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Algebraic Aspects of the Hirzebruch Signature Operator and Applications to Transitive Lie Algebroids

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Abstract. The index of the classical Hirzebruch signature operator on a manifold M is equal to the signature of the manifold. The examples of Lusztig ([10], 1972) and Gromov ([4], 1985) present the Hirzebruch signature operator for the cohomology (of a manifold) with coefficients in a flat symmetric or symplectic vector bundle. In [6], we gave a signature operator for the cohomology of transitive Lie algebroids.

In this paper, firstly, we present a general approach to the signature operator, and the above four examples become special cases of a single general theorem.

Secondly, due to the spectral sequence point of view on the signature of the cohomology algebra of certain filtered DG-algebras, it turns out that the Lusztig and Gromov examples are important in the study of the signature of a Lie algebroid. Namely, under some natural and simple regularity assumptions on the DG-algebra with a decreasing filtration for which the second term lives in a finite rectangle, the signature of the second term of the spectral sequence is equal to the signature of the DG algebra. Considering the Hirzebruch–Serre spectral sequence for a transitive Lie algebroid A over a compact oriented manifold for which the top group of the real cohomology of A is nontrivial, we see that the second term is just identical to the Lusztig or Gromov example (depending on the dimension). Thus, we have a second signature operator for Lie algebroids.

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1. PRELIMINARIES CONCERNING LIE ALGEBROIDS, AND THE SIGNATURE OF TRANSITIVE LIE ALGEBROIDS

The Lie algebroids arose as infinitesimal objects associated with Lie groupoids, principal fiber bundles, vector bundles (Pradines, 1967), TC-foliations and nonclosed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalents are known as Lie pseudo-algebras (Herz, 1953), which are also referred to as Lie–Rinehart algebras (Huebschmann, 1990).

A *Lie algebroid* on a manifold M is a triple $A=(A,[\![\cdot\,,\cdot]\!],\#_A)$, where A is a vector bundle on M, $(\operatorname{Sec}A,[\![\cdot\,,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, $\#_A : A\to TM$ is a linear homomorphism (the so-called *anchor*) of vector bundles, and the Leibniz condition $[\![\xi,f\cdot\eta]\!]=f\cdot[\![\xi,\eta]\!]+\#_A(\xi)(f)\cdot\eta$ (where $f\in C^\infty(M)$ and $\xi,\eta\in\operatorname{Sec}A$) is satisfied.

The anchor is bracket-preserving, $\#_A\circ[\![\xi,\eta]\!]=[\#_A\circ\xi,\#_A\circ\eta]$. A Lie algebroid is said to be *transitive* if the anchor $\#_A$ is an epimorphism. For a transitive Lie algebroid A,

(1) the Atiyah sequence holds,

$$0\longrightarrow \mathbf{g}\hookrightarrow A\xrightarrow{\#_A} TM\longrightarrow 0,\qquad \mathbf{g}:=\ker\#_A, \tag{1}$$

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(2) the fiber $\mathbf{g}_x$ of the bundle $\mathbf{g}$ at the point $x \in M$ is the Lie algebra (the so-called isotropy Lie algebra of A at $x \in M$) with the commutator operation $[v,w] = [\![\xi,\eta]\!](x)$, $\xi,\eta \in \operatorname{Sec} A$, $\xi(x) = v, \eta(x) = w$, $v,w \in \mathbf{g}_x$,
(3) the vector bundle $\mathbf{g}$ is a Lie algebra bundle (briefly, LAB), the so-called *adjoint* of A, and the fibers are isomorphic Lie algebras.

The tangent bundles to manifolds and finite-dimensional Lie algebras are simple examples of transitive Lie algebroids.

