions, flows, and other structures on manifolds were obtained. Almost all of such solutions are formulated by using their invariants which are some combinatorial objects (graphs, chord diagrams, shakes etc.) For example in [2] the authors constructed spin graphs that were used for the classification of Morse-Smale flows on closed surfaces. In this note we use the so called combinatorial diagrams (being graphs with additional data) for the classification of pseudoharmonic functions on 2-disk.

Let $D^2 \subset \mathbb{C}$ be a oriented closed $2-$disk and $f : D^2 \to \mathbb{R}$ be a pseudoharmonic function. Recall that a function f is pseudoharmonic if there is a preserving orientation homeomorphism φ of a domain D^2 onto itself such that $f \circ \varphi$ is harmonic.

Let us characterize all types of points of $IntD^2$ and ∂D^2.

A point $z_0 \in IntD^2$ is called *a regular* point of f if there is a $U(z_0)$ in which f equals to $f = Rez + f(z_0)$, otherwise z_0 is critical.

A point $z_0 \in \partial D^2$ is called *a boundary regular* point of f if there exist $U(z_0) \in D^2$ and a homeomorphism $h : U \to D^2_+$ of it onto a semidisk $D^2_+ = D^2 \cap (\mathbb{R} \times [0, +\infty))$ such that a function $h \circ f \circ h^{-1} : D^2_+ \to \mathbb{R}$ equals to $Rez + f(z_0)$.

Points of ∂D^2 that are neither regular nor isolated points of their level curves will be called *boundary critical.*

A value c of f is called critical (regular) if the connected components of level curves of $f^{-1}(c)$ contain critical points (don't contain critical points and are homeomorphic to a disjoint union of segments that intersect the ∂D^2 in their endpoints).

Definition 1.1. A value c of f is semiregular if it is neither regular nor critical.

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It is known that the level curves of a critical value of a pseudoharmonic function are trees (in general, a disjoint union of trees) [4].

By $f|_{\partial D^2}$ we denote a restriction of f on the boundary ∂D^2. We will be interested in a case when $f|_{\partial D^2}$ *has a finite number of local extrema.* In other words, the pseudoharmonic function defined on $D^2 \subset C$ is a continuous function satisfying the following conditions [4, 5, 6, 8]:

- $f|_{\partial D^2}$ is a continuous function with a finite number of local extrema;
- f has a finite number of critical points in the interior of a disk and each of them is a saddle point (in the neighborhood of them the function f has a representation like $f = Rez^n + const$, $n \geq 2$, where $z = x + iy$).

At first we will construct its combinatorial invariant (called a combinatorial diagram) and formulate a necessary and sufficient condition of a topological equivalence of such function. After that we will be interested in the converse problem. Suppose we are given a graph, to which conditions it should satisfy to be a combinatorial invariant of a pseudoharmonic function.

Let us recall some definitions. We will say that $f, g : D^2 \to \mathbb{R}$ are topologically equivalent if there exist preserving orientation homeomorphisms $h_1 : D^2 \to D^2$ and $h_2 : \mathbb{R} \to \mathbb{R}$ such that $f = h_2^{-1} \circ g \circ h_1$.

By a graph G we mean a topological graph (i.e., a CW - complex with 0 and 1 – cells, where $0-$cells are its vertices and $1-$ cells are its edges). Let T be a tree (i.e., a $CW-$complex such that it does not contain cycles). Denote by V the set of its vertices and by $V_{ter}(T)$ the subset of V consisting of all terminal vertices. A disjoint union of trees is called a forest and denoted by $F = \bigcup_i \Psi_i$, where every Ψ_i is a tree. By a path $P(v_i, v_k)$ which connects vertices v_i and v_k of G we mean a sequence of edges e_j such that each of them belongs to it once. It is known that for any two vertices of a tree there is a unique path connecting them. Two vertices of a graph are *adjacent* if they are the ends of the same edge. A graph G is embedded into R^2 if there exists a bijection $\phi : G \to \mathbb{R}^2$ such that two points $\phi(x)$ and $\phi(y)$ are joined by a segment iff x and y are joined by an edge of G and no two distinct open segments have a point in common.

2 Theorem of a topological equivalence

Since a disk is a manifold with a boundary and the problem of a construction of a invariant of such manifolds is unsolved we need to obtain a special structure.

First construct the Kronrod-Reeb graph (Γ_{K-R}) of $f|_{\partial D^2}$. $\Gamma_{K-R}(f|_{\partial D^2})$ is isomorphic to a circle with even number of vertices. Then, we add to $\Gamma_{K-R}(f|_{\partial D^2})$ collections of connected components $\widehat{f}^{-1}(a_i) \subseteq f^{-1}(a_i)$ and $\widehat{f}^{-1}(c_i) \subseteq f^{-1}(c_i)$

which contain only critical and boundary critical points, where every a_i is a critical value and every c_j is a semiregular value. Put

$$P(f) = \Gamma_{K-R}(f|_{\partial D^2}) \bigcup_i \widehat{f}^{-1}(a_i) \bigcup_j \widehat{f}^{-1}(c_j),$$

where every a_i (c_j) is a critical value (a semiregular value). By using the values of f, we can put a partial order on vertices of $P(f)$ in the following way: $v_1 > v_2 \iff f(x_1) > f(x_2)$, where $v_1, v_2 \in P(f)$, and $x_1, x_2 \in D^2$ are the corresponding points of vertices v_1, v_2. If $f(x_1) = f(x_2)$ then v_1 and v_2 are non comparable.

Since this partial order is irreflexive, asymmetric, and transitive, it is *the strict partial order*.

The graph $P(f)$ with a strict partial order will be called the combinatorial diagram of f.

Definition 2.1. A subgraph $q(f)$ of $P(f)$ will be called a $\mathcal{C}r$ – subgraph if it satisfies the following conditions:

- $q(f)$ is a simple cycle;
- every pair of adjacent vertices $v_i, v_{i+1} \in q(f)$ is comparable.

It is easy to show that $P(f)$ has the following properties:

A1) there exists a $\mathcal{C}r$-subgraph $q(f) \in P(f)$;

A2) $\overline{P(f) \setminus q(f)}$ is a disjoint union of finitely many subtrees $\Psi_1, \Psi_2, \dots, \Psi_n$ such that for every i any two vertices $v', v'' \in \Psi_i$ are non comparable;

A3) there is an embedding $\psi : P(f) \to D^2$ such that $\psi(P(f)) \subset D^2, \psi(q(f)) = \partial D^2$, and $\psi(P(f) \setminus q(f)) \subset IntD^2$;

A4) a set $\Theta = D^2 \setminus \psi(P(f))$ is a disjoint union of θ_i such that every θ_i is a simply connected domain and $\partial\overline{\theta}_i$ contains either one or two arcs of ∂D^2.

By A1 and A2 a $\mathcal{C}r$-subgraph is unique. The following lemma holds true.

Lemma 2.2. *If $P(f) \subset \mathbb{R}^3$ is a diagram of a pseudoharmonic function f then an embedding ψ such that $\psi(P(f)) \subset D^2, \psi(q(f)) = \partial D^2$ and $\psi(P(f) \setminus q(f)) \subset IntD^2$ is unique up to homeomorphism of D^2 onto itself.*

Theorem 2.3. *Two pseudoharmonic functions f and g are topologically equivalent iff there exists a preserving order isomorphism between their combinatorial diagrams $\varphi : P(f) \to P(g)$.*

Proof. Necessity. Suppose that two pseudoharmonic functions $f : D^2 \to \mathbb{R}$ and $g : D^2 \to \mathbb{R}$ are topologically equivalent and let $P(f)$, $P(g)$ be their diagrams. By Lemma 2.2 for $P(f)$ $(P(g))$ there exists a unique embedding ψ_1 (ψ_2) such that $\psi_1(P(f)) \subset D^2$ $(\psi_2(P(g)) \subset D^2)$. Then we put $\varphi = \psi_2^{-1} \circ h_1 \circ \psi_1$.

Sufficiency. Let $\varphi : P(f) \to P(g)$ be a preserving order isomorphism. We will construct a homeomorphism $h_1 : D^2 \to D^2$ such that $f = g \circ h_1$. Notice that $f|_{\partial D^2} = g \circ \varphi|_{\partial D^2}$, therefore we may define h_1 on ∂D^2 by $\varphi|_{\partial D^2}$.

By Lemma 2.2 let consider an embedding ψ_1 (ψ_2) of $P(f) \in \mathbb{R}^3$ $(P(g) \in \mathbb{R}^3)$. In accordance with A4 a homeomorphism $h_1 = \psi_2 \circ \varphi \circ \psi_1^{-1}$ is defined on the boundary of θ_i (θ'_j). Then there exists an extension of h_1 to the interior of D^2 such that $f = g \circ h_1$. $\square$

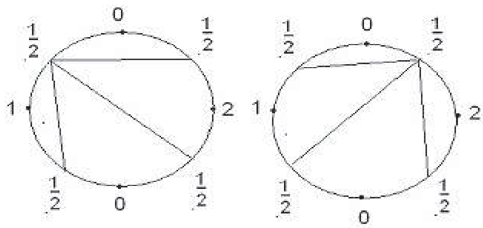

Figure 1: Diagrams of two topologically nonequivalent pseudoharmonic functions.

In Figure 1 on the right a boundary critical point lies between 0 and 2 but on the left between 0 and 1.

3 Realization of a graph as the diagram of a pseudoharmonic function

At first we recall that a nonempty set X is called a cyclically ordered set if there is a ternary relation C on the set X with the following propeties:

1) if $C(x, y, z)$ then $x \neq y \neq z \neq x$;

2) if $x \neq y \neq z \neq x$ then either $C(x, y, z)$ or $C(x, z, y)$ is true;

3) if $C(x, y, z)$ then $C(y, z, x)$;

4) if $C(x, y, z)$ and $C(x, z, t)$ then $C(x, y, t)$.

Two elements are neighboring with respect to a fixed cyclic order C if for any $x \neq x', x''$ we have either $C(x', x'', x)$ or $C(x'', x', x)$. Let fix a subset V^* of vertices of a tree T such that $V_{ter}(T) \subseteq V^* \subseteq V$ and put a cyclic order on them. The notation $C(v_{j_1}, v_{j_2}, ..., v_{j_k})$ means that $C(v_{j_1}, v_{j_2}, ..., v_{j_k}) = C(v_{j_k}, v_{j_1}, v_{j_2}, ..., v_{j_{k-1}}) = \ldots = C(v_{j_2}, ..., v_{j_k}, v_{j_1})$, where $v_{j_i} \neq v_{j_k}$, $k \neq i$, $v_{j_i} \in V^*$.

Definition 3.1. A tree T with a fixed cyclic order of vertices $v_i \in V^*$ is called a $\mathfrak{D}$-planar if there exists an embedding φ such that $\varphi(T) \subset D^2, \varphi(V^*) \subset \partial D^2$, $\varphi(T \setminus V^*) \subset IntD^2$, $C(v_{j_1}, v_{j_2}, ..., v_{j_k})$= $C(\varphi(v_{j_1}), \varphi(v_{j_2}), ..., \varphi(v_{j_k}))$, where a cyclic order $C(\varphi(v_{j_1}), \varphi(v_{j_2}), ..., \varphi(v_{j_k}))$ is generated by the orientation of ∂D^2 and $V_{ter}(T) \subseteq V^* \subseteq V$.

The next theorem is a criteria of the $\mathfrak{D}$-planarity of a tree.

Theorem 3.2. *If $\sharp V^* = 2$ then a tree T is a $\mathcal{D}$-planar.*

If $\sharp V^* \geq 3$ then the $\mathcal{D}$-planarity of a tree T is equivalent to the following condition that has to be satisfied: for every edge e there are exactly two paths in T such that they pass through an edge e and connect adjacent vertices of V^*.

We leave the proof to the reader.

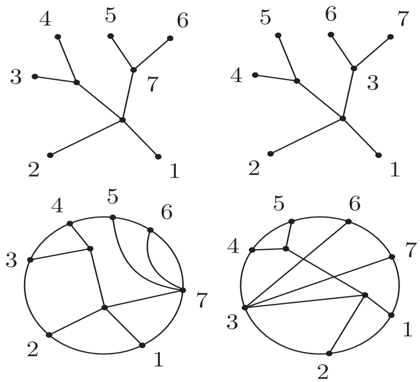

Figure 2: On the right a tree is not a $\mathfrak{D}$– planar.

Let G be a finite connected graph with a strict partial order and every of its vertex has a degree greater than 1.

Definition 3.3. A simple cycle $\gamma \subset G$ will be called a Cr – cycle if every pair of adjacent vertices of γ are comparable.

Suppose that for a graph G the following conditions hold:

C1) a graph G has a unique Cr-cycle γ;;

C2) $\overline{G \setminus \gamma} = F = \bigcup_{i=1}^{k} \Psi_i$ where F is a forest such that

- any two vertices v_j and v_k of every connected component $\Psi_i \subset F$ are non comparable;
- for any vertex $v \in G \setminus \gamma$ it is true that $deg(v) = 2s \geq 4$;

C3) *Condition on a strict order for Cr – cycle γ:*

for any vertex v of γ and its adjacent vertices v_1 and v_2 such that $v_1, v_2 \in \gamma$ one of the following conditions holds true:

- if $deg(v) = 2$ then $deg(v_1) > 2$, $deg(v_2) > 2$ and there exists a unique index i such that $v_1, v_2 \in \Psi_i$;
- if $deg(v) = 2s > 2$ $(deg(v) = 2s + 1)$ then $v_1 \lesseqgtr v \gtreqless v_2$ $(v_1 \lesseqgtr v \lesseqgtr v_2)$;

C4) *Condition on a strict order for G*: if for some vertices of G we have that $v < v'$ $(v > v')$ and $v', v'' \in \Psi_i \subseteq \overline{G \setminus \gamma}$ then $v < v''$ $(v > v'')$;.

By C2 there exists a nonempty subset of vertices V^* of F such that $V^* \in \gamma$. It is clear that a subset V^* of F is divided into subsets V_i^* such that $V^* = \bigcup_i V_i^*$ and $V_i^* \subset \Psi_i \subset F$.

Definition 3.4. A graph $G \subset R^3$ satisfying C1 and C2 is called a $\mathfrak{D}-$planar if there exists an embedding $\varphi : G \to D^2$ such that $\varphi(\gamma) = \partial D^2$, $\varphi(G \setminus \gamma) \subset IntD^2$.

Theorem 3.5. *A graph G satisfying C1 and C2 is a $\mathfrak{D}$ – planar iff every tree Ψ_i with a subset of vertices V_i^* that has a cyclic order defined by γ is a $\mathfrak{D}$-planar and for any index $m \neq n$ a set V_n^* belongs to a unique connected component of $\gamma \setminus V_m^*$.*

Let v_1 and v_2 be any two vertices $v_1, v_2 \in V_i^* \subseteq \Psi_i \subseteq G$ satisfying C1 and C2. A set $\gamma \setminus (v_1 \cup v_2)$ consists of a disjoint union of sets γ_1 and γ_2.

Definition 3.6. A pair of vertices $v_1, v_2 \in V_i^*$ is called boundary if there exists an index k such that γ_k does not contain vertices of a set V_i^* and at least one vertex of $V^* \setminus V_i^*$ belongs to γ_k, $k = \overline{1,2}$.

Denoted by $\omega(v_1, v_2)$ a boundary pair and by α a set γ_k from Definition 3.6. It is clear that for every vertex v_i of $\omega(v_1, v_2)$ there exists an adjacent vertex $\tilde{v}_i$ such that $\tilde{v}_i \in \alpha$ where $i = \overline{1,2}$.

Definition 3.7. A graph G is called special if the following conditions hold:

S1) G satisfies C1 and C2;

S2) G is a $\mathfrak{D}$ – planar;

S3) for any boundary pair $\omega(v_1, v_2) \in V_i^*$ the pair $\tilde{v}_1, \tilde{v}_2$ belongs to a unique set V_k^*, where $V_k^* \subset V^* \setminus V_i^*$.

Lemma 3.8. *If a graph G is special then the set $\Theta = D^2 \setminus \phi(G)$ consists of a disjoint union of domains θ_i such that every $\partial\overline{\theta}_i$ contains either one or two arcs of a boundary ∂D^2, where $\phi : G \to D^2$ is an embedding such that $\phi(\gamma) = \partial D^2$, $\phi(G \setminus \gamma) \subset IntD^2$.*

Definition 3.9. A special graph G is called a Δ – graph if it satisfies C3.

Let us there are two partial order $<$ and $<'$ on a set A. We will say that a partial order $<$ can be extended to $<'$ if a map $Id : (A, <) \to (A, <')$ is monotonous.

Theorem 3.10. *If graph is a combinatorial diagram of a pseudoharmonic function f then G is a Δ – graph.*

If a graph G is a Δ – graph, then a strict partial order on $V(G)$ can be extended to one so that a graph G with new partial order on a set of vertices will be isomorphic to the combinatorial diagram of some pseudoharmonic function f. A strict partial order of a graph G coincides with a strict partial order of a combinatorial diagram $P(f)$ of a pseudoharmonic function f iff G satisfies:

if vertices v', v'' are non comparable then from $v > v'$ follows $v > v''$ where $v \in G$, $v \neq v'$, $v \neq v''$.

We briefly give main ideas of a proof.

Proof. Necessity. This follows from A1-A4.

Sufficiency. Let G be a Δ – graph. By Definition 3.7 and Lemma 3.8 there exists an embedding $\phi : G \to D^2$ such that $\phi(\gamma) = \partial D^2$, $\phi(G \setminus \gamma) = IntD^2$, and the set $\Theta = D^2 \setminus \phi(G)$ consists of a disjoint union of domains θ_i such that every $\partial\overline{\theta}_i$ contains either one or two arcs of a boundary ∂D^2. For every θ_i let us define a foliation:

Case 1: Let $\theta_{k_1} \subset \Theta$ and $\partial\theta_{k_1}$ contains one arc $S_i \subset \partial D^2$. We consider $\phi^{-1}(S_i)$. Since γ is a Cr – cycle satisfying C3, the set $\phi^{-1}(S_i)$ consists of two edges e_1 and e_1 such that they are adjacent to vertex v (its degree equals to 2) and another their end vertices are non comparable. We define a foliation by curves Γ_i such that for any index i it is true that $\Gamma_i \bigcap S_i = x_1^i \cup x_2^i$, $\lim\limits_{i \to n} \Gamma_i = \partial\theta_{k_1} \setminus S_i$, and $\bigcap\limits_i \Gamma_i = \varnothing$, $\Gamma_0 = \partial\varepsilon(\phi(v))$, where $x_1^i \in \phi^{-1}(e_1)$, $x_2^i \in \phi^{-1}(e_2)$, $\varepsilon(\phi(v))$ is a neighborhood of $\phi(v)$ (Figure 3).

Case 2: Let $\theta_{k_2} \subset \Theta$ be such that $\partial\theta_{k_2}$ contains two boundary arcs $S_{n_i} \subset \partial D^2$ and $S_{m_i} \subset \partial D^2$. We consider $\phi^{-1}(S_{n_i})$ and $\phi^{-1}(S_{m_i})$. Since γ is Cr – cycle satisfying C3, the set $\phi^{-1}(S_{n_i})$ ($\phi^{-1}(S_{m_i})$) consists of an edge e_1 (e_2) such that its

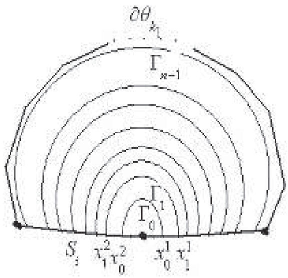

Figure 3: A foliation of a domain with one boundary arc.

end vertices are comparable. It is clear that $\partial\theta_{k_1} \setminus (S_{n_i} \cup S_{m_i}) = \lambda_1 \cup \lambda_2$, where $\phi^{-1}(\lambda_1) \in \Psi_l$, $\phi^{-1}(\lambda_2) \in \Psi_k$. We define a foliation by curves Γ_i such that for any index i it is true that $\Gamma_i \bigcap S_{n_i} = x_1^i$, $\Gamma_i \bigcap S_{m_i} = x_2^i$ and $\lim\limits_{i\to n} \Gamma_i = \lambda_2$, $\bigcap\limits_i \Gamma_i = \varnothing$, $\Gamma_0 = \lambda_1$, where $x_1^i \in \phi^{-1}(e_1)$, $x_2^i \in \phi^{-1}(e_2)$ (Figure 4).

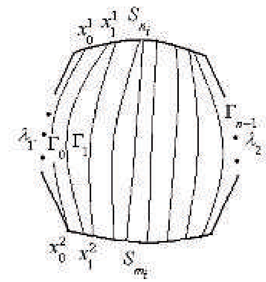

Figure 4: A foliation of a domain with two boundary arcs.

A family of curves $\{\Gamma_j\}$ is a regular and there exists a continuous function such that this family is the level curves of it [7]. Therefore for every $\overline{\theta}_i$ there is a continuous function f_i. Since $\overline{\theta}_i \cap \overline{\theta}_j \neq \varnothing$ and $f_i(x) = f_j(x)$, where $x \in \overline{\theta}_i \cap \overline{\theta}_j$, there exist a continuous function f such that $f|_{\overline{\theta}_i} = f_i$. By C2 all critical points of $IntD^2$ are saddle.

Let consider the restriction of f onto ∂D^2. The points $\phi(v_i)$ corresponding to vertices v_i such that $v_i \in \gamma$ and $deg(v_i) = 2k$ are local extrema of it. It is obvious that there is a finite number of them. By C3 it is easy to prove that after local minima be next a local maxima on ∂D^2.

□

Corollary 3.11. *Let G is a Δ – graph. G satisfies $C4$ iff a strict partial order of G coincides with a strict partial order of a combinatorial diagram $P(f)$ of some pseudoharmonic function f corresponding it.*

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Hyperspaces of max-plus and max-min convex sets[1]

by Lydia Bazylevych

Abstract

It is proved that the hyperspace of max-plus convex compact subsets in a domain U of $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the Q-manifold $U \times Q \times [0, 1)$, where Q denotes the Hilbert cube.

1 Introduction

The topology of hyperspaces of compact and closed convex sets was investigated by many authors. The classical result by Nadler, Quinn and Stavrokas [9] asserts that the hyperspace of convex compact subsets in $\mathbb{R}^n$, $n \geq 2$, (endowed with the Hausdorff metric) is a contractible Q-manifold. Recall that a Q-manifold is a manifold modeled on the Hilbert cube $Q = [0, 1]^\omega$. The mentioned result found many applications in convex geometry, for example in the proof that the hyperspace of all compact strictly convex bodies is homeomorphic to the separable Hilbert space ℓ^2 (see [2]). L. Montejano [8] generalized the mentioned result of Nadler, Quinn and Stavrokas and proved that, for every open subset U of $\mathbb{R}^n$, $n \geq 2$, the hyperspace of compact convex subsets contained in U is homeomorphic to the Q-manifold $U \times Q \times [0, 1)$.

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. Given $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we denote by $x \oplus y$ the coordinatewise maximum of x and y and by $\lambda \odot x$ the vector obtained from x by adding λ to every its coordinate. A subset A in $\mathbb{R}^n$ is said to be *max-plus convex* if $\alpha \odot a \oplus \beta \odot b \in A$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$ (see Fig. 1, which demonstrates that, even in the two-dimensional case, the geometry of max-convex sets is reacher then that of convex sets; in particular, there are three types of segments in this case).

The max-plus convexity (or tropical convexity, in another terminology) is a natural counterpart of convexity in the so-called idempotent mathematics; see, e.g., [7]. A related notion of $\mathbb{B}$-convex set is considered in [3].

We denote by $\mathrm{mpcc}(\mathbb{R}^n)$ the hyperspace of all nonempty max-plus convex compact subsets in $\mathbb{R}^n$. Note that every max-plus convex compact subset in $\mathbb{R}^n$ is a subsemilattice of in $\mathbb{R}^n$ with respect to the operation $\oplus$. In particular, $\max A \in A$, for any max-plus convex compact subset A in $\mathbb{R}^n$.

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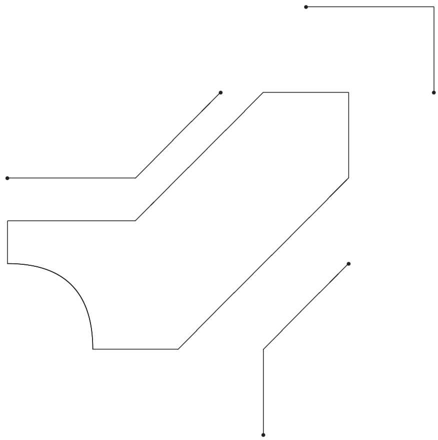

Figure 1: A max-plus convex set and typical segments in the plane

The aim of this paper is to show that there exists a max-plus counterpart of the Montejano theorem. We prove that the hyperspace of max-plus convex compact sets in an open set U of $\mathbb{R}^n$, $n \geq 2$, is the Q-manifold $U \times Q \times [0, 1)$.

In order to keep a reasonable length of this publication we only outline the proofs and omit technical details. They will appear elsewhere.

2 Preliminaries

If $a \in \mathbb{R}^n$, we write $a = (a_1, \ldots, a_n)$. We endow $\mathbb{R}^n$ with the L_∞-metric d, $d(a, b) = \max_i |a_i - b_i|$. The set $\mathbb{R}_{\max}$ is endowed with the metric ϱ, $\varrho(x, y) = |e^x - e^y|$ (convention: $e^{-\infty} = 0$).

By $\exp X$ we denote the set of all nonempty compact subsets of a metric space (X, d). We endow $\exp X$ with the Hausdorff metric d_H:

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}$$

(hereafter, $O_r(C)$ denotes the r-neighborhood of $C \in \exp X$; also, by $\bar{O}_r(C)$ we denote the closed r-neighborhood of C). We denote by max the maximum operation in $\mathbb{R}^n$. Note that, for any nonempty compact subset A in $\mathbb{R}^n$, the element $\max(A)$ is well-defined.

Lemma 2.1. *The maximum map* $\max\colon \mathrm{mpcc}(\mathbb{R}^n) \to \mathbb{R}^n$ *is continuous.*

Proof. Suppose that $A, B \in \mathrm{mpcc}(\mathbb{R}^n)$ and $d_H(A, B) = \varepsilon \geq 0$. Let $\max(A) = (p_1, \ldots, p_n)$, $\max(B) = (q_1, \ldots, q_n)$. Suppose that $d(\max(A), \max(B)) > \varepsilon$. Without loss of generality, we may assume that $q_i > p_i + \varepsilon$, for some i. Then

$O_{q_i-p_i}(\max(B))$ does not intersect A, whence $d_H(A,B) \geq q_i - p_i > \varepsilon$. The obtained contradiction demonstrates that $d(\max(A), \max(B)) \leq d_H(A,B)$. Thus d_H is nonexpanding and therefore continuous. □

Note also that $\max(A) \in A$, for every $A \in \mathrm{mpcc}(\mathbb{R}^n)$.

For any subset U of $\mathbb{R}^n$, by $\mathrm{mpcc}(U)$ we denote the set

$$\{A \in \mathrm{mpcc}(\mathbb{R}^n) \mid A \subset U\}.$$

Recall that a map is said to be *proper* if the preimage of any compact subset is compact.

By AR (respectively ANR) we denote the class of absolute (neighborhood) retracts for the class of metrizable spaces.

The following notion is introduced in [6]. A *c-structure* on a topological space X is an assignment to every nonempty finite subset A of X a contractible subspace $F(A)$ of X such that $F(A) \subset F(A')$ whenever $A \subset A'$. A pair (X,F), where F is a c-structure on X is called a *c-space*. A subset E of X is called an *F-set* if $F(A) \subset E$ for any finite $A \subset E$. A metric c-space (X,d) is said to be a *metric l.c.-space* if all the open balls are F-sets and all open r-neighborhoods of F-sets are also F-sets. It is known (see [6]) that any complete metric $l.c.$-space is an absolute retract.

We say that a metric space X satisfies the *disjoint approximation property* (DAP) if for every continuous function $\varepsilon\colon X \to (0,\infty)$ there exist continuous maps $f_1, f_2\colon X \to X$ such that $d(f_i(x),x) < \varepsilon(x)$, for every $x \in X$, $i = 1,2$, and $f_1(X) \cap f_2(X) = \emptyset$.

We will need the following characterization theorem for Q-manifolds.

Theorem 2.2 (Toruńczyk [10])**.** *A locally compact ANR X is a Q-manifold if and only if X satisfies the DAP.*

A Q-manifold M is said to be $[0,1)$-*stable* if M is homeomorphic to $M \times [0,1)$ (see [5]). The following result of R.Y.T. Wong [11] is often used in infinite-dimensional topology: a Q-manifold X is $[0,1)$-stable if and only if there is a proper homotopy $H\colon X \to [0,1) \to X$ such that $H(x,0) = x$ for every $x \in X$ (recall that a map is *proper* if the preimage of every compact set is compact).

We will use the following Classification theorem for $[0,1)$-stable Q-manifolds; see [5, Theorem 21.2].

Theorem 2.3. *Two $[0,1)$-stable Q-manifolds are homeomorphic if and only they are homotopy equivalent.*

3 Main result

The main result of this section is the following theorem.

Theorem 3.1. *Let U be an open subset of $\mathbb{R}^n$, where $n \geq 2$. Then the hyperspace* $\mathrm{mpcc}(U)$ *is homeomorphic to* $U \times Q \times [0, 1)$.

Proof. First we are going to prove that the hyperspace $\mathrm{mpcc}(\mathbb{R}^n)$ is an absolute retract. We introduce a c-structure on $\mathrm{mpcc}(\mathbb{R}^n)$ as follows. First, if $\{A_1, \ldots, A_k\}$ is a finite subset of $\mathrm{mpcc}(\mathbb{R}^n)$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\max}$ with $\lambda_1 \oplus \cdots \oplus \lambda_k = 0$, then we let

$$\bigoplus_{i=1}^{k} \lambda_i \odot A_i = \{\oplus_{i=1}^{k} \lambda_i \odot a_i \mid a_i \in A_i,\ i = 1, \ldots, k\}.$$

Note that $\bigoplus_{i=1}^{k} \lambda_i \odot A_i \in \mathrm{mpcc}(\mathbb{R}^n)$.

Next, let

$$F(\{A_1, \ldots, A_k\}) = \{\oplus_{i=1}^{k} \lambda_i \odot A_i \mid \lambda_i \in \mathbb{R}_{\max},\ i = 1, \ldots, k,\ \oplus_{i=1}^{k} \lambda_i = 0\}.$$

Note that every set of the form $\mathcal{A} = F(\{A_1, \ldots, A_k\})$ is contractible. Indeed, let $A_0 \in \mathcal{A}$. Define the map $F\colon \mathcal{A} \times [-\infty, 0] \to \mathcal{A}$ by the formula

$$F(A, \alpha) = A \oplus \alpha \odot A_0,\ (A, \alpha) \in A \times [-\infty, 0].$$

Also, we define the map $G\colon \mathcal{A} \times [-\infty, 0] \to \mathcal{A}$ by the formula

$$G(A, \alpha) = \alpha \odot A \oplus A_0,\ (A, \alpha) \in A \times [-\infty, 0].$$

Gluing the maps F and G, define the map $H\colon \mathcal{A} \times [-\infty, \infty] \to \mathcal{A}$ by the formula

$$H(A, t) = \begin{cases} F(A, t), & \text{if } t \in [-\infty, 0], \\ G(A, -t), & \text{if } t \in [0, \infty]. \end{cases}$$

Then it can be easily proved that H is a homotopy that connects the identity map and a constant map.

A direct verification that F determines a c-structure and that the space so obtained is a metric *l.c.*-space is left to the reader.

We are going to demonstrate that the space $\mathrm{mpcc}(\mathbb{R}^n)$ satisfies the DAP. Let $\varepsilon\colon \mathrm{mpcc}(\mathbb{R}^n) \to (0, \infty)$ be a continuous function. Define $f_1\colon \mathrm{mpcc}(\mathbb{R}^n) \to \mathrm{mpcc}(\mathbb{R}^n)$ by the formula $f_1(A) = \bar{O}_{\varepsilon(A)/2}(A)$. It is easy to see that the map f_1 is continuous and $d(f_1(A), A) \leq \varepsilon(A)/2 < \varepsilon(A)$.

Let

$$f_2(A) = A \cup \{\lambda \odot \max A \mid \lambda \in [0, \varepsilon(A)/3]\}.$$

Then f_2 is a continuous map, $d(f_1(A), A) < \varepsilon(A)$, and simple geometric arguments show that

$$f_1(\mathrm{mpcc}(\mathbb{R}^n)) \cap f_2(\mathrm{mpcc}(\mathbb{R}^n)) = \emptyset$$

(indeed, for any $A \in f_1(\mathrm{mpcc}(\mathbb{R}^n))$, every neighborhood of the point $\max A$ is n-dimensional, while, for any $A \in f_2(\mathrm{mpcc}(\mathbb{R}^n))$, the point $\max A$ has 1-dimensional neighborhoods).

By Toruńczyk's characterization theorem, the space $\mathrm{mpcc}(\mathbb{R}^n)$ is a Q-manifold.

Note that the hyperspace $\mathrm{mpcc}(U)$ is an open subset of the hyperspace $\mathrm{mpcc}(\mathbb{R}^n)$ and therefore is also a Q-manifold.

In order to prove that the space $\mathrm{mpcc}(U)$ is $[0,1)$-stable we have to construct a proper homotopy

$$H\colon \mathrm{mpcc}(U) \times [1,\infty) \to \mathrm{mpcc}(U)$$

such that $H(A,1) = A$, for every $A \in \mathrm{mpcc}(U)$ (see, e.g. [4]). For any $A \in \mathrm{mpcc}(U)$, let

$$\varphi(A) = \sup\{t \in [1,\infty) \mid h_\tau(A) \subset U \text{ for every } \tau \in [1,t)\}$$

(note that $\varphi(A)$ can be equal to ∞), where $h_\tau(A)$ is the image of A under the homothety map centered at $\max(A)$ with dilation coefficient τ (note also that the homothety maps preserve the max-plus convexity). That the function φ is lower-semicontinuous follows from the fact that U is open in $\mathbb{R}^n$. In its turn, this implies that the set

$$W = \{(A,t) \in \mathrm{mpcc}(U) \times [1,\infty) \mid t < \varphi(A)\}$$

is open in $\mathrm{mpcc}(U) \times [1,\infty)$. There exists a map $g\colon \mathrm{mpcc}(U) \times [1,\infty) \to [1,\infty)$ such that the map $f\colon \mathrm{mpcc}(U)\times[1,\infty) \to W$ defined by $f(A,t) = (A, g(A,t))$, for every $A \in \mathrm{mpcc}(U)$, is a homeomorphism. Indeed, the function φ being a lower-semicontinuous is a pointwise limit of a sequence of continuous functions $\varphi_1 \le \varphi_2 \le \dots$. Without loss of generality, one may assume that $1 < \varphi_1 < \varphi_2 < \dots$. Given $A \in \mathrm{mpcc}(U)$, define a function $\Phi_A\colon [1,\infty) \to [1,\varphi(A))$ by the conditions: $\Phi_A(1) = 1$; $\Phi_A(n) = \varphi_{n-1}(A)$, for every $n = 2,3,\dots$; Φ_A is linear on every segment $[1,\varphi_1(A)]$ and $[\varphi_n(A),\varphi_{n+1}(A)]$, $n \in \mathbb{N}$. Finally, let $g(A,t) = \Phi_A(t)$.

Then define a required homotopy H by the formula $H(A,t) = h_{g(A,t)}(A)$.