To an arbitrary (transitive or not) Lie algebroid A we assign the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \operatorname{Sec} \bigwedge A^* \text{ is the space of cross-sections of } \bigwedge A^*, \quad d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A),$$

$$\begin{aligned}(d_A\omega)(\xi_0,\dots,\xi_k) &= \sum_{j=0}^{k}(-1)^j\,(\#_A \circ \xi_j)\,(\omega(\xi_0,\dots,\hat{\jmath},\dots,\xi_k)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\,\omega([\![\xi_i,\xi_j]\!],\xi_0,\dots,\hat{\imath},\dots,\hat{\jmath},\dots,\xi_k),\end{aligned} \tag{2}$$

$\omega \in \Omega^k(A)$, and $\xi_i \in \operatorname{Sec} A$. The operators d_A^k satisfy the condition $d_A(\omega\wedge\eta) = d_A\omega\wedge\eta + (-1)^k\omega\wedge d_A\eta$; thus, they are of first order, and the symbol of d_A^k is equal to $S\left(d_A^k\right)_{(x,v)} : \bigwedge^k A_x^* \to \bigwedge^{k+1} A_x^*$, where $\left(d_A^k\right)_{(x,v)}(u) = (v\circ(\#_A)_x)\wedge u$, $0 \neq v \in T_x^*M$. This implies the following assertion.

Proposition 1.1. *The sequence* $\bigwedge^k A_x^* \xrightarrow{S(d_A^k)_{(x,v)}} \bigwedge^{k+1} A_x^* \xrightarrow{S(d_A^{k+1})_{(x,v)}} \bigwedge^{k+2} A_x^*$ *is exact if and only if A is transitive. Therefore, the complex* $\left\{d_A^k\right\}$ *is* elliptic *provided that A is transitive.*

Proof. The composition is zero. If $0 \neq v \in T_x^*M$ and A is transitive, then $\tilde{v} = (v\circ(\#_A)_x) \neq 0$ and $S\left(d_A^{k+1}\right)_{(x.v)} = \rho_{\tilde{v}}$, $\rho_{\tilde{v}}(u) = \tilde{v}\wedge u$. By the properties of exterior algebras, the sequence of symbols is exact. If A is not transitive, then there is a covector $0 \neq v \in T_x^*M$ such that $\tilde{v} = (v\circ(\#_A)_x) = 0$. Hence, $\sigma\left(d_A^k\right)_{(x,v)} = \rho_{\tilde{v}} = 0$ for any k, and the sequence of symbols is not exact.

For the trivial Lie algebroid TM (the tangent bundle of the manifold M), the differential d_{TM} is the ordinary de Rham differential d_M of differential forms on M, whereas, for $L = \mathfrak{g}$ (a Lie algebra), the differential $d_{\mathfrak{g}}$ is the usual Chevalley–Eilenberg differential, $d_{\mathfrak{g}} = \delta_{\mathfrak{g}}$.

Theorem 1.2 [Kubarski–Mishchenko [8], 2004]**.** *For any transitive Lie algebroid* $(A, [\![\cdot,\cdot]\!], \#_A)$ *with the Atiyah sequence* (1) *over a connected compact oriented manifold M, the following conditions are equivalent* ($m = \dim M$ *and* $n = \dim \mathbf{g}_{|x}$, *i.e.,* $\operatorname{rank} A = m+n$):

(1) $\mathbf{H}^{m+n}(A) \neq 0$,
(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$,
(3) *A is invariantly oriented, i.e., there is a global nonsingular cross-section ε of the vector bundle* $\bigwedge^n \mathbf{g}$, $0 \neq \varepsilon_x \in \bigwedge^n \mathbf{g}_{|x}$, *invariant with respect to the adjoint representation of A in the vector bundle* $\bigwedge^n \mathbf{g}$ (*extending the adjoint representation ad_A of A on $\mathbf{g}$ given by* $(ad_A)(\xi)\colon \operatorname{Sec}\mathbf{g} \to \operatorname{Sec}\mathbf{g}$, $\nu \longmapsto [\![\xi,\nu]\!]$).