In order to finish the proof of the theorem, we have, according to the classification theorem for the $[0,1)$-stable Q-manifolds, to show that the space $\mathrm{mpcc}(U)$ is homotopy equivalent to U. This can be demonstrated as follows. Let

$$G\colon \mathrm{mpcc}(U) \times [0,1] \to \mathrm{mpcc}(U)$$

be the homotopy defined by the formula

$$G(A,t) = \{\ln t \odot \max(A) \oplus a \mid a \in A\}, (A,t) \in \mathrm{mpcc}(U) \times [0,1]$$

(convention: $\ln 0 = -\infty$). We see that the subspace $G(\mathrm{mpcc}(U) \times \{1\})$, which is naturally homeomorphic to U is a deformation retract of $\mathrm{mpcc}(U)$.

$\square$

4 max-min-convex sets

For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and any $\lambda \in (-\infty, \infty]$, let

$$\lambda \otimes x = (\min\{\lambda, x_1\}, \dots, \min\{\lambda, x_n\}).$$

A subset A of $\mathbb{R}^n$ is called max-min-*convex* if, for any $x, y \in A$ and any $\lambda, \mu \in [-\infty, \infty]$ with $\lambda \oplus \mu = \infty$, we have $(\lambda \otimes x) \oplus (\mu \otimes y) \in A$. The geometry of the max-min-convex sets is considerably more complicated than that of convex sets and even max-plus convex sets.

We denote by $\mathrm{mmcc}(\mathbb{R}^n)$ the hyperspace of nonempty compact max-min-convex subsets in $\mathbb{R}^n$.

The following result is a counterpart of Theorem 3.1 for the max-min convex sets.

Theorem 4.1. *Let U be an open subset of $\mathbb{R}^n$, where $n \geq 2$. Then the hyperspace $\mathrm{mmcc}(U)$ is homeomorphic to $U \times Q \times [0, 1)$.*

The proof of this theorem follows the line of that of Theorem 3.1. Here we only note that a c-structure on the hyperspace $\mathrm{mmcc}(\mathbb{R}^n)$ can be defined as follows: given a finite subset $\{A_1, \dots, A_k\}$ is of $\mathrm{mmcc}(\mathbb{R}^n)$ we let

$$F(\{A_1, \dots, A_k\}) = \{\oplus_{i=1}^k \lambda_i \otimes A_i \mid \lambda_i \in [-\infty, \infty],\ i = 1, \dots, k,\ \oplus_{i=1}^k \lambda_i = \infty\},$$

where

$$\bigoplus_{i=1}^k \lambda_i \otimes A_i = \{\oplus_{i=1}^k \lambda_i \otimes a_i \mid a_i \in A_i,\ i = 1, \dots, k\}.$$

5 Remarks and open questions

Recall that $\mathbb{R}^\infty$ is the direct limit of the sequence

$$\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \dots,$$

where every arrow denotes the natural embedding

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0) \colon \mathbb{R}^n \to \mathbb{R}^{n+1}.$$

Define $\mathrm{mpcc}(\mathbb{R}^\infty)$ is the direct limit of the sequence

$$\mathrm{mpcc}(\mathbb{R}) \hookrightarrow \mathrm{mpcc}(\mathbb{R}^2) \hookrightarrow \mathrm{mpcc}(\mathbb{R}^3) \hookrightarrow \dots.$$

For any subset U of $\mathbb{R}^\infty$, we let $\mathrm{mpcc}(U) = \{A \in \mathrm{mpcc}(\mathbb{R}^\infty) \mid A \subset U\}$.

We conjecture that, for any open subset U of $\mathbb{R}^\infty$, the spaces $\mathrm{mpcc}(U)$ and $\mathrm{mmcc}(U)$ are homeomorphic to U.

Let h denote one of the hyperspaces mpcc or mmcc. We denote by p-$h(X)$ the subspace of the space $h(X)$ consisting of all subpolyhedra. Applying the technique of absorbing sets in Q-manifolds (see, e.g. [1]) one can prove that the hyperspace of compact convex polyhedra in an open set U of $\mathbb{R}^n$ is homeomorphic to the σ-manifold $U \times \sigma$, where

$$\sigma = \{(x_i) \in Q = [0,1]^\omega \mid x_i = 0 \text{ for all but finitely many } i\}.$$

We conjecture that the hyperspaces p-mpcc(U) or p-mmcc(U) are also homeomorphic to $U \times \sigma$. Moreover, it looks plausible that the pairs $(\mathrm{mpcc}(U), \text{p-mpcc}(U))$ and $(\mathrm{mmcc}(U), \text{p-mmcc}(U))$ are homeomorphic to the pair

$$(U \times Q \times [0,1), U \times \sigma \times [0,1)).$$

The following two questions are suggested by the referee.

Problem 5.1. The max-plus convex subsets can be naturally defined also for the space $(\mathbb{R} \cup \{-\infty\})^n$. Does Theorem 3.1 still hold for open subsets in $(\mathbb{R} \cup \{-\infty\})^n$, for $n \geq 2$?

Also, max-min convex sets can be naturally defined in the space $[-\infty, \infty]^n$. Does Theorem 4.1 still hold for open subsets in $[-\infty, \infty]^n$, for $n \geq 2$?

Problem 5.2. Let $C(X, \mathbb{R})$ denote the Banach lattice of continuous functions on a compact metrizable space X. The notion of max-plus convexity can be naturally defined for $C(X, \mathbb{R})$. One can ask whether the hyperspace of compact max-plus convex subsets in $C(X, \mathbb{R})$ is homeomorphic to the separable Hilbert space ℓ^2, for infinite X.

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Manifolds modeled on countable direct limits of absolute extensors[1]

by Oryslava Shabat and Michael Zarichnyi

Abstract

This is a survey of results on topology of manifolds modeled on countable direct limits of absolute extensors. We also prove a characterization theorem for manifolds modeled on countable direct limits of Tychonov cubes.

1 Introduction

We consider manifolds modeled on countable direct limits of absolute extensors. A survey of some known theories of such manifolds is presented in Section 3. In Section 4 we consider generalizations of manifolds modeled on the countable direct limits of the Hilbert cubes and prove a characterization theorem for such manifolds. Section 5 is devoted to universal maps of model spaces. Some applications are presented in Section 6 and in concluding Section 7 we formulate a short collection of open problems.

2 Preliminaries

By the ANR-spaces we mean the absolute neighborhood retracts for metrizable spaces.

By $\mathrm{w}(X)$ we denote the weight of a topological space X.

By $\mathcal{MC}$ (respectively $\mathcal{MC}(n)$) we will denote the class of metrizable compacta (respectively the class of metrizable compacta of dimension $\leq n$). A space X is said to be an absolute extensor for the class $\mathcal{MC}$ (respectively $\mathcal{MC}(n)$) if, given any map $f\colon B \to X$ defined on a closed subset B of a space $Y \in \mathcal{MC}$ (respectively $Y \in \mathcal{MC}(n)$), there exists a continuous extension of f onto Y. Replacing, in the last phrase, the words "onto Y" by "onto a neighborhood of B in Y" we obtain the notion of absolute neighborhood extensor for the class $\mathcal{MC}$ (respectively $\mathcal{MC}(n)$).

Given a class $\mathcal{C}$ of topological spaces, we denote by $\mathcal{C}^\infty$ the class of spaces which can be represented as countable direct limits of sequences of spaces $X_1 \hookrightarrow X_2 \hookrightarrow \dots$, where $X_i \in \mathcal{C}$ and X_i is a closed subset of X_{i+1} for every i.

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By Q we will denote the Hilbert cube, $Q = \prod_{i=1}^{\infty}[-1,1]_i$. Let Q^∞ denote the direct limit of the sequence

$$Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \to \dots .$$

By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R} \to \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \dots .$$

A closed set A in a space X is called a *Z-set* (respectively *strong Z-set*) if there exist maps $f\colon X \to X$ arbitrarily close to the identity map of X and such that the image $f(X)$ (respectively the closure of $f(X)$) misses A. An embedding is called a *Z-embedding* if its image is a Z-set.

3 Manifolds

3.1 $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds

The manifolds modeled on the spaces $\mathbb{R}^\infty$ and Q^∞ (i.e. $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds) are investigated in many publications (most of the literature on the list of references are devoted to these manifolds).

In particular, the topological characterization of these manifolds is very simple (see [26]).

Theorem 3.1. *Let X be a countable direct limit of a sequence of finite-dimensional compact metrizable spaces. The following conditions are equivalent:*

1. *X is an $\mathbb{R}^\infty$-manifold;*
2. *for every compact metrizable finite-dimensional pair (A, B) and every embedding $f\colon B \to X$ there exists an embedding $\bar{f}\colon U \to X$ of a neighborhood U of B in A.*

Deleting the words "finite-dimensional" from the above theorem one obtains a characterization of Q^∞-manifolds.

The spaces $\mathbb{R}^\infty$ and Q^∞ admit a weaker metrizable topologies. Namely, identifying every $\mathbb{R}^n$ with the subspace $\{(x_i) \in \ell^2 \mid x_i = 0 \text{ for all } i > n\}$ of the separable Hilbert space ℓ^2 one can regard the set $\mathbb{R}^\infty$ as the subspace

$$\sigma = \{(x_i) \in \ell^2 \mid x_i = 0 \text{ for all but finitely many } i\}$$

of ℓ^2.

A characterization of the bitopological space $(\mathbb{R}^\infty, \sigma)$ and bitopological manifolds modeled on these spaces is obtained in [6].

Theorem 3.2. *A bitopological space (M_w, M_m) is an $(\mathbb{R}^\infty, \sigma)$-manifold if and only if M_w is an $\mathbb{R}^\infty$-manifold, the identity map* $\mathrm{id}\colon M_w \to M_m$ *is a fine homotopy equivalence and every compact subset in M_m is a strong Z-set.*

We can identify Q with the standard Hilbert cube $\prod_{i=1}^\infty[-1/i; 1/i]$ in ℓ^2. Then each $nQ = \prod_{i=1}^\infty[-n/i; n/i]$ is a Z-set in $(n+1)Q = \prod_{i=1}^\infty[-(n+1)/i; (n+1)/i]$. The subspace $\cup_{n=1}^\infty nQ \subset \ell^2$ is the linear hull of Q in ℓ^2, which is denoted by Σ. By the characterization of Q^∞, it can be shown that Q^∞ is homeomorphic to the direct limit of the sequence $Q \hookrightarrow 2Q \hookrightarrow 3Q \hookrightarrow \dots$. Thus, the set Q^∞ is identified with Σ. The bitopological space (Q^∞, Σ) and (Q^∞, Σ)-manifolds are characterized similarly to Theorem 3.2.

The following theorems can be proved for both $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds (note that we do not formulate these results in full generality; e.g., see [26, 27, 31, 32, 33, 15, 19]):

1. (Open Embedding) Every connected $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold) admits an open embedding into the model space;

2. (Closed Embedding) Every connected $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold) admits a closed embedding into the model space;

3. (Stability) Let M be an $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold). Then $M \times \mathbb{R}^\infty$ (resp. $M \times Q^\infty$) is homeomorphic to M;

4. (Classification) Every homotopy equivalence between $\mathbb{R}^\infty$-manifolds (resp. Q^∞-manifolds) is homotopic to a homeomorphism;

5. (Product) Let $X \in \mathcal{MC}^\infty$ (resp. $X \in \mathcal{C}^\infty$, where $\mathcal{C} = \cup_{n=1}^\infty \mathcal{MC}(n)$) be an absolute neighborhood extensor. Then $X \times Q^\infty$ (resp. $X \times \mathbb{R}^\infty$) is a Q^∞-manifold (resp. $\mathbb{R}^\infty$-manifold);

6. (Triangulation) For every $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold) X there exists a locally finite simplicial complex K such that X is homeomorphic to $|K| \times \mathbb{R}^\infty$ (resp. $|K| \times Q^\infty$).

3.2 Manifolds modeled on direct limits of Menger compacta

The standard universal n-dimensional Menger compactum μ_n is defined as follows (e.g., see [12]). Let $\mathcal{F}_i$, $i = 0, 1, 2, \dots,$ be the family of 3^{mi} congruent cubes obtained by means of partition of the unit m-dimensional cube I^m, $m \geq n$, by $(m-1)$-dimensional affine subspaces in $\mathbb{R}^m$ given by the equations $x_j = k/3^i$, $j = 1, 2, \dots, m$ and $0 \leq k \leq 3^i$. For a collection $\mathcal{K}$ of cubes, denote by $S_n(\mathcal{K})$ the union of all faces of dimension $\leq n$ of the cubes in $\mathcal{K}$. Taking $\mathcal{F}_0 = \{I^m\}$ and $F_0 = \cup\mathcal{F}_0$ and assuming that $\mathcal{F}_i$ and F_i are already defined for all $i < k$, set

$$\mathcal{F}_k = \{K \in \mathcal{F}_{k-1} \mid K \cap (\cup S_n(\mathcal{F}_{k-1})) \neq \emptyset\},\ F_k = \cup\mathcal{F}_k.$$

Finally, let $\mu_n^m = \cap_{i=0}^{\infty} F_i \subset I^m$.

For $m \geq 2n+1$ and n fixed, all spaces μ_n^m are homeomorphic [8]. Let $\mu_n = \mu_n^{2n+1}$.

Denote by μ_n^∞ the direct limit of the sequence

$$\mu_n^{(1)} \hookrightarrow \mu_n^{(2)} \hookrightarrow \mu_n^{(3)} \hookrightarrow \dots, \tag{3.1}$$

in which all spaces $\mu_n^{(i)}$ are topological copies of μ_n and all embeddings are Z-embeddings.

A paracompact space X is called a μ_n^∞-*manifold* if X is locally homeomorphic to open subsets in the space μ_n^∞. The theory of μ_n^∞-manifolds is developed in [25]. In particular, a characterization theorem in the spirit of Theorem 3.1 is proved.

Other theories of manifolds modeled on countable direct limits of absolute extensors related to transfinite and cohomological dimension theories are considered in [23].

4 Manifolds modeled on countable direct limits of Tychonov cubes

By $\mathbb{I}$ we denote the unit segment $[0,1]$. Further, ω denotes the smallest infinite ordinal number. The Tychonov product $\mathbb{I}^\tau$, where τ is a cardinal number, is called a *Tychonov cube.*

In [30], we consider the space $\mathbb{I}^{(\alpha)}$, which is the countable direct limit

$$\mathbb{I}^{\tau_0} \longrightarrow \mathbb{I}^{\tau_0} \times \{0\} \hookrightarrow \mathbb{I}^{\tau_0} \times \mathbb{I}^{\tau_1} \hookrightarrow \dots,$$

where $\alpha = (\tau_0, \tau_1, \dots)$ is a sequence of ordinal numbers such that $\omega < \tau_0 \leq \tau_1 \leq \dots$.

The space $\mathbb{I}^{(\alpha)}$ admits a topological characterization, in some sense analogical to the characterization of the space $Q^\infty = \varinjlim Q^n$, given by K. Sakai [26]. It is also proved in [30] that the space $\mathbb{I}^{(\alpha)}$ is topologically homogeneous and self-similar. Therefore, the space $\mathbb{I}^{(\alpha)}$ can be considered as a model space for a class of manifolds.

Let $\alpha = (\tau_0, \tau_1, \dots)$ be a sequence described above. Denote $\tau(\alpha) = \sup\limits_i \tau_i^+$.

Let us introduce the following classes:

$$\begin{aligned}\mathcal{C}^{(\infty)} =& \{U \mid U \cong \varinjlim\ K_i,\ \text{where } K_i \\ & \quad \text{are compact Hausdorff spaces and } K_1 \subset K_2 \subset \dots\} \\ \mathcal{C}^{(\alpha)} =& \{U \in \mathcal{C}^{(\infty)} \mid \mathrm{w}(K_i) < \tau(\alpha)\ , i = 1, 2, \dots\}.\end{aligned}$$

Definition 4.1. A paracompact space X is called an $\mathbb{I}^{(\alpha)}$*-manifold* if there is an open cover $\mathcal{U}$ of the space X such that:

1. $\mathcal{U}$ is either finite or countable locally finite;
2. every $U \in \mathcal{U}$ is homeomorphic to an open subset in $\mathbb{I}^{(\alpha)}$ and, moreover, $U \in \mathcal{C}^{(\infty)}$.

Note that if the manifolds under consideration are connected then the first condition from the definition can be dropped.

Theorem 4.2 (Characterization). *Let X be a topological space, $X \in \mathcal{C}^{(\alpha)}$. The following are equivalent:*

1. *X is an $\mathbb{I}^{(\alpha)}$-manifold;*
2. *For every compact Hausdorff pair (A, B), where $\mathrm{w}(A) \leq \tau(\alpha)$, and every embedding $f\colon B \to X$ there exists a neighborhood U of the set B in A and an embedding $\overline{f}\colon U \to X$ such that $\overline{f}|_B = f$.*

We first need the following statement.

Proposition 4.3. *Let X be an $\mathbb{I}^{(\alpha)}$-manifold. Then $X \in \mathcal{C}^{(\alpha)}$.*

Proof. Since X is an $\mathbb{I}^{(\alpha)}$-manifold, there exists a countable locally finite cover $\{U_i\}$ of the space X, where U_i is homeomorphic to some open set in $\mathbb{I}^{(\alpha)}$ for every i and $U_i = \varinjlim_j K_{ij}$, where K_{ij} are compact subspaces, $K_{i1} \subset K_{i2} \subset \dots$.

Let us define the compact subsets L_i as follows:

$$L_1 = K_{11}, L_2 = K_{12} \cup K_{22}, \dots, L_n = K_{1n} \cup K_{2n} \cup \dots \cup K_{nn}, \dots.$$

Then, clearly, $X = \bigcup_{k=1}^{\infty} L_k$ and $L_1 \subset L_2 \subset \dots$. It remains to prove that $X = \varinjlim K_i$, i.e., for any subset $W \subset X$ the following holds: $W \underset{\mathrm{op}}{\subset} X$ if and only if $W \cap L_k \underset{\mathrm{op}}{\subset} L_k$ for all k.

If $n \leq k$, then the set $W \cap K_{nk} = (W \cap L_k) \cap K_{nk}$ is open in K_{nk}, therefore the set $W \cap U_n$ is open in U_n and hence in X, for all n. Thus, W is open.

□

Proof of Theorem. 1) $\Rightarrow$ 2). Consider a compact Hausdorff pair (A, B), where $\mathrm{w}(A) < \tau$ and let $f\colon B \to X$ be an embedding. Again we assume that $\{U_i\}$ is a countable locally finite cover of X, where U_i is homeomorphic to some open subset of the space $\mathbb{I}^{(\alpha)}$ for every i and $U_i = \varinjlim_j K_{ij}$, where K_{ij} are compact subsets, $K_{i1} \subset K_{i2} \subset \dots$; denote by $h_i\colon U_i \to h(U_i) \underset{\mathrm{op}}{\subset} \mathbb{I}^{(\alpha)}$ the corresponding homeomorphism. There exists $n < \omega$ such that $f(B) \subset \bigcup_{i=1}^{n} U_i$. One may assume that $B = \bigcup_{i=1}^{n} B_i$, where B_i are closed subsets of B and $f(B_i) \subset U_i$.

By induction, define compact neighborhoods V_i of the sets $B_1 \cup \dots \cup B_i$ in A and embeddings $g_i \colon V_i \to X$, $i = 1, 2 \dots n$ such that $B \subset B_i$ and $g_i|_{V_{i-1}} = g_{i-1}$, where $V_0 = B$ and $g_0 = f$.

There exists a closed neighborhood $\widetilde{B_1}$ of the set B_1 in B for which $f(\widetilde{B_1}) \subset U_1$. Then $f|_{\widetilde{B_1}} \colon \widetilde{B_1} \to U_1 \xrightarrow{h_1} h_1(U_1) \underset{\text{op}}{\subset} \mathbb{I}^{(\alpha)}$. By the characterization theorem for the space $\mathbb{I}^{(\alpha)}$ (see [30][Theorem 3.1]), for the pair $(A, \widetilde{B_1})$, the embedding $h_1 \circ (f|_{\widetilde{B_1}}) \colon \widetilde{B_1} \to \mathbb{I}^{(\alpha)}$ can be extended to an embedding $\overline{f_1} \colon A \to \mathbb{I}^{(\alpha)}$. Choose a closed neighborhood C_1 of the set B_1 such that $C_1 \subset W$, $\overline{f_1}(C_1) \subset h_1(U_1)$ and $C_1 \cap B \subset \widetilde{B_1}$. Denote $h_1^{-1} \circ \overline{f_1} \equiv \widetilde{f_1}$.

Consider $V_1 = B \cup C_1$ and the embedding $g_1 \colon V_1 \to X$ which satisfies the conditions $g_1|_B = f$, $g_1|_{V_1} = \widetilde{f_1}$. In n steps we obtain an embedding $g_n \colon V_n \to X$, where V_n is a neighborhood of the set B in A.

2) $\Rightarrow$ 1). Let $X = \varinjlim K_i$, where $K_1 \subset K_2 \subset \dots$ are compact Hausdorff spaces with $\mathrm{w}(K_i) < \tau$. Write $\mathbb{I}^{(\alpha)} = \varinjlim \{Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \dots\}$, where Y_i are compact Hausdorff spaces from the definition of $\mathbb{I}^{(\alpha)}$, i.e. $Y_i = \mathbb{I}^{\tau_0} \times \dots \times \mathbb{I}^{\tau_i}$.

Let $n_1 = 1$. There exists an embedding $f_1 \colon K_{n_1} \to \mathbb{I}^{(\alpha)}$. Since $f_1(K_{n_1})$ is compact, $f_1(K_{n_1}) \subset Y_{m_1}$, for some m_1. Consider the pair $(Y_{m_1}, (f_1(K_{n_1}))$ and the embedding $f_1^{-1} \colon f_1(K_{n_1}) \to X$. By the condition, there exists a neighborhood U_1 on which one can extend the embedding $g_1 \colon U_1 \to X$, where U_1 is the closed neighborhood of the set $f(K_{n_1})$ in Y_{m_1}. Since the set $g_1(U_1)$ is compact, there exists $n_2 > n_1$ such that $g_1(U_1) \subset K_{n_2}$. The embedding $g_1^{-1} \colon g_1(U_1) \to \mathbb{I}^{(\alpha)}$ can be extend to an embedding $f_2 \colon K_{n_2} \to \mathbb{I}^{(\alpha)}$, by [30, Theorem 3.1]. Since $f_2(K_{n_2})$ is a compact set, there exists $m_2 > m_1$ such that $f_2(K_{n_2}) \subset Y_{m_2}$. Proceeding as above, we obtain the following commutative diagram:

$$\begin{tikzcd}
K_{n_1} \arrow[r, hook] \arrow[d, "f_1"] & K_{n_2} \arrow[r, hook] \arrow[d, "f_2"] & K_{n_3} \arrow[r, hook] \arrow[d, "f_3"] & \dots \\
f_1(K_{n_1}) \arrow[d, hook] & f_2(K_{n_2}) \arrow[d, hook] & f_3(K_{n_3}) \arrow[d, hook] & \dots \\
U_1 \arrow[r, hook] \arrow[uur, "g_1"] \arrow[d, hook] & U_2 \arrow[r, hook] \arrow[uur, "g_2"] \arrow[d, hook] & U_3 \arrow[r, hook] \arrow[uur, "g_3"] \arrow[d, hook] & \dots \\
Y_{m_1} \arrow[r, hook] & Y_{m_2} \arrow[r, hook] & Y_{m_3} \arrow[r, hook] & \dots
\end{tikzcd}$$

By the construction above, the maps $g_i f_i$ and $f_{i+1} g_i$ are inclusions for each $i = 1, 2, \dots$ or this might be simply written $g_i f_i = \mathrm{id}$ and $f_{i+1} g_i = \mathrm{id}$.

We therefore obtain that

$$
\begin{aligned}
X = & \varinjlim_j K_j = \varinjlim \{K_{n_1} \hookrightarrow K_{n_2} \hookrightarrow \dots\} \\
= & \varinjlim \{K_{n_1} \xrightarrow{f_1} U_1 \xrightarrow{g_1} K_{n_2} \xrightarrow{f_2} U_2 \xrightarrow{g_2} \dots\} \\
= & \varinjlim \{U_1 \hookrightarrow U_2 \hookrightarrow \dots\} = U
\end{aligned}
$$

is an open subset in the space $\mathbb{I}^{(\alpha)}$. Therefore X is an $\mathbb{I}^{(\alpha)}$-manifold.

□

Note that, actually, we have proved a stronger result.

Theorem 4.4 (Open embedding)**.** *Every $\mathbb{I}^{(\alpha)}$-manifold X admits an open embedding into the model space $\mathbb{I}^{(\alpha)}$.*

5 Universal maps

The projection maps $\mathrm{pr}\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ and $\mathrm{pr}\colon Q^\infty \times Q^\infty \to Q^\infty$ are characterized in [2].

The second named author [37, 39], using the construction of Dranishnikov map $f_n\colon \mu_n \to Q$ (see [11]), defined a universal map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$. This map can be characterized as follows:

Definition 5.1. A map $f\colon X \to Y$ is said to be *strongly* (ω, ∞)*-universal* if for every finite-dimensional compact metrizable pair (Z, A), and a metrizable compactum C, every embedding $\alpha\colon A \to X$ and maps $\beta\colon Z \to C$, $\gamma\colon C \to Y$ such that $f \circ \alpha = \gamma \circ \beta | A$, there exists an embedding $\bar\alpha\colon Z \to X$ such that $\bar\alpha | A = \alpha$ and $f \circ \bar\alpha = \gamma \circ \beta$ (i.e., the diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
A \arrow[u, "\alpha"] \subset & Z \arrow[ul, dashed, "\bar\alpha"'] \arrow[r, "\beta"'] & C \arrow[u, "\gamma"']
\end{tikzcd}
$$

is commutative).

Theorem 5.2. *There exists a unique (up to homeomorphisms) strongly* (ω, ∞)*-universal map* $\varphi\colon \mathbb{R}^\infty \to Q^\infty$.

It is natural to conjecture that the universal map φ is locally self-similar, i.e. for every $x \in \mathbb{R}^\infty$ and every neighborhood U of x there exists a neighborhood $V \subset U$ of x such that the restriction map $\varphi | V\colon V \to \varphi(V)$ is homeomorphic to φ. It is proved in [5] that this is not the case.

Universal maps $\mu_n^\infty \to \mu_n^\infty$, $\mu_n^\infty \to \mathbb{R}^\infty$ and $\mu_n^\infty \to Q^\infty$ are constructed and characterized in [25]. In the theory of μ_n^∞-manifolds, the universal map $\psi_n\colon \mu_n^\infty \to \mu_n^\infty$ plays a role analogical to the projection map $\mathrm{pr}\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ in the theory of $\mathbb{R}^\infty$-manifolds. In particular, a counterpart of Stability Theorem looks as follows (see the proof of Theorem 4.5 in [25]): for any μ_n^∞-manifold $M \subset \mu_n^\infty$, the preimage $\psi_n^{-1}(M)$ is homeomorphic to M .

Note also that there exists a metrizable counterpart of the universal map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$, namely, a map $\sigma \to \Sigma$ defined in [40].

6 Applications

The $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds naturally arise in different areas of topology and other areas of mathematics.

Given a sequence (X_i, x_i) of separable metrizable locally compact ANR-spaces with base points, one can construct the direct limit

$$X_1 \to X_1 \times \{x_2\} \hookrightarrow X_1 \times X_2 \to X_1 \times X_2 \times \{x_3\} \hookrightarrow \dots$$

It is proved in [22] that, under some natural conditions, the mentioned direct limit is an $\mathbb{R}^\infty$-manifold. Another situations when the countable direct limits of ANR-spaces are either $\mathbb{R}^\infty$- or Q^∞-manifolds are considered in [36].

It is proved in [34] that the free topological groups of compact metrizable (finite-dimensional) ANR-spaces are Q^∞-manifolds (respectively $\mathbb{R}^\infty$-manifolds). Recall that the free topological group of a Tychonov space X is a topological group $F(X)$ which is characterized by the following conditions:

1. X is a closed subspace of $F(X)$;
2. any continuous map of $f\colon X \to G$ into a topological group G admits an extension $\bar{f}\colon F(X) \to G$, where $\bar{f}$ is a continuous homomorphism.

An analogous result is proved in [34] for the infinite symmetric power functor SP^∞. Recall that the space $\mathrm{SP}^\infty(X, *)$ is defined as the countable direct limit $\varinjlim \mathrm{SP}^n(X)$, where the embedding $\mathrm{SP}^n(X) \to \mathrm{SP}^{n+1}(X)$ is defined by the formula $[x_1, \dots, x_n] \mapsto [x_1, \dots, x_n, *]$. The condition that X is an ANR-space is not necessary in order that $\mathrm{SP}^\infty(X, *)$ be an $\mathbb{R}^\infty$-manifold; a corresponding example is constructed in [35]. The infinite symmetric powers are examples of topological semigroups. In [3], one can find results when the free topological semigroups in some classes are $\mathbb{R}^\infty$-manifolds. In [7] it is proved that the free topological semilattices over suitable spaces are $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds. In particular, such a free semilattice over a space $X \in \mathcal{MC}^\infty$ is a Q^∞-manifold if and only if every compact subspace of X is contained in a Q-manifold.

In [39], it is proved that for any topological group G whose underlying space is from the class $\mathcal{MC}^\infty$ there exists a continuous epimorphism $: H \to G$, where

H is a topological group whose underlying space is an $\mathbb{R}^\infty$-manifold (called an $\mathbb{R}^\infty$-resolution of G). The result is based on existence of a counterpart of the universal map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$ in the theory of topological groups.

Let us turn our attention to the theory of topological linear spaces. It is known in [16] that the space Q^∞ is homeomorphic to the space (ℓ^2, bw), where bw stands for the bounded-weak topology. In [1], there are characterized the linear topological spaces which are homeomorphic to $\mathbb{R}^\infty$. In [5], it is proved that the universal map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$ can be realized as a linear map of linear topological spaces. A related result that the universal map φ can be realized as a map of suitably topologized spaces of probability measures with finite supports is obtained in [38].

The following application of the topology of Q^∞-manifolds to function spaces is obtained in [29]. Let X, Y be metric spaces and for each k let k-$\mathrm{LIP}(X,Y)$ be the space of all k-Lipschitz maps from X to Y endowed with the compact-open topology. In [29], it is proved that, under some natural conditions on X and Y, the direct limit of the sequence $1\text{-}\mathrm{LIP}(X,Y) \hookrightarrow 2\text{-}\mathrm{LIP}(X,Y) \hookrightarrow \cdots$ is a Q^∞-manifold

The Banach-Mazur compactum $\mathrm{BM}(n)$ is the space of norms on Euclidean n-space $\mathbb{R}^n$ modulo the natural action of the linear group $\mathrm{GL}(n)$. In [4], a natural chain of embeddings

$$\mathrm{BM}(1) \hookrightarrow \mathrm{BM}(2) \hookrightarrow \cdots \hookrightarrow \mathrm{BM(n)} \hookrightarrow \cdots$$

is defined and it is proved that the direct limit $\varinjlim \mathrm{BM}(n)$ of this sequence is homeomorphic to the space Q^∞.

Because of fractal nature of the n-dimensional universal Menger compactum μ_n, one can expect that the μ_n^∞-manifolds will be applied in fractal geometry.

7 Remarks and open problems

In connection to the results on existence of universal maps $\mathbb{R}^\infty \to Q^\infty$ and $\sigma \to \Sigma$ the following problem seems to be natural.

Problem 7.1. Is there a bitopological universal map $(\mathbb{R}^\infty, \sigma) \to (Q^\infty, \Sigma)$? If the answer is affirmative, find a topological characterization of such a map.

One can naturally introduce the notion of differentiable structure on every $\mathbb{R}^\infty$-manifold (see, e.r. [24]).

As it is proven in [34] the free topological groups of suitable ANR-spaces are $\mathbb{R}^\infty$-manifolds.

Problem 7.2. Is there a differentiable structure on the free topological group $F(X)$ for which both the multiplication operation and the inversion are differentiable?

The results of Section 4 lead to the following question:

Problem 7.3. Is it possible to extend Theorem 4.2 to the case of sequence $\alpha = (\tau_1, \tau_2, \tau_3, \dots)$ such that $\tau_1 > \omega$ and $\tau_1 > \tau_i$, for all $i > 1$?

Some results in this direction can be found in [10].

Problem 7.4. Is the free topological group of the Tychonov cube $\mathbb{I}^\tau$, where $\tau > \omega$, an $\mathbb{I}^{(\alpha)}$-manifold, for some α?

Problem 7.5. Are there representations of $\mathbb{I}^{(\alpha)}$-manifolds "in nature" (e.g., as function spaces or linear topological spaces)?

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Second order variational problem and 2-dimensional concircular geometry[1] [2]

by Roman Matsyuk

Abstract

It is proved that the set of geodesic circles in two dimensions may be given a variational description and the explicit form of it is presented. In the limit case of the Euclidean geometry a certain claim of uniqueness of such description is proved. A formal notion of 'spin' force is discovered as a by-product of the variation procedure involving the acceleration.

1 Introduction

The concircular geometry deals with geodesic circles in (pseudo)-Riemannian space. Geodesic circles in two dimensions are those curves in 2-dimensional (pseudo)-Riemannian space who preserve the Frenet curvature along them. In relativity theory this coincides with the definition of the uniformly accelerated one-dimensional motion of a test particle. The ordinary differential equation to govern such curves has order three [1]. Thus the Lagrange function should involve second derivatives and, at the same time, it should depend linearly on them.

Aiming at simplification of the exposition and of the accompanying notations, let us agree not to be confused with such notions as vector or bivector norm in pseudo-Riemannian geometry. Although the outcome of present investigation lucidly concerns both the proper Riemannian and the pseudo-Riemannian geometries, for the sake of prudence one may restrict oneself to the case of proper Riemannian space, and it still will remain evident, wherein the results will be valid in actually the pseudo-Riemannian framework as well. Thus hereinafter we shall somewhat vaguely use the terms *Riemannian* and *Euclidean,* keeping in mind that strictly speaking, some details of pure technical developments can in fact apply only to proper Riemannian case.

Consider the following Lagrange function in 2–dimensional Euclidean space:

$$L = L_{II} + L_I = \frac{\epsilon_{ij} u^i \dot{u}^j}{\|\mathbf{u}\|^3} - m\|\mathbf{u}\| \,, \tag{1}$$

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with ϵ_{ij} denoting the skew-symmetric covariant Levi–Civita symbol. The first addend, L_{II}, is the so-called signed first Frenet curvature of a path. Further in this contribution we show that the expression (1) as a candidate for the Lagrange function is very tightly defined by the conditions of the symmetry of corresponding equation of motion and by the request that the Frenet curvature be preserved along the extremal curves.

Formula (1) clearly suggests accepting same Lagrange function also for general Riemannian case,

$$L^R = k - m\|\mathbf{u}\| \,. \tag{2}$$

To prove the preservation of curvature k along the extremals of (2) we need some further tools as introduced below.