Condition (3) implies that the structure Lie algebras $\mathbf{g}_{|x}$ are unimodular. Lie algebroids satisfying (3) arose in 1996 [5] under the title "TUIO-Lie algebroids" (transitive unimodular invariantly oriented). Since M is connected, any invariant cross-section ε is uniquely determined up to constant factor. The fiber integral operator $\fint_A\colon \Omega^k(A) \to \Omega_{dR}^{k-n}(M)$, $k \geqslant n$, given by $\left(\fint_A \omega\right)_x(w_1,\dots,w_{k-n}) = (-1)^{kn}\omega_x(\varepsilon_x,\tilde{w}_1,\dots,\tilde{w}_{k-n})$ with $\#_A(\tilde{w}_i) = w_i$, commutes with the differentials d_A and d_M if and only if ε is invariant. In this case, the fiber integral defines a homomorphism in cohomology, $\fint_A^{\#} : \mathbf{H}^\bullet(A) \to \mathbf{H}_{dR}^{\bullet-n}(M)$ and we obtain an isomorphism $\fint_A^{\#} : \mathbf{H}^{m+n}(A) \xrightarrow{\cong} \mathbf{H}_{dR}^m(M) = \mathbb{R}$. The Poincaré inner product $\mathcal{P}_A^k : \mathbf{H}^k(A)\times\mathbf{H}^{m+n-k}(A) \to \mathbb{R}$, $([\omega],[\eta]) \longmapsto \int_A \omega\wedge\eta =$

$\int_M \left(\fint_A \omega \wedge \eta\right)$, is nondegenerate and, if $m+n=4p$, then $\mathcal{P}_A^{2p} : \mathbf{H}^{2p}(A) \times \mathbf{H}^{2p}(A) \to \mathbb{R}$ is nondegenerate and symmetric. Therefore, its signature is well defined; it is referred to as the *signature* of A and is denoted by $\operatorname{Sig}(A)$.

Problem 1.3. *Evaluate the signature* $\operatorname{Sig}(A)$ *and give conditions for the relation* $\operatorname{Sig}(A) = 0$.

There are examples for which $\operatorname{Sig}(A) \neq 0$. In them the example of flat bundle over surfaces with nonzero signature is used [4, $8\frac{2}{7}$].

In [6], a Hirzebruch signature operator for the cohomology $\mathbf{H}(A)$ was constructed. Below, we look at this operator from a more general point of view, and also present a general mechanism to calculate the signature via spectral sequences, see [8], in which we use two kinds of spectral sequences associated with Lie algebroids, namely,

a) the spectral sequence of the Čech–de Rham complex,
b) the Hochschild–Serre spectral sequence.

2. GENERAL APPROACH TO SIGNATURE VIA SPECTRAL SEQUENCES

The idea to use spectral sequences, to find the signature comes from Chern, Hirzebruch, and Serre [1]. Using spectral sequences, the authors proved the following assertion.

Theorem 2.1. *Let $E \to M$ be a fiber bundle with the typical fiber F and such that the following two conditions hold*:

(1) *E, M, and F are compact connected oriented manifolds*;
(2) *the fundamental group $\pi_1(M)$ acts trivially on the cohomology ring $\mathbf{H}^*(F)$ of F.*

In this case, if E, M, and F are oriented coherently (in such a way that the orientation of E is induced by those of F and M), then the index of E is the product of the indices of F and M, i.e., $\operatorname{Sig}(E) = \operatorname{Sig}(F)\cdot \operatorname{Sig}(M)$.

The authors consider the Leray spectral sequence $E_s^{p,q}$ for the cohomology of the bundle $E \to B$ with real coefficients. By condition (2), the term E_2 is the bigraded algebra $E_2^{p,q} \cong \mathbf{H}^p(M; \mathbf{H}^q(F)) \cong \mathbf{H}^p(M) \otimes \mathbf{H}^q(F)$. Therefore, $E_2^{p,q} = 0$ for $p > m$ or $q > n$. Clearly, E_2 is a *Poincaré algebra* by assumption (1). Using the spectral sequence argument, the authors noted that $(E_s, d_s, \cdot)$, $s \geqslant 2$, are Poincaré algebras with *Poincaré differentiation*. The infinite term $(E_\infty, \cdot)$ is also a Poincaré algebra, and the coincidence of signatures $\operatorname{Sig} E_2 = \operatorname{Sig} E_3 = \cdots = \operatorname{Sig} E_\infty$ holds. The last step $\operatorname{Sig} E_\infty = \operatorname{Sig} \mathbf{H}(E)$ is also proved. We note that it is by no means trivial since the algebras E_∞ and $\mathbf{H}(E)$ are not isomorphic in general (although $E_\infty \cong \mathbf{H}(E)$ as bigraded spaces).

Recall that a finitely graded algebra $\left(A^* = \bigoplus_{0\leqslant r\leqslant N} A^N, \cup\right)$ is referred to as a *Poincaré algebra* if

(1) $\dim A^N = 1$,
(2) $x \cup y = (-1)^{ij} y \cup x$ if $x \in A^i$, $y \in A^j$, i.e., $(A, \cup)$ is an anticommutative algebra,
(3) if $\xi \neq 0$ is a base element of A^N, then the bilinear form $\langle \cdot\,, \cdot \rangle \colon A^r \times A^{N-r} \to \mathbb{R}$ related to ξ (i.e., $\langle x, y\rangle \xi = x \cup y$) is nondegenerate. Therefore, $A^r \cong \left(A^{N-r}\right)^*$ and $\dim A^r = \dim A^{N-r}$.

The key to the further investigation is the notion of *Poincaré differentiation*, i.e., of a linear homomorphism $d\colon A \to A$ satisfying the following conditions:

(1) $d^2 = 0$,
(2) $d\,[A^r] \subset A^{r+1}$,
(3) d is an antiderivation,
(4) $d\left[A^{N-1}\right] = 0$ (in particular, if $x \in A^r$ and $y \in A^{N-r-1}$, then $dx \cup y = -(-1)^r x \cup dy$).

By analogy with the signature of an oriented manifold, the *signature* of a finite-dimensional Poincaré algebra $(A = \bigoplus A^r, \cup)$ related to ξ, $0 \neq \xi \in A^N$, is introduced. It must be zero if $N \neq 0$ (mod 4) and, if $N = 4k$, then $\operatorname{Sig} A$ is the signature of the symmetric nondegenerate function $\langle \cdot\,, \cdot \rangle^{2k,2k} : A^{2k} \times A^{2k} \to \mathbb{R}$ defined relative to ξ, $\operatorname{Sig} A = \operatorname{Sig}\langle \cdot\,, \cdot \rangle^{2k,2k}$.

The following lemma is very useful below.

Lemma 2.2 [1]. *If $(A, \cup, d)$ is a finite-dimensional Poincaré algebra with Poincaré differentiation, then the graded cohomology algebra $(\mathbf{H}^*(A), \cup)$ is a Poincaré algebra, and* $\operatorname{Sig} A = \operatorname{Sig} \mathbf{H}(A)$ *relative to the same element ξ, $0 \neq \xi \in A^N = \mathbf{H}^N(A, d)$.*

Example 2.3. (1) Let E be any finite-dimensional vector space. Then the exterior algebra $\bigwedge E$ is a Poincaré algebra. Its signature is zero.

(2) Let $\mathfrak{g}$ be any real Lie algebra. Then the system $(\wedge\mathfrak{g}^*, \wedge, \delta_{\mathfrak{g}})$ with the Chevalley–Eilenberg differentiation $\delta_{\mathfrak{g}}$ is a Poincaré algebra with Poincaré differentiation if and only if $\mathfrak{g}$ is unimodular. By the above lemma, if $\mathfrak{g}$ is unimodular, then the cohomology algebra $\mathbf{H}(\mathfrak{g})$ is a Poincaré algebra and $\operatorname{Sig} \mathbf{H}(\mathfrak{g}) = \operatorname{Sig} \bigwedge \mathfrak{g}^* = 0$.

It turns out that the Chern–Hirzebruch–Serre arguments used to prove the above theorems on the signature of the total space of the bundle $E \to M$ are purely algebraic and lead to the following general theorems [8].