2 Means from higher order mechanics of Ostrohrads'kyj

2.1 Parametric homogeneity

Let $T^qM = \{x^j,\, u^j,\, \dot u^j,\, \ddot u^j, \ldots,\, \overset{(q-1)}{u}{}^j\}$ denote the manifold of q^{th}-order Ehresmann velocities to the base manifold M of dimension n. The prolonged reparametrization group $Gl^q_n = J^q_0\,(\mathbb{R},\,\mathbb{R})_0$ acts on the manifold $T^qM = J^q_0\,(\mathbb{R},\,M)$ by composition of jets (in our case $n = 2$). As far as the Lagrange function (2) depends on the derivatives of at most second order, it lives on the space T^2M. The infinitesimal counterpart of the above mentioned parameter transformations of T^2M (we put $q = 2$) is given by so-called fundamental fields (for arbitrary order consult [2, 3]):

$$\zeta_1 = u^i \frac{\partial}{\partial u^i} + 2\,\dot u^i \frac{\partial}{\partial \dot u^i}\,, \quad \zeta_2 = u^i \frac{\partial}{\partial \dot u^i}\,.$$

If a function F defined on T^2M does not change under arbitrary parameter transformations discussed above, then it with necessity satisfies the following sufficient conditions:

$$\zeta_1 F = 0\,, \quad \zeta_2 F = 0\,. \tag{3}$$

On the other hand, if a function L on T^2M defines a parameter-independent autonomous variational problem with the action functional

$$\int L\left(x^j,\, u^j,\, \dot u^j\right) d\varsigma\,,$$

then it also with necessity satisfies the so-called Zermelo sufficient conditions [4, 5]:

$$\zeta_1 L = L\,, \quad \zeta_2 L = 0\,. \tag{4}$$

The generalized momenta are being conventionally introduced by the next expressions:

$$p_i^{(2)} = \frac{\partial L}{\partial \dot u^i}\,, \quad p_i^{(1)} = \frac{\partial L}{\partial u^i} - \frac{d}{d\varsigma} p_i^{(2)}\,,$$

while the Hamilton function reads:

$$H = p_i^{(2)} \dot u^i + p_i^{(1)} u^i - L\,.$$

This Hamilton function may also be expressed in different way [3, 8]:

$$H = \zeta_1 L + \frac{d}{d\varsigma} \zeta_2 L - L\,. \tag{5}$$

As the Hamilton function is a constant of motion, from (3), (4), and (5) we immediately obtain the following proposition:

Proposition 2.1. *Let a function L_{II} be parameter-independent, and let another function L_{II} define a parameter-independent variational problem on T^2M. Then L_{II} is constant along the extremals of the variational problem, defined by the Lagrange function*

$$L = L_{II} + L_I. \tag{6}$$

This holds because $L_{II} = -H$ with H corresponding to (6).

Frenet curvature is constant along the extremals of (2), so by the Proposition 2.1 we have right to state:

Claim 2.1 ([6, 7])**.** *The Lagrange function(2) constitutes the variational principle for the geodesic circles.*

Now we wish to provide evidence that in the limit case of Euclidean space the corresponding Euler-Poisson equation may be specified by means of symmetry considerations together with the curvature preservation requirement. This means that the inverse variational problem tools should be applied.

2.2 The generalized Helmholtz conditions and symmetry.

Following Tulczyjew (see [9, 3]), let us introduce some operators, acting in the graded algebra of differential forms who live on manifolds T^qM of varying order q of jets:

1. The total derivative:

$$d_T f = u^i \frac{\partial f}{\partial x^i} + \dot u^i \frac{\partial f}{\partial u^i} + \ddot u^i \frac{\partial f}{\partial \dot u^i} + \cdots + \overset{q}{u}{}^i \frac{\partial f}{\partial \overset{q-1}{u}{}^i}\,, \quad d d_T = d_T d\,;$$

2. For each of $r \leq q$ the derivations of zero degree:

$$\begin{aligned} i_0(\omega) &= \deg(\omega)\,\omega\,, & i_r(f) &= 0\,, \quad i_r(dx^i) = 0\,, \\ i_r(d\overset{k}{u}{}^i) &= \tfrac{(k+1)\,!}{(k-r+1)\,!}\, d\overset{k-r}{u}{}^i, & i_r(d\overset{k}{u}{}^i) &= 0\,, \quad \text{if} \quad k < r-1\,; \end{aligned}$$

3. The Lagrange derivative:

$$\delta = \left(i_0 - d_T i_1 + \frac{1}{2}\, {d_T}^2 i_2 - \frac{1}{6}\, {d_T}^3 i_3 + \cdots + \frac{(-1)^q}{q!}\, {d_T}^q i_q \right) d\,.$$

It is of common knowledge that the Euler–Poisson expressions constitute a covariant object.

Lemma 2.1 ([9])**.** *Let a system of some differential expressions of the third order form a covariant object—the differential one-form*

$$\varepsilon = E_i \left(x^j,\, u^j,\, \dot u^j,\, \ddot u^j \right)\, dx^i. \tag{7}$$

Then $\varepsilon = \delta\,(L)$ for some (local) L if and only if

$$\delta\,(\varepsilon) = 0\,. \tag{8}$$

Developing the criterion (8) amounts to establishing a general pattern for the expression (7),

$$E_i = A_{ij}(x^l, u^l)\, \ddot u^j + \dot u^p \frac{\partial}{\partial u^p} A_{ij}(x^l, u^l)\, \dot u^j + B_{ij}(x^l, u^l)\, \dot u^j + q_i(x^l, u^l), \tag{9}$$

and to some generalized Helmholtz conditions [8, 10, 11], cast in the form of a system of partial differential equations, imposed on the coefficients $A_{ij} = -A_{ji}$, B_{ij}, and q_i:

$$\begin{aligned} &\partial_{u^{[i}} A_{jl]} = 0 \\ &2\, B_{\,[ij]} - 3\, \mathbf{D_1} A_{ij} = 0 && \text{(10a)} \\ &2\, \partial_{u^{[i}} B_{j]\,l} - 4\, \partial_{x^{[i}} A_{j]\,l} + \partial_{x^l} A_{ij} + 2\, \mathbf{D_1} \partial_{u^l} A_{ij} = 0 && \text{(10b)} \\ &\partial_{u^{(i}}\, q_{j)} - \mathbf{D_1} B_{(ij)} = 0 \\ &2\, \partial_{u^l} \partial_{u^{[i}}\, q_{j]} - 4\, \partial_{x^{[i}} B_{j]\,l} + {\mathbf{D_1}}^2 \partial_{u^l} A_{ij} + 6\, \mathbf{D_1} \partial_{x^{[i}} A_{jl]} = 0 \\ &4\, \partial_{x^{[i}}\, q_{j]} - 2\, \mathbf{D_1} \partial_{u^{[i}}\, q_{j]} - {\mathbf{D_1}}^3 A_{ij} = 0\,, \end{aligned}$$

where the notation $\mathbf{D_1} = u^p \partial_{x^p}$ was introduced.

The Euclidean symmetry means that everywhere on the submanifold E defined by the system of equations $E_l = 0$ the shifted system $X\,(E_l)$ vanishes too, where

X denotes the prolonged generator of (pseudo)-orthogonal transformations. We denote this criterion as

$$X(E_l)\,|_E = 0\,. \tag{11}$$

That we tend to embrace nothing more but only the *geodesic circles* as extremals, falls into similar condition:

$$(d_T k)\,|_E = 0\,. \tag{12}$$

As far as in two-dimensional space ($\dim M = 2$) the skew-symmetric matrix A_{ij} is invertible, it is not difficult to implement conditions (11) and (12).

If one wishes to include in the set of extremals all those Euclidean geodesics that refer to the natural parameter, one has to imply one more condition:

$$E_l\,|_{\dot{\mathbf{u}}=\mathbf{0}}\,. \tag{13}$$

Theorem 2.1. *Let a third order autonomous dynamical equation* $\mathbf{E} = \mathbf{0}$ *in two-dimensional space obey conditions:*

1. $$\delta(\varepsilon) = 0;$$
2. *The system of ODEs* $\{E_j = 0\}$ *possesses the Euclidean symmetry;*
3. *The system* $\{E_j = 0\}$ *possesses the first integral — the Frenet curvature* k*, and includes all curves of constant curvature as its solutions;*
4. *It also includes the strait lines with natural parametrization,* $\dot{\mathbf{u}} = \mathbf{0}$.

Then

$$E_i = \frac{\epsilon_{ij}\ddot{u}^j}{\|\mathbf{u}\|^3} - 3\,\frac{(\dot{\mathbf{u}}\cdot\mathbf{u})}{\|\mathbf{u}\|^5}\,\epsilon_{ij}\dot{u}^j + m\,\frac{\|\mathbf{u}\|^2\dot{u}_i - (\dot{\mathbf{u}}\cdot\mathbf{u})\,u_i}{\|\mathbf{u}\|^3}\,.$$

The Lagrange function is given by (1).

Remarks.

- If, for instance, we took $L = k\sqrt{u_i u^i}$, then $H = 0$ for this Lagrange function, and the Proposition 2.1 wouldn't work.
- Because of the non-degeneracy of the matrix A_{ij}, there cannot exist a parameter-invariant variational problem in two dimensions that would produce strictly the third order Euler–Poisson equation. But, if we omit the first addend k in (2), then what remains defines the conventional parameter-invariant problem for the Riemannian projective geodesic paths. So, what fixes the parameter along the extremal in our case, is the Frenet curvature k in (2).

One should confer with [12] and [13] on these remarks.

2.3 Proof of the Theorem 2.1

Before passing to the proof of the above Theorem let us notice two simplification formulæ which hold at specific occasion of two dimensions. Namely, for arbitrary vectors $\mathbf{a}$, $\mathbf{c}$, $\mathbf{v}$, and $\mathbf{w}$ it is true that

$$(14)\quad \|\mathbf{a}\wedge\mathbf{c}\| = \sqrt{|\det[g_{ij}]|}\,|\epsilon_{ij}a^i c^j| \quad \text{and} \quad \|\mathbf{a}\wedge\mathbf{c}\|\|\mathbf{v}\wedge\mathbf{w}\| = |(\mathbf{a}\wedge\mathbf{c})\cdot(\mathbf{v}\wedge\mathbf{w})|\ ,$$

where, as usual, $(\mathbf{a}\wedge\mathbf{c})\cdot(\mathbf{v}\wedge\mathbf{w}) = (\mathbf{a}\cdot\mathbf{v})(\mathbf{c}\cdot\mathbf{w}) - (\mathbf{c}\cdot\mathbf{v})(\mathbf{a}\cdot\mathbf{w})$ [14]. Also, let us agree to postpone the proof of the second part of statement 3 of Theorem 2.1 until more general Riemannian case proved in Section 3.2.

Proof of the necessity implication of Theorem 2.1 assumptions. In order to meet the condition 4 of the Theorem 2.1 in the form(13), we have to remove the array $\mathbf{q}$ from (9). Next we write down the first part of the statement 3 given by means of (12). Starting with the expression

$$(15)\qquad k = \frac{\|\mathbf{u}\wedge\dot{\mathbf{u}}\|}{\|\mathbf{u}\|^3}$$

of the Frenet curvature we substitute $\ddot{\mathbf{u}}$ in

$$d_T k = \frac{(\mathbf{u}\wedge\dot{\mathbf{u}})\cdot(\mathbf{u}\wedge\ddot{\mathbf{u}})}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\dot{\mathbf{u}}\|} - 3\,\frac{\|\mathbf{u}\wedge\dot{\mathbf{u}}\|(\mathbf{u}\cdot\dot{\mathbf{u}})}{\|\mathbf{u}\|^5}$$

by $\ddot{\mathbf{u}} = -A^{-1}(\dot{\mathbf{u}}.\boldsymbol{\partial}_u)\,A\dot{\mathbf{u}} - A^{-1}B\dot{\mathbf{u}}$ of (9) and then split the expression (12) by the powers of $\dot{\mathbf{u}}$ to obtain separately

$$(16a)\qquad (\mathbf{u}\cdot\mathbf{u})(\mathbf{u}\wedge\dot{\mathbf{u}})\cdot\big(\mathbf{u}\wedge\big(A^{-1}(\dot{\mathbf{u}}.\boldsymbol{\partial}_u)A\dot{\mathbf{u}}\big)\big) + 3\,(\mathbf{u}\cdot\dot{\mathbf{u}})\|\mathbf{u}\wedge\dot{\mathbf{u}}\|^2 = 0$$

$$(16b)\qquad (\mathbf{u}\wedge\dot{\mathbf{u}})\cdot\big(\mathbf{u}\wedge\big(A^{-1}B\dot{\mathbf{u}}\big)\big) = 0\,.$$

Let us recall that the covariant and the contravariant Levi–Civita symbols are related by $\epsilon_{ij}e^{jl} = -\delta_i{}^l$ and also let matrix A be expressed as $A_{ij} = A_{12}\epsilon_{ij}$. With these agreements the first addend in (16a) becomes

$$\frac{1}{A_{12}}\|\mathbf{u}\|^2\|\mathbf{u}\wedge\dot{\mathbf{u}}\|^2(\dot{\mathbf{u}}.\boldsymbol{\partial}_u)A_{12}\,,$$

thus reducing (16a) by means of (14) to the partial differential equation

$$\|\mathbf{u}\|^2(\dot{\mathbf{u}}.\boldsymbol{\partial}_u)A_{12} + 3\,A_{12}(\mathbf{u}\cdot\dot{\mathbf{u}}) = 0$$

that in turn yields the solution

$$A_{12} = \alpha\|\mathbf{u}\|^{-3}.$$

Now we see that matrix A satisfies the relations

$$\dot{\mathbf{u}}.\boldsymbol{\partial}_u\, A = -3\,\frac{\mathbf{u}{\cdot}\dot{\mathbf{u}}}{\|\mathbf{u}\|^2}\, A\,, \tag{17}$$

and, evidently,

$$e^{ij} u_i \frac{\partial}{\partial u^j}\, A = 0\,, \tag{18}$$

with the help of which the Euler–Poisson expression (9) becomes

$$\mathbf{E} = A\ddot{\mathbf{u}} - 3\,\frac{\mathbf{u}{\cdot}\dot{\mathbf{u}}}{\|\mathbf{u}\|^2} A\dot{\mathbf{u}} + B\dot{\mathbf{u}}\,, \tag{19}$$

so that the submanifold $\mathbf{E} = \mathbf{0}$ is now defined by the equation

$$\ddot{\mathbf{u}} = 3\,\frac{\mathbf{u}{\cdot}\dot{\mathbf{u}}}{\|\mathbf{u}\|^2} A^{-1} B\dot{\mathbf{u}}\,. \tag{20}$$

Again with the help of (14) the equation (16b) takes the shape

$$\left\|\mathbf{u}\wedge\left(A^{-1}B\dot{\mathbf{u}}\right)\right\| = 0, \quad \text{or} \quad \epsilon_{ij} e^{jp} B_{pl} u^i \dot{u}^l = 0\,,$$

from where it follows that

$$u^p B_{pl} = 0. \tag{21}$$

The generators of the Euclidean symmetry are enumerated by an arbitrary constant ϖ and an arbitrary constant array $\boldsymbol{\chi} = \{\chi^i\}$ and they read:

$$\boldsymbol{\chi}.\boldsymbol{\partial}_x \left(\equiv \chi^i \frac{\partial}{\partial x^i}\right)\;; \tag{22a}$$

$$\varpi e^{ij}\left(x_i\frac{\partial}{\partial x^j} + u_i\frac{\partial}{\partial u^j} + \dot{u}_i\frac{\partial}{\partial \dot{u}^j} + \ddot{u}_i\frac{\partial}{\partial \ddot{u}^j}\right). \tag{22b}$$

Applying criterion (11) with $X = \boldsymbol{\chi}.\boldsymbol{\partial}_x$ and taking into account the substitution (20) ends in

$$-\frac{\boldsymbol{\chi}.\boldsymbol{\partial}_x\,\alpha}{\alpha}\, B\dot{\mathbf{u}} + \boldsymbol{\chi}.\boldsymbol{\partial}_x\, B\dot{\mathbf{u}} = \mathbf{0}. \tag{23}$$

Applying criterion (11) with X equal to (22b) and again calling to mind the substitution (20) with the help of

$$A_{lj} e^{ij} {A^{-1}}_i{}^p = \frac{1}{A_{12}} {A^{-1}}_l{}^p = -g_{il} e^{ip}$$

ends in

$$g_{ij}e^{il}B_{lp}\dot{u}^p + e^{il}u_i\frac{\partial}{\partial u^l}\,B_{jp}\dot{u}^p + e^{il}B_{jl}\dot{u}_i = 0\,, \quad \text{identically with respect to } \dot{u}^p,$$

from where we conclude:

$$e^{il}u_i\frac{\partial}{\partial u^l}\,B_{jp} + g_{ij}e^{il}B_{lp} + g_{ip}e^{il}B_{jl} = 0\,. \tag{24}$$

We may deduce from (24) that the skew-symmetric part of B should satisfy the equation:

$$e^{ij}u_i\,\frac{\partial}{\partial u^j}\,B_{[lp]} + g_{il}e^{ij}B_{[jp]} + g_{ip}e^{ij}B_{[lj]} = 0. \tag{25}$$

Let the skew-symmetric part of matrix B be presented as $\beta\epsilon_{ij}$. Then equation (25) confirms that β should be a differential invariant:

$$e^{ij}u_i\,\frac{\partial}{\partial u_j}\,\beta = 0. \tag{26}$$

But the variationality condition (10a) now says:

$$2\beta = 3\,\mathbf{u}.\boldsymbol{\partial}_x\alpha\,. \tag{27}$$

Applying the left hand side operator of (26) to (27) along with equation (18) produces

$$\epsilon_{ji}e^{ip}\,\frac{\partial}{\partial x^p}\,\alpha = 0.$$

Thus α does not depend on x^i. Looking back at (27) immediately implies $\beta = 0$, matrix B being symmetric thus. In addition, we see that matrix B also should not depend on x^i by the reason of relation (23).

Now it is time to turn back to the constraint (21) Of course, we could have used it much earlier, but we prefer to unleash it now. So, the two equations, contained there, allow us to prescribe the shape to the matrix B as follows (independent of its virtual symmetry). Let $B_{12} = b_1u_2$, $B_{21} = b_2u_1$. Then from (21) one has:

$$B_{ij} = b_iu_j - (\mathbf{b}{\cdot}\mathbf{u})g_{ij}\,.$$

But we already know that $B_{[ij]} = 0$. This immediately implies that $\mathbf{b}$ and $\mathbf{u}$ must be collinear, $\mathbf{b} = \mu\,\mathbf{u}$, thus suggesting the following form of matrix B:

$$B_{ij} = \mu\,\left(u_iu_j - (\mathbf{u}\cdot\mathbf{u})\,g_{ij}\right)\,. \tag{28}$$

Let us again act on (28) with the operator $e^{ij}u_i\frac{\partial}{\partial u^j}$ and make use of (24). After some simplifications we get:

$$e^{ij}u_i\frac{\partial}{\partial u^j}\,\mu = 0\,,$$

what suggests that μ depends on u^i exclusively via $\mathbf{u}{\cdot}\mathbf{u}$.

The definite step consists in applying the second valid variational criterion, that of (10b). It is efficient to make contraction of (10b) with u^i on the left and in meanwhile not to forget about the constraint (21). One obtains:

$$u^i \frac{\partial}{\partial u^i} B_{jp} = -B_{jp}\,.$$

Together with the guise (28) this produces

$$\left(2\,(\mathbf{u}\cdot\mathbf{u})\,\frac{\partial\mu}{\partial\|\mathbf{u}\|^2} + 3\,\mu\right)(u_i u_j - (\mathbf{u}\cdot\mathbf{u})g_{ij}) = 0\,,$$

what clearly has the solution $\mu = \frac{m}{(\mathbf{u}\cdot\mathbf{u})^{3/2}}$ and so says the finite appearance of B:

$$B_{ij} = \frac{m}{(\mathbf{u}\cdot\mathbf{u})^{3/2}}\,(u_i u_j - (\mathbf{u}\cdot\mathbf{u})\,g_{ij})\,.$$

□

3 The variational description of geodesic circles

3.1 The variational equation

Before calculating the variation of the integrand in the functional expression $\int k\, d\varsigma$ let us agree on some basic formulæ. If υ denotes the infinitesimal shift of the path $x^i(\varsigma)$ and if $\tilde{D}$ stands for the covariant differentiation operator according to that shift, then the covariant variation of any vector field ξ along this path is given by

(29) $$\langle \upsilon, \tilde{D}\xi\rangle^i = \langle \upsilon, d\xi^i\rangle + \Gamma^i_{lj}\xi^j \upsilon^l.$$

Let the covariant derivative of a vector field be notated by prime. And let us introduce a special designation for the evaluation of Riemannian curvature on velocities as follows:

$$\sigma^l{}_j = R_{ji,p}{}^l u^i u^p.$$

The vector differential one-form $\boldsymbol{\sigma} = [\sigma^l{}_j]$ is semi-basic when the projection $TM \to M$ is considered: $\langle \upsilon, \boldsymbol{\sigma}\rangle^l = \sigma^l{}_j \upsilon^j$. Let $\boldsymbol{\theta}$ denote the vector one-form representing the identity: $\boldsymbol{\theta} = [\delta^l{}_j]$. Next formulæ replace then the usual interchange rule between infinitesimal variation and ordinary differentiation:

(30) $$\begin{aligned} \tilde{\mathbf{D}}\mathbf{u} &= \boldsymbol{\theta}' \quad \text{[this recapitulates definition (29)]},\\ \tilde{\mathbf{D}}(\mathbf{u}') &= (\tilde{\mathbf{D}}\mathbf{u})' - \boldsymbol{\sigma} \quad \text{[this recapitulates the definition of the tensor } R_{ji,p}{}^l\text{]}. \end{aligned}$$

Further on we shall find escape from highly tangled and tedious calculations in the truth of the following relation (valid in two dimensions only):

(31) $$(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{v}\wedge\mathbf{c}){\cdot}(\mathbf{v}\wedge\mathbf{c}) - (\mathbf{a}\cdot\mathbf{v})\,(\mathbf{v}\wedge\mathbf{c}){\cdot}(\mathbf{a}\wedge\mathbf{c}) + (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{v}\wedge\mathbf{c}){\cdot}(\mathbf{a}\wedge\mathbf{v}) = 0\,,$$

along with the simplification formulæ (14).

The above formal and highly symbolic notations save place and time and help to avoid unessential calculative details, whereas keeping the skeleton of the variational procedure untouched and faithfully tracing the logical outlines of our development as well as producing the correct final result.

With these prerequisites we calculate the covariant variation of the Frenet curvature (15), discarding terms which present total covariant derivatives:

$$\tilde{D}k = \frac{(\mathbf{u}\wedge\mathbf{u}')\cdot\left(\tilde{\mathbf{D}}\mathbf{u}\wedge\mathbf{u}'\right)}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\mathbf{u}'\|} - 3\,\frac{\|\mathbf{u}\wedge\mathbf{u}'\|}{\|\mathbf{u}\|^5}\left(\mathbf{u}\cdot\tilde{\mathbf{D}}\mathbf{u}\right) + \frac{(\mathbf{u}\wedge\mathbf{u}')\cdot\left(\mathbf{u}\wedge\tilde{\mathbf{D}}\mathbf{u}'\right)}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\mathbf{u}'\|} =$$

[by (14), (30), and Leibniz rule]

$$\begin{aligned}
&= 2\,\frac{\|\tilde{\mathbf{D}}\mathbf{u}\wedge\mathbf{u}'\|}{\|\mathbf{u}\|^3} - 3\,\frac{\|\mathbf{u}\wedge\mathbf{u}'\|}{\|\mathbf{u}\|^5}\left(\mathbf{u}\cdot\tilde{\mathbf{D}}\mathbf{u}\right) - 3\,\frac{\|\tilde{\mathbf{D}}\mathbf{u}\wedge\mathbf{u}\|}{\|\mathbf{u}\|^5}\,(\mathbf{u}\cdot\mathbf{u}') \\
&\qquad - \frac{(\mathbf{u}\wedge\mathbf{u}')\cdot(\mathbf{u}\wedge\boldsymbol{\sigma})}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\mathbf{u}'\|} \\
&= -\frac{\|\tilde{\mathbf{D}}\mathbf{u}\wedge\mathbf{u}'\|}{\|\mathbf{u}\|^3} - \frac{(\mathbf{u}\wedge\mathbf{u}')\cdot(\mathbf{u}\wedge\boldsymbol{\sigma})}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\mathbf{u}'\|} \qquad \text{[by (31)]} \\
&= \frac{\|\boldsymbol{\theta}\wedge\mathbf{u}''\|}{\|\mathbf{u}\|^3} - 3\,\frac{\|\boldsymbol{\theta}\wedge\mathbf{u}'\|}{\|\mathbf{u}\|^5}\,(\mathbf{u}\cdot\mathbf{u}') - \frac{(\mathbf{u}\wedge\mathbf{u}')\cdot(\mathbf{u}\wedge\boldsymbol{\sigma})}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\mathbf{u}'\|} \quad \text{[by Leibniz rule again].}
\end{aligned}$$

Let us introduce one more succinct notation:

$$\mathcal{R}_j = \frac{\sqrt{|\det[g_{ij}]|}}{\|\mathbf{u}\|^3}\,\epsilon_{il}R_{jn,p}{}^{l}u^i u^p u^n$$

The relation between this scalar semi-basic one form $\mathcal{R}_j dx^j$ and previously introduced vector semi-basic one form $\sigma^i{}_j dx^j$ is obvious:

$$\sqrt{|\det[g_{ij}]|}\,\frac{\epsilon_{il}u^i\sigma^l{}_j}{\|\mathbf{u}\|^3} = \mathcal{R}_j.$$

Both quantities satisfy the constraint imposed on the contraction with velocity:

$$\mathcal{R}_j u^j = 0\,, \tag{32}$$

along with

$$u_i\sigma^i{}_j = 0\,. \tag{33}$$

Now the Euler–Poisson equation for the complete Lagrange function (2) may be expressed in the form, valid in each case of different signature of metric tensor g_{ij} with the help of Hodge star operator:

$$\mathbf{E}^R = -\frac{*\,\mathbf{u}''}{\|\mathbf{u}\|^3} + 3\,\frac{(\mathbf{u}\cdot\mathbf{u}')}{\|\mathbf{u}\|^5}\,*\,\mathbf{u}' + m\,\frac{(\mathbf{u}\cdot\mathbf{u})\,\mathbf{u}' - (\mathbf{u}'\cdot\mathbf{u})\,\mathbf{u}}{\|\mathbf{u}\|^3} - \boldsymbol{\mathcal{R}} = \mathbf{0} \tag{34}$$

Remark. The force $\boldsymbol{\mathcal{R}}$ may be given another shape thanks to the relation (33):

$$\mathcal{R}_l dx^l = \frac{(\mathbf{u}\wedge\mathbf{u}')\cdot(\mathbf{u}\wedge\boldsymbol{\sigma})}{\|\mathbf{u}\|^3\|\mathbf{u}\wedge\mathbf{u}'\|} = \frac{\boldsymbol{\sigma}\cdot\mathbf{u}'}{\|\mathbf{u}\|\|\mathbf{u}\wedge\mathbf{u}'\|} = \frac{1}{2}\, R_{lj,pi} u^j S^{pi} dx^l\,,$$

where $S^{pi} = \dfrac{(\mathbf{u}\wedge\mathbf{u}')^{pi}}{\|\mathbf{u}\|\|\mathbf{u}\wedge\mathbf{u}'\|}$ is a formally introduced 'spin' tensor.

3.2 Completeness of variational description of geodesic circles

It remains to prove that every geodesic circle may be given a consistent parametrization, which makes it an extremal of the variational problem with the Lagrange function (2).

The governing equation for the geodesic circles. With the intention to derive a dynamical differential equation, governing the motion along a geodesic path, we put equal to zero the derivative of the Frenet curvature function k in terms of natural parametrization by $ds = \sqrt{u_i u^i}\, d\varsigma$:

(35) $$\mathbf{u}'_s\cdot\mathbf{u}''_s = 0.$$

To it we add the obvious constraint

(36) $$\mathbf{u}'_s\cdot\mathbf{u}'_s + \mathbf{u}_s\cdot\mathbf{u}''_s = 0,$$

which merely presents the differential consequence of

(37) $$\mathbf{u}_s\cdot\mathbf{u}'_s = 0.$$

Next we solve the system of equations (35) and (36) for $\mathbf{u}''_s$ to obtain

(38) $$(u''_s)_l = \frac{\epsilon_{li}(u'_s)^i}{\epsilon_{ij}(u'_s)^i(u_s)^j}\,\mathbf{u}'_s\cdot\mathbf{u}'_s\,.$$

We leave it to the Reader to check with the help of (37) and of $\mathbf{u}_s\cdot\mathbf{u}_s = 1$ that in two–dimensional space the late (38) by means of the relation $\epsilon_{il}(u'_s)^i = (u'_s)_l\epsilon_{ij}(u'_s)^i(u_s)^j$ reduces to the well known governing equation of geodesic circles

(39) $$\mathbf{u}''_s + (\mathbf{u}'_s\cdot\mathbf{u}'_s)\,\mathbf{u} = 0.$$

In order to dispense with the constraint $\mathbf{u}_s\cdot\mathbf{u}_s = 1$ we recalculate the derivatives in (39) by the reparametrization from s to an arbitrary elapsed parameter ς along the path of a geodesic circle to see at last that geodesic circles accept characterization as the integral curves of the following parameter-homogeneous differential equation:

(40) $$\frac{\mathbf{u}''}{\|\mathbf{u}\|^3} = \frac{\mathbf{u}\cdot\mathbf{u}''}{\|\mathbf{u}\|^5}\,\mathbf{u} + 3\,\frac{\mathbf{u}\cdot\mathbf{u}'}{\|\mathbf{u}\|^5}\,\mathbf{u}' - 3\,\frac{(\mathbf{u}\cdot\mathbf{u}')^2}{\|\mathbf{u}\|^7}\,\mathbf{u}\,.$$

Proof of the exhaustivenes of extremal set. Let us complement equation (40) by the following additional one, which is consistent with the equation (34) (as its consequence) and will play the role of the means to fix the way of parametrization along the extremal curve:

$$\frac{\mathbf{u}{\cdot}\mathbf{u}''}{\|\mathbf{u}\|^3} - 3\,\frac{(\mathbf{u}\cdot\mathbf{u}')^2}{\|\mathbf{u}\|^5} = \left(\frac{m}{\|\mathbf{u}\|}\,\mathbf{u}\wedge\mathbf{u}' - \mathbf{u}\wedge\boldsymbol{\mathcal{R}}\right). \tag{41}$$

For the sake of efficiency, let us evaluate the Euler–Poisson expression (34) on some arbitrary vector $\boldsymbol{v}$:

$$\mathbf{E}^R.\,\boldsymbol{v} = \frac{*\,(\boldsymbol{v}\wedge\mathbf{u}'')}{\|\mathbf{u}\|^3} - 3\,\frac{\mathbf{u}{\cdot}\mathbf{u}'}{\|\mathbf{u}\|^5}\;*\;(\boldsymbol{v}\wedge\mathbf{u}') + \frac{m}{\|\mathbf{u}\|^3}\,(\mathbf{u}\wedge\mathbf{u}'){\cdot}(\mathbf{u}\wedge\boldsymbol{v}) - \boldsymbol{\mathcal{R}}\,.\,\boldsymbol{v}\,.$$

If now we substitute $\mathbf{u}''$ in this equation with the expression from (40) and simultaneously take into account the additional equation (41), we will get:

$$\begin{aligned}\mathbf{E}^R.\,\boldsymbol{v} &= -\frac{*\,(\boldsymbol{v}\wedge\mathbf{u})\;*\;(\boldsymbol{v}\wedge\boldsymbol{\mathcal{R}})}{\|\mathbf{u}\|^2} + \frac{m}{\|\mathbf{u}\|^3}\;*\;(\boldsymbol{v}\wedge\mathbf{u})\;*\;(\mathbf{u}\wedge\mathbf{u}') \\ &\qquad + \frac{m}{\|\mathbf{u}\|^3}\,(\mathbf{u}\wedge\mathbf{u}'){\cdot}(\mathbf{u}\wedge\boldsymbol{v}) - \boldsymbol{\mathcal{R}}\,.\,\boldsymbol{v} \\ &= -\frac{(\boldsymbol{v}\cdot\mathbf{u})\,(\mathbf{u}\cdot\boldsymbol{\mathcal{R}})}{\|\mathbf{u}\|^2} + \boldsymbol{v}{\cdot}\boldsymbol{\mathcal{R}} - \boldsymbol{\mathcal{R}}\,.\,\boldsymbol{v} \equiv 0\end{aligned}$$

because of (32) □

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Complex of Nijenhuis and KV cohomology [1]

by Michel Nguiffo Boyom

Abstract

Various cohomology theories on the same locally flat manifold and their relationships are discussed. In particular a spectral sequence is used to relate the cohomology of superorder differential forms following Koszul [19] and a pioneering complex initiated by Nijenhuis [32]. The moduli of holomorphic structures which are adapted to a symplectic structure is discussed. A kahlerian alter ego of the Weinstein-Marsden symplectic reduction is studied as well as some relevant applications.

1 Introduction

In a locally flat manifold (M, D) we plan dealing with three cochain complexes. The first one has been constructed in [32]. This work of Nijenhuis is really the pioneering work on the cohomology of locally flat manifolds. The second complex we are interested in is the complex of superorder differential forms. It is a subcomplex of the Chevalley-Eilenberg complex of the Lie algebra $A(M)$ of smooth vector fields with coefficents in the vector space of tensors in M [19]. The third complex we intend to deal with is called KV complex [29]. From the theory of deformations of linear connections viewpoint [34] the KV complex of a locally flat manifold (M, D) is the solution of the following conjecture of Gerstenhaber.
Every restrict theory of deformation generates its proper cohomology theory [7]. Really the pioneering work of Nijenhuis has been motivated by the Lie group versus of this conjecture. Indeed, Albert Nijenhuis and Jean-Louis Koszul were interested in the deformations of (Koszul-) Vinberg algebras. These algebras are closely related to left invariant locally flat structures in Lie groups. At the other side compact locally flat hyperbolic maniflods are non rigid, viz they always admit non trivial deformations [17] [18]. The homological versus of this non rigidity property might be the non vanishing property of the second cohomolgy space of some deformation complex. Through section 8 we discuss relationships between various cohomology complexes on a locally flat manifold (M, D). For instance we shall prove that the cohomology of the pioneering complex constructed by Nijenhuis and the cohomology of the complex of superorder differential forms constructed

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by Koszul coincide with the term E_1 of a spectral sequence $E = (E_r^{p,q})$ which converges to the KV cohomology of (M, D). To start we shall recall the definitions of cochain complexes we are intrested in.

2 Complex of superorder differential forms

Let $A(M)$ be the Lie algebra of smooth vector fields in a smooth manifold M. We denote by θ the representation of $A(M)$ which extends its adjoint in the space $\tau(M)$ of tensors in M. Every subspace $T_s^r(M)$ of homogeneous tensors of type (r, s) is a $A(M)$-module. Let

$$C(A(M), T_s^r(M) = \oplus_k C^k(A(M), T_s^r(M))$$

be the $T_s^r(M)$-valued Chevalley-Eilenberg complex of $A(M)$. Elements of

$$C^k = C^k(A(M), T_s^r(M))$$

are k-multilinear maps from $A(M)$ to $T_s^r(M)$. We set

$$C^k = 0$$

whenever k is a negative integer,

$$C^0 = T_s^r(M).$$

Let ℓ be a non negative integer. Following [19] a cochain $f \in C^k$ is of *order* $\leq \ell$ if for $X_1, .., X_k \in A(M)$ and $x \in M$ the value $f(X_1, .., X_k)(x)$ depends on the ℓ-jets $j_x^\ell X_1, .., j_x^\ell X_k$. The set of cochains of *order* $\leq \ell$ is denoted by $C_\ell(A(M), \tau(M))$. Let us set

$$C_\infty(A(M), \tau(M)) = \cup_\ell C_\ell(A(M), \tau(M)).$$

Then $C_\infty(A(M), \tau(M))$ is called the complex of $\tau(M)$-valued superorder differential forms in M. This complex contains some canonical non vanishing cohomology classes. Here are two examples given in [19].(i) The divergence class is defined in an oriented manifold equipped with a volume form v. The divergence of an element $X \in A(M)$ is defined by the formula

$$div(X)v = \theta(X)v.$$

Let ∇ be the covariant derivation of a linear connection in M. Then consider the $T_1^2(M)$-valued 1-cochain α defined by

$$\alpha(X) = \theta(X)\nabla.$$

For $(Y, Z \in A(M)$ $\alpha(X)(Y, Z)$ is defined by

$$\theta(X)\nabla)(Y, Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z].$$

This cocycle α is of order ≤ 1. Its cohomology class $[\alpha]$ never vanishes. Furthermore this cohomology class $[\alpha]$ doesn't depend on the choice of the linear connection ∇. Some interpretations of those canonical classes are given in [19].

3 Pioneering cochain complex of Nijenhuis

Let (M, D) a locally flat manifold. This means that D is the covariant derivation of a torsion free linear connection whose curvature tensor vanishes identically. Therefore the Lie algebra $A(M)$ is the commutator (Lie) algebra of the Koszul-Vinberg algebra A whose multiplication is defined by $XY = D_X Y \ \forall X, Y \in A(M)$. Thus the vector space $X(M)$ of smooth vector fields is a two-sided A-module.
We recall that a two-sided module of a Koszul-Vinberg algebra A is a vector space V with two-sided actions of A which are denoted by av and by va respectively. These actions must satisfy the following axioms: $\forall a, b \in A, \forall v \in V$

$$a(bv)) - (ab)v = b(av)) - (ba)v,$$

$$a(vb)) - (av)b = v(ab)_{(}va)b.$$

The tensor product $V \otimes W$ of A-two-sided modules V and W is a two-sided module over the KV algebra A under the following actions

$$a(v \otimes w) = av \otimes w + v \otimes aw,$$

$$(v \otimes w)a = v \otimes wa.$$

The linear space $Hom(V, W)$ is a A-two-sided module under the following actions of A: $\forall f \in Hom(V, W), a \in A, v \in V$

$$(af)(v) = a((fv)) - f(av),$$

$$(fa)(v) = (f(v))a.$$

Thus if W is a left module over A then $Hom(V, W)$ is a left module over A.
Let us go back to the case of the locally flat manifold (M, D). The extension of D_X in $\tau(M)$ is denoted by D_X as well. Then A and $\tau(M)$ are left modules over the Lie algebra $A(M)$. The vector space $Hom_R(A, \tau(M))$ is endowed with the structure of left module over $A(M)$ which is defined by

$$(Xf)(Y) = D_X(f(Y)) - f(D_X Y)$$

Now we consider the Chevalley-Eilenberg complex $C_{ce}(A(M), Hom_R(A, \tau(M)))$ of $Hom_R(A, \tau(M))$-valued cochains of $A(M)$. Its k^{th} homogeneous subspace is denoted by $C^k_N(A, \tau(M)$. It consists of $Hom_R(A, \tau(M))$-valued skew symmetric k-multilinear maps of $A(M)$. Its coboundary operator d_{ce} is defined as it follows.

If $f \in C^k_N(A(M), Hom_R(A, \tau(M))$ then $d_{ce}f \in C^{k+1}_N(A(M), Hom_R(A, \tau(M))$ is defined as it follows

$$(d_{ce}f)(X_1, .., X_{k+1}) = \sum_{j\leq k+1}(-1)^{j+1}D_{X_j}(f(X_1, ..\hat{X}_j, ..; X_{k+1}))+$$

$$\sum_{i\leq j, i\neq j}(-1)^{i+j}f([X_i, X_j], ., \hat{X}_i, ., \hat{X}_j, ., X_{k+1}).$$

The symbol $\hat{X}_j$ means that the variable X_j is missing. The k^{th} cohomology space of this complex is denoted by $H^k_N(A(M), Hom_R(A, \tau(M)))$. Following [32] the graded vector space $\oplus_k C^k_N(A(M), Hom_R(A, \tau(M)))$ is the complex of $\tau(M)$-valued cochains of the Koszul-Vinberg algebra A. Thereby A. Nijenhuis defines the k^{th} cohomology space $H^k_N(A, \tau(M))$ of the Koszul-Vinberg algebras A as it follows

$$H^k(A, \tau(M)) = H^{k-1}_{CE}(A(M), Hom_R(A, \tau(M))).$$

4 The KV complex of $(\mathbf{M}, \mathbf{D})$

Let (M, D) be a locally flat manifold. As before we consider the Koszul-Vinberg algebra A whose multiplication is defined by $XY = D_XY$ $\forall X, Y \in X(M)$. The vector space of contravariant tensors is a two-sided module over A. Both left action and right action are defined as it follows. Let $\xi = Y_1 \otimes Y_2.. \otimes Y_p$ and let $X \in A$. We set

$$X\xi = \sum_{j\leq p} Y_1 \otimes X_2.. \otimes XY_j.. \otimes Y_p,$$

$$\xi X = Y_1 \otimes Y_2.. \otimes Y_{p-1} \otimes Y_pX.$$

The subspace of covariant tensors is a left module over A. Indeed we regard a covariant p-tensor f as a p-multilinear map from the vector space A to the vector space of real valued smooth functions. Then the left action Xf is defined by

$$(Xf)(Y_1 \otimes Y_2... \otimes Y_p) = X(f(Y_1 \otimes .. \otimes Y_p)) - \sum_{j\leq p} f(Y_1 \otimes Y_2..XY_j \otimes ...Y_p).$$

Let $J(\tau(M)) \subset \tau(M)$ be the subspace consisting of $\xi \in \tau(M)$ such that $X(Y\xi)) = (XY)\xi$ $\forall X, Y \in A$. The subspace of k-multilinear maps from A to $\tau(M)$ is denoted by $C^k_{KV}(A, \tau(M))$. Let Z be the group of integers. We build the Z-graded vector space $C_{KV}(A, \tau(M)) = \oplus_k C^k_{KV}(A, \tau(M))$ by setting

$$C^k_{KV}(A, \tau(M)) = 0$$

if k is a negative integer,

$$C^0_{KV}(A, \tau(M)) = J(\tau(M)),$$

$$C^k(A,\tau(M)) = Hom_R(A^{\otimes k},\tau(M))$$

if k is a positive integer. We define the coboundary operator

$$\delta : C^k(A,\tau(M)) \to C^{k+1}(A,\tau(M)).$$

Given $f \in C^0(A,\tau(M))$ then $\delta f \in C^1(A,\tau(M))$ is defined by

$$(\delta f)(X) = -Xf + fX.$$

Let $f \in C^k_{KV}(A,\tau(M))$. Then $\delta f \in C^{k+1}_{KV}(A,\tau(M))$ is defined as it follows

$$(\delta f)(X_1 \otimes X_2.. \otimes X_{k+1}) = \sum_{j\leq k}(-1)^j[(X_j f)(X_1 \otimes X_2..\hat{X}_j.. \otimes X_{k+1})+$$

$$(f(X_1 \otimes X_2..\hat{X}_j)...X_k \otimes X_j)X_{K+1}].$$

The couple $(C_{KV}(A,\tau(M)),\delta)$ is a cochain complex. It is called the tensor-valued KV complex of the locally manifold (M,D). Its cohomology is denoted by $H_{KV}(A,\tau(M))$. The ultimate goal of the next sections is the study of relationships between the KV cohomology and the following three deeply significant cochain complexes of (M,D): the complex $C_N(A,\tau(M))$ initiated by Nijenhuis [32], the complex of superorder differential forms $C(A(M),\tau(M))$ initiated by Koszul [19] and the classical de Rham complex of differential form $C_{dR}(M,R)$. That are reasons why we plan performing some spectrale sequences which converge to the cohomolgy of the KV complex $C_{KV}(A,\tau(M))$. To make short S-sequence stands for spectral sequence.

5 An S-sequence converging to $\mathbf{H_{KV}(A,\tau(M))}$

Let (j,ℓ) be a couple of non negative integers. Let $C^{j,\ell}_{KV}(A,\tau(M))$ be the vector space consisting of $f \in C^{j+\ell}_{KV}(A,\tau(M))$ such that $f(X_1,..,X_j,Y_1,..,Y_\ell)$ is skew symmetric w.r.t. $X_1,..,X_j$. We set

$$F^jC_{KV}(A,\tau(M)) = \oplus_\ell C^{j,\ell}_{KV}(A,\tau(M)).$$

These subspaces have the following properties

$$F^{j+1}C_{KV}(A,\tau(M)) \subset F^jC_{KV}(A,\tau(M)),$$

$$\delta F^jC_{KV}(A,\tau(M)) \subset F^jC_{KV}(A,\tau(M)),$$

The filtration of $C^k_{KV}(A,\tau(M))$ by $F^jC_{KV}(A,\tau(M)) \cap C^k_{KV}(A,\tau(M))$ is bounded. From these properties one derives a spectral sequence $E = E^{p,q}_r$. Furthermore one deduces that this spectral sequence E_r converges to $H_{KV}(A,\tau(M))$. In particular we have the decomposition

$$H^k_{KV}(A,\tau(M)) = \oplus_{p+q=k}E^{p,q}_\infty.$$

6 Relationships between scalar KV cohomology, cohomology of superorder differential forms, and the Nijenhuis complex

The spectral sequence E_r of the last section is an efficient tool to relate the cohomology $H_N(A, \tau(M))$ of Nijenhuis complex to the cohomology $H(A(M), \tau(M))$ of superorder differential forms following Koszul [19]. This section is devoted to the case of $T^0(M)$-valued KV complex of A. The subspace $T^0(M)$ is nothing else than the space of real valued smooth functions. It is a left A-module. At the ohter side we may identify the vector space $T^1_0(M)$ with the vector space $Hom_R(A, T^0(M))$. Thus according to our previous notation the vector space $C^k_N(A(M), T^1_0(M))$ is a subspace of the vector space $C^{k+1}_{KV}(A, T^0(M))$. The following result is a key step towards our goal.

Theorem 6.1. *[20] The graded vector space*

$$C_N = \oplus_k C^k_N(A(M), T^1_0(M))$$

is a subcomplex of the scalar KV complex $C_{KV}(M)$. Its $(k)^{th}$ cohomology space $H^k_N(A(M), T^1_0(M))$ after Nijenhuis [32] coincides with its $(k+1)^{th}$ KV cohomology space $H^{k+1}_{KV}(C_N)$.

Really Theorem 6.1 is a corollary of the following general feature. Let V be a two-sided module over a Koszul-Vinberg algebra A. Then the vector space $C^k_N(A, V) = Hom(\wedge^k A, Hom(A, V)$ is obviously a subspace of the vector space $C^{k+1}_{KV}(A, V)$. Therefore let $f \in C^k_N(A, V) \subset C^{k+1}_{KV}(A, V)$. As we did before the coboundary operator of Nijenhuis and the KV coboundary operator are denoted by d_{ce} and by δ respectively. Given $X_1, .., X_{k+2} \in A$, according to the definition of the operator δ one has

$$\delta f(X_1, ., X_{k+2}) = \sum_{j \leq (k+1)} (-1)^j [(X_j f)(X_1, .\hat{X}_j, .X_{k+2}) +$$

$$(f X_{k+2})(X_1, ..\hat{X}_j, .., X_{k+1}, X_j)].$$

At the right side of the formula above $X_j f$ and $f X_{k+2}$ must be understood as it follows

$$(X_j f)(a_1, .., a_{k+1}) = X_j(f(a_1, .., a_{k+1})) - \sum_i (f(a_1, .., X_j a_i, .., a_{k+1})),$$

$$(f X_{k+2})(a_1, .., a_{k+1}) = (f(a_1, .., a_{k+1}))X_{k+2}.$$

It is easy to check that $(\delta f)(X_1, ..X_{k+2})$ is skew symmetric w.r.t. $X_1, .., X_{k+1}$. Thus the graded space $C_N(A, V) = \oplus Hom(\wedge^k A, Hom(A, V))$ is preserved by the KV coboundary operator δ. Thereby the graded space $C_N(A,V)$ has two cochain complex structures: $(C_N(A,V),\delta)$ and $(C_N(A,V),d_{ce})$. Nevertheless the cohomology spaces which are derived from those cochain complex structures don't conicide. Theorem 6.1 is a straight consequence of the following

Theorem 6.2. *Suppose V to be a left module over the KV algebra A. Then the coboundary operators δ and d_{ce} we have defined are related by the formula*

$$\delta f(X_1,..,X_{k+2}) = -(d_{ce}f(X_1,..,X_{k+1}))(X_{k+2})$$

$\forall f \in C_N^k(A,V)$.

Theorem 6.2 help to end the sketch of proof of Theorem 6.1. Now we go back to the previous filtration. Namely $F^jC_{KV}(A,T^0(M)) \subset C_{KV}(A,T^0(M))$. It gives rise to the filtration of $H_{KV}(A,T^0(M))$ by the subspaces

$$F^jH_{KV}(A,T^0(M)) = \iota^\star(H_{KV}(F^jC_{KV}(A,T^0(M)))).$$

The mapping $\iota : F^jC_{KV}C_{KV}(A,T^O(M)) \to C_{KV}(A,T^0(M))$ is the inclusion map. From Theorem 6.2 the cohomology defined by Nijenhuis [32] is closely related to the cohomology of superorder differential forms defined by Koszul [19]. Indeed we are in position to impliment the following identification

$$H^{k+1}_{KV}(F^kC_{KV}(A,T^O(M))) = H^{k+1}_N(A,T^0(M)) = H^k(A(M),T^1_O(M)).$$

Really the vector space $H^k(A(M),T^1_0(M))$ is nothing but the $T^1_0(M)$- valued k^{th} cohomology of superorder differential forms [19]. Owning this argument we have the following equalities

$$F^kH_{KV}(A,T^0(M)) \cap H^{k+1}_{KV}(A,T^0(M)) = \iota^\star(H^{k+1}_N(A(M),T^0(M))).$$

To conclude we use the spectral sequence E_r which is derived from the filtration $F^jC_{KV}(A,T_0(M))$. Then it becomes easy to check that both $H^{j-1}(A(M),T^1_0(M))$ and $H^j_N(A,T^0(M))$ defined by Nijenhuis [32] and by Koszul [19] respectively are connected to the spectral sequence E_r by of the identifications

$$H^j_N(A,T^0(M)) = H^j_{KV}(F^{j-1}C^\star(A,T^0(M))) = H^{j-1}(A(M),T^1_O(M)).$$

7 Relation to the de Rham complex of M

In this section we shall use the filtration $F^jC_{KV}(A,T^0(M))$ to point out some relationships between the KV cohomology and the de Rham cohomolgy of a locally flat manifold (M,D). Let us consider the vector space $C^{p,q}_{KS}(A,T^0(M))$ whose elements f are $(p+q)$-multilinear maps $f(X_1,..,X_p,Y_1,..,Y_q)$ from A to $T^0(M)$

subject to the following requirements

(i) $f(X_1, .., X_p, Y_1, .., Y_q)$ is skew symmetric w.r.t. the variables $X_1, .., X_p$;

(ii) $f(X_1, .., X_p, Y_1, .., Y_q)$ is symmetric w.r.t. the variables $Y_1, .., Y_q$.

Thus we can write $f(X_1 \wedge X_2 \wedge .. \wedge X_p, Y_1 \vee Y_2 \vee .. \vee Y_q)$ where $X_1 \wedge X_2 \wedge .. \wedge X_p$ is the exterior (tensor) product of $X_1, .., X_p$ and $Y_1 \vee Y_2 \vee .. \vee Y_q$ is the symmetric (tensor) product of $Y_1, .., Y_q$. So, one has

$$C_{KS}(M, R) = \sum_{p,q} C^{p,q}_{KS}(A, T^0(M)).$$

Consider the Koszul-Spencer coboundary operator

$$\partial : C^{p,q}_{KS}(M, R) \to C^{p+1,q-1}_{KS}(M, R)$$

[10]. Let $f \in C^{p,q}_{KS}(M, R)$. Let $\xi = X_1 \wedge X_2 \wedge .. \wedge X_{p+1}$ and $\zeta = Y_1 \vee Y_2 \vee .. \vee Y_{q-1}$. Then $\partial f \in C^{p+1,q-1}_{KS}(M, R)$ is defined by

$$(\partial f)(\xi, \zeta) = \sum_{j=1}^{p+1} (-1)^j (f(X_1 \wedge ..\hat{X}_i ... \wedge X_{p+1}, X_i \vee Y_1 \vee ..Y_{q-1}).$$

It is easy to check that $\partial^2 = 0$. Thus one gets the Koszul-Spencer complex

$$\to C^{p-1,q+1}_{KS}(M, R) \to C^{p,q}_{KS}(M, R) \to C^{p+1,q-1}_{KS}(M, R) \to$$

whose cohomology at the level $C^{p,q}_{KS}(M, R)$ is denoted by $H^{p,q}_{KS}(M, R)$. This complex is positively acyclic in the sense that $H^{p,q}_{KS}(M, R)$ vanishes for every pair (p, q) of positive integers. Furthermore the map

$$\partial : C^{p,1}_{KS}(M, R) \to C^{p+1,0}_{KS}(M, R)$$

is surjective. The vector space $C^{p,0}_{KS}(M, R)$ is the space of p-multilinear skew symmetric maps from the vector space A to the vector space $T^0(M)$. The vector space $\Omega^p(M)$ of ordinary differential p-forms is a subspace of $C^p_{KS}(M, R)$. Now let

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

be the classical exterior differential operator. The space $C^{p,1}_{KS}(M, R)$ is nothing but $C^p_N(A, T^1_O(M)) \subset C^{p+1}_{KV}(A, T^0(M))$. Then the KV coboundary operator δ maps $C^{p,1}_{KS}(M, R)$ in $C^{p+1,0}_{KS}(M, R)$. Theorem 6.2 is a particular case of the following statement (where KS coboundary operator means Koszul-Spencer coboundary operator).

Theorem 7.1. *Let $f \in C^{p,1}_{KS}(M,R)$. The de Rham operator d, the K-S coboundary operator ∂ and the KV operator δ are related as it follows. Given $f \in C^{p,1}_{KS}(M,R)$ we have*

$$d(\delta f) = (-1)^{p-1}\partial(\delta f).$$

Clearly tensorial cochains are superorder differential forms of order $\leq O$. They are preserved by all of the coboundary operators d, ∂ and δ. Now we deal with tensorial cochains of $C_{KS}(M,R)$. This means that we are concerned with cochains which are $T^0(M)$-linear. Then Theorem 7.1 shows that the Koszul-Spencer operator gives rise to the following short exact sequence of cochain complexes

$$O \to \partial(C^{p-1,2}_{KS}(M,R)) \to C^{p,1}_{KS}(M,R) \to \Omega^{p+1}(M) \to 0.$$

At the level $\partial(C^{p-1,2}_{KS}(M,R))$ the KV cohomology of $\sum_p \partial(C^{p-1,2}_{KS}(M,R)$ will be denoted by $h^{p-1,2}$. On one hand we already know that $H^p_N(A, T^1_0(M))$ coincides with the $(p+1)^{th}$ real valued KV cohomology space $H^{p+1}(A)$ of $C_{KV}(M,R)$ at the level of $C^{p,1}_{KS}(M,R)$. On another hand we know that the cohomology of $(\Omega(M),d)$ is nothing else than the real de Rham cohomology $H_{dR}(M,R)$ of the manifold M. This discussion yields the long exact cohomology sequence which relates the KV cohomology to the real de Rham cohomology, namely

$$\to h^{p-1,2} \to H^{p+1}(A) \to H^{p+1}_{dR}(M,R) \to h^{p,2} \to$$

8 Weinstein- Marsden reduction of adapted holomorphic structures

This section is devoted to foliated symplectic manifolds and their reductions under hamiltonian actions of Lie [22]. Hamiltonian actions we deal with have equivariant momenta maps. Main definitions are given below.

Definition 8.1. A FL structure in a smooth manifold M is a triple (M,ω,L) where L is a lagrangian foliation in a symplectic manifold (M,ω).

Definition 8.2. An automorphism of (M,ω,L) is a L-preserving symplectomorphism of (M,ω)

Let $\Phi = (\phi_t)_{t\in R}$ be a flow of automorphisms of (M,ω,L) which is generated by a non singular vector field X. The flow Φ is assumed to be a hamiltonian flow of (M,ω) with a proper momentum map which is denoted by μ. Let us suppose that zero is a regular value of μ. Then $\mu^{-1}(0)$ is a submanifold of M. By the virtue of the classical Weinstein-Marsden theorem $\Phi\backslash\mu^{-1}(0)$ admits a symplectic form ω_o such that

$$\pi^\star(\omega_o) = \iota^\star(\omega)$$

where π is the canonical projection

$$\mu^{-1}(0) \to \Phi\backslash\mu^{-1}(0)$$

and ι is the inclusion map of $\mu^{-1}(0)$ in M.

Definition 8.3. An almost complex tensor $J \in Aut(TM)$ is said to be adapted to (M,ω) if at every point $x \in M$ $\omega_x(JX,JY) = \omega(X,Y)$ and if the quadratic form $\omega(JX,X)$ is positive definite.

There are symplectic maniflolds without any adapted holomorphic structure [31], [36] If J is adapted to (M,ω) then $J(L)$ is a lagrangian distribution which is transverse to L. Then arises the following question: (Q_1): Is the distribution $J(L)$ completely integrable? When the answer of question (Q_1) is affirmative the couple $(L, J(L))$ defines a bilagrangian structure in (M,ω). Thereby there exists a unique torsion free linear connection whose covariant derivation D satisfies the followings requirements:

$$D_X(L) = L,$$
$$D_X(J(L)) = J(L),$$
$$D_X\omega = 0$$

for every smooth vector field X.
Let $X(M)$ be the vector space of smooth vector fields in M. We consider the bilinear map ∇ from $X(M)$ to itself which is definied by

$$\omega(\nabla_X Y, Z) = X(\omega(Y,Z)) - \omega(Y,[X,Z])$$

$\forall X,Y,Z \in X(M)$. The bilinear ∇ has the following remarkable property. Every lagrangian foliation L in (M,ω) is preserved by ∇. Furthermore if F is a leaf of L then the restriction of ∇ in F is a torsion free linear connection whose curvature tensor vanishes identically [39]. We decompose the tangent bundle TM as $TM = L \oplus J(L)$. Then every $X \in X(M)$ is decomposed as $X = (X_1, X_2) \in L \times J(L)$. Suppose that $J(L)$ is completely integrable. Then the covariant derivation of the unique torsion free symplectic connection D defined by the couple $(L, J(L))$ is given by

$$D_{X_1,X_2}(Y_1,Y_2) = (\nabla_{X_1}Y_1 + [X_2,Y_1]_1, \nabla_{X_2}Y_2 + [X_1,Y_2]_2).$$

Now we suppose $J(L)$ to be completely integrable. Araises the question (Q_2) to know whether the connection D is locally flat. To answer this question Q_2 one may perform the following relevant theorem by H. Hess [12]. A system $(q_1,..,q_m,p_1,..,p_m))$ of local coordinate functions will be denoted simply by $q,p)$.

Theorem 8.4. *Suppose that $J(L)$ is completely integrable. Then the following statements are equivalent*
(1) The curvature tensor of D vanishes identically.
(2) In a neighbourhood of each point of M there exists a system (q,p) of local coordinate functions subject to the following requirements
(a) $\omega = \sum_i dq_i \wedge dp_i$,
(b) L is generated by the hamiltonian vector fields X_i of the functions p_i,
(c) $J(L)$ is generated by the hamiltonian vector fields Y_i of the functions q_i.

Let us go back to the Weinstein-Marsden reduction $(\Phi\backslash\mu^{-1}(0), \omega_o)$. Since the foliation L is preserved by Φ let us assume that L is subordinated to $\mu^{-1}(0)$ in the following meaning. If a leaf F of L meets $\mu^{-1}(0)$ then $F \subset \mu^{-1}(0)$. Thereby the foliation L gives rise to a lagrangian foliation L_o in the reduction $(\Phi\backslash\mu^{-1}(0), \omega_o)$. The third question we are interested in is (Q_3): does J give rise to an almost complex tensor $J_o \in Aut(T(\Phi\backslash\mu^{-1})(0))$ which is adapted to the reduction $(\Phi\backslash\mu^{-1}(0), \omega)$? The last statement relating J and J_o has the following meaning. Let g_o be the Riemannian metric in $\mu^{-1}(0)$ which is defined by the quadratic form $\omega(J(X), X)$. We suppose that $\Phi\backslash\mu^{-1}(0)$ is simply connected. Then there exists a complex analytic submanifold W of (M, J) satisfying the following conditions
(i) $W \subset \mu^{-1}(0)$
(ii) W is a transversal of the foliation of $\mu^{-1}(0)$ by the orbits of Φ.
The main concern of the sequel is to (briefly) discuss questions (Q_1), (Q_2) and (Q_3). We suppose once for all that the following property (DH) holds in (M, ω, L). In a neighbourhood of every point $x \in M$ there exists a system $(q_1, .., q_m, p_1, .., p_m)$ of local coordinate functions such that (DH_1): $\omega = \sum_j dq_j \wedge dp_j$, (DH_2): L is generated by the hamiltonian vector fields of the functions p_j. Using property (DH) one defines a local complex analytic structure whose complex coordinates functions are

$$z_j = q_j + \sqrt{-1}p_j.$$

The almost complex tensor J_{DH} of this local holomorphic structure is defined by

$$J_{DH}(S_j) = Y_j,$$

$$J_{DH}(Y_j) = -S_j$$

where S_j and Y_j are the hamiltonian vector fields of p_j and of q_j respectively. In regard to the question Q_1 the following theorem is useful.

Theorem 8.5. *Let (M, J) be an integrable almost complex structure which is adapted to (M, ω). Let (q_j, p_j) be local coordinate functions with property (DH). Then the hamiltonian vector fields S_j satisfy the following properties : (a) $[S_i, J(S_j)]$ is a local section of $J(L)$, (b) $[J(S_i), J(S_j)]$ is a local section of L.*

Proof. We suppose that the local functions q_i, p_i are defined in an open $U \subset M$ and that the tangent bundle TM is trivial over U. Let L_U be the restriction of L over U. Then we define the map θ from L_U to $U \times R^{2m}$ by setting

$$\theta(x, v) = (x, \omega_x(J(S_1), v), .., \omega_x(J(S_m), v), \omega_x(J(Y_1), v), .., \omega_x(J(Y_m), v)).$$

Now let $(\sigma_1, .., \sigma_m$ be a basis of local sections of L_U such that $\theta(\sigma_j) = constant$. Clearly the system $(S_1, .., S_m, Y_1, .., Y_m)$ is an orthonormal basis of the local Riemannian metric g_{DH} which is defined by

$$g_{DH}(X, Y) = \omega(J_{DH}X, Y).$$

Then let us perform the Riemannian metric g_{DH} to get the following g_{DH}-othogonal decomposition of $J(\sigma_j)$

$$J(\sigma_j) = \sum \omega(J_{DH}S_i, J(\sigma_j))S_i + \sum \omega(J_{DH}Y_i, J(\sigma_j))Y_i.$$

Since the local functions $\omega(JS_i, \sigma_j)$ and $\omega(J(Y_i), \sigma_j)$ are constant, all of the brackets $[S_i, \sigma_j]$ vanish identically. Of course the Nijenhuis tensor N_J vanishes identically as well. One combines the last arguments with the fact that S_i and Y_j are hamiltonian vector fields to end the proof of Theorem 8.5.
We have assumed that the flow Φ preserves the lagrangian foliation L. Without loss of generality we can suppose that S_1 coincides with the generator of Φ. We know that $[S_1, J(S_1]$ is a local section of $J(L)$. The complex line bundle generated by span$(S_1\ ,J(S_1\ ))$ is completely integrable. According to our hypothesis the triple (M, ω, J) is a Kaehler manifold. Let T_o be the orthogonal of span$(S_1,\ J(S_1))$ w.r.t. the Riemannian metric defined by $\omega(JX, Y)$. Then both span$(S_1\ ,J(S_1))$ and T_o are complex analytic distributions. According to [13], [14] T_o is completely integrable (resp totally geodesic) if and only span$(S_1,\ J(S_1))$ is completely integrable (resp. totally geodesic). These arguments allow one to prove that $J(L)$ is completely integrable. To make this claim more precise the following remark may be useful. Every leaf F_o of T_o is a Kaehlerian submanifold of (M, ω, J). Moreover such a leaf F_o is a covering of the symplectic reduction $(\Phi\backslash\mu^{-1}(0), \omega_o)$. Let us suppose that $\Phi\backslash\mu^{-1}(0)$ is simply connected. Then we may identify $\Phi\backslash\mu^{-1}(0)$ with the Kaehlerian manifold (F_o, ω, J). Thus every integrable almost complex structure (M, J) which is adapted to (M, ω) has a reduction in $(\Phi\backslash\mu^{-1}(0), \omega_o, L_o)$. □

We conclude the discussion above by the following statement

Theorem 8.6. *Every holomorphic structure (M, J) which is adapted to (M, ω) has a reduction $(\Phi\backslash\mu^{-1}(0), J_o)$ which is adapted to $(\Phi\backslash\mu^{-1}(0), \omega_o, L_o)$.*

Theorem 8.6 above helps to prove inductively (w.r.t. $dim(M)$) that $J(L)$ is completely integrable [25].

9 The flatness problem for (L,J(L))

From now on we suppose that $J(L)$ is completely integrable. We keep the previous notation. Let us recall that the couple $(L, J(L))$ defines a unique $(L, J(L))$-preserving torsion free symplectic connection whose covariant derivative D is defined by

$$D_{X_1,X_2}(Y_1, Y_2) = (\nabla_{X_1}Y_1 + [X_2, Y_1]_1, \nabla_{X_2}Y_2 + [X_1, Y_2]_2).$$

The curvature tensor of D may be different from zero. For instance let one consider the euclidian space R^2 with the symplectic form $\omega(x, y) = exp(xy)dx \wedge dy$. Let L and L' be the foliations whose leaves are the straight lines $y = constant$ and $x = constant$ respectively. These foliations are lagrangian. The torsion free symplectic connection defined by (L, L') is not locally flat, viz its curvature tensor doesn't vanish identically. In constrast to this, let one take $\omega(x, y) = (exp(x) + exp(y))dx \wedge dy$. The foliations L and L' are the same as above. Then the curvature tensor of the torsion free symplectic connection defined by (L, L') vanishes identically.

We go back to general situation. We consider the couple $(L, J(L))$ given by a holomorphic structure (M, J) which is adapted to (M, ω). We suppose that (M, ω, L) has property (DH). Let (q_i, p_i) be a system of local Darboux coordinate functions with property (DH). Thus, the functions $p_1, .., p_m$ are local first integrals of the lagrangian foliation L. We fix a basis $\alpha_1, .., \alpha_m$ of local sections of L subject to the following requirements: $\omega(JS_i, \alpha_j) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker symbol. We use property (DH) and Theorem 8.6 to prove that for every couple (i, j) the local vector fields $[S_i, J(S_j)]$ and $[JS_i, JS_j]$ are local sections of $J(L)$ and of L respectively. Thereby the complete integrability of $J(L)$ implies that $[JS_i, JS_j] = 0$. We combine these results to prove that the local vector fields $J\alpha_j$ are hamiltonian vector fields. Moreover the system $(S_1, .., S_m, J\alpha_1, .., J\alpha_m)(x)$ is a symplectic basis of (T_xM, ω) satisfying the following conditions

$$[S_i, S_j] = 0,$$
$$[S_i, J\alpha_j] = 0,$$
$$[J\alpha_i, J\alpha_j] = 0.$$

Now for every index j let y_j be a fixed local primitive of the closed differential 1-form $i_{J\alpha_j}\omega$. In regard to the bilagrangian structure $(M, \omega, J, J(L))$ the system $(y_1, .., y_m, p_1, .., p_m)$ of local coordinate functions satisfy the properties of the theorem of Hess. In other words the following statements hold

(a) $\omega = \sum dy_i \wedge dp_i$,

(b) L is spanned by the hamiltonian vector fields S_j of p_j;

(c) $J(L)$ is spanned by the hamiltonian vector fields $J\alpha_j$ of y_j.

Let D be the torsion free symplectic connection defined by the couple $(L, J(L))$.

According to the theorem of Hess the conditions (a), (b) and (c) hold if and only if the curvature tensor of D vanishes identically. Thus we can state the following

Theorem 9.1. *Let (M, ω, L) be a FL-structure in a symplectic manifold (M, ω). Then, every complex analytic structure (M, J) which is adapted to (M, ω) defines a locally flat structure (M, D) satisfying the following conditions*
(i) $D_X \omega = 0$,
(ii) $D_X(L) = L$ and $D_X(J(L)) = J(L)$ for every vector field X.

The flatness problem for $(L, J(L))$ that we just discussed is but a particular case of the general extension problem. More precisely suppose a submanifold $N \subset M$ (respectively suppose the leaves of a foliation of M) to carry a particular geometrical structure (S). The problem is to know whether (S) can be extended in the ambiant manifold M. For instance, we consider a FL-structure (M, ω, L). The leaves of L are locally flat manifolds. In the discussion above we have been concerned with the extension problem for the locally flat structures of leaves of L and of leaves of $J(L)$ respectively. About the reduction of (M, ω, L) by the flow Φ our discussion yields the following statement.

Theorem 9.2. *Let us consider data (M, ω, L), Φ and μ such that the symplectic reduction $(\Phi\backslash\mu^{-1}(0), \omega_o)$ is simply connected. If (M, ω, L) admits an adapted holomorphic structure (M, J) then its reduction $(\Phi\backslash\mu^{-1}(0), \omega_o, L_o)$ admits an adapted holomorphic structure $(\Phi\backslash\mu^{-1}(0), J_o)$.*

The Kaehlerian reduction theorem above yields other relevant corollaries some of which have been proved using different techniques [25]. Let one state (without proof) two such theorems.

Theorem 9.3. *[8]. We assume that the manifold M is simply connected. Suppose (M, ω, L) is homogeneous under an effective action of a completely solvable Lie group G. If a G-homogeneous holomorphic structure (M, J) is adapted to (M, ω) then the following statements hold.*
(i) The action of G in M is free.
(ii) The manifold M is the total space of a fiber bundle whose base is a Hessian manifold and whose fibers are simply connected homogeneous Kaehlerian submanifolds.

Theorem 9.3 we just stated has been proved by Gindikin S.G, Piatecckii-Sapiro I.I. and Vinberg E.B. [8] See also [4], [5] and [9]. Theorem 9.3 is a special case of a theorem by Dorfmeister and Nakjima (the fundamental conjecture) [5]. For other cases of the conjecture see [3], [23].
The second statement we have in mind is a particular case of a conjecture of Benson and Gordon [1], [2].

Theorem 9.4. *We assume that (M, ω, L) is homogeneous under the action of a completely solvable Lie group G. Suppose that a holomorphic structure (M, J) is* *adapted to (M, ω). If the Kaehlerian structure (M, ω, J) is invariant under the action of a cocompact lattice $\Gamma \subset G$ then the compact manifold $\Gamma \backslash G$ is diffeomorphic to the flat torus $T^{2m} = Z^{2m} \backslash R^{2m}$.*

Really the statement above is a particular solution of a conjecture which has been stated by C. Benson and C. Gordon see [2],[11],[25], [33], [37]. See also [20].

10 Lagrangian webs

Let (M, ω, L) be a FL-structure. We plan pointing out another relevant corollary of the existence of holomorphic structures which are adapted to (M, ω).

Theorem 10.1. *Let M be a simply connected manifold with a FL-structure (M, ω, L). Then every holomorphic structure (M, J) which is adapted to (M, ω) gives rise to a continuous one parameter family $L(t)$ of lagrangian foliations of (M, ω) which have the following properties.*
(i) These foliations $L(t)$ are pairwise transverse everywhere and $(L(t), L(t'))$ is locally flat $\forall (t, t') \in R^2 - \Delta R^2$.
(ii) The locally flat symplectic connection $D_{t,t'}$ defined by $(L(t), L(t'))$ doesn't depend on the choice of the point $(t, t') \in R^2 - \Delta R^2$.