Theorem 2.4. *Let $((A, \langle\,,\rangle), A^r, \cup, D, A_j)$ be any DG-algebra with grading A^r and decreasing filtration A_j and let $(E_s^{p,q}, d_s)$ be its spectral sequence. Assume that there are positive integers m and n such that:*

- *$E_2^{p,q} = 0$ for $p > m$ and $q > n$, $m + n = 4k$,*
- *E_2 is a Poincaré algebra with respect to the total grading and the top group $E_2^{(m+n)} = E_2^{m,n}$.*

Then each term $(E_s^{(*)}, \cup, d_s)$, $2 \leqslant s < \infty$, is a Poincaré algebra with Poincaré differentiation, and the infinite term $(E_\infty^{(*)}, \cup)$ is also a Poincaré algebra with $\operatorname{Sig} E_2 = \operatorname{Sig} E_3 = \cdots = \operatorname{Sig} E_\infty$.

If m and n are odd, then $\operatorname{Sig} E_2 = 0$. *If m and n are even, then* $\operatorname{Sig} E_2 = \operatorname{Sig}\big(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\big) = \operatorname{Sig}\big(E_2^{m/2,n/2} \times E_2^{m/2,n/2} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\big)$.

It remains to prove the relation $\operatorname{Sig} E_\infty = \operatorname{Sig} \mathbf{H}(A)$. The same arguments as in the original work [1] give the following general theorem.

Theorem 2.5. *Let $(A, A^r, \cup, D, A_j)$ be any DG-algebra with a grading A^r and with a decreasing filtration A_j compatible with the DG structure, i.e., $A_i A_j \subset A_{i+j}$, $D(A_j) \subset A_j$, $A_j = \bigoplus_r A^r \cap A_j$, and satisfying the regularity condition $A_0 = A$, $A = A_0 \supset \cdots \supset A_j \supset A_{j+1} \supset \cdots$. Let $(E_s^{p,q}, d_s)$ be the spectral sequence associated with this graded differential filtered algebra A. Assume that*

— *the infinite term $E_\infty^{p,q}$ lives in the rectangle $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant n$,*
— $\dim E_\infty^{m,n} = 1$,
— *E_∞ is a Poincaré algebra with respect to the total grading; in particular,* $\dim E_\infty$ *is finite.*

Under the above assumptions on the graded differential filtered algebra A, the cohomology algebra $\mathbf{H}(A)$ satisfies the following conditions:

(1) $\mathbf{H}^{m+n}(A) \cong E_\infty^{m,n}$; *in particular,* $\dim \mathbf{H}^{m+n}(A) = 1$;
(2) *the algebra $\mathbf{H}(A) = \bigoplus_{r=0}^{m+n} \mathbf{H}^r(A)$ is a Poincaré algebra;*
(3) *the signature of the cohomology of $\mathbf{H}(A)$ is equal to the signature of the term E_∞,* $\operatorname{Sig} E_\infty = \operatorname{Sig} \mathbf{H}(A)$, *under a suitable choice of generators of the top groups.*

Therefore, under natural simple regularity assumptions on the DG-algebra A^r, if $E_2^{p,q}$ is a Poincaré algebra living in a finite rectangle, then $\operatorname{Sig}(E_2) = \operatorname{Sig}(\mathbf{H}(A))$.

Let us apply this mechanism

(a) to the spectral sequence for the Čech–de Rham complex of the Lie algebroid A [8],
(b) to the Hochshild–Serre spectral sequences [9].

(a) For details, see [8]. Let $\mathcal{H}^*(A) = \big(U \longmapsto \mathbf{H}^*\left(A_{|U}\right)\big)$ be the Leray-type cohomology presheaf (locally constant on a good covering $\mathfrak{U}$) with values in the cohomology algebra $\mathbf{H}^*(\mathfrak{g})$ of the structural Lie algebra $\mathfrak{g}$. Then $E_1^{p,q} = C^p(\mathfrak{U}, \mathcal{H}^q(A))$ and $d_1 = \delta^\# : E_1^{p,q} \to E_1^{p+1,q}$, where δ is the coboundary homomorphism, and $E_2^{p,q} = \mathbf{H}^p_{\delta^\#}(\mathfrak{U}, \mathcal{H}^q(A))$. If the monodromy representation $\rho\colon \pi_1(M) = \pi_1(N(\mathfrak{U})) \to \operatorname{Aut}(\mathbf{H}(\mathfrak{g}))$ of the presheaf $\mathcal{H}(A)$ is trivial, then $E_2^{p,q} \cong \mathbf{H}^p_{dR}(M) \otimes \mathbf{H}^q(\mathfrak{g})$ (the isomorphisms are canonical isomorphisms of bigraded algebras). Therefore, $\operatorname{Sig} E_2 =$