Proof. We start with the locally flat structure (M, D) which is defined by $(L, J(L))$. For every $t \in R$ the distribution $L(t)$ is spanned by the vector fields $X + tJ(X)$ where X runs over the set of local sections of L. The complete integrability of $J(L)$ implies the complete integrability of $L(t)$. Obviously the lagrangian foliations $L(t)$ and $L(t')$ are transverse whenever $t \neq t'$.
Let $D_{t,t'}$ be the torsion free symplectic connection defined by the couple $(L(t), L(t'))$. Let $(x_1, .., x_m, y_1, .., y_m)$ be a system of local coordinates functions satisfying the conditions
(a) $\omega = \sum dx_i \wedge dy_i$,
(b) L is generated by the hamiltonian vector fields of $y_1, .., y_m)$
(c) $J(L)$ is generated by the hamiltonian vector fields of $(x_1, .., x_m)$.
Then the functions x_j, y_j are local affine coordinate functions of the locally flat structure (M, D). In other words, these functions satisfy the following conditions

$$D(dx_j) = 0,$$

$$D(dy_j) = 0.$$

Let $(t, t') \in R^2 - \Delta R^2$. Then we set

$$x_j(t, t') = \frac{1}{t - t'}(y_j + tx_j),$$

$$y_j(t, t') = y_j + t'x_j$$

for $j := 1, .., m$.
Now it is easy to check that data

$$D(t,t') = ((x_1(t,t'), ., x_m(t,t'), y_1(t,t'), ., y_m(t,t')), \omega, (L(t), L(t')))$$

satisfy conditions (a), (b) and (c). Therefore, by the virtue of Theorem 8.4 the curvature tensor of $D_{t,t'}$ vanishes identically. The property (i) is proved.
At one side the local functions $x_j(t,t'), y_j(t,t')$ satisfy the conditions

$$D_{t,t'}(dx_j(t,t')) = 0,$$

$$D_{t,t'}(dy_j(t,t')) = 0.$$

At the other side the same functions $x_j(t,t')$, $y_j(t,t')$ satisfy the requirements $D(dx_j(t,t')) = 0$ and $D(dy_j(t,t')) = 0$. As conclusion we get $D_{t,t'} = D$. □

A straightforth consequence of Theorem 10.1 is that (M, ω) admits lagrangian k-web for every positive integer k. The study of these lagrangian webs is not the purpose of this paper. Nevertheless we intend to show that those lagrangian webs are locally linearizable.

11 Linearization problem for lagrangian webs

Let $F_j, j := 1, .., k$ be a k-web in a smooth manifold M.
Let $m = dimM$ and $n = dimF_j$. Then $m = 2n$. Let $x \in M$. Let $F_j(x)$ be the leaf of F_j through the point x and let $V_j = T_xF_j$.
We shall say that the web F_j is linearizable near the point $x \in M$ if there exists a local chart of M, namely (U, ϕ), satisfying the following conditions

$$\phi(U \cap F_j) \subset (d\phi)(x)(V_j)$$

$\forall j := 1, .., k$. The linearization problem for webs is a difficult problem. Our concern is the linearization problem for lagrangian webs defined by a holomorphic structure (M, J) which is adapted to a symplectic manifold with a FL-structure (M, ω, L)
From Theorem 10.1 above we deduce that the symplectic manifold (M, ω) admits a lagrangian k-web for every positive integer k. Such a k-web is defined by a k-sequence $(t_1, .., t_k) \in R^k$ with $t_j \neq t_{j'}$ whener $j \neq j'$.

Theorem 11.1. *The family $L(t)$ is continuously locally linearizable.*

Proof. Actually, every k-web $(L(t_1), .., L(t_k))$ is locally linearizable. Indeed, in a neighbourhood U of $x \in M$ one considers a system of local coordinate functions, namely $x_1, .., x_m, y_1, .., y_m$ which satisfy conditions (a), (b) and (c) (see the proof of Theorem 10.1). Let us suppose that $x_i(p) = 0$, $y_i(p) = 0$. The lagrangian foliation

$L(t_j)$ is locally defined by the system $y_i + t_j x_i = c_i$, with $c_i \in R\ \forall i := 1,..,m$. Let X_i and Y_i be the hamiltonian vector fields of y_i and of x_i respectively. Condition (a) implies that

$$[X_i, X_{i'}] = 0,$$

$$[X_i, Y_{i'}] = 0,$$

$$[Y_i, Y_{i'}] = 0.$$

Thus conditions (b) and (c) imply that $span((Y_1 + t_j X_1, .., Y_m + t_j X_m)) = L(t_j)$. From this discussion we deduce that the local chart whose local coordinate functions are $x_1, .., x_m, y_1, .., y_m$ yields a linearization of the web $(L(t_1), .., L(t_k))$. It is clear that our proof walks continuously w.r.t. the parameter $t \in R$. This ends the proof of Theorem 11.1. $\square$

Now we have an application which assigns to each couple $(t, t') \in R^2 - \Delta R^2$ the locally flat structure $(M, D_{t,t'})$ which is defined by $(L(t), L(t'))$. We know that

$$(M, D_{t,t'}) = (M, D_{0,1}) = (M, D).$$

Then arises the question to know whether (M, D) is complete or hyperbolic. Before discussing those questions let us recall the notions we are concerned with. Let $c : [0, 1] \to M$ be a smooth curve. The parallel transport (w.r.t. D) along c is denoted by τ. Let $s \in [0, 1]$, we denote by τ_s the parallel transport from $T_{c(0)}M$ to $T_{c(s)}M$. Then let one set

$$Q(c) = \int_O^1 (\tau_s^{-1}(\dot{c}(s)))ds$$

where $\dot{c}(s)$ is the velocity of c at the time s. Since (M, D) is locally flat $Q(c)$ depends only on the homotopy class of c. Thus, let us identify the universal covering $\tilde{M}$ with the set $[[0, 1], M]_{x_o}$ of homotopy class of curves c with $c(0) = x_o \in M$. Then, we get a local (affine) diffeomorphism Q from $\tilde{M}$ to the vector space $T_{xo}M$. The couple (τ, Q) defines an affine representation of the fundamental group $\pi_1(M)$ in the affine space $T_{x_o}M$. The linear part $\tau : \pi_1(M) \to GL(T_oM$ is the linear holonomy representation and $\tau(\pi_1(M))$ is the linear holonomy group of (M, D). Let us go back to the map which goes from $(M, L(t), L(t'))$ to $(M, D_{(t,t')})$. Now, $(M, D_{(t,t')})$ defines the linear holonomy group $H \subset GL(T_{x_o})$ which doesn't depend on (t, t'). This holonomy group H is a discrete subgroup of the linear group $GL(T_{x_o}M)$. The locally flat structure (M, D) is called complete if Q is a global diffeomorphim onto $T_{x_o}M$. In constrast to the completeness property a locally flat structure (M, D) is called hyperbolic if Q is a diffeomorphism onto a convex domain not containing any straight line. Taking into account the discussion concerning the holonomy group H, we have the following result. Let $Aff(M, \omega, D)$ be the Lie group of (ω, D)-preserving diffeomorphisms of M.

Theorem 11.2. *Suppose that the manifold M is simply connected. If the Lie group $Aff(M, \omega, D)$ contains a discrete nilpotent subgroup Γ such that $\Gamma \backslash M$ is compact then Q is a diffeomorphism onto $T_{x_o}M$.*

Theorem 11.2 is a corrolary of a theorem due to David Fried, Bill Goldman and Michael Hirsh [6]. Regarding the completeness problem for affinely flat manifolds see also [16], [24]. Really (M, D) is the universal covering of a locally flat structure $(\Gamma \backslash M, D')$ whose fundamental group is Γ. The linear holonomy group H of $(\Gamma \backslash M, D')$ is unimodular and nilpotent. According to [6] the conjecture of Markus holds for H. In other words both (M, D) and $(\Gamma \backslash M, D')$ are complete.
Let (M, ω, L, N) be an arbitrary locally flat bilagrangian structure. Let D be the torsion free symplectic connection defined by (L, N). We know that every local Darboux coordinate system $(q_1, .., q_m, p_1, .., p_m)$ which satisfies the conditions of the theorem of Hess are actually affine coordinates functions of (M, D) [26]. Such a local coordinate system is called Darboux-Hess coordinate funtions.
Let us observe that when (M, D) is hyperbolic all of the geodesics of the linear connection D are bounded. Now suppose that $(L, N) = (L, J(L))$. Every system $(x_1, .., x_m, y_1, .., y_m)$ of Darboux-Hess coordinate funtions of $(L, J(L))$ gives rise to a system $(x_i(t, t'), y_i(t, t'))$ of Darboux-Hess coordinate functions of $(L(t), L(t'))$, namely

$$x_i(t, t') = \frac{1}{t - t'}(y_i + tx_i),$$

$$y_i(t, t') = y_i + t'x_i.$$

We have observed that these functions $x_i(t, t')$, $y_i(t, t')$ are affine functions of (M, D).

Theorem 11.3. *We assume that M is simply connected. Suppose that the group $Aff(M, D)$ of D-preserving diffeomorphisms contains a discrete subgroup Γ such that $(\Gamma \backslash M, D)$ is a compact locally flat manifold. Then the following statements are equivalent. (i) $(\Gamma \backslash M, D)$ is hyperbolic. (ii) there exist a real valued smooth function $h \in C^\infty(M)$ such that dh is Γ-equivariant and the quadratic form $D(dh)$ is positive definite.*

Theorem 11.3 is a reformulation of a result of Koszul [18]. According to the theorem of Koszul a compact locally flat maniflod (M, D) is hyperbolic if and only if it admits a locally hessian Riemannian metric [35]. In other words, a compact locally flat manifold (M, D) is hyperbolic if and only if there is a closed differential 1-form $\theta \in \Omega_1(M, R)$ whose covariant derivative $D\theta$ is positive definite.
J-L Koszul has also proved that a compact hyperbolic manifold (M, D) always admits non trivial deformations.
The KV cohomology theory provides a good framework for the study of hyperbolic locally flat manifolds. This cohomology also helps to handle the non rigidity theorem proved by Koszul. In the next section we plan to summarize some other

applications of the KV cohomology of locally flat manifolds. In this note we have highlighted the pioneering paper of Nijehnuis. The reader is refered to [29], [32]. In terms of KV cohomology a hessian manifold is a triple (M, D, g) where (M, D) is a locally flat manifold and (M, g) is a Riemannian manifold whose metric tensor $g \in C_2(M, R)$ is a 1-cocycle of the real valued KV complex $(C(M, R), \delta)$. In other words it satifies the condition $\delta(g) = 0$. The condition $\delta g = 0$ implies that in a neighbourhood of every point of M there exists a local real valued smooth function h such that $g = \delta(dh)$. Let us go back to the FL-structure (M, ω, L) admitting an adapted holomorphic structure (M, J). Let (M, D) be the locally flat structure defined by the bilagrangian structure $(L, J(L))$. We are going to build a spectral sequence $E_r^{p,q}$ which is analogous to the Hochschild-Serre spectral sequence and which converges to the KV cohomology $H^\star_{KV}(M, R)$. The notation is the same as in section 4. So A is the KV algebra $(X(M), D)$ of (M, D). Let $A_L \subset A$ be the subspace of smooth vector fields which are tangent to leaves of L. We endow the cochain complex $\oplus C_k(M, R)$ with the filtration

$$F^j C = \oplus_k (F^j C) \cap C_k(M, R).$$

By definition $f \in (F^j C) \cap C_k(M, R)$ if $f(X_1 \otimes X_2 .. \otimes X_k) = 0$ whenever more than $k + 1 - j$ arguments between $X_1, .., X_k$ are elements of the subspace A_L. We put $F^0 C \cap C_k(M, R) = C_k(M, R)$. It is easy to check that all of the subspaces $F^j C$ have the following properties

$$F^{j+1} C \subset F^j C,$$

$$\delta(F^j C) \subset F^j C$$

$$F^{k+1} C \cap C_k(M, R) = 0.$$

Conditions (i), (ii) and (iii) imply that the spectral sequence $E = E_r^{p,q}$ which is derived from the filtration $F^j C$ converges to the cohomology space $H^\star_{KV}(M, R)$. Roughly speaking one has

$$E_\infty = \lim_r E_r = H_{KV}(M).$$

This implies that

$$H^k_{KV}(M) = \sum_{p+q=k} E^{p,q}_\infty.$$

The spectral sequence $E_r^{p,q}$ we just built is the alter ego of the Hochschild-Serre spectral sequence of a pair $(h \subset g)$ of Lie algebras.

12 KV cohomology and hyperbolicity

Our discussions show that the hyperbolicity problem for locally flat manifolds is closely related to the KV cohomology. The same cohomology helps to handle the deformations of linear connections on a locally flat manifold (M, D). The reader is refereed to [34] for other aspect of deformation of torsion free linear connections on Riemannian manifolds. Regarding the hyperbolicity problem the reader is refereed to [15], [18], [38]. The hyperbolicity property is in contrast with the following rigidity theorem

Theorem 12.1. *Let (M, D) be a locally flat manifold. If $H^2_{KV}(M) = 0$ then every deformation of (M, D) is trivial.*

Thus the non rigidity of compact hyperbolic locally flat manifolds (M, D) implies the non vanishing of $H^2_{KV}(M) = H^2_{KV}(A, A)$. An simple example has been worked in detail in [29].

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The Kepler and the two-center problems in spaces of constant curvature: on the regularization problem; Lie algebra of first integrals[1]

by Tatiana G. Vozmishcheva

Abstract

The Kepler and the two-center problems in spaces of constant curvature are investigated. The statement of problems of dynamics in spaces of constant curvature is often rather nontrivial. One of the main question is the description of the potential which is generated by a gravitational center. There exist several approaches to the generalization of the classical problems for curved spaces. Moreover, the systems under consideration in celestial mechanics have singularities and, formally speaking, are not integrable in the sense of Liouville (the vector fields generated by the Hamiltonians are not complete). In this paper we describe a regularization of the problem under consideration; after this regularization the vector fields become complete and smooth.

1 Dynamics in spaces of constant curvature

The problem on the regularization takes one of the central places in celestial mechanics. A solution of the equations of motion in the classical Kepler problem at $\gamma > 0$ has a singularity corresponding to a material particle falling on the gravitational center. At the point of singularity the velocity of motion is infinite (i.e., the vector fields are not complete). However this singularity can be eliminated after a suitable regularization.

We describe two approaches to the generalization of the Newtonian potential to the case of curved spaces using the example of spaces of constant curvature.

1. One approach to determine the Newtonian potential V in the planar case consists in solving the Poisson equation, which reduces to the Laplace equation for zero mass density. Except of the constants, there exist only the following spherically-symmetrical solutions of the Laplace equation in n-dimensional Euclidean space (with accuracy up to the multiplication by a constant): r^{2-n} for

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$n \geq 3$ and $\ln r$ for $n = 2$. Thus, in the three-dimensional Euclidean space, the function $\frac{1}{r}$ is a harmonic function. If we consider the plane motion under the action of the gravitational potential field (for instance, in the classical Kepler problem) as a reduction of the three-dimensional problem, then one can determine the Newtonian potential as $-\frac{1}{r}$ in the two-dimensional case as well.

The solutions of the Laplace equation (invariant with respect to rotations about the attracting center) are the functions $(\tan r)^{2-n}$ and $(\tanh r)^{2-n}$ for $n \geq 3$ and the functions $\ln \tan r$ and $\ln \tanh r$ for $n = 2$ (here r is a distance to the attracting center, the constant, which enters into the solution of the Laplace equation, is unessential) for the n-dimensional sphere (of constant curvature 1) and for the n-dimensional Lobachevsky space (of constant curvature -1) respectively. As in the plane case, for the two-dimensional sphere an analog of the Newtonian potential can be thought of (with accuracy up to a coefficient) as the function $-\operatorname{cotan} r$, i.e., a harmonic function on a three-dimensional sphere. For the two-dimensional Lobachevsky space an analog of the Newtonian potential is the function $-\operatorname{cotanh} r$ (with accuracy up to a coefficient).

2. Another approach to the determination of the Newtonian potential consists in the fact that only this potential (and the potential of an elastic spring) generates a central field where all bounded orbits are closed.

Let a material particle p of unit mass move in a field of force with the potential V depending only on a distance between the particle and the fixed gravitational center P in a three-dimensional space of constant curvature. We consider it as a sphere $\mathbb{S}^3$ or the upper sheet of hyperboloid $\mathbb{H}^3$ (depending on a sign of curvature) embedded in $\mathbb{R}^4$ or in the Minkowski space M^4 (with the coordinates q_0, q_1, q_2, q_3) in a standard way. We will attend to the case of the negative curvature when the equation of hyperboloid has the form

$$q_0^2 - \mathbf{q}^2 = R^2, \quad R^2 = \frac{1}{\lambda},$$

making the necessary remarks for the case of the positive curvature. Here, λ is the curvature, R is the curvature radius. For a sphere the corresponding equation has the form

$$q_0^2 + \mathbf{q}^2 = R^2.$$

The problem under consideration is an analog of the classical problem on the motion in the central field. Let θ be the length of the arc of the hyperbola, connecting the points p and P (θ is measured in radians). Let us place the gravitational center at the vertex of hyperboloid. Then the potential V is a function depending only on the angle θ

$$V = -\frac{\gamma}{R}\operatorname{cotanh}\theta + \mathrm{const}\,. \tag{1.1}$$

The constant is unessential, the parameter γ plays the role of the gravitational constant.

In the case of positive curvature we must replace the hyperbolic functions by the corresponding trigonometric functions in formula (1.1).

The function V has the singular point of Newtonian type at $\theta = 0$. For the sphere the potential is antisymmetric between the two hemispheres. If γ is positive, then we have the attracting singularity at $\theta = 0$ (the north pole) and the equal repelling singularity at the antipodal point, i.e., at $\theta = \pi$ (the south pole). These two singularities can be treated as a source and a sink, since the phase flow through the boundary of any closed region, which does not contain gravitational centers, equals zero.

The Lagrangian function in the problem considered is

$$L = \frac{1}{2}[-(\dot{q}_0)^2 + (\dot{q}_1)^2 + (\dot{q}_2)^2 + (\dot{q}_3)^2] - V.$$

The Lagrangian is defined in the ambient space where the metric is indefinite. It has to be restricted to the tangent space to $\mathbb{H}^3$. The metric induced by the pseudo-Euclidean metric in $\mathbb{H}^3$ is positive definite and so the kinetic energy is also positive definite. The metric signature of the Minkowski space is $g(-1, 1, 1, 1)$. If the metric signature in the Minkowski space is defined as $g(1, -1, -1, -1)$, then the induced metric in $\mathbb{H}^3$ is negative definite. In this case, in order to obtain a positive definite kinetic energy, we have to take the induced metric with the reversed sign.

Let us pass to the pseudospherical coordinates. The transformation formulas are the following

$$\begin{array}{ll} q_0 = R\cosh\theta, & q_1 = R\sinh\theta\cos\varphi, \\ q_2 = R\sinh\theta\sin\varphi\cos\psi, & q_3 = R\sinh\theta\sin\varphi\sin\psi. \end{array} \tag{1.2}$$

Here, θ defines the length of hyperbola ("meridian") in the pseudo-Euclidean metric going from the pole of the upper sheet of the hyperboloid to a variable point, that is, the pseudospherical coordinates are analogous to the spherical coordinates (for the spherical coordinates, θ is the length of a meridian, great circle, going from the north pole of the sphere to a variable point). The metric induced in the space $\mathbb{H}^3$ (relative to the coordinates R, θ, φ, ψ) is

$$ds^2 = R^2(d\theta^2 + \sinh^2\theta d\varphi^2 + \sinh^2\theta\sin^2\varphi d\psi^2).$$

The Lagrangian L is given by

$$L = \frac{1}{2}R^2(\dot{\theta}^2 + sh^2\theta(\dot{\varphi}^2 + \sin^2\varphi\dot{\psi}^2) - V. \tag{1.3}$$

(In the case of positive curvature the function $\sinh\theta$ in formula (1.3) must be replaced by the function $\sin\theta$.)

It appears that for the potential function of type (1.1) (we suppose that $\gamma > 0$) all bounded orbits of a mass point are closed.

It is well known that for the Newtonian potential all finite orbits are closed (they are ellipses). As was proved by G. Bertrand (1873), along with Newtonian, there exists exactly one more central potential field for which all finite trajectories are closed. This is the field generated by the Hook potential $V = kr^2$, where k is a positive constant.

For the sphere and the Lobachevsky space, the problem of describing potentials for which all finite trajectories are closed was solved in [5, 6, 4, 7].

2 Hamiltonian vector fields

Let us presents some definitions.

Let H be a smooth function on a symplectic manifold (M^{2n}, w). We define the vector of *skew-symmetric gradient* $\operatorname{sgrad} H$ for this function by using the following identity:

$$w = (v, \operatorname{sgrad} H)\,,$$

where v is an arbitrary tangent vector, $v(H)$ is the derivative of the function H along v.

In local coordinates $x^1, ..., x^{2n}$, we obtain the following expression:

$$(\operatorname{sgrad} H)^i = \omega^{ij} \frac{\partial H}{\partial x^j}\,.$$

Here, ω^{ij} are components of the inverse matrix to the matrix Ω (the summation is taken over recurring superscripts and subscripts), Ω is the matrix of the canonical form

$$\Omega = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$

Definition 1. The vector field $\operatorname{sgrad} H$ is called a *Hamiltonian vector field*. The function H is called the *Hamiltonian* (or energy function) of the vector field $\operatorname{sgrad} H$.

In local symplectic coordinates $q^1, ..., q^n,\ p_1, ..., p_n$, which, according to the Darboux theorem, always exist in a neighborhood of any point of manifold, the Hamiltonian system is written in the following form:

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{\partial p_i}{\partial t} = -\dfrac{\partial H}{\partial q^i} \end{cases} \qquad i = 1, , \ldots , n$$

$$H(q, p) = H(q^1\,, ..., q^n\,, p_1\,, ..., p_n)\,,$$

that is, the components of the Hamiltonian vector field in appropriate (canonical) coordinates have the form:

$$\operatorname{sgrad} H = \left(\frac{\partial H}{\partial p_1}\,, ..., \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q^1}\,, ..., -\frac{\partial H}{\partial q^n} \right).$$

Definition 2.1. A function f on a manifold is called a first integral of the vector field v, if it is constant along all trajectories of the system, i.e.,

$$f(\gamma(t)) = const\,,$$

where $\gamma(t)$ is the integral trajectory of the system under consideration.

3 Regularization of the Kepler problem on a sphere

Let us carry out the regularization of the Kepler problem on a two-dimensional sphere [8]. We shall consider the gravitating center placed at the pole of the sphere $q_0^2 + q_1^2 + q_2^2 = \frac{1}{\lambda}$ with the coordinates $(q_0, q_1, q_2) = (\frac{1}{\sqrt{\lambda}}, 0, 0)$, and locally pass to the gnomonic coordinates x_1, x_2.

Let us introduce the complex variable $z = x_1 + ix_2$. Then in the gnomonic variables the energy of the system can be presented in the form

$$h = (1 + \lambda r^2)(\dot{z}^2 + \lambda(\mathbf{z}, \dot{\mathbf{z}})^2) + V(r)\,. \tag{3.1}$$

Let us make the change of variable z and time t by the formulas

$$\begin{aligned} &z = \omega^2, \quad t' = \frac{dt}{d\tau} = 4|\omega^2| = 4|z|;\\ &\dot{z} = 2\omega\dot{\omega} = \frac{1}{2|\omega^2|}\omega\omega';\\ &r = |z|; \quad V(r) = \frac{\gamma}{r}. \end{aligned} \tag{3.2}$$

Thus, the expression for h can be rewritten as

$$\frac{1}{2}\left(1 + \lambda|\omega^2|^2\right)\left\{|\omega'|^2 + \lambda|\omega^2|\left[Re(\omega\overline{\omega}')\right]^2\right\} + 4\gamma = 4h|\omega^2|\,. \tag{3.3}$$

Clearly, after the change of coordinates and time, the equations of motion do not have any singularity.

4 Lie Algebra of first integrals

As was noted above the Kepler problem in Euclidean space admits 5 independent first integrals. In the Kepler problem all the bounded orbits are closed. Because of this it is possible to construct integrals of motion which define the orientation of an orbit in its plane. These integrals are the cartesian components of the Laplace-Runge-Lenz vector, which at every point of a Keplerian orbit lies in its plane and is parallel to the major axis of the orbit

$$A_i = -L_{ij}p_j + \frac{\gamma x_i}{r}. \tag{4.1}$$

The generalization of expression (4.1) for a sphere was obtained in [5]. The first term in the expression for A_i is conserved in the motion of a free particle. It is constructed from the generators L_{ij} and momenta p_j. In free particle motion on the sphere, the law of conservation of linear momentum is replaced by the law of conservation of the following vector

$$\pi = \mathbf{p} + \lambda \mathbf{x}(\mathbf{x} \cdot \mathbf{p}). \tag{4.2}$$

The components are proportional to the corresponding generators of the geometrical symmetry group $SO(N+1)$. These generators are the components of angular momentum in the ambient space

$$\pi_i = \lambda^{1/2} L_{0i}\,.$$

Making the same replacement in expression (4.1), one can obtain the required generalization of the Runge-Lenz-Laplace vector in the Kepler problem on the sphere

$$A_i = -L_{ij}\pi_j + \frac{\gamma x_i}{r}. \tag{4.3}$$

The length of the Runge-Lenz-Laplace vector is defined from

$$\mathbf{A}^2 = \gamma^2 + 2H\mathbf{L}^2 - \lambda(\mathbf{L}^2)^2. \tag{4.4}$$

The Hamiltonian is written in the form

$$H = \frac{1}{2}(\pi^2 + \lambda \mathbf{L}^2) - \frac{\gamma}{r}. \tag{4.5}$$

The commutative relations for $\mathbf{L}$ and $\mathbf{A}$ are the following [5], [2], [3]

$$\begin{aligned} \{A_i, A_j\} &= -2\varepsilon_{ijk} L_k (H - \lambda \mathbf{L}^2)\,, \\ \{L_i, L_j\} &= \varepsilon_{ijk} L_k\,, \quad \{L_i, A_j\} = \varepsilon_{ijk} A_k\,, \end{aligned} \tag{4.6}$$

(here, we use the notation $L_{ij} = \varepsilon_{ijk} L_k$). In the space of the negative constant curvature the sign in the brackets is positive.

5 Description of the system on a sphere. Reduction

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be radius-vectors (issuing from the center of the sphere) of stationary attracting centers, and let $\mathbf{r}$ be the radius-vector of a mass point. Then the potential of the two-center problem on a sphere has the form

$$V = -\frac{\gamma_1}{R} \operatorname{cotan} \theta_1 - \frac{\gamma_2}{R} \operatorname{cotan} \theta_2\,, \tag{5.1}$$

where R is the radius of the sphere, γ_1 and γ_2 are positive constants characterizing the force of attraction, and θ_i is the angle between the vectors $\mathbf{r}_i$ and $\mathbf{r}$.

Let P_1 and P_2 denote the attracting centers, and let Q_1 and Q_2 be the points which are diametrically opposite to them. It is seen from the formula for the potential V that at the points P_1, P_2 this potential has singularities of the type $-\dfrac{1}{r}$ and at the points Q_1, Q_2 it has singularities of the type $\dfrac{1}{r}$, i.e., for the Newtonian potential on a sphere, the existence of the attracting center leads to the appearance of an additional repelling center (at the antipodal point).

We consider a three-dimensional sphere $\mathbb{S}^3$ of radius R. It is embedded in a standard way in the space $\mathbb{R}^4$ that is described in the Cartesian coordinates q_0, q_1, q_2, q_3. The equation of this sphere has the form $(q_0)^2+(q_1)^2+(q_2)^2+(q_3)^2 = R^2$. Let the attracting centers P_1 and P_2 be located on the sphere at the points with the coordinates $\mathbf{r}_1 = (\alpha, \beta, 0, 0)$ and $\mathbf{r}_2 = (-\alpha, \beta, 0, 0)$, $\alpha > 0, \beta > 0, \alpha^2 + \beta^2 = R^2$, under the action of the Newtonian attraction of which a mass point moves.

We introduce the spherical coordinate system. The transition formulas are written as

$$\begin{aligned} q_0 &= R\cos\theta, & q_1 &= R\sin\theta\cos\varphi, \\ q_2 &= R\sin\theta\sin\varphi\cos\psi, & q_3 &= R\sin\theta\sin\varphi\sin\psi. \end{aligned}$$

The metric that is induced on the three-dimensional sphere is written as

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\varphi^2 + \sin^2\theta\sin^2\varphi d\psi^2). \tag{5.2}$$

Relative to the spherical coordinates R, θ, φ, ψ, the Lagrangian L is given by

$$L = \frac{1}{2}R^2(\dot\theta^2 + \sin^2\theta(\dot\varphi^2 + \sin^2\varphi\dot\psi^2)) - V. \tag{5.3}$$

Without any loss in generality, we consider a unit sphere.

Theorem 5.1. *A material point in the two-center problem on the three-dimensional sphere $\mathbb{S}^3$ moves in the same way as in the two-dimensional system (on the unit two-dimensional sphere $\mathbb{S}^2 : y^2 + x^2 + z^2 = 1$) with the energy*

$$h = \frac{1}{2}(\dot y^2 + \dot x^2 + \dot z^2) + V_{eff},$$

where the effective potential energy is given by the expression

$$V_{eff} = -\gamma_1 \mathrm{cotan}\,\theta_1 - \gamma_2 \mathrm{cotan}\,\theta_2 + \frac{p_\varphi^2}{2z^2}.$$

Proof. Let a mass point move on a three-dimensional sphere in the field of two fixed attracting centers. We pass to the new variables $q_0 = x, q_1 = y, q_2 = z\cos\varphi$

and $q_3 = z\sin\varphi$. The Lagrange function with respect to the new variables is given by

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + z^2\dot{\varphi}^2) - V\,. \tag{5.4}$$

Since the Lagrangian does not depend on the variable φ, then the system admits the symmetry group $g^\alpha\colon \varphi \to \varphi + \alpha, x \to x, y \to y, z \to z$. The vector field $\partial/\partial\varphi$ corresponds to this symmetry group. According to the Noether theorem, the value $p_\varphi = \dfrac{\partial L}{\partial \dot{\varphi}}$ is preserved. In mechanics the coordinate φ is called a cyclic coordinate. Now we can exclude it using the Routh method, i.e.,

$$\frac{\partial L}{\partial \dot{\varphi}^2} = p_\varphi = \text{const}\,, \tag{5.5}$$

where p_φ is a generalized momentum corresponding to the coordinate φ and depending on the initial conditions. As a result, we have the Routh function

$$R = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V_{eff}, \tag{5.6}$$

where V_{eff} is the reduced potential of the form

$$V_{eff} = -\gamma_1 \text{cotan}\,\theta_1 - \gamma_2 \text{cotan}\,\theta_2 + \frac{p_\varphi^2}{2z^2}\,. \tag{5.7}$$

Thus, we have reduced our problem on the motion of a material particle on a three-dimensional sphere in the field of two fixed centers to the two-dimensional case, that is, to the motion on a two-dimensional sphere $x^2 + y^2 + z^2 = 1$ in the field with the reduced potential (5.7). □

6 Integrals of the system

The coordinates on a sphere in which the Hamiltonian of the two-center problem has the Liouville form were pointed out in [7]. These are spheroconical coordinates ξ, η. They are determined in the following way. Consider the equation (relative to λ)

$$\frac{x^2}{\lambda - \alpha^2} + \frac{y^2}{\lambda + \beta^2} + \frac{z^2}{\lambda} = 0\,.$$

It is easily verified that its roots have different signs. Denoting them by ξ^2 and $-\eta^2$, we obtain the coordinates (ξ, η), where $0 \le \xi \le \alpha$ and $0 \le \eta \le \beta$.

The coordinate lines of this coordinate system are the lines ($\xi = \text{const}\,, \eta = \text{const}\,$) of intersection of the sphere with two families of conical cones with vertices at the center of the sphere, i.e., the quadrics. The geometric properties of these lines on the sphere are in many respects similar to that of ordinary ellipses and hyperbolae on the plane. In addition, they are the "Keplerian" orbits in the problem on motion of a point on a sphere in the field generated by a single center (see [4]).

Formulas expressing the Cartesian coordinates through the spheroconical ones have the form

$$x^2 = \frac{1}{\alpha^2}(\alpha^2 - \xi^2)(\alpha^2 + \eta^2), \;\; y^2 = \frac{1}{\beta^2}(\beta^2 + \xi^2)(\beta^2 - \eta^2), \;\; z^2 = \frac{R^2}{\alpha^2\beta^2}\xi^2\eta^2 .$$

Obviously, extracting the roots, we obtain different signs for x and y depending on what half of the sphere (right or left, upper or lower) we consider. Therefore, we obtain the following expressions for the coordinate transformation formulas:

$$\begin{aligned} x &= \frac{1}{\alpha}\operatorname{sgn}(x)\sqrt{(\alpha^2 - \xi^2)(\alpha^2 + \eta^2)}, \\ y &= \frac{1}{\beta}\operatorname{sgn}(y)\sqrt{(\beta^2 + \xi^2)(\beta^2 - \eta^2)}, \\ z &= \frac{R}{\alpha\beta}\xi\eta. \end{aligned} \tag{6.1}$$

(In order to obtain the limit cases of the problems under consideration we consider here the radius of the sphere to be equal to R, in [7] the radius is 1. This means that if $R \to \infty$ we must obtain the plane case.)

Introduced curvilinear coordinates are orthogonal and satisfy the following conditions:

$$\xi^2 \le \alpha^2, \quad \eta^2 \le \beta^2. \tag{6.2}$$

Let us consider the potential (5.1)

$$V = \frac{-\gamma_1 \cos\theta_1 \sin\theta_2 - \gamma_2 \cos\theta_2 \sin\theta_1}{R \sin\theta_1 \sin\theta_2} .$$

It is clear that

$$\cos\theta_i = \pm\alpha x + \beta y, \quad \sin^2\theta_i = 1 - \cos^2\theta_i .$$

The Hamiltonian function relative to the new coordinates is given by the expression

$$
\begin{aligned}
&H = T + V_{eff},\\
&T = \frac{(\alpha^2 - \xi^2)(\beta^2 + \xi^2)}{2(\xi^2 + \eta^2)} p_\xi^2 + \frac{(\alpha^2 + \eta^2)(\beta^2 - \eta^2)}{2(\xi^2 + \eta^2)} p_\eta^2,\\
&V_{eff} = \frac{-\operatorname{sgn}(y)(\gamma_1 + \gamma_2)\sqrt{(\alpha^2 + \eta^2)(\beta^2 - \eta^2)}}{R(\xi^2 + \eta^2)} +\\
&\frac{-\operatorname{sgn}(x)(\gamma_1 - \gamma_2)\sqrt{(\alpha^2 - \xi^2)(\beta^2 + \xi^2)}}{R(\xi^2 + \eta^2)} + \frac{p_\varphi^2 \alpha^2 \beta^2 (\xi^{-2} + \eta^{-2})}{2R(\xi^2 + \eta^2)},
\end{aligned}
\tag{6.3}
$$

where T and V_{eff} are the kinetic and reduced potential energies, respectively, p_ξ and p_η are generalized momenta corresponding to the coordinates ξ and η. (The coefficient γ_1 corresponds to the attracting center with the coordinates $(\alpha, \beta, 0, 0)$, and the coefficient γ_2 corresponds to the attracting center with the coordinates $(-\alpha, \beta, 0, 0)$). The functions $\operatorname{sgn}(x)$ and $\operatorname{sgn}(y)$ describe the potential in the quarter of the sphere, i.e., $x > 0,\ y > 0$; $x < 0,\ y < 0$; $x > 0,\ y < 0$; $x < 0,\ y > 0$.