$\operatorname{Sig}(\mathbf{H}(M)\otimes\mathbf{H}(\mathfrak{g}))=\operatorname{Sig}\mathbf{H}(M)\cdot\operatorname{Sig}\mathbf{H}(\mathfrak{g})$. Hence, since the isotropy Lie algebra $\mathfrak{g}$ is unimodular, i.e., $\dim\mathbf{H}^n(\mathfrak{g})=1$, it follows that $\operatorname{Sig}\mathbf{H}(\mathfrak{g})=\operatorname{Sig}\bigwedge\mathfrak{g}^*=0$, and therefore, $\operatorname{Sig}(A)=\operatorname{Sig}H(A)=\operatorname{Sig}E_2=\operatorname{Sig}(M)\cdot\operatorname{Sig}(\mathfrak{g})=0$.

Example 2.6. The triviality condition for the monodromy (in the situation $\operatorname{Sig}(A)=0$) holds provided that

- M is simply connected,
- $\operatorname{Aut}G=\operatorname{Int}G$, where G is a simply connected Lie group with the Lie algebra $\mathfrak{g}$ (for example, if $\mathfrak{g}$ is a simple Lie algebra of type $B_l, C_l, E_7, E_8, F_4, G_2$),
- the adjoint Lie algebra bundle $\mathbf{g}$ is trivial in the category of flat bundles (the cohomology bundle $\mathbf{H}(\mathbf{g})$ of isotropy Lie algebras with typical fiber $H(\mathfrak{g})$ possesses a canonical flat covariant derivative, which is of importance when studying the Hochshild–Serre spectral sequence). For example, $\mathbf{g}$ is trivial, in the category of flat bundles, for the Lie algebroid $A(G;H)$ of the TC-foliation of left cosets of a nonclosed Lie subgroup H in any Lie group G.

(b) Following Hochschild and Serre [2], for a pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g},\mathfrak{k})$, one can consider a graded cochain group of $\mathbb{R}$-linear alternating functions $A_{\mathbb{R}}(P)=\bigoplus_{k\geqslant0}A^k(P)$, $A^k(P)=C^k(\mathfrak{g},P)$, with values in a $\mathfrak{g}$-module P, $\mathfrak{g}\times P\to P$, with the standard $\mathbb{R}$-differential operator d of degree 1 and the Hochschild–Serre filtration $A_j\subset A_{\mathbb{R}}(P)$ as follows:

— $A_j=A_{\mathbb{R}}(P)$ for $j\leqslant 0$,

— if $j>0$, then $A_j=\bigoplus_{k\geqslant j}A_j^k$, $A_j^k=A_j\cap A^k$, where A_j^k consists of all k-cochains f for which $f(\gamma_1,\dots,\gamma_k)=0$ whenever $k-j+1$ arguments γ_i belong to $\mathfrak{k}$.