In the coordinates (ξ, η) the Hamiltonian has the Liouville form, which makes it possible to write down the integrals of the system under consideration, namely, the energy integral

$$
h = T + V_{eff}, \tag{6.4}
$$

and the two Liouville integrals (they are dependent)

$$
\begin{aligned}
&I_1 = \frac{1}{2}(\alpha^2 - \xi^2)(\beta^2 + \xi^2)\dot{p}_\xi^2 -\\
&-\frac{\operatorname{sgn}(x)(\gamma_1 - \gamma_2)}{R}\sqrt{(\alpha^2 - \xi^2)(\beta^2 + \xi^2)} + \frac{p_\varphi^2 \alpha^2 \beta^2 \xi^{-2}}{2R} - h\xi^2,\\
&I_2 = \frac{1}{2}(\alpha^2 + \eta^2)(\beta^2 - \eta^2)\dot{p}_\eta^2 -\\
&-\frac{\operatorname{sgn}(y)(\gamma_1 + \gamma_2)}{R}\sqrt{(\alpha^2 + \eta^2)(\beta^2 - \eta^2)} + \frac{p_\varphi^2 \alpha^2 \beta^2 \eta^{-2}}{2R} - h\eta^2.
\end{aligned}
\tag{6.5}
$$

The additional integral can be also written in the following symmetrical form

$$
\begin{aligned}
&L = \frac{(\alpha^2 - \xi^2)(\beta^2 + \xi^2)\eta^2}{2(\xi^2 + \eta^2)} p_\xi^2 - \frac{(\alpha^2 + \eta^2)(\beta^2 - \eta^2)\xi^2}{2(\xi^2 + \eta^2)} p_\eta^2 +\\
&+\frac{(\operatorname{sgn}(y)(\gamma_1 + \gamma_2)\sqrt{(\alpha^2 + \eta^2)(\beta^2 - \eta^2)} - p_\varphi^2 \alpha^2 \beta^2 \eta^{-2}/2)\xi^2}{R(\xi^2 + \eta^2)} +\\
&\frac{(-\operatorname{sgn}(x)(\gamma_1 - \gamma_2)\sqrt{(\alpha^2 - \xi^2)(\beta^2 + \xi^2)} + p_\varphi^2 \alpha^2 \beta^2 \xi^{-2}/2)\eta^2}{R(\xi^2 + \eta^2)}.
\end{aligned}
$$

In this problem the potential has a singularity at the point with the coordinates $(\xi = 0, \eta = 0)$. Instead of t, we introduce a new independent variable τ (the new time increases monotonically with the growth of t)

$$dt = \frac{\xi^2 + \eta^2}{\sqrt{2(\gamma_1 + \gamma_2)}} d\tau$$

Making the change of variables $h \to \dfrac{h}{\gamma_1 + \gamma_2}$, $\quad l \to \dfrac{l}{\gamma_1 + \gamma_2}$, we reduce the problem to quadratures

$$\frac{d\xi}{d\tau} = \sqrt{R(\xi)}, \quad \frac{d\eta}{d\tau} = \sqrt{S(\eta)}, \tag{6.6}$$

where $R(\xi)$ and $R(\eta)$ are the irrational functions

$$R(\xi) = (\alpha^2 - \xi^2)(\beta^2 + \xi^2)R^*(\xi),$$
$$S(\eta) = (\alpha^2 + \eta^2)(\beta^2 - \eta^2)S^*(\eta),$$

where

$$\begin{aligned} R^*(\xi) &= l + h\xi^2 + \frac{\operatorname{sgn}(x) K \sqrt{(\alpha^2 - \xi^2)(\beta^2 + \xi^2)}}{R}, \\ S^*(\eta) &= -l + h\eta^2 + \frac{\operatorname{sgn}(y) \sqrt{(\alpha^2 + \eta^2)(\beta^2 - \eta^2)}}{R}. \end{aligned} \tag{6.7}$$

As was already noted, the properties of the coordinate lines of the coordinate system (ξ, η) are similar to that of ellipses and hyperbolae on the plane. For instance, for each point of the coordinate line $\{\eta = \text{const}\}$, the sum of distances (on the sphere) from this point to the attracting centers P_1 and P_2, and also the difference of distances from this point to the points P_1 and Q_2 are constant. The same property holds also for the coordinate lines $\{\xi = \text{const}\}$. Using this fact, we can write the Hamiltonian and the integral of the problem under consideration in more descriptive coordinates q_1 and q_2, where

$$q_1 = \theta_2 - \theta_1\,, \qquad q_2 = \theta_2 + \theta_1\,,$$

and θ_1 and θ_2 are the angular sizes of arcs connecting the point under consideration with the centers P_1 and P_2.

Let $p_\varphi = 0$. Introducing the momenta p_1 and p_2 corresponding to the coordinates q_1 and q_2, we can write the integral and the Hamiltonian in the following form:

$$\begin{aligned} H = {} & \frac{2(\cos q_1 - \cos\delta)}{\cos q_1 - \cos q_2} p_1^2 + \frac{2(\cos\delta - \cos q_2)}{\cos q_1 - \cos q_2} p_2^2 - \\ & - \frac{(\gamma_1 - \gamma_2)\sin q_1 + (\gamma_1 + \gamma_2)\sin q_2}{R(\cos q_1 - \cos q_2)}, \\ L = {} & \frac{2\cos q_2(\cos\delta - \cos q_1)}{\cos q_1 - \cos q_2} p_1^2 - \frac{2\cos q_1(\cos\delta - \cos q_2)}{\cos q_1 - \cos q_2} p_2^2 + \\ & + \frac{\gamma_1 \sin(q_1 + q_2) + \gamma_2 \sin(q_2 - q_1)}{R(\cos q_1 - \cos q_2)}, \end{aligned}$$

where δ denotes the angular size of the arc connecting the centers P_1 and P_2.

Naturally, the coordinates q_1, q_2 (like the coordinates ξ, η) are not global coordinates on the sphere (no such coordinates exist on a sphere). The coordinate system (q_1, q_2) has singularities at the points of intersection of the sphere with the plane containing the centers. In particular, with the points of the sphere symmetric about this plane, one associates identical coordinates q_1, q_2.

However, if the variables are separated, then their coordinate lines are defined uniquely (if the system is nonresonant) by the system itself because they bound the projections of the Liouville tori on the configuration space. Therefore, for the system under consideration, any other "good" (i.e., separable) variables must have the form of the functions $\tilde{q}_1(q_1)$ and $\tilde{q}_2(q_2)$.

We describe the coordinates u, v on the sphere (with the same coordinate lines), which are more convenient for the calculation of the topological invariants of the system.

It is known that a two-dimensional sphere can be presented as a two-sheeted branching covering by a two-dimensional torus with four branch points. We can choose the covering such that the branch points are the attracting points P_1, P_2 and the antipodal points Q_1, Q_2, and the preimages of "ellipses" $\{q_1 = \text{const}\}$ and $\{q_2 = \text{const}\}$ are the coordinate lines of the angle coordinates u, v on the torus.

This covering can be described in terms of the elliptic Jacobi functions. Consider the mapping of the torus T^2 with the angle coordinates u, v into the space $\mathbb{R}^3$ with the Cartesian coordinates x, y, z, given by the formulas

$$\begin{aligned} x &= R\,\mathrm{sn}\,(u, k_1)\,\mathrm{dn}\,(v, k_2)\,, \\ y &= R\,\mathrm{sn}\,(v, k_2)\,\mathrm{dn}\,(u, k_1)\,, \\ z &= R\,\mathrm{cn}\,(u, k_1)\,\mathrm{cn}\,(v, k_2)\,. \end{aligned} \tag{6.8}$$

Here, $\mathrm{sn}\,(u, k_1), \mathrm{cn}\,(u, k_1), \mathrm{dn}\,(u, k_1)$ are the Jacobi function with the module $k_1 = \frac{\alpha}{R} = \sin\frac{\delta}{2}$, and $\mathrm{sn}\,(v, k_2), \mathrm{cn}\,(v, k_2), \mathrm{dn}\,(v, k_2)$ are the Jacobi function with the module $k_2 = \frac{\beta}{R} = \cos\frac{\delta}{2}$ (where δ is the angle value of the arc between the centers P_1 and P_2). In what follows, we will not indicate the module for brevity assuming that for the Jacobi functions of the variable u, the module is equal to k_1, and for the Jacobi functions of the variable v, the module is equal to k_2.

Using the properties of the Jacobi functions it can be easily verified that under the mapping given by formulas (6.8), the image of any point of the torus (u, v) is a point on the sphere $\{x^2 + y^2 + z^2 = R^2\}$. Moreover, two points of the torus are mapped into each point of the sphere (except for the points P_1, P_2, Q_1, and Q_2). Thus, the mapping (6.8) is the two-sheeted covering $\mathbb{T}^2 \to \mathbb{S}^2$ branched at four points.

It is convenient to present this covering in the following way. The first two of the formulas (6.8) specify the continuous one-to-one mapping of the rectangle $\{|u| \le K_1,\, |v| \le K_2\}$ onto the circle $\{x^2 + y^2 \le R^2\}$, where K_1 and K_2 are complete elliptic integrals of the first kind corresponding to the modules k_1 and

k_2. Here, the corners of the rectangle go into the points of the boundary circle with the coordinates $(\pm\alpha, \pm\beta)$. This mapping is extended to the whole plane $\mathbb{R}^2(u,v)$ by symmetry about the sides of the rectangle. Taking into account the third formula in (6.8), we obtain the mapping of the plane $\mathbb{R}^2(u,v)$ onto the sphere $\{x^2+y^2+z^2=R^2\}$.

In Fig. 1, the partition of the plane $\mathbb{R}^2(u,v)$ into rectangles with the sides $2K_1$ and $2K_2$ is depicted; each of these rectangles is mapped onto the hemisphere (shaded rectangles are mapped onto the "top" hemisphere $\{z \geq 0\}$, and unshaded rectangles are mapped onto the "bottom" hemisphere $\{z \leq 0\}$). Here, the "black" corners of rectangles are mapped into attracting centers P_1 and P_2, while the "white" are mapped into repelling centers Q_1 and Q_2. Since the functions $\operatorname{sn} u$ and $\operatorname{cn} u$ have the period $4K_1$, and the functions $\operatorname{sn} v$ and $\operatorname{cn} v$ have the period $4K_2$, the described mapping of the plane $\mathbb{R}^2(u,v)$ onto onto the sphere $\mathbb{S}^2$ specifies the mapping $\mathbb{T}^2 \to \mathbb{S}^2$, where the torus $\mathbb{T}^2$ can be represented as a rectangle in the plane $\mathbb{R}^2(u,v)$ with the sides $4K_1$ and $4K_2$ (consisting of two shaded and two unshaded rectangles with common vertex); the pairs of opposite sides of this rectangle are identified via translations.

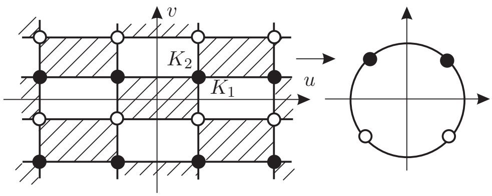

Figure 1.

The central symmetry of the plane $\mathbb{R}^2(u,v)$ about any of the corners of rectangles (see Fig. 1) specifies the involution $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$ with four fixed points. The quotient space $\mathbb{T}^2/\sigma$ is the configuration space $\mathbb{S}^2$ of the problem under consideration. Therefore, instead of the motion of the point on the sphere $\mathbb{S}^2$, one can consider the motion of the point on the torus $\mathbb{T}^2$, next taking into account the action of the involution σ. This procedure is described more precisely in Sec. 4.

The Hamiltonian and the additional integral in the variables u, v (i.e., describing the motion of the point on the torus $\mathbb{T}^2$) have the form

$$H = \frac{p_u^2 + p_v^2}{2\left(\sin^2\delta \operatorname{cn}^2 u + \cos^2\delta \operatorname{cn}^2 v\right)} - \\ - \frac{(\gamma_1 - \gamma_2)\sin\delta \operatorname{sn} u \operatorname{dn} u + (\gamma_1 + \gamma_2)\cos\delta \operatorname{sn} v \operatorname{dn} v}{R\left(\sin^2\delta \operatorname{cn}^2 u + \cos^2\delta \operatorname{cn}^2 v\right)},$$

$$L = \frac{\operatorname{cotan}\delta \operatorname{cn}^2 v p_u^2 - \tan\delta \operatorname{cn}^2 u p_v^2}{2\left(\tan\delta \operatorname{cn}^2 u + \operatorname{cotan}\delta \operatorname{cn}^2 v\right)} - \\ - \frac{(\gamma_1 - \gamma_2)\cos\delta \operatorname{sn} u \operatorname{dn} u \operatorname{cn}^2 v - (\gamma_1 + \gamma_2)\sin\delta \operatorname{sn} v \operatorname{dn} v \operatorname{cn}^2 u}{R\left(\tan\delta \operatorname{cn}^2 u + \operatorname{cotan}\delta \operatorname{cn}^2 v\right)},$$

where p_u and p_v are the momenta corresponding to the coordinates u, v.

7 Regularization

The potential V of the problem under consideration has singularities at four points on the sphere (the attracting centers P_1 and P_2 and repelling centers Q_1 and Q_2). Moreover, at the points P_1 and P_2, the function V tends to $-\infty$, and at the points Q_1 and Q_2 it tends to $+\infty$. Since the kinetic energy T is always positive and the total energy $H = T + V$ is constant along the trajectories of the system, this implies that the particle moving on the sphere in the field generated by the potential V never reaches the points Q_1 and Q_2. For the points P_1 and P_2, the situation is just opposite: for any location of a particle on the sphere, we can set an initial velocity such that this particle reaches the attracting center for a finite time. Moreover, the velocity of the particle "at the instant of falling on the attracting center" becomes infinitely large (because $T + V = \text{const}$).

Thus, the two-center problem on a sphere is described by the Hamiltonian system on the cotangent bundle to the two-dimensional sphere $T^*\mathbb{S}^2$ with the Hamiltonian $H = T + V$, where T is a function quadratic in momenta (the standard metric on the sphere), and V is the function on the sphere given by formula (5.1). However, in this approach, the phase space of this system is not the whole manifold $T^*\mathbb{S}^2$, because the function V is not defined at four points P_1, P_2, Q_1, and Q_2 of the sphere (and, therefore, the function H is not defined on four planes which are fibers of the cotangent bundle $T^*\mathbb{S}^2$ over these four points).

Denote by S_0 the sphere $\mathbb{S}^2$ with four points P_1, P_2, Q_1, and Q_2 removed. Then the phase space of the system is T^*S_0. As was already noted above, the Hamiltonian vector field $w = \operatorname{sgrad} H$ on T^*S_0, which specifies the system, is not complete. Therefore, although the system has the additional integral L (see Sec. 3), it is not Liouville integrable. Nevertheless, as will be shown below, after a certain regularization the qualitative behavior of the system will be the same as that of Liouville integrable Hamiltonian systems (almost all trajectories are conditionally periodic windings of tori).

Note that the method for regularizing the system described below is similar to the regularization proposed by T. Levi-Civita for the classical Kepler problem.

Consider the Hamiltonian H as a function of the variables (u, v, p_u, p_v). A system with such a Hamiltonian can be considered as a Hamiltonian system on the cotangent bundle to the torus $\mathbb{T}^2$. Introduce the notation

$$\lambda(u,v) = \sin^2\delta \operatorname{cn}^2 u + \cos^2\delta \operatorname{cn}^2 v\,. \tag{7.1}$$

Then the Hamiltonian has the form

$$H = \frac{p_u^2 + p_v^2}{2\lambda(u,v)} - \frac{(\gamma_1-\gamma_2)\sin\delta \operatorname{sn} u \operatorname{dn} u + (\gamma_1+\gamma_2)\cos\delta \operatorname{sn} v \operatorname{dn} v}{R\cdot\lambda(u,v)}\,,$$

and the coordinates of the field $W = \operatorname{sgrad} H$ on $T^*\mathbb{T}^2$ are equal to

$$\left(\frac{\partial H}{\partial p_u}, \frac{\partial H}{\partial p_v}, -\frac{\partial H}{\partial u}, -\frac{\partial H}{\partial v}\right).$$

In the phase space $T^*\mathbb{T}^2$, the vector field W has singularities at the points where $\lambda(u,v) = 0$, i.e., at the points $(\pm K_1, \pm K_2, p_u, p_v)$. Consider the vector field $\tilde{W} = \lambda(u,v)\cdot \operatorname{sgrad} H$. In the coordinates (u, v, p_u, p_v) it has the form

$$\begin{aligned} &\left(p_u, p_v, \sin\delta \operatorname{cn} u \left(\frac{\gamma_1-\gamma_2}{R}(2\operatorname{dn}^2 u - 1) - 2\sin\delta \operatorname{sn} u \operatorname{dn} u \cdot h\right),\right. \\ &\qquad\left.\cos\delta \operatorname{cn} v\left(\frac{\gamma_1+\gamma_2}{R}(2\operatorname{dn}^2 v - 1) - 2\cos\delta \operatorname{sn} v \operatorname{dn} v\cdot h\right)\right), \end{aligned} \tag{7.2}$$

where $h = H(u, v, p_u, p_v)$ is the value of the Hamiltonian at the point (u, v, p_u, p_v). The vector field $\tilde{W}$ also has singularities at the points $(\pm K_1, \pm K_2, p_u, p_v)$ because the Hamiltonian H is not defined at the points where $\lambda(u,v) = 0$. Denote by W_h the restriction of the vector field $\tilde{W}$ to the isoenergy surface $Q_h = \{H = h\} \subset T^*\mathbb{T}^2$. The vector field W_h already has no singularities (but it is defined only on the three-dimensional surface Q_h). It is specified by formula (7.2) and, in particular, is defined at the points $(\pm K_1, \pm K_2, p_u, p_v)$ lying on the surface Q_h, i.e.,

$$W_h(\pm K_1, \pm K_2, p_u, p_v) = (p_u, p_v, 0, 0)\,.$$

It is clear that integral trajectories of the field W_h coincide (with an accuracy up to the change of a parameter) with integral trajectories of the initial vector field $W = \operatorname{sgrad} H$ on $T^*\mathbb{T}^2$, since the multiplication of the field W by the function $\lambda(u,v)$ can be treated as the change of time: $\dfrac{dt}{d\tau} = \lambda(u(t), v(t))$, where $(u(t), v(t), p_u(t), p_v(t))$ is the trajectory of the field W.

On the other hand, the vector field W_h on the surface Q_h coincides with the restriction to this surface of a certain vector field defined on the whole phase space

$T^*\mathbb{T}^2$. It is clear that such a continuation is not unique. For instance, as a such field, we can take the field sgrad F_h, where

$$\begin{aligned}F_h = \lambda(H-h) = \frac{p_u^2+p_v^2}{2} - h\left(\sin^2\delta \operatorname{cn}^2 u + \cos^2\delta \operatorname{cn}^2 v\right) - \\ -\frac{\gamma_1-\gamma_2}{R}\sin\delta \operatorname{sn} u \operatorname{dn} u - \frac{\gamma_1+\gamma_2}{R}\cos\delta \operatorname{sn} v \operatorname{dn} v\,.\end{aligned} \tag{7.3}$$

Then $\operatorname{sgrad} F_h = \lambda \operatorname{sgrad} H + (H-h)\operatorname{sgrad}\lambda$. Since $\{F_h = 0\} = \{H = h\}$, the vector field sgrad F_h is tangent to the surface Q_h and coincides with the field W_h on it.

The integral L of the initial system is, obviously, the integral of the Hamiltonian system with the Hamiltonian F_h on the surface $\{F_h = 0\}$. Therefore, after the described regularization the topological properties of the system with the Hamiltonian H on $T^*\mathbb{T}^2$ on each isoenergy surface Q_h, will be similar to those of ordinary integrable Hamiltonian systems. In particular, nonsingular invariant manifolds of the system are Liouville tori and surgeries of these tori can be described via the Fomenko–Ziecshang invariants.

Up to now, we, in fact, spoke about the regularization of the system on $T^*\mathbb{T}^2$, which appeared from the consideration of the (branched) covering of the sphere $\mathbb{S}^2$ by the torus $\mathbb{T}^2$. This covering is defined by the involution $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$, described in Sec. 3. The involution σ extends to an involution $\sigma^*\colon T^*\mathbb{T}^2 \to T^*\mathbb{T}^2$ in a natural way. Now, in order to return to the system on the sphere (and exactly this system is the main subject of our study), it is necessary to take into account the action of the involution σ^* on $T^*\mathbb{T}^2$.

Since the involution $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$ is generated by the central symmetry of the plane $\mathbb{R}^2(u,v)$ about the point (K_1, K_2) (or any other corner of rectangles in Fig. 1), in the coordinates (u, v, p_u, p_v), the involution σ^* has the form

$$\sigma^*\colon (u, v, p_u, p_v) \to (2K_1 - u, 2K_2 - v, -p_u, -p_v)\,. \tag{7.4}$$

Therefore, the involution σ^* has exactly 4 fixed points $(\pm K_1, \pm K_2, 0, 0)$. Note that the quotient space $T^*\mathbb{T}^2/\sigma^*$ is not a manifold.

Now, we fix a certain value of h and consider the function F_h given by formula (7.3). It is easy to see that the surface $\{F_h = 0\}$ is invariant with respect to the involution σ^* and does not contain the points $(\pm K_1, \pm K_2, 0, 0)$ (a direct calculation shows that at these 4 points, the values of the function F_h are equal to $\pm\dfrac{\gamma_i}{R}\sin\delta$, where $i = 1, 2$). Thus, the quotient spaces $\{F_h = 0\}/\sigma^*$ can be considered as the isoenergy surfaces of the initial system on the sphere after the regularization.

Moreover, it is also easily verified that the vector field W_h on the surface $\{F_h = 0\}$ is likewise invariant with respect to the involution σ^*, which makes it possible to consider the vector field $w_h = W_h/\sigma^*$ as a result of the regularization of the initial vector field $w = \operatorname{sgrad} H$ on the isoenergy surface $Q_h = \{H = h\} \subset T^*\mathbb{S}^2$.

Thus, the above argument leads to the following statement describing the regularization of the two-center problem on the sphere, i.e., the Hamiltonian system $w = \operatorname{sgrad} H$ on the cotangent bundle to the sphere $T^*\mathbb{S}^2$ with the Hamiltonian $H = T + V$, where the function T quadratic in momenta is defined by the standard metric on the sphere of radius R in $\mathbb{R}^3$, and the function V is given by formula (5.1).

Theorem 7.1 (on the regularization)**.** *Let h be a regular value of the Hamiltonian H, and let $Q_h = \{H = h\} \subset T^*\mathbb{S}^2$ be the corresponding isoenergy surface. On the surface Q_h, we consider the vector field $w_h = \lambda \operatorname{sgrad} H$, where λ is function (7.1) on the sphere $\mathbb{S}^2$.*

Let $f\colon \mathbb{T}^2 \to \mathbb{S}^2$ be the (branched) two-sheeted covering (6.8), let $\sigma^*\colon T^*\mathbb{T}^2 \to T^*\mathbb{T}^2$ be the corresponding involution (7.4), and let F_h be the function defined by formula (7.3), on the cotangent bundle to the torus $T^*\mathbb{T}^2$. On the surface $\{F_h = 0\} \subset T^*\mathbb{T}^2$, consider the vector field $W_h = \operatorname{sgrad} F_h$.

Then we have the following:

(1) *the surface $\{F_h = 0\} \subset T^*\mathbb{T}^2$ is a closed three-dimensional manifold on which the involution σ^* acts without fixed points*;

(2) *on the surface $\{F_h = 0\}$, the vector field W_h has no singularities and is invariant with respect to the involution σ^**;

(3) *mapping (6.8) induces a diffeomorphism of the quotient space* (*with respect to the involution σ^**) *of the surface $\{F_h = 0\}$ without points lying in* 4 *fibers over the branch points of the mapping f onto the surface $\{H = h\}$; moreover, this diffeomorphism transforms the vector field W_h into the vector field w_h.*

Note that the irregular values h of the Hamiltonian H are explicitly written in the construction of bifurcation diagrams.

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Sur la désintégration de la représentation de conjugaison des groupes de Lie nilpotents[1]

by Fatma Khlif

Résumé

Nous présentons dans ce papier une désintégration en irréductibles de la représentation de conjugaison d'un groupe de Lie nilpotent connexe et simplement connexe. Nous construisons aussi un opérateur d'entrelacement explicite pour cette désintégration.

Classification Mathématique par Matières. 22E27.

Mots-clef. Représentation de conjugaison, opérateur d'entrelacement, double-classe.

1 Introduction

Soit G un groupe localement compact de fonction module Δ_G. On définit la représentation de conjugaison $(\gamma_G, L^2(G))$ de G par

$$\gamma_G(x)\xi(y) = \Delta_G^{\frac{1}{2}}(x)\xi(x^{-1}yx), \xi \in L^2(G), x, y \in G.$$

Il est clair que γ_G est triviale sur le centre $Z(G)$ de G. Soient $(\pi, \mathcal{H}_\pi)$ et $(\pi', \mathcal{H}_{\pi'})$ deux représentations unitaires de G. On appelle opérateur d'entrelacement entre π et π' tout opérateur linéaire et borné $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ tel que $U \circ \pi(g) = \pi'(g) \circ U$ pour tout $g \in G$. L'espace des opérateurs d'entrelacement est noté $\mathrm{Hom}(\pi, \pi')$. On dit que π et π' sont équivalentes et on note $\pi \simeq \pi'$ s'il existe un opérateur unitaire dans $\mathrm{Hom}(\pi, \pi')$. Soient maintenant S et T deux ensembles de représentations de G,. On dit que S est faiblement contenu dans T, qu'on note $S \prec T$, si $\cap_{\sigma\in S}\ker(\sigma) \supseteq \cap_{\tau\in T}\ker(\tau)$. On dit aussi que S et T sont faiblement équivalents et on note $S \sim T$, si $S \prec T$ et $T \prec S$. Le support d'une représentation ρ de G est défini par

$$\mathrm{supp}\rho = \{\pi \in \hat{G}, \pi \prec \rho\}.$$

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Alors il est clair que $\mathrm{supp}\gamma_G \subset \widehat{G/Z(G)}$, et que l'inclusion peut être stricte (Voir [4], [6] et [7]). Cette égalité a été établie pour les groupes de Lie compacts et connexes (voir [11]), les groupes de Lie simplement connexes nilpotents de classe maximale, les produits semi-directs de la forme $\mathbb{R} \ltimes \mathbb{R}^n$ (voir [9]).

Dans [6], E. Kaniuth a montré que

$$\mathrm{supp}\gamma_G \sim \{\pi \otimes \bar{\pi}, \pi \in \hat{G}\}$$

pour les groupes σ-compacts et moyennables.

Dans ce papier, on se propose dans une première étape de montrer que si G est un groupe de Lie nilpotent connexe et simplement connexe, alors γ_G est équivalente à $\int_{\hat{G}}^{\oplus} \pi_{l,-l} d\mu(\pi_l)$, pour une certaine mesure $d\mu$ sur $\hat{G}$ et certaines représentations unitaires $\pi_{l,-l}$ spéciales de G (voir 3.4). Ensuite, on écrit une désintégration en irréductibles de la représentation γ_G en faisant intervenir des formules de désintégrations des représentations induites [2] et des restrictions des représentations unitaires et irréductibles des groupes de Lie nilpotents [1]. Finalement, on construit un opérateur d'entrelacement explicite pour cette désintégration.

2 Définitions et rappels

2.1 Mesures sur les sous-groupes de Lie nilpotents

Soit G un groupe de Lie connexe, simplement connexe et nilpotent d'algèbre de Lie $\mathfrak{g}$. En notant exp l'application exponentielle, on écrit $G = \exp \mathfrak{g}$. Soit $B = \exp \mathfrak{b}$ un sous-groupe connexe fermé de G.

Nous appelons base de Jordan-Hölder de $\mathfrak{b}$ toute base $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ de $\mathfrak{b}$ telle que le sous-espace $\mathfrak{b}_i := \mathrm{vec}\{Y_i, ..., Y_k\}$ soit un idéal de $\mathfrak{b}$ pour $i = 1, ..., k$. La suite d'idéaux $(\mathfrak{b}_i)_{i=1}^{k+1}$, telle que $\mathfrak{b}_{k+1} = \{0\}$, s'appelle suite de Jordan-Hölder de $\mathfrak{b}$ associée à $\mathcal{Y}$.

Une base de Malcev de $\mathfrak{g}$ relative à $\mathfrak{b}$ est par définition toute famille libre de vecteurs $\mathcal{X} = \{X_1, ..., X_{n-k}\}$ telle que $\mathrm{vec}\{X_j, ..., X_{n-k}, \mathfrak{b}\}$ soit une sous-algèbre de dimension $n - j + 1$ pour tout $j = 1, ..., n - k$. Alors les applications

$$\mathrm{Exp}_{\mathfrak{b},\mathcal{Y}} : \mathbb{R}^k \to B, \quad (t_1, \ldots, t_k) \mapsto \exp t_1 Y_1 \cdots \exp t_k Y_k$$

et

$$\mathrm{Exp}_{\mathfrak{b}}^{\mathfrak{g}} : \mathbb{R}^{n-k} \to G/B, \quad (t_1, \ldots, t_{n-k}) \mapsto \exp t_1 X_1 \cdots \exp t_{n-k} X_{n-k} \cdot B$$

sont des difféomorphismes.

L'espace de Schwartz $\mathcal{S}(G)$ de G est donné par l'ensemble des fonctions $\varphi : G \to \mathbb{C}$ vérifiant $\varphi \circ \mathrm{Exp}_{\mathfrak{g},\mathcal{Z}} \in \mathcal{S}(\mathbb{R}^n)$, où $\mathcal{Z}$ est une base de Jordan-Hölder de $\mathfrak{g}$ et où $\mathcal{S}(\mathbb{R}^n)$ désigne l'espace de Schwartz ordinaire des fonctions C^∞ à décroissance rapide sur l'espace vectoriel réel $\mathbb{R}^n$.

On choisit une mesure db sur B de la manière suivante. Si $\eta \in C_c(B)$, l'espace des fonctions continues sur B à support compact, on pose

$$\int_B \eta(b)db := \int_{\mathbb{R}^k} \eta(\exp t_1 Y_1 \cdots \exp t_k Y_k) dt_1 \dots dt_k.$$

Cette mesure est invariante à gauche, donc une mesure de Haar. D'autre part si $\mathcal{X} = \{X_1, \dots, X_{n-k}\}$ une base de Malcev de $\mathfrak{g}$ relative à $\mathfrak{b}$, alors la mesure $d_{\mathcal{X}}$ sur G/B définie pour toute fonction continue à support compact η sur G/B par

$$\int_{G/B} \eta(g) d_{\mathcal{X}}(g) := \int_{\mathbb{R}^{n-k}} \eta(\mathrm{Exp}_{\mathfrak{b}}^{\mathfrak{g}}(t)) dt$$

est aussi G-invariante.

2.2 Représentations induites

Soit G un groupe de Lie nilpotent connexe et simplement connexe d'algèbre de Lie $\mathfrak{g}$. Soit l une forme linéaire sur $\mathfrak{g}$ et b_l la forme bilinéaire définie par $b_l(X,Y) = \langle l, [X,Y] \rangle$, $X, Y \in \mathfrak{g}$. On dit qu'une sous-algèbre de Lie $\mathfrak{h}$ de $\mathfrak{g}$ est subordonnée à l si elle est totalement isotrope pour b_l, c.à.d $b_l(\mathfrak{h}, \mathfrak{h}) = \{0\}$. On note $S(l, \mathfrak{g})$ l'ensemble de telles sous-algèbres et $M(l, \mathfrak{g})$ celui des sous-algèbres qui sont en même temps des sous-espaces totalement isotropes pour b_l et maximaux. Un élément $\mathfrak{h}$ de $M(l, \mathfrak{g})$ sera appelé polarisation en l dans $\mathfrak{g}$, et par abus de language, le sous-groupe de Lie associé $H = \exp \mathfrak{h}$ sera aussi appelé polarisation en l dans G. Soit $\mathcal{Z}$ une base de Jordan-Hölder de $\mathfrak{g}$. On obtient une polarisation $\mathfrak{p}(l)$ en $l \in \mathfrak{g}^*$ particulière, appelée polarisation de Vergne relative à $\mathcal{Z}$, de la façon suivante. Soit $(\mathfrak{g}_j)_{j=1}^{n+1}$ la suite de Jordan-Hölder de $\mathfrak{g}$ associée à $\mathcal{Z}$. Alors

$$\mathfrak{p}(l) := \sum_{j=1}^{n} \mathfrak{g}_j(l_{|\mathfrak{g}_j}).$$

Ici pour une algèbre de Lie $\mathfrak{g}$ et une forme linéaire $l \in \mathfrak{g}^*$, le symbole $\mathfrak{g}(l)$ désigne le stabilisateur de l dans $\mathfrak{g}$, c. à d. $\mathfrak{g}(l) = \{X \in \mathfrak{g}; \langle l, [X, \mathfrak{g}] \rangle = \{0\}\}$.

Soit $\mathfrak{h} \in S(l, \mathfrak{g})$ et $H = \exp \mathfrak{h}$. On définit le caractère χ_l de H par $\chi_l(\exp X) = \mathrm{e}^{-i\langle l, X\rangle}$, $X \in \mathfrak{h}$. On construit la représentation induite $\tau_l := \tau_{l,H} := \mathrm{Ind}_H^G \chi_l$ de G de la façon suivante. Soit

$$\begin{aligned} \mathcal{S}(G/H, \chi_l) \quad := \quad & \{\xi : G \to \mathbb{C}; \xi(gh) = \chi_l(h^{-1})\xi(g), \text{ pour tout } g \in G, h \in H, \\ & \xi \circ \mathrm{Exp}_{\mathfrak{h}}^{\mathfrak{g}} \in \mathcal{S}(\mathbb{R}^{n-k})\}, \end{aligned}$$

pour une base (et alors pour toute base) de Malcev $\mathcal{X} = \{X_1, \dots, X_{n-k}\}$ de $\mathfrak{g}$ relative à $\mathfrak{h}$. Nous munissons $\mathcal{S}(G/H, \chi_l)$ de la norme

$$\|\xi\|_2^2 := \int_{G/H} |\xi(g)|^2 d_{\mathcal{X}}(g), \quad \xi \in \mathcal{S}(G/H, \chi_l).$$

Comme la mesure $d_{\mathcal{X}}$ est G-invariante, la translation à gauche dans $\mathcal{S}(G/H, \chi_l)$ est isométrique et définit la représentation τ_l sur le complété $L^2(G/H, \chi_l)$ de $\mathcal{S}(G/H, \chi_l)$ pour la norme $\|.\|_2$.

D'après la théorie de Kirillov, la représentation $\tau_{l,H}$ est irréductible, si et seulement si H est une polarisation en l dans G. Dans ce cas-là, on écrira π_l au lieu de τ_l. En outre, les représentations π_{l,B_l} et π_{l,B'_l} sont équivalentes si et seulement si B_l et B'_l sont deux polarisations en l dans G.

2.3 Représentation conjuguée

Soit G un groupe localement compact et $(\pi, \mathcal{H}_\pi)$ une représentation de G. On note $\mathcal{H}'_\pi$ le dual $\mathbb{C}$-linéaire de $\mathcal{H}_\pi$. D'après le théorème de Riesz, l'application $\mathcal{H}_\pi \to \mathcal{H}'_\pi$, $\xi \mapsto f_\xi$, définie pour tout $\eta \in \mathcal{H}_\pi$ par $f_\xi(\eta) = \langle \xi, \eta \rangle$, est bijective et anti-linéaire. Munissons $\mathcal{H}'_\pi$ du produit scalaire $\langle f_\xi, f_\eta \rangle = \langle \xi, \eta \rangle$. La représentation conjuguée $\overline{\pi}$ de G agit sur $\mathcal{H}'_\pi$ par $\overline{\pi}(g) f_\xi = f_{\pi(g)\xi}$ pour tout $g \in G$.