In this way, we have obtained a graded filtered differential $\mathbb{R}$-vector space $\left(A_{\mathbb{R}}=\bigoplus_{k\geqslant0}A^k,d,A_j\right)$. Let us use its spectral sequence $\left(E_s^{p,q},d_s\right)$. For a transitive Lie algebroid $A=(A,[\![\cdot,\cdot]\!],\#_A)$ with the Atiyah sequence $0\to\mathbf{g}\hookrightarrow A\overset{\#_A}{\to}TM\to0$, consider the pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g},\mathfrak{k})$, where $\mathfrak{g}=\operatorname{Sec}(A)$, $\mathfrak{k}=\operatorname{Sec}(\mathbf{g})$. Following Mackenzie (1987) (see [13]), Itskov, Karasev, and Vorobjev (1998) (see [11]), and Kubarski and Mishchenko (2004) (see [9]), consider the $C^\infty(M)$-submodule of $C^\infty(M)$-linear alternating cochains $\Omega^k(A)\subset C^k(\mathfrak{g},C^\infty(M))$ with values in the trivial $\mathfrak{g}$-module $C^\infty(M)$ (i.e., with respect to the trivial representation $\partial_\xi(X)=\#_A(\xi)(X)$) and the induced filtration $\Omega_j=\Omega_j(A)=A_j\cap\Omega(A)$ of $C^\infty(M)$-modules. In this way, we obtain a graded filtered differential space $\left(\Omega(A)=\bigoplus_k\Omega^k(A),d_A,\Omega_j\right)$ and its spectral sequence $\left(E_{A,s}^{p,q},d_{A,s}\right)$.

Assume as above that $m=\dim M$ and $n=\dim\mathbf{g}_{|x}$, i.e., $\operatorname{rank}A=m+n$. The multiplication $\wedge$ and differentiation d_A of the differential forms (defined by (2)) preserves gradings and filtrations, $\wedge\colon\Omega_j^k\times\Omega_i^r\to\Omega_{j+i}^{k+r}$ and $d_A\colon\Omega_j^k\to\Omega_j^{k+1}$. We have $E_{A,0}^{p,q}=\Omega_p^{p+q}(A)/\Omega_{p+1}^{p+q}$ and $d_{A,0}^{p,q}\colon E_{A,0}^{p,q}\to E_{A,0}^{p,q+1}$, $[\omega]\longmapsto[d_A\omega]$. Taking an arbitrary connection $\lambda\colon TM\to A$ on the Lie algebroid A, we obtain an isomorphism of $C^\infty(M)$-modules [9, Con. 5.2], $a_A^{p,q}\colon E_{A,0}^{p,q}\xrightarrow{\cong}\Omega^p\left(M;\wedge^q\mathbf{g}^*\right)$, $a_A^{p,q}([\omega])_x(v_1,\dots,v_p)(\sigma_1,\dots,\sigma_q)=\omega_x(\sigma_1,\dots,\sigma_q,\lambda v_1,\dots,\lambda v_p)$, where $v_i\in T_xM$ and $\sigma_i\in\mathbf{g}_{|x}$. Since $\omega\in\Omega_j^{p+q}$, the isomorphism $a_A^{p,q}$ does not depend on λ.