2.4 Intégrale directe d'espaces de Hilbert

Soient G un groupe localement compact et X un espace localement compact, soit $\{(\pi_x, \mathcal{H}_{\pi_x}), x \in X\}$ un champ de représentations unitaires de G et μ une mesure sur X. On note $\mathcal{H}_X = \cup_{x \in X} \mathcal{H}_{\pi_x}$ et $L_0^2 = L_0^2(X, \mathcal{H}_X, \mu)$ l'ensemble des applications $f : X \to \mathcal{H}_X$ telles que pour tout $x \in X$, $f(x) \in \mathcal{H}_{\pi_x}$ et les fonctions $x \to \|f(x)\|_{\mathcal{H}_{\pi_x}}$ et $(g, x) \to \langle \pi_x(g)[f(x)], f(x) \rangle$ soient continues à support compact sur X resp. sur $G \times X$. On suppose que pour tout $x \in X$, le sous-espace $\{f(x), f \in L_0^2\}$ est dense dans $\mathcal{H}_{\pi_x}$. On munit L_0^2 de la norme

$$\|f\|_2 := \sqrt{\int_X \|f(x)\|^2_{\mathcal{H}_x} d\mu(x)}.$$

On appelle alors intégrale directe de $(\mathcal{H}_{\pi_x})_{x \in X}$, notée $\int_X^\oplus \mathcal{H}_{\pi_x} d\mu(x)$, l'adhérence de L_0^2 pour la norme $\|.\|_2$ et on définit la représentation intégrale directe $\pi = \int_X^\oplus \pi_x d\mu(x)$ de G agissant sur $\mathcal{H}_\pi = \int_X^\oplus \mathcal{H}_{\pi_x} d\mu(x)$ par $[\pi(g)f](x) = \pi_x(g)[f(x)]$ pour tout $g \in G, x \in X$ et $f \in \mathcal{H}_\pi$.

2.5 Espace de Garding

Soient G un groupe de Lie et du la mesure de Haar sur G, $(\pi, \mathcal{H}_\pi)$ une représentation unitaire de G, $\xi \in \mathcal{H}_\pi$ et $\varphi \in C_c^\infty(G)$, l'espace des fonctions infiniment différentiables à support compact sur G. On appelle vecteur de Garding le vecteur $\pi(\varphi)\xi$ défini par

$$\pi(\varphi)\xi = \int_G \varphi(u)\pi(u)\xi \, du.$$

2.6 Opérateurs de Hilbert-Schmidt

Soient $\mathcal{H}$ un espace de Hilbert, $\mathcal{B}(\mathcal{H})$ l'ensemble des opérateurs bornés de $\mathcal{H}$ et $A \in \mathcal{B}(\mathcal{H})$. On dit que A est un opérateur de Hilbert-Schmidt si l'opérateur borné A^*A est traçable. L'ensemble des opérateurs de Hilbert-Schmidt sera noté $HS(\mathcal{H})$, sur lequel on définit la norme suivante

$$\|A\|_{HS}^2 = \mathrm{tr(A^*A)}.$$

En particulier pour une représentation unitaire $(\pi, \mathcal{H}_\pi)$ d'un groupe de Lie connexe, simplement connexe et nilpotent $G = \exp \mathfrak{g}$ et $\varphi \in \mathcal{S}(G)$, l'opérateur $\pi(\varphi)$ est toujours de Hilbert-Schmidt. De plus, si $\pi = \pi_l \in \hat{G}$, $l \in \mathfrak{g}^*$ et $\varphi \in \mathcal{S}(G)$, alors l'opérateur $\pi_l(\varphi)$ est à noyau, c'est-à-dire on peut écrire

$$\pi_l(\varphi)\xi(x) = \int_{G/B_l} K_{\pi_l(\varphi)}(x, g)\xi(g)dg, \ \xi \in \mathcal{H}_{\pi_l}, \ x \in G,$$

où B_l est une polarisation en l dans G et $K_{\pi_l(\varphi)}$ est le noyau de $\pi_l(\varphi)$, qui est donné par

$$K_{\pi_l(\varphi)}(x, y) = \int_{B_l} \varphi(xby^{-1})\chi_l(b)db, \quad x, y \in G.$$

De plus $K_{\pi_l(\varphi)} \in \mathcal{S}(G/B_l \times G/B_l, \chi_l \times \chi_{-l})$ et l'application

$$\mathcal{S}(G) \to \mathcal{S}(G/B_l \times G/B_l, \chi_l \times \chi_{-l}); \quad \varphi \mapsto K_{\pi_l(\varphi)},$$

est continue et surjective (voir [5]).

2.7 Mesure de Plancherel

Soient $G = \exp \mathfrak{g}$ un groupe de Lie nilpotent connexe et simplement connexe, $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ une base de Jordan-Hölder de $\mathfrak{g}$ et $(\mathfrak{g}_j)_{j=1}^{n+1}$ la suite de Jordan-Hölder de $\mathfrak{g}$ associée à $\mathcal{Z}$. On note pour tout $j \in \{1, \ldots, n+1\}$, $d^j(l) = \dim(G \cdot l_{|\mathfrak{g}_j})$ et $d^j = \max_{l \in \mathfrak{g}^*} d^j(l)$. Soit $\mathcal{E} = \{l \in \mathfrak{g}^*, \ d^j(l) = d^j, j \in \{1, \ldots, n+1\}\}$ l'ouvert $G-$invariant des orbites génériques. On considère les ensembles d'indices suivants

$$S = \{j \in \{1, \ldots, n\}; d^j \neq d^{j+1}\}, \qquad T = \{1, \ldots, n\} \setminus S.$$

Si on pose $\mathfrak{g}_S^* = \mathrm{vec}\{Z_j^*, j \in S\}$ et $\mathfrak{g}_T^* = \mathrm{vec}\{Z_j^*, j \in T\}$, on aura $\mathfrak{g}^* = \mathfrak{g}_S^* \oplus \mathfrak{g}_T^*$. De plus, chaque $G-$orbite dans $\mathcal{E}$ rencontre $\mathfrak{g}_T^*$ en un seul point, en particulier $\mathcal{E} \cap \mathfrak{g}_T^*$ est un ouvert de Zariski non vide dans $\mathfrak{g}_T^*$. D'autre part, pour tout $l \in \mathcal{E}$, la forme bilinéaire b_l, définie par $b_l(X, Y) = \langle l, [X, Y] \rangle$, $X, Y \in \mathfrak{g}$, est non dégénérée. Il existe alors une fonction polynomiale, appelée le Pfaffien de l, qui vérifie $\mathrm{Pf}(l)^2 = \det\big(\langle l, [Z_i, Z_j] \rangle\big)_{i,j \in S}$. Soit K la bijection de Kirillov définie par

$$K : \mathfrak{g}^*/G \ni \mathrm{Ad}_G^*(l) \mapsto [\pi_l] \in \hat{G}.$$

Si dl_T est la mesure euclidienne sur $\mathfrak{g}_T^*$, alors l'application $K : (\mathcal{E} \cap \mathfrak{g}_T^*)/G \to \hat{G}$ envoie la mesure $dl = |\mathrm{Pf}(l)|dl_T$ en la mesure de Plancherel $d\mu$ sur $\hat{G}$. Cette mesure intervient dans la désintégration de la représentation régulière gauche de G. La formule de Plancherel dit que

$$\|\varphi\|_2^2 = \int_{\hat{G}} \|\pi(\varphi)\|_{\mathrm{HS}}^2 d\mu(\pi) = \int_{\mathcal{E}\cap\mathfrak{g}_T^*} \|\pi_l(\varphi)\|_{\mathrm{HS}}^2 |\mathrm{Pf}(l)|dl_T, \quad \varphi \in L^2(G).$$

Par abus d'écriture nous mettons aussi

$$\int_{\mathfrak{g}^*/G} d\mu(\pi_l) = \int_{\hat{G}} d\mu(\pi).$$

Nous définissons l'espace Hilbertien $L^2(\hat{G})$ comme intégrale hilbertienne

$$L^2(\hat{G}) := \oint_{\hat{G}} HS(\mathcal{H}_\pi) d\mu(\pi).$$

2.8 Ensemble des double-classes

Soient $H = \exp\mathfrak{h}$ et $B = \exp\mathfrak{b}$ deux sous-groupes fermés connexes d'un groupe de Lie $G = \exp\mathfrak{g}$ nilpotent, connexe et simplement connexe. On définit la double-classe relative à H et B d'un élément $g \in G$, et on la note $\tilde{g} = H \cdot g \cdot B$, comme étant le sous-ensemble de G donné par $H \cdot g \cdot B = \{h \cdot g \cdot b,\ (h, b) \in H \times B\}$. L'ensemble de toutes ces doubles-classes sera noté $H\diagdown G\diagup B$. Pour décrire cet ensemble, on peut se référer au travail de Abdennadher et Ludwig [1]. Dans la suite, on adopte leurs notations pour rappeler leur résultat. Soit $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ une base de Jordan-Hölder de $\mathfrak{g}$. L'idéal engendré par $\{Z_i, \dots, Z_n\}$ est noté $\mathfrak{g}_i$ et G_i son sous-groupe de Lie. On considère maintenant les sous-ensembles d'indices suivants

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} = \{i \in \{1, \dots, n\}, Z_i \notin \mathfrak{h} + \mathfrak{g}_{i+1}\}, \quad \mathcal{I} = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}}.$$

On note aussi $\mathcal{I}(\mathfrak{h}, \mathfrak{b})$ le sous-ensemble de $\mathcal{I}$ caractérisé par

$$i \in \mathcal{I}(\mathfrak{h}, \mathfrak{b}) \Longleftrightarrow \begin{cases} \text{il existe un ouvert de Zariski non vide } \mathcal{U}_i \subset G \\ \text{tel que pour tout } g \in \mathcal{U}_i, Z_i \notin \mathfrak{h} + \mathrm{Ad}_g\mathfrak{b} + \mathfrak{g}_{i+1}. \end{cases}$$

On désigne par $\mathcal{W}_i,\ i \in \mathcal{I}$, la partie de G donnée par

$$\mathcal{W}_i = \begin{cases} \{g \in G, Z_i \notin \mathfrak{h} + \mathrm{Ad}_g\mathfrak{b} + \mathfrak{g}_{i+1}\} & \text{si } i \in \mathcal{I}(\mathfrak{h}, \mathfrak{b}) \\ \{g \in G, Z_i \in \mathfrak{h} + \mathrm{Ad}_g\mathfrak{b} + \mathfrak{g}_{i+1}\} & \text{sinon.} \end{cases}$$

Ces ensembles ont la propriété $(\mathcal{P})$ de stabilité par la multiplication à droite avec les éléments de B et à gauche avec les éléments de H. De plus, chaque $\mathcal{W}_i$ contient un ouvert de Zariski maximal $\mathcal{U}_i$ non vide de G, ayant la propriété $(\mathcal{P})$. On pose

$$\mathcal{U} = \bigcap_{i\in\mathcal{I}} \mathcal{U}_i,$$

qui est aussi un ouvert de Zariski de G vérifiant $(\mathcal{P})$. L'ensemble d'indices $\mathcal{I}(\mathfrak{h},\mathfrak{b})$, caractérisant l'ensemble des double-classes $H\diagdown G\diagup B$, est alors

$$\mathcal{I}(\mathfrak{h},\mathfrak{b}) = \{i \in \mathcal{I},\ \forall g \in \mathcal{U},\ Z_i \notin \mathfrak{h} + \mathrm{Ad}_g\mathfrak{b} + \mathfrak{g}_{i+1}\}.$$

On pose $\mathcal{I}(\mathfrak{h},\mathfrak{b}) = \{i_1 < \cdots < i_d\}$ et on considère les applications suivantes

$$\begin{array}{cccc} \phi : & \mathbb{R}^d & \rightarrow & G \\ & (t_1,\ldots,t_d) & \mapsto & \exp t_1 Z_{i_1} \cdots \exp t_d Z_{i_d} \end{array}$$

et

$$\begin{array}{cccc} P : & G & \rightarrow & H\diagdown G\diagup B \\ & g & \mapsto & \tilde{g} = H\cdot g\cdot B. \end{array}$$

Ainsi $\mathcal{V} = \phi^{-1}(\mathcal{U})$ est un ouvert de Zariski de $\mathbb{R}^d$ et $\tilde{\mathcal{U}} = P(\mathcal{U})$ est l'ensemble des double-classes de presque tous les éléments de G. Avec cette construction l'application

$$\begin{array}{cccc} \tilde{\phi} : & \mathcal{V} & \rightarrow & \tilde{\mathcal{U}} \\ & (t_1,\ldots,t_d) & \mapsto & H\cdot \exp t_1 Z_{i_1} \cdots \exp t_d Z_{i_d}\cdot B \end{array}$$

est un homéomorphisme. Dans la suite on notera $\Phi : \tilde{\mathcal{U}} \rightarrow G$ tel que $\Phi = \phi \circ \tilde{\phi}^{-1}$.

Dans le même travail, Abdennadher et Ludwig [1] ont désintégré la mesure invariante sur G/B en une intégrale sur l'ensemble des double-classes. Si on prend des bases de Malcev $\mathcal{Y}(g,\mathfrak{h},\mathfrak{b})$ et $\mathcal{X}$ de $\mathfrak{h}$ relative à $\mathfrak{h}\cap\mathrm{Ad}_g\mathfrak{b}$ respectivement de $\mathfrak{g}$ relative à $\mathfrak{b}$ qui varient rationnellement en $g \in G$, alors il existe une fonction rationnelle F définie sur l'ensemble des double-classes telle que la mesure $d\gamma$, définie pour toute fonction $\psi : H\diagdown G\diagup B \rightarrow \mathbb{C}$ continue et à support compact dans $\tilde{\mathcal{U}}$, par la relation

$$\int_{H\diagdown G\diagup B} \psi(\tilde{g})d\gamma(\tilde{g}) := \int_{\mathcal{V}} \psi(H\cdot \exp t_1 Z_{i_1}\cdots \exp t_d Z_{i_d}\cdot B)|F(t_1,\ldots,t_d)|dt_1\ldots dt_d,$$

vérifie

(2.1)
$$\int_{G/B}\varphi(g)d_{\mathcal{X}}(g) = \int_{H\diagdown G\diagup B}\Big(\int_{H/H\cap\Phi(\tilde{g})\cdot B\cdot\Phi(\tilde{g})^{-1}} \varphi(h\cdot\Phi(\tilde{g})\cdot B)d_{\mathcal{Y}(\Phi(\tilde{g}),\mathfrak{h},\mathfrak{b})}(h)\Big)d\gamma(\tilde{g})$$

pour $\varphi \in C_c(G/B)$.

2.9 Désintégration des restrictions de représentations

Soient G un groupe de Lie nilpotent connexe et simplement connexe d'algèbre de Lie $\mathfrak{g}$, $l \in \mathfrak{g}^*$ et $\pi_l \in \hat{G}$. Soient $B_l = \exp\mathfrak{b}_l$ une polarisation de G en l et $H = \exp\mathfrak{h}$ un sous-groupe fermé connexe de G. La désintégration de $\pi_{l|H}$ sur l'ensemble des double-classes $H\diagdown G\diagup B_l$ est donnée par Mackey dans [10] par

$$\pi_{l|H} \simeq \int_{H\diagdown G\diagup B_l}^{\oplus} \mathrm{Ind}_{H\cap\Phi(\tilde{g})\cdot B_l\cdot\Phi(\tilde{g})^{-1}}^{H} \chi_{\mathrm{Ad}^*_{\Phi(\tilde{g})}(l)} d\gamma(\tilde{g})$$

avec une mesure γ sur $H\diagdown G\diagup B_l$. Un opérateur d'entrelacement explicite pour cette désintégration est donné par Abdennadher et Ludwig dans [1] par

$$S_{\mathrm{rest}} : \mathcal{S}(G/B_l, \chi_l) \longrightarrow \int_{H\diagdown G\diagup B_l}^{\oplus} L^2(H/H\cap\Phi(\tilde{g})\cdot B_l\cdot\Phi(\tilde{g})^{-1}, \chi_{\mathrm{Ad}^*_{\Phi(\tilde{g})}(l)}) d\gamma(\tilde{g})$$

tel que

$$S_{\mathrm{rest}}(\xi)(\tilde{g})(h) = \xi(h\cdot\Phi(\tilde{g})), \tag{2.2}$$

où $\xi \in \mathcal{S}(G/B_l, \chi_l)$, $\tilde{g} \in H\diagdown G\diagup B_l$ et $h \in H$.

2.10 Désintégration des représentations induites

Soient G un groupe de Lie nilpotent connexe simplement connexe d'algèbre de Lie $\mathfrak{g}$, $l \in \mathfrak{g}^*$, $H = \exp \mathfrak{h}$ un sous-groupe de Lie fermé de G tel que $\langle l, [\mathfrak{h}, \mathfrak{h}]\rangle = \{0\}$ et soit $\tau_l = \mathrm{Ind}_H^G \chi_l$ la représentation monomiale induite à partir du caractère unitaire χ_l. D'après Corwin, Greenleaf [3] et Lipsman [8], la désintégration de τ_l en irréductibles s'obtient comme suit

$$\tau_l \simeq \int_{(l+\mathfrak{h}^\perp)/H}^{\oplus} \pi_f d(f),$$

$d(f)$ étant une certaine mesure naturelle sur l'espace $(l+\mathfrak{h}^\perp)/H$ des H-orbites. Une désintégration concrète de τ_l est donnée par Baklouti et Ludwig [2], par la formule

$$\tau_l \simeq \int_{\mathcal{V}_l^{R,\mathcal{B}}}^{\oplus} \pi_f d\lambda_l^{R,\mathcal{B}}(f), \tag{2.3}$$

où $\mathcal{V}_l^{R,\mathcal{B}}$ est un sous-espace affine construit comme suit. Soit $\mathfrak{s} = (\mathfrak{g}_j)_{0\le j\le n}$ une suite d'idéaux de $\mathfrak{g}$ vérifiant

$$\{0\} = \mathfrak{g}_{n+1} \subset \mathfrak{g}_n \subset \ldots \subset \mathfrak{g}_1 = \mathfrak{g}$$

et $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ la base de Jordan-Hölder associée à $\mathfrak{s}$, c.à.d $Z_j \in \mathfrak{g}_j/\mathfrak{g}_{j+1}$ et $\mathfrak{g}_j = \{Z_j, \ldots, Z_n\}$. On note $\mathfrak{h}_j = \mathfrak{g}_j + \mathfrak{h}$, $j = 1, \ldots, n$ et $1 \le j_1 < \ldots < j_k \le n$ les indices tels que $\mathfrak{h}_j \ne \mathfrak{h}_{j+1}$. Ainsi on construit à partir de $\mathcal{Z}$ une base de Malcev $\mathcal{B} = \{X_1, \ldots, X_k\}$ de $\mathfrak{g}$ relative à $\mathfrak{h}$, dite extraite de la base $\mathcal{Z}$, telle que $X_i = Z_{j_i}$. On pose alors

$$\mathcal{V}_l^{R,\mathcal{B}} = \{l + \sum_{i=1}^{r} R_i(t)X_i^*, t \in \mathbb{R}^k\} \subset l + \mathfrak{h}^\perp \tag{2.4}$$

où $R = (R_1, ..., R_r)$ est une famille de fonctions affines sur $\mathbb{R}^k$, pour un certain $r \in \mathbb{N}$, qui seront décrites en détail dans (4.1). De plus $d\lambda_l^{R,\mathcal{B}}$ est la mesure de Lebesgue sur $\mathcal{V}_l^{R,\mathcal{B}}$. Dans leur même article, Baklouti et Ludwig ont construit un opérateur d'entrelacement pour cette équivalence, en choisissant pour tout $f \in \mathcal{V}_l^{R,\mathcal{B}}$ et pour sa polarisation de Vergne relative à $\mathcal{Z}$ $B_f = \exp \mathfrak{b}_f$, certaines bases de Malcev $\mathcal{Y}(f)$ de $\mathfrak{b}_f$ relative à $\mathfrak{b}_f \cap \mathfrak{h}$, qui varient rationnellement avec f, de sorte que l'opérateur

$$S_{\mathrm{Ind}} : \mathcal{S}(G/H, \chi_l) \longrightarrow \int_{\mathcal{V}_l^{R,\mathcal{B}}}^{\oplus} L^2(G/B_f, \chi_f) d\lambda_l^{R,\mathcal{B}}(f) \tag{2.5}$$

défini pour tout $\xi \in \mathcal{S}(G/H, \chi_l)$, $f \in \mathcal{V}_l^{R,\mathcal{B}}$ et $g \in G$ par

$$S_{\mathrm{Ind}}(\xi)(f)(g) = \int_{B_f/B_f \cap H} \xi(gb)\chi_f(b) d_{\mathcal{Y}(f)}(b)$$

se prolonge sur $L^2(G/H, \chi_l)$ en un opérateur d'entrelacement unitaire pour l'équivalence (2.3).

3 Désintégration de la représentation de conjugaison

Il s'agit de donner dans cette partie une désintégration en irréductibles de la représentation de conjugaison γ_G d'un groupe de Lie nilpotent connexe et simplement connexe G. Nous devons d'abord introduire certaines nouvelles représentations du groupe G.

Définition 3.1. Soit $l \in \mathfrak{g}^*$ et B_l une polarisation en l. On définit la représentation $(\pi_{l,-l})$ de G dans l'espace hilbertien $L^2(G/B_l \times G/B_l, \chi_l \times \chi_{-l})$ par

$$\pi_{l,-l}(g)\eta(x, y) = \eta(g^{-1}x, g^{-1}y), x, y, g \in G.$$

Définition 3.2. Soit $(\pi, \mathcal{H}_\pi)$ une représentation unitaire et irréductible de G et $A \in HS(\mathcal{H}_\pi)$, alors l'opérateur $\pi(g) \circ A \circ \pi(g^{-1})$ est aussi dans $HS(\mathcal{H}_\pi)$ pour $g \in G$. On définit alors la représentation $(C_\pi, HS(\mathcal{H}_\pi))$ de G par

$$C_\pi(g)A = \pi(g) \circ A \circ \pi(g^{-1}), \quad g \in G,\ A \in HS(\mathcal{H}_\pi).$$

En outre, la représentation $(C, L^2(\hat{G}))$ de G est définie comme étant l'intégrale hilbertienne

$$C := \int_{\hat{G}}^{\oplus} C_\pi d\mu(\pi).$$

Notre premier résultat s'énonce comme suit.

Lemme 3.3. *Soit $l \in \mathfrak{g}^*$ et B_l une polarisation de G en l. Alors pour $\pi_l = \pi_{l,B_l}$ on a que*

$$C_{\pi_l} \simeq \pi_{l,-l}.$$

Démonstration. Notons $HS_0(\mathcal{H}_{\pi_l}) = \{\pi_l(\xi), \xi \in \mathcal{S}(G)\}$, qui est un sous-espace dense de $HS(\mathcal{H}_{\pi_l})$. Nous savons d'après [5], que l'application linéaire

$$U : HS_0(\mathcal{H}_{\pi_l}) \;\to\; L^2(G/B_l \times G/B_l, \chi_l \times \chi_{-l})$$

définie par

$$U(\pi_l(\xi)) := K_{\pi_l(\xi)}, \quad \xi \in \mathcal{S}(G),$$

envoie $HS_0(\mathcal{H}_{\pi_l})$ sur l'espace $\mathcal{S}(G/B_l \times G/B_l, \chi_l \times \chi_{-l})$. Elle s'étend par continuité en un opérateur d'entrelacement unitaire entre C_{π_l} et $\pi_{l,-l}$. En effet, pour $\xi \in \mathcal{H}_{\pi_l}$, on a

$$\|U(\pi_l(\xi))\|^2_{L^2(G/B_l \times G/B_l, \chi_l \times \chi_{-l})} = \int_{G/B_l \times G/B_l} |K_{\pi_l(\xi)}(x,y)|^2 dxdy = \|\pi_l(\xi)\|^2_{HS(\mathcal{H}_{\pi_l})}.$$

D'autre part, pour $x, y \in G$ on a

$$\begin{aligned} U \circ C_{\pi_l}(g)(\pi_l(\xi))(x,y) &= K_{C_{\pi_l}(g)(\pi_l(\xi))}(x,y) = K_{\pi_l(\gamma_G(g)(\xi))}(x,y) = K_{\pi_l(\xi)}(g^{-1}x, g^{-1}y) \\ &= \pi_{l,-l}(g)(K_{\pi_l(\xi)})(x,y) = \pi_{l,-l}(g) \circ U(\pi_l(\xi))(x,y). \end{aligned}$$

□

Théorème 3.4. *Soit G un groupe de Lie nilpotent connexe et simplement connexe. Soit μ la mesure de Plancherel de $\hat{G}$. Alors*

$$\gamma_G \simeq \int_{\mathfrak{g}^*/G}^{\oplus} \pi_{l,-l} d\mu(\pi_l).$$

Démonstration. Il suffit de se rappeler que d'après le théorème de Plancherel, les espaces hilbertiens $L^2(G)$ et $L^2(\hat{G}) = \oint_{\hat{G}} HS(\mathcal{H}_\pi) d\mu(\pi)$ sont isométriquement isomorphes et la transformation de Fourier

$$F(\xi)(\pi) := \pi(\xi), \quad \xi \in L^2(G) \cap L^1(G),$$

nous donne un tel opérateur unitaire. En outre, F entrelace les représentations γ_G et C. En effet

$$F(\gamma_G(g)\xi)(\pi) = \pi(g)\pi(\xi)\pi(g^{-1}) = C_\pi(g)F(\xi)(\pi), \quad \xi \in L^2(G),\ g \in G.$$

Alors on a d'après le lemme 3.3

$$\gamma_G \simeq \int_{\hat{G}}^{\oplus} C_\pi d\mu(\pi) \simeq \int_{\mathfrak{g}^*/G}^{\oplus} \pi_{l,-l} d\mu(\pi_l).$$

□

L'étape suivante dans cette section consiste à écrire une formule de désintégration en irréductibles de la représentation de conjugaison. Commençons par prouver la proposition suivante.

Proposition 3.5. *On a :*

$$\gamma_G \simeq \int_{\mathfrak{g}^*/G}^{\oplus} \mathrm{Ind}_{B_l}^G(\chi_{l|B_l} \otimes \pi_{-l|B_l}) d\mu(\pi_l).$$

Démonstration. D'après le théorème 3.4, il suffit de montrer l'équivalence

$$\pi_{l,-l} \simeq \mathrm{Ind}_{B_l}^G(\chi_{l|B_l} \otimes \pi_{-l|B_l}),$$

où $l \in \mathfrak{g}^*$ et B_l une polarisation de G en l. On note $\nu_l := \mathrm{Ind}_{B_l}^G(\chi_{l|B_l} \otimes \pi_{-l|B_l})$, donc ν_l est une représentation de G qui agit sur l'espace hilbertien

$$\mathcal{H}_{\nu_l} := \left\{ \begin{array}{c} \psi : G \to L^2(G/B_l, \chi_{-l}), \psi \text{ mesurable}, \\ \psi(gb) = \chi_l(b)^{-1}\pi_{-l}(b)^{-1}(\psi(g)), g \in G, b \in B_l, \\ \int_{G/B_l} \|\psi(g)\|^2_{\mathcal{H}_{\pi_{(-l)}}} d\dot{x}(g) < \infty \end{array} \right\}.$$

Nous vérifions que si $\eta \in L^2(G/B_l \times G/B_l, \chi_l \times \chi_{-l})$, alors la fonction $y \mapsto \eta(x, xy) =: V(\eta)(x)(y)$ est dans $L^2(G/B_l, \chi_{-l})$ pour presque tout $x \in G$ et que

$$V(\eta)(xb)(y) = \eta(xb, xby) = \chi_l(b)^{-1}\pi_{-l}(b)^{-1}(V(\eta(x)))(y), \quad y \in G,\ b \in B_l.$$

On peut donc considérer l'opérateur

$$V : \mathcal{H}_{\pi_{l,-l}} \longrightarrow \mathcal{H}_{\nu_l}$$

défini par

$$V(\eta)(x)(y) = \eta(x, xy), \quad x, y \in G, \eta \in \mathcal{H}_{\pi_{l,-l}}.$$

Il est clair que V est une isométrie. D'autre part, on vérifie facilement que $V \circ \pi_{l,-l}(g) = \nu_l(g) \circ V$. En effet, pour $x, y \in G$, on a

$$\begin{aligned}(V \circ \pi_{l,-l}(g))\eta(x)(y) &= \pi_{l,-l}(g)\eta(x, xy) = \eta(g^{-1}x, g^{-1}xy) \\ &= V(\eta)(g^{-1}x, y) = (\nu_l(g) \circ V)\eta(x)(y).\end{aligned}$$

Par conséquent, les représentations $\pi_{l,-l}$ et ν_l sont équivalentes. Le théorème 3.4 permet de conclure.

□

En ce qui suit, on donne finalement la formule de la désintégration de γ_G en irréductibles.

Théorème 3.6. *Soient* $\mathfrak{g}^* \ni l \mapsto B_l = \exp \mathfrak{b}_l$ *un choix mesurable de polarisations en* l *dans* G *et soit* $d\gamma^l, l \in \mathfrak{g}^*$, *la mesure sur l'ensemble des double-classes* $B_l \diagdown G \diagup B_l =: G\diagup\diagup B_l$ *décrite dans 2.8. Pour* $g \in G$, *on note* $l(g) := l - \mathrm{Ad}^*_{\Phi(\tilde{g})}(l)$, $\mathcal{V}^{R,\mathcal{B}}_{l(g)}$ *le sous-espace affine de* $l(g) + (\mathfrak{b}_l \cap \mathrm{Ad}_{\Phi(\tilde{g})}\mathfrak{b}_l)^\perp$ *défini comme dans (2.4) et* $d\lambda^{R,\mathcal{B}}_{l(g)}$ *la mesure de Lebesgue associée. On a alors*

$$\gamma_G \simeq \int^{\oplus}_{\mathfrak{g}^*/G} \int^{\oplus}_{G\diagup\diagup B_l} \int^{\oplus}_{\mathcal{V}^{R,\mathcal{B}}_{l(g)}} \pi_{l(g)+R(t)} d\lambda^{R,\mathcal{B}}_{l(g)}(t) d\gamma^l(\tilde{g}) d\mu(\pi_l). \tag{3.1}$$

Démonstration. Pour tout $l \in \mathfrak{g}^*$ et B_l une polarisation en l dans G on a

$$\begin{aligned}
\mathrm{Ind}^G_{B_l}(\chi_{l_{|B_l}} \otimes \pi_{-l|B_l}) &\simeq \mathrm{Ind}^G_{B_l}(\chi_{l_{|B_l}} \otimes \int^{\oplus}_{G\diagup\diagup B_l} \mathrm{Ind}^{B_l}_{B_l\cap\Phi(\tilde{g})\cdot B_l\cdot\Phi(\tilde{g})^{-1}} \chi_{\mathrm{Ad}^*_{\Phi(\tilde{g})}(-l)} d\gamma^l(\tilde{g}))\\
&\simeq \mathrm{Ind}^G_{B_l} \int^{\oplus}_{G\diagup\diagup B_l} \mathrm{Ind}^{B_l}_{B_l\cap\Phi(\tilde{g})\cdot B_l\cdot\Phi(\tilde{g})^{-1}} \chi_{l(g)} d\gamma^l(\tilde{g})\\
&\simeq \int^{\oplus}_{G\diagup\diagup B_l} \mathrm{Ind}^{G}_{B_l\cap\Phi(\tilde{g})\cdot B_l\cdot\Phi(\tilde{g})^{-1}} \chi_{l(g)} d\gamma^l(\tilde{g})\\
&\simeq \int^{\oplus}_{G\diagup\diagup B_l} \int^{\oplus}_{\mathcal{V}^{R,\mathcal{B}}_{l(g)}} \pi_{l(g)+R(t)} d\lambda^{R,\mathcal{B}}_{l(g)}(t) d\gamma^l(\tilde{g}).
\end{aligned}$$

La proposition 3.5 nous permet de conclure. □

Corollaire 3.7. *Soit* G *un groupe de Lie nilpotent connexe et simplement connexe et* γ_G *la représentation de conjugaison associée, alors si* $L : \hat{G} \to \mathfrak{g}^*/G$ *désigne l'application réciproque de la bijection de Kirillov, on a*

$$L(\mathrm{supp}\gamma_G) \subset \{(\Omega_\pi - \Omega_\pi)/G, \pi \in \hat{G}\},$$

où Ω_π *est l'orbite coadjointe de* π.

Démonstration. Ce résultat découle immédiatement de la désintégration de la représentation γ_G décrite dans le théorème 3.6. En effet, pour tout $l \in \mathfrak{g}^*$ et $g \in G$ on a

$$\begin{aligned}
l - \mathrm{Ad}^*_g(l) + (\mathfrak{b}_l \cap \mathrm{Ad}_g\mathfrak{b}_l)^\perp &= l - \mathrm{Ad}^*_g(l) + (\mathfrak{b}_l)^\perp + (\mathrm{Ad}_g\mathfrak{b}_l)^\perp\\
&= (l + (\mathfrak{b}_l)^\perp) - (\mathrm{Ad}^*_g(l) + (\mathrm{Ad}_g\mathfrak{b}_l)^\perp) \subset \Omega_{\pi_l} - \Omega_{\pi_l}.
\end{aligned}$$

□

4 Espace de désintégration

Dans cette partie, on écrit une désintégration concrète de γ_G. Le lemme suivant est un outil indispensable dans la construction explicite de l'espace de désintégration.

Lemme 4.1. (voir [1]) *On considère une famille $\mathcal{F}(t) = \{V^1(t), \ldots, V^k(t)\}$ de k vecteurs de $\mathbb{R}^n$ qui dépendent polynomialement d'un paramètre $t \in \mathbb{R}^n$. Soit $r = \max\{\mathrm{rang}(\mathcal{F}(t)),\ t \in \mathbb{R}^n\}$ et soit $\{e_1, \ldots, e_n\}$ la base canonique de $\mathbb{R}^n$. On a alors le résultat suivant*

1. *$\mathcal{V}_r = \{t \in \mathbb{R}^n, \mathrm{rang}(\mathcal{F}(t)) = r\}$ est un ouvert de Zariski de $\mathbb{R}^n$.*
2. *Pour tout $i \in \{1, \ldots, n\}$, l'ensemble $\{t \in \mathbb{R}^n,\ e_i \notin \mathrm{vec}\{\mathcal{F}(t)\}\}$ ou bien son complémentaire, contient un ouvert de Zariski non vide de $\mathbb{R}^n$.*

Fixons une base de Jordan-Hölder $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ de $\mathfrak{g}$ et adoptons les notations de 2.7. On a vu que chaque $G-$orbite dans $\mathcal{E}$ rencontre $\mathfrak{g}_T^*$ en un seul point, en particulier $\mathcal{E} \cap \mathfrak{g}_T^*$ est un ouvert de Zariski non vide dans $\mathfrak{g}_T^*$. Soit q le cardinal de l'ensemble indice $T = \{T_1 < \cdots < T_q\}$. Identifions $\mathfrak{g}_T^*$ avec $\mathbb{R}^q$ à l'aide de l'application

$$w = (w_1, \ldots, w_q) \to \sum_{j=1}^{q} w_j Z_{T_j}^* =: l_w.$$

Ainsi il existe un ouvert de Zariski $\mathcal{O}$ de $\mathbb{R}^q$, tel que

$$\gamma_G \simeq \int_{\mathcal{O}}^{\oplus} \mathrm{Ind}_{B_{l_w}}^G \big((\chi_{l_w})_{|B_{l_w}} \otimes (\pi_{-l_w})_{|B_{l_w}}\big)|\mathrm{Pf}(l_w)|dw.$$

Dans la suite, on donnera une désintégration en irréductibles de $(\pi_{-l_w})_{|B_{l_w}}$ en se basant sur les résultats de Abdennadher et Ludwig [1]. Notons que l'existence des différents ouverts de Zariski dans cette construction est due au lemme 4.1. On note $B_w := B_{l_w}$ et $\mathfrak{b}_w := \mathfrak{b}_{l_w}$ la polarisation de Vergne dans $\mathfrak{g}$ en l_w relative à la base de Jordan-Hölder $\mathcal{Z}$. Soit $w \in \mathcal{O}$ et $i \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{b}_w}$, alors $Z_i \in \mathfrak{b}_w + \mathfrak{g}_{i+1}$. Il existe $Z_i(w) = Z_i + \sum_{k=i+1}^n a_{i,k}(w) Z_k \in \mathfrak{b}_w$ tels que $a_{i,k}(w)_{i+1 \leq k \leq n}$ soient les solutions du système d'équations linéaires donné par $\langle l_w, [Z_i(w), Z_k]\rangle = 0$ pour tout $k \in \{i, \ldots, n\}$. En réduisant si nécessaire $\mathcal{O}$ on peut supposer que pour tout w dans l'ouvert de Zariski $\mathcal{O}$ on a

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}_w} = \{j_1, ..., j_p\}.$$

D'autre part, d'après le lemme 4.1, il existe un ouvert de Zariski $\mathcal{W}$ de $\mathbb{R}^q \times G$ et un ensemble d'indices $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ tel que pour tout $w \in \mathbb{R}^q$ verifiant $(\{w\} \times G) \cap \mathcal{W} \neq \emptyset$,

$$\mathcal{I}(\mathfrak{b}_w, \mathfrak{b}_w) = \{i \in I^{\mathfrak{g}/\mathfrak{b}_w}; Z_i \notin \mathfrak{b}_w + \mathrm{Ad}_g \mathfrak{b}_w + \mathfrak{g}_{i+1}, \forall (w, g) \in \mathcal{W}\} = \{i_1, \ldots, i_d\}.$$

Notons aussi pour $w \in \mathbb{R}^q$, tel que $(\{w\} \times G) \cap \mathcal{W} \neq \emptyset$

$$\phi = \phi_w : \mathbb{R}^d \to G; \phi(s) = \prod_{k=1}^{d} \exp s_k Z_{i_k},$$

$$\mathcal{W}_w := \{g \in G; (w,g) \in \mathcal{W}\} \text{ et } \mathcal{V}_w = \phi_w^{-1}(\mathcal{W}_w).$$

On obtient alors

$$\pi_{l_w|B_w} \simeq \int_{\mathcal{V}_w}^{\oplus} \mathrm{Ind}_{B_w \cap \phi(s)\cdot B_w \cdot \phi(s)^{-1}}^{B_w} \chi_{\mathrm{Ad}^*_{\phi(s)}(l_w)} |F_w(s)| ds, \quad s \in \mathcal{V}_w$$

et d'après l'égalité (2.1), $F_w(s)$ est une fonction rationnelle en s et en w, donc elle est continue en (w,s) pour tout $(w,\phi(s))$ dans l'ouvert de Zariski $\mathcal{W}$ et qui détermine la mesure sur l'ensemble des double-classes.

Passons maintenant à la construction d'un opérateur d'entrelacement pour cette équivalence. On a vu que $\{Z_{j_1}, ..., Z_{j_p}\}$ est une base de Malcev de $\mathfrak{g}$ relative à $\mathfrak{b}_w$, pour tout $w \in \mathcal{O}$. De plus, si on note ${}^c\mathcal{I}^{\mathfrak{g}/\mathfrak{b}_w} = \{1, ..., n\} \backslash \mathcal{I}^{\mathfrak{g}/\mathfrak{b}_w} = \{\alpha_1, ..., \alpha_{n-p}\}$, alors on trouve une famille $\{Z_{\alpha_1}(w), ..., Z_{\alpha_{n-p}}(w)\}$ de vecteurs de $\mathfrak{g}$ qui forment une base de Jordan-Hölder de $\mathfrak{b}_w$ pour tous les w dans un ouvert de Zariski qu'on peut supposer être égal à $\mathcal{O}$, en réduisant si nécessaire $\mathcal{O}$, et qui varient rationnellement et continûment en $w \in \mathcal{O}$.

On considère maintenant pour tout (w,s) tel que $(w,\phi(s)) \in \mathcal{W}$ l'ensemble d'indices

$$\mathcal{I}^{\mathfrak{b}_w/\mathfrak{b}_w \cap \mathrm{Ad}_{\phi(s)}\mathfrak{b}_w} = \{\alpha_i,\ i \in \{1, ..., n-p\};\ Z_{\alpha_i}(w) \notin \mathfrak{b}_w \cap \mathrm{Ad}_{\phi(s)}\mathfrak{b}_w + (\mathfrak{b}_w)_{i+1}\},$$

avec $(\mathfrak{b}_w)_{i+1} = \mathrm{vec}\{Z_{\alpha_{i+1}}(w), ..., Z_{\alpha_{n-p}}(w)\},\ i = 1, ..., n-p$.
Comme précédement, en réduisant si nécessaire $\mathcal{W}$, on peut supposer qu'il existe une partie $\{\beta_1, ..., \beta_e\} \subset \{\alpha_1, ..., \alpha_{n-p}\}$ telle que

$$\mathcal{I}^{\mathfrak{b}_w/\mathfrak{b}_w \cap \mathrm{Ad}_{\phi(s)}\mathfrak{b}_w} = \{\beta_1, ..., \beta_e\} \quad \text{pour tout } (w,\phi(s)) \in \mathcal{W}.$$

Ainsi $\{Z_{\beta_1}(w), ..., Z_{\beta_e}(w)\}$ est une base de Malcev de $\mathfrak{b}_w$ relative à $\mathfrak{b}_w \cap \mathrm{Ad}_{\phi(s)}\mathfrak{b}_w$ pour tout $(w,\phi(s)) \in \mathcal{W}$. D'après l'équation (2.2), l'opérateur d'entrelacement pour la restriction est

$$S_{\mathrm{rest}}(w) : L^2(G/B_w, \chi_{l_w}) \to \int_{\mathcal{V}_w \subset \mathbb{R}^d}^{\oplus} L^2(\mathbb{R}^e, \chi_{\mathrm{Ad}^*_{\phi(s)}(l_w)}) F_w(s) ds,$$

défini par

$$\begin{aligned} &S_{\mathrm{rest}}(w)(\xi)(s_1, \ldots, s_d)(z_1, \ldots, z_e) \\ &\qquad = \xi(\exp z_1 Z_{\beta_1}(w) \cdots \exp z_e Z_{\beta_e}(w) \cdot \exp s_1 Z_{i_1} \cdots s_d Z_{i_d}). \end{aligned}$$

Il s'agit maintenant de désintégrer la représentation

$$\mathrm{Ind}_{B_w\cap\phi(s)\cdot B_w\cdot\phi(s)^{-1}}^{G}\chi_{l_w+\mathrm{Ad}^*_{\phi(s)}(-l_w)}$$

en se basant sur les résultats de Baklouti et Ludwig [2] dans le cadre de la désintégration concrète des représentations induites des groupes de Lie nilpotents. Soit $(w,\phi(s))\in\mathcal{W}$, on note

$$l_w(s) := l_w + \mathrm{Ad}^*_{\phi(s)}(-l_w), \quad \mathfrak{b}(w,s) := \mathfrak{b}_w \cap \mathrm{Ad}_{\phi(s)}\mathfrak{b}_w.$$

Nous voyons que $l_w(s)$ varie polynomialement en (w,s). Il existe donc un ouvert de Zariski de $\mathbb{R}^q\times G$, qu'on peut supposer être égal à $\mathcal{W}$, et un ensemble d'indices $\{\theta_1,...,\theta_k\}\subset\{1,...,n\}$, tel que $\mathcal{I}^{\mathfrak{g}/\mathfrak{b}(w,s)}=\{\theta_1,...,\theta_k\}$ pour tout $(w,\phi(s))\in\mathcal{W}$. On note pour tout $1\le j\le k$, $X_j=Z_{\theta_j}$. Ainsi $\mathcal{B}=\{X_1,...,X_k\}$ est une base de Malcev de $\mathfrak{g}$ relative à toutes les sous-algèbres de Lie $\mathfrak{b}(w,s)$. Ecrivons

$$\mathfrak{b}_i(w,s) := \mathfrak{b}(w,s) + \mathrm{vec}\,\{X_i,...,X_k\} \text{ et } B_i(w,s) := \exp\mathfrak{b}_i(w,s), \quad (w,\phi(s))\in\mathcal{W}.$$

On note pour tout $j\in\{1,\dots,k\}$, et $(w,\phi(s))\in\mathcal{W}$, $\Gamma_{w,s}:=l_w(s)+\mathfrak{b}(w,s)^{\perp}$ et $d_j(w,s)$ le maximum des dimensions des $B_j(w,s)$-orbites dans $l_w(s)_{|\mathfrak{b}_j(w,s)}+\mathfrak{b}(w,s)^{\perp}\subset[\mathfrak{b}_j(w,s)]^*$.
Cette dimension maximale est donnée par le rang d'une matrice qui varie polynomialement en w et s. Soit

$$d_j=\max_{(w,\phi(s))\in\mathcal{W}} d_j(w,s),\ d_0=0 \text{ et } \ \mathcal{L}=\{j\in\{1,...,k\}; d_j=d_{j-1}\}=\{\lambda_1,...,\lambda_r\}.$$

Alors, il existe un ouvert de Zariski, qu'on peut supposer être égal à $\mathcal{W}$, tel que

$$\forall(w,\phi(s))\in\mathcal{W}, \quad d_j(w,s)=d_j, \quad j=1,...,k.$$

Soit

$$R=\{R_1,\dots,R_k\}:\mathbb{R}^r\to\mathbb{R}^k \tag{4.1}$$

la famille de fonctions affines définies sur $\mathbb{R}^r$ de la façon suivante. Soit $t=(t_1,...,t_r)\in\mathbb{R}^r$,

$$\begin{aligned} R_j(t) &= t_i,\ \text{si } j=\lambda_i\in\mathcal{L},\\ &= 0,\ \text{si } j\notin\mathcal{L}. \end{aligned}$$

L'espace de désintégration de $\mathrm{Ind}_{B(w,s)}^{G}\chi_{l_w(s)}$ est donné par

$$\mathcal{V}_{l_w(s)}^{R,\mathcal{B}}=\{l_w(s,t); t\in\mathbb{R}^r\}\subset\Gamma_{w,s}$$

où

$$l_w(s,t) := l_w(s) + \sum_{j=1}^{k} R_j(t)X_j^*.$$

Ainsi pour tout $(w,\phi(s)) \in \mathcal{W}$, on trouve la désintégration suivante

$$\mathrm{Ind}_{B(w,s)}^{G}\chi_{l_w(s)} \simeq \int_{\mathbb{R}^r}^{\oplus} \pi_{l_w(s,t)} dt, \tag{4.2}$$

avec $\pi_{l_w(s,t)} = \mathrm{Ind}_{B(w,s,t)}^{G}\chi_{l_w(s,t)}$, $B(w,s,t) = \exp \mathfrak{b}(w,s,t)$ et $\mathfrak{b}(w,s,t)$ est la polarisation de Vergne en $l_w(s,t)$ dans $\mathfrak{g}$ relative à la base de Jordan-Hölder $\mathcal{Z}$.

Pour la construction d'un opérateur d'entrelacement pour cette équivalence, on se propose tout d'abord de construire une base de Malcev de $\mathfrak{b}(w,s,t)$ relative à $\mathfrak{b}(w,s,t) \cap \mathfrak{b}(w,s)$. On note comme avant, pour $(w,\phi(s),t) \in \mathcal{W} \times \mathbb{R}^r$

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}(w,s,t)} = \{i \in \{1,...,n\}, Z_i \notin \mathfrak{b}(w,s,t) + \mathfrak{g}_{i+1}\}.$$

De nouveau, on peut supposer qu'il existe un ouvert de Zariski $\mathcal{D}$ de $\mathcal{W} \times \mathbb{R}^r$ et un ensemble indice $\{m_1,...,m_v\} \subset \{1,...,n\}$ tel que pour tout $(w,\phi(s),t) \in \mathcal{D}$ on ait

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}(w,s,t)} = \{\kappa_1,...,\kappa_v\}.$$

On ordonne aussi l'ensemble indice

$${}^{c}\mathcal{I}^{\mathfrak{g}/\mathfrak{b}(w,s,t)} = \{1,...,n\} \setminus \mathcal{I}^{\mathfrak{g}/\mathfrak{b}(w,s,t)} = \{\zeta_1,...,\zeta_{n-v}\}$$

Ainsi $\{Z_{\kappa_1},...,Z_{\kappa_v}\}$ est une base de Malcev de $\mathfrak{g}$ relative à $\mathfrak{b}(w,s,t)$ pour tout $(w,\phi(s),t) \in \mathcal{D}$. D'autre part, pour $(w,\phi(s),t) \in \mathcal{D}$, on prend une base de Jordan-Hölder de $\mathfrak{b}(w,s,t)$

$$Z_{\zeta_i}(w,s,t) = Z_{\zeta_i} + \sum_{j=\zeta_i+1}^{n} a'_{i,j}(w,s,t) Z_j, \quad i = 1,...,n-v$$

qui varie rationnellement et même, quitte à réduire $\mathcal{D}$, continûment sur $\mathcal{D}$ et on pose

$$(\mathfrak{b}(w,s,t))_i = \mathrm{vec}\{Z_{\zeta_i}(w,s,t),...,Z_{\zeta_{n-v}}(w,s,t)\}, \quad i = 1,...,n-v,$$

et aussi

$$\begin{aligned}&\mathcal{I}^{\mathfrak{b}(w,s,t)/\mathfrak{b}(w,s,t)\cap\mathfrak{b}(w,s)}\\ &\quad = \{\zeta_i, i \in \{1,...,n-v\}; Z_{\zeta_i}(w,s,t) \notin \mathfrak{b}(w,s,t) \cap \mathfrak{b}(w,s) + (\mathfrak{b}(w,s,t))_{i+1}\}.\end{aligned}$$

On peut supposer de nouveau que pour tout $(w,\phi(s),t) \in \mathcal{D}$, on a

$$\mathcal{I}^{\mathfrak{b}(w,s,t)/\mathfrak{b}(w,s,t)\cap\mathfrak{b}(w,s)} = \{\vartheta_1,...,\vartheta_u\}.$$

Ainsi $\{Z_{\vartheta_1}(w,s,t),...,Z_{\vartheta_u}(w,s,t)\}$ est une base de Malcev de $\mathfrak{b}(w,s,t)$ relative à $\mathfrak{b}(w,s,t) \cap \mathfrak{b}(w,s)$ pour tout $(w,\phi(s),t) \in \mathcal{D}$.

Soit $b = (b_1, ..., b_u) \in \mathbb{R}^u$, on note

$$\mathrm{Exp}_{\mathfrak{b}(w,s,t)\cap\mathfrak{b}(w,s)}^{\mathfrak{b}(w,s,t)}(b) = \exp(b_1 Z_{\vartheta_1}(w,s,t)) \cdots \exp(b_u Z_{\vartheta_u}(w,s,t)).$$

D'autre part, d'après [2] et 2.10, il existe pour (w, s) fixé, un choix continu en t de bases de Malcev $\mathcal{Y}(w, s, t)$ de $\mathfrak{b}(w, s, t)$ relative à $\mathfrak{b}(w, s, t) \cap \mathfrak{b}(w, s)$, tel que l'opérateur

$$S_{\mathrm{Ind}}(w,s) : \mathcal{S}(G/B(w,s), \chi_{l_w(s)}) \to \int_{\mathbb{R}^r}^{\oplus} \mathcal{H}_{\pi_{l_w(s,t)}} dt,$$

$$S_{\mathrm{Ind}}(w,s)\xi(t)(g) = \int_{B(w,s,t)/B(w,s,t)\cap B(w,s)} \xi(gb)\chi_{l_w(s,t)}(b) d_{\mathcal{Y}(w,s,t)}(b), g \in G,$$

réalise une isométrie qui se prolonge en un opérateur d'entrelacement unitaire pour l'équivalence (4.2). En fait les bases construites dans l'article [2] varient de façon rationnelle en t et il est facile de vérifier que cette variation est aussi rationnelle en (w, s). Donc finalement les bases $\mathcal{Y}(w, s, t)$ sont continues en (w, s, t) pour tout $(w, \phi(s), t)$ dans un ouvert de Zariski, qu'on peut supposer être égal à $\mathcal{D}$.

Si on remplace la base $\mathcal{Y}(w, s, t)$ par une autre base $\mathcal{Y}'(w, s, t)$, qui varie continûment en w, s et t, alors on obtient une fonction continue $C(w, s, t)$ strictement positive, telle que

$$d_{\mathcal{Y}(w,s,t)} = C(w,s,t) d_{\mathcal{Y}'(w,s,t)},$$

Donc pour notre choix de la base $\mathcal{Y}'(w, s, t) = \{Z_{\vartheta_1}(w, s, t), ..., Z_{\vartheta_u}(w, s, t)\}$ il existe une fonction strictement positive continue $C(w, s, t)$, pour laquelle l'opérateur d'entrelacement

$$S_{\mathrm{Ind}}(w,s) : \mathcal{S}\big(G/B(w,s), \chi_{l_w(s)}\big) \longrightarrow \int_{\mathbb{R}^r}^{\oplus} L^2\big(G/B(w,s,t), \chi_{l_w(s,t)}\big) C(w,s,t) dt,$$

défini par

$$\begin{aligned} &S_{\mathrm{Ind}}(w,s)(\xi)(t)(g) \\ &\quad = \int_{\mathbb{R}^u} \xi\big(g.Exp_{\mathfrak{b}(w,s,t)\cap\mathfrak{b}(w,s)}^{\mathfrak{b}(w,s,t)}(b)\big)\chi_{l_w(s,t)}(Exp_{\mathfrak{b}(w,s,t)\cap\mathfrak{b}(w,s)}^{\mathfrak{b}(w,s,t)}(b)) d_{\mathcal{Y}'(w,s,t)}(b), \quad g \in G, \end{aligned}$$

réalise une isométrie qui se prolonge en opérateur unitaire pour l'équivalence (4.2).

5 Opérateur d'entrelacement

On rassemble maintenant les résultats des sections précédentes pour obtenir la forme finale de notre opérateur d'entrelacement concret pour l'équivalence (3.1).

Théorème 5.1. *Avec les notations plus haut, pour tout $(w, \phi(s), t) \in \mathcal{D}$, il existe une base de Malcev $\mathcal{Y}(w, s, t)$ de $\mathfrak{b}(w, s, t)$ relative à $\mathfrak{b}(w, s, t) \cap \mathfrak{b}(w, s)$, pour laquelle l'opérateur défini par*

$$S : \mathcal{S}(G) \longrightarrow \int_{\mathbb{R}^q}^{\oplus} \int_{\mathbb{R}^d}^{\oplus} \int_{\mathbb{R}^r}^{\oplus} L^2(G/B(l_w(s,t)), \chi_{l_w(s,t)}) C(w,s,t)|F_w(s)|\,|\mathrm{Pf}(l_w)| dtdsdw,$$

tel que :

$$\begin{aligned} &S(\xi)(w,s,t)(x) \\ &\quad = \int_{B(w,s,t)/B(w,s,t)\cap B(w,s)} K_{\pi_{l_w}(\xi)}(xb, xb\phi(s))\chi_{l_w(s,t)}(b) d_{\mathcal{Y}(w,s,t)}(b) \\ &\quad = \int_{B(w,s,t)/B(w,s,t)\cap B(w,s)} \int_{B_w} \xi(xbb'\phi(s)^{-1}b^{-1}x^{-1})\chi_{l_w}(b')db'\chi_{l_w(s,t)}(b) d_{\mathcal{Y}(w,s,t)}(b). \end{aligned}$$

est une isométrie qui s'étend en un opérateur unitaire qui entrelace γ_G et sa désintégration en irréductiles.

6 Exemples

Exemple 6.1. Soit $\mathfrak{g} = \mathrm{vec}\{X, Y_1, ..., Y_n, Z\}$ l'algèbre de Lie filiforme de dimension $n+2$ tel que $n \geq 1$ et de crochets non nuls

$$[X, Y_i] = Y_{i+1}, \quad i \in \{1, ..., n-1\}, \quad [X, Y_n] = Z.$$

On peut identifier le groupe de Lie $G = \exp \mathfrak{g}$ avec le produit semi-direct $\exp \mathbb{R}X \ltimes \exp \mathfrak{a}$, avec $\mathfrak{a} = \mathrm{vec}\{Y_1, ..., Y_n, Z\}$. La représentation de conjugaison γ_G est donnée, pour tout $(x, a), (t, b) \in G$ et $\xi \in L^1(G)$, par

$$\begin{aligned} \gamma_G(t,b)\xi(x,a) &= \xi((t,b)^{-1} \cdot (x,a) \cdot (t,b)) = \xi(b^{-1} \cdot t^{-1} \cdot x \cdot a \cdot t \cdot b) \\ &= \xi(x \cdot x^{-1} \cdot b^{-1} \cdot x \cdot t^{-1} \cdot a \cdot t \cdot b) = \xi(x, b - \mathrm{Ad}_{x^{-1}} b + \mathrm{Ad}_{t^{-1}} a). \end{aligned}$$

Dans cet exemple on écrit dans une première étape une désintégration en irréductibles de la represéntation $\pi_{l,-l}$, $l \in \mathfrak{g}^*$ et on donne l'opérateur d'entrelacement associé. Ensuite, on en déduit une désintégration en irréductibles de γ_G. Soit $l \in \mathfrak{g}^*$ satisfaisant $l(Z) \neq 0$, on montre que

$$\pi_{l,-l} \simeq \int_{\mathbb{R}}^{\oplus} \pi_{(\mathrm{I}-\mathrm{Ad}^*_{\exp sX})(l)} ds =: \rho_l. \tag{6.1}$$

Soit $\mathcal{Z} = \{X, Y_1, ..., Y_n, Z\}$ une base de Jordan-Hölder de $\mathfrak{g}$ et $\mathfrak{b}_l = \mathrm{vec}\{Y_1, ..., Y_n, Z\}$ la polarisation de Vergne relative à $\mathcal{Z}$ en tout $l \in \mathfrak{g}^*$ telle que $l(Z) \neq 0$ et $B_l = \exp \mathfrak{b}_l$. On montre que l'opérateur

$$S_1 : \mathcal{S}(G/B_l \times G/B_l, \chi_l \times \chi_{-l}) \to \int_{\mathbb{R}}^{\oplus} \mathcal{H}_{\pi_{(\mathrm{I}-\mathrm{Ad}^*_{\exp sX})(l)}} ds$$

défini pour $\eta \in \mathcal{S}(G/B_l \times G/B_l, \chi_l \times \chi_{-l})$ par

$$S_1(\eta)(s)(\exp uX) = \eta(\exp uX, \exp(u+s)X), \quad u, s \in \mathbb{R}$$

se prolonge par continuité sur $\mathcal{H}_{\pi_{l,-l}}$ en un opérateur qui entrelace les représentations de l'équivalence (6.1). Dans la suite on note $\mathrm{e}^{uX} := \exp uX$ et

$$\delta_u(a) := \exp uX \cdot a \cdot \exp(-uX).$$

Soit $l \in \mathfrak{g}^*$ et $g = \mathrm{e}^{wX}.a$ tels que $w \in \mathbb{R}$ et $a = \exp A \in \exp \mathfrak{a}$. On a

$$\begin{aligned}
(S_1 \circ \pi_{l,-l}(g))(\eta)(s)(\mathrm{e}^{uX}) &= \pi_{l,-l}(g)(\eta)(\mathrm{e}^{uX}, \mathrm{e}^{(u+s)X}) = \eta(g^{-1}\mathrm{e}^{uX}, g^{-1}\mathrm{e}^{(u+s)X}) \\
&= \eta(a^{-1}\mathrm{e}^{(u-w)X}, a^{-1}\mathrm{e}^{(u-w+s)X}) \\
&= \eta(\mathrm{e}^{(u-w)X}\delta_{w-u}(a^{-1}), \mathrm{e}^{(u+s-w)X}\delta_{w-u-s}(a^{-1})) \\
&= \chi_l(\delta_{(w-u)}(a))\chi_{-l}(\delta_{(w-u-s)}(a))\eta(\mathrm{e}^{(u-w)X}, \mathrm{e}^{(u+s-w)X}) \\
&= \mathrm{e}^{-i<l,\mathrm{Ad}_{\mathrm{e}^{(w-u)X}}A>}\mathrm{e}^{i<l,\mathrm{Ad}_{\mathrm{e}^{(w-u-s)X}}A>}\eta(\mathrm{e}^{(u-w)X}, \mathrm{e}^{(u+s-w)X}) \\
&= \mathrm{e}^{-i<(\mathrm{I}-\mathrm{Ad}^*_{\mathrm{e}^{sX}})(l),\mathrm{Ad}_{\mathrm{e}^{(w-u)X}}A>}\eta(\mathrm{e}^{(u-w)X}, \mathrm{e}^{(u-w+s)X}) \\
&= \chi_{(\mathrm{I}-\mathrm{Ad}^*_{\mathrm{e}^{sX}})(l)}(\delta_{(w-u)}(a))(S_1(\eta))(s)(\mathrm{e}^{(u-w)X}) \\
&= (S_1(\eta))(s)(g^{-1}\mathrm{e}^{uX}) = \pi_{(\mathrm{I}-\mathrm{Ad}^*_{\mathrm{e}^{sX}})(l)}(g)(S_1(\eta))(s)(\mathrm{e}^{uX}) \\
&= (\rho_l(g) \circ S_1)(\eta)(s)(\mathrm{e}^{uX}).
\end{aligned}$$

D'autre part on a

$$\begin{aligned}
\|S_1(\eta)\|^2_{\mathcal{H}_{\rho_l}} &= \int_{\mathbb{R}} \|S_1(\eta)(s)\|^2_{\mathcal{H}_{\pi_{(\mathrm{I}-\mathrm{Ad}^*_{\mathrm{e}^{sX}})(l)}}} ds = \int_{\mathbb{R}}\int_{\mathbb{R}} \|S_1(\eta)(s)(\mathrm{e}^{uX})\|^2 duds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \|\eta(\mathrm{e}^{uX}, \mathrm{e}^{(u+s)X})\|^2 duds = \|\eta\|^2_{L^2(G/B_l \times G/B_l, \chi_l \times \chi_{-l})}.
\end{aligned}$$

Par suite, d'après (3.4), on a

$$\gamma_G \simeq \int_{\mathfrak{g}^*/G}^{\oplus} \pi_{l,-l} d\mu(\pi_l) \simeq \int_{\mathbb{R}^{n-1}\times\mathbb{R}^*}^{\oplus} \int_{\mathbb{R}}^{\oplus} \pi_{(\mathrm{I}-\mathrm{Ad}^*_{\mathrm{e}^{sX}})l_{(y,z)}} |z| ds dy_1 \dots dy_{n-1} dz,$$

avec $l_{(y,z)} = \sum_{i=1}^{n-1} y_i Y_i^* + zZ^*$. Posons $y_n = 0$, alors on a

$$\mathrm{Ad}^*_{\mathrm{e}^{sX}} l_{(y,z)} = \sum_{j=1}^{n} \Big(\sum_{i=j}^{n} \frac{(-s)^{i-j}}{(i-j)!} y_i + \frac{(-s)^{n-j+1}}{(n-j+1)!} z \Big) Y_j^* + zZ^*.$$

Par conséquent

$$l_{(y,z)}(s) := (\mathrm{I} - \mathrm{Ad}^*_{\mathrm{e}^{sX}})l_{(y,z)} = \sum_{j=1}^{n-1}\Big(\sum_{i=j+1}^{n} -\frac{(-s)^{i-j}}{(i-j)!}y_i - \frac{(-s)^{n-j+1}}{(n-j+1)!}z\Big)Y_j^*.$$

Ainsi on trouve

$$\gamma_G \simeq \int_{\mathbb{R}^{n-1}\times\mathbb{R}^*}^{\oplus}\int_{\mathbb{R}}^{\oplus} \pi_{l_{(y,z)}(s)}|z|ds\ dy\ dz.$$

L'opérateur d'entrelacement est donné par

$$S : \mathcal{S}(G) \to \int_{\mathbb{R}^{n-1}\times\mathbb{R}^*}^{\oplus}\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}, \chi_{l_{(y,z)}(s)})|z|ds\ dy\ dz,$$

tel que

$$\begin{aligned} S(\xi)(y,z)(s)(x) &= S_1(K_{\pi_{l_{(y,z)}(s)}(\xi)})(s)(x) = K_{\pi_{l_{(y,z)}(s)}(\xi)}(\mathrm{e}^{xX}, \mathrm{e}^{(x+s)X}) \\ &= \int_{\mathbb{R}^n\times\mathbb{R}} \xi(\mathrm{e}^{xX}\mathrm{e}^{a_1Y_1}\ldots\mathrm{e}^{a_nY_n}\mathrm{e}^{tZ}\mathrm{e}^{-(x+s)X})\chi_{l_{(y,z)}(s)}(\mathrm{e}^{a_1Y_1}\ldots\mathrm{e}^{a_nY_n}\mathrm{e}^{tZ})\ da_1\ldots da_n\ dt \\ &= \int_{\mathbb{R}^n\times\mathbb{R}} \xi\big(\mathrm{e}^{-sX}\cdot\prod_{k=1}^{n}\exp(\sum_{j=0}^{k-1}\frac{(x+s)^j}{j!}a_{k-j}\big)Y_k\cdot\exp\big(t+\sum_{j=1}^{n}\frac{(x+s)^j}{j!}a_{n-j+1})Z\big) \\ &\quad \mathrm{e}^{-i\Big(\sum_{j=1}^{n-1}(\sum_{i=j+1}^{n} -\frac{(-s)^{i-j}}{(i-j)!}y_i - \frac{(-s)^{n-j+1}}{(n-j+1)!}z)a_j + tz\Big)} da_1\ldots da_n dt. \end{aligned}$$

Exemple 6.2. Soit $\mathfrak{g}$ l'algèbre de Lie de dimension 6 engendrée par les vecteurs $\{A,B,C,U,V,Z\}$ et de crochets non nuls

$$[A,B] = Z, [B,C] = U \text{ et } [C,A] = V.$$

D'après le théorème (3.4), on a

$$\gamma_G \simeq \int_{(\mathbb{R}\times\mathbb{R}^*)^2}^{\oplus} \pi_{l_{(a,u,v,z)},-l_{(a,u,v,z)}}|u|da\ du\ dv\ dz$$

avec $l_{(a,u,v,z)} = aA^* + uU^* + vV^* + zZ^*$. La polarisation en $l_{(a,u,v,z)}$ dans $\mathfrak{g}$ relative à la base de Jordan-Hölder $\{A,B,C,U,V,Z\}$ est $\mathfrak{b}_{(a,u,v,z)} = \mathrm{vec}\{A + \frac{v}{u}B, C, U, V, Z\}$ qui est un idéal de $\mathfrak{g}$. Ainsi

$$\gamma_G \simeq \int_{(\mathbb{R}\times\mathbb{R}^*)^2}^{\oplus}\int_{\mathbb{R}}^{\oplus} \mathrm{Ind}_{B_{(a,u,v,z)}}^{G}\chi_{l_{(a,u,v,z)}(s)}|u|ds\ da\ du\ dv\ dz$$

avec $l_{(a,u,v,z)}(s) = (\mathrm{I} - \mathrm{Ad}^*{}_{\exp sB})l_{(a,u,v,z)} = -szA^* + suC^*$ et $u, z \in \mathbb{R}^*$. Par suite

$$\gamma_G \simeq \int_{(\mathbb{R}\times\mathbb{R}^*)^2}^{\oplus}\Big(\int_{\mathbb{R}^2}^{\oplus}\chi_{-szA^*+suC^*+t(B^*-\frac{v}{u}A^*)}|u|dt\ ds\Big)da\ du\ dv\ dz. \tag{6.2}$$

D'après (5.1), l'opérateur d'entrelacement pour cette désintégration est donné, pour tout $\xi \in \mathcal{S}(G)$ et $(a,u,v,z,s,t) \in \mathbb{R} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^2$, par

$$S(\xi)(a,u,v,z)(s,t) = \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{bB}, \mathrm{e}^{(b+s)B})\mathrm{e}^{-itb} db.$$

Dans la suite on vérifie avec un calcul explicite que S entrelace les représentations de l'équivalence (6.2).

$$\begin{aligned}
\|S(\xi)\|_2^2 &= \int_{(\mathbb{R}\times\mathbb{R}^*)^2\times\mathbb{R}^2} \Big| \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{bB}, \mathrm{e}^{(b+s)B})\mathrm{e}^{-itb} db\Big|^2 \ |u| dt\ ds\ da\ du\ dv\ dz \\
&= \int_{(\mathbb{R}\times\mathbb{R}^*)^2} \int_{\mathbb{R}^2} |K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{tB}, \mathrm{e}^{sB})|^2 \ |u| dt\ ds\ da\ du\ dv\ dz \\
&= \int_{(\mathbb{R}\times\mathbb{R}^*)^2} \|K_{\pi_{(a,u,v,z)}(\xi)}\|^2_{L^2(G/B_{(a,u,v,z)}\times G/B_{(a,u,v,z)})} |u|\ da\ du\ dv\ dz \\
&= \int_{(\mathbb{R}\times\mathbb{R}^*)^2} \|\pi_{(a,u,v,z)}(\xi)\|^2_{HS(\mathcal{H}_{\pi_{(a,u,v,z)}})} |u|\ da\ du\ dv\ dz \\
&= \|\xi\|^2_{L^2(G)}.
\end{aligned}$$

D'autre part pour tout $g = \mathrm{e}^{\beta B} h \in G$ tel que $\beta \in \mathbb{R},\ h = \mathrm{e}^{\alpha(A+\frac{v}{u}B)+\lambda C} \mathrm{mod}\ Z(G)$, on note $\delta_\beta(h) = \mathrm{e}^{\beta B} h \mathrm{e}^{-\beta B}$. Alors on a

$$\begin{aligned}
(S\circ\gamma_G(g))(\xi)(a,u,v,z)(s,t) &= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\gamma_G(g)\xi)}(\mathrm{e}^{bB}, \mathrm{e}^{(b+s)B})\mathrm{e}^{-itb} db \\
&= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(g^{-1}\mathrm{e}^{bB}, g^{-1}\mathrm{e}^{(b+s)B})\mathrm{e}^{-itb} db \\
&= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(h^{-1}\mathrm{e}^{(b-\beta)B}, h^{-1}\mathrm{e}^{(b-\beta+s)B})\mathrm{e}^{-itb} db \\
&= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{bB}\delta_{-b}(h^{-1}), \mathrm{e}^{(b+s)B}\delta_{-(b+s)}(h^{-1})) \\
&\qquad \cdot \mathrm{e}^{-itb} db\ \mathrm{e}^{-it\beta} \\
&= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{bB}, \mathrm{e}^{(b+s)B})\chi_{l_{(a,u,v,z)}}(\delta_{-b}(h)) \\
&\qquad \cdot \chi_{l_{(a,u,v,z)}}(\delta_{-(b+s)}(h^{-1}))\mathrm{e}^{-itb} db\ \mathrm{e}^{-it\beta} \\
&= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{bB}, \mathrm{e}^{(b+s)B})\mathrm{e}^{-itb} db\ \mathrm{e}^{-it\beta}\mathrm{e}^{-i(-s\alpha z+su\lambda)} \\
&= \int_{\mathbb{R}} K_{\pi_{(a,u,v,z)}(\xi)}(\mathrm{e}^{bB}, \mathrm{e}^{(b+s)B})\mathrm{e}^{-itb} db \\
&\qquad \cdot \chi_{-szA^*+suC^*+tB^*}(\mathrm{e}^{\beta B}\mathrm{e}^{\alpha A+\lambda C})
\end{aligned}$$

$$= S(\xi)(a,u,v,z)(s,t)\chi_{-szA^*+suC^*+t(B^*-\frac{v}{u}A^*)}(g).$$

D'où le résultat.

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