Through the isomorphism $a_A^{p,q}$, the differential $d_{A,0}:E_{A,0}^{p,q}\to E_{A,0}^{p,q+1}$ can be identified with the differentiation $\tilde d_{A,0}^{p,*}:\Omega^p\left(M;\wedge^q\mathbf{g}^*\right)\to\Omega^p\left(M;\wedge^{q+1}\mathbf{g}^*\right)$ of differential forms with values in $\wedge^q\mathbf{g}^*$ with respect to the Chevalley–Eilenberg differential at any point for the isotropy Lie algebra $\mathbf{g}_{|x}$, $\mathbf{H}\left(\Omega^p\left(M;\bigwedge^\bullet\mathbf{g}^*\right),\tilde d_{A,0}\right)=\Omega^p(M;\mathbf{H}^\bullet(\mathbf{g}))$. Therefore, $E_{A,1}^{p,\bullet}\cong\mathbf{H}\left(E_{A,0}^{p,\bullet},d_{A,0}\right)\overset{b^{p,\bullet}}{\cong}\Omega^p(M;\mathbf{H}^\bullet(\mathbf{g}))$, where the isomorphism $b^{p,q}$ is given by $[\omega]\longmapsto[\bar\omega_p]$ for $\omega\in\Omega_p^{p+q}$, $d_A\omega\in\Omega_{p+1}^{p+q+1}$, and $\bar\omega_p\in\Omega^p\left(M;\bigwedge^q\mathbf{g}^*\right)$ is equal to $\bar\omega_p(v_1,\dots,v_p)(\sigma_1,\dots,\sigma_q)=\omega(\sigma_1,\dots,\sigma_q,\lambda v_1,\dots,\lambda v_p)$. Transfer the differentials $d_{A,1}^{p,q}\colon E_{A,1}^{p,q}\to E_{A,1}^{p+1,q}$ to $\Omega^p(M;\mathbf{H}^q(\mathbf{g}))$ by using the isomorphisms $b^{p,q}$. There is a flat covariant derivative ∇^q on the vector bundle $\mathbf{H}^q(\mathbf{g})$ such that $d_{A,1}^{*,q}=(-1)^q d_{\nabla^q}$ [9, Prop. 5.9], and ∇^q is defined by $\nabla_X^q[f]=[\mathcal{L}_X f]$ for $f\in\Omega^p\left(M;Z\left[\bigwedge^q\mathbf{g}^*\right]\right)$, $[f]\in\Omega^p(M;\mathbf{H}^q(\mathbf{g}))$, where $(\mathcal{L}_X f)(\sigma_1,\dots,\sigma_q)=\partial_X(f(\sigma_1,\dots,\sigma_q))-\sum_{i=1}^q f(\sigma_1,\dots,[\![\lambda X,\sigma_i]\!],\dots,\sigma_q)$ (recall that $\lambda:TM\to A$ is an arbitrary auxiliary connection in A). Therefore, $E_{A,2}^{p,q}\cong\mathbf{H}^p(E_{A,1}^{\bullet,q},d_{A,1}^{\bullet,q})\cong\mathbf{H}_{\nabla^q}^p(M;\mathbf{H}^q(\mathbf{g}))$, where the second isomorphism is given by $[\omega] \longmapsto [[\bar{\omega}]]$.

Summing up, we obtain the following assertion.

Theorem 2.7. *If A is a TUIO-Lie algebroid such that $m+n=4p$ (with $m=\dim M$ and $n=\dim \mathbf{g}_{|x}$), then the following assertions hold.*

a) *If m and n are odd, then $\operatorname{Sig} A = 0$.*

b) *If m and n are even, then $\operatorname{Sig} A = \operatorname{Sig} E_2 = \operatorname{Sig}\big(E_2^{(2p)} \times E_2^{(2p)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\big) = \operatorname{Sig}\big(E_2^{m/2,n/2} \times E_2^{m/2,n/2} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\big)$, where $E_2^{m/2,n/2} = \mathbf{H}^{m/2}_{\nabla^{n/2}}\big(M;\mathbf{H}^{n/2}(\mathbf{g})\big)$ and $\mathbf{H}^{m/2}_{\nabla^{n/2}}\big(M;\mathbf{H}^{n/2}(\mathbf{g})\big) \times \mathbf{H}^{m/2}_{\nabla^{n/2}}\big(M;\mathbf{H}^{n/2}(\mathbf{g})\big) \to \mathbf{H}^{m}_{\nabla^{n}}\left(M;\mathbf{H}^{n}(\mathbf{g})\right) = \mathbb{R}$ is defined by the usual multiplication of differential forms with respect to the multiplication of cohomology class for Lie algebras, $\phi\colon \mathbf{H}^{n/2}(\mathbf{g}) \times \mathbf{H}^{n/2}(\mathbf{g}) \to \mathbf{H}^{n}(\mathbf{g}) = M \times \mathbb{R}$. Note that, if $n/2$ is even, then ϕ is symmetric and nondegenerate (in this way, we obtain the Lusztig example), whereas, if $n/2$ is odd, then ϕ is symplectic (in this way, we obtain the Gromov example). However,*

$$\mathbf{H}^{m/2}(M;\mathbf{H}^{n/2}(\mathbf{g})) \times \mathbf{H}^{m/2}(M;\mathbf{H}^{n/2}(\mathbf{g})) \to \mathbb{R}$$

is always symmetric and nondegenerate.

3. ALGEBRAIC ASPECTS OF THE HIRZEBRUCH SIGNATURE OPERATOR

Below, we present a common algebraic approach to the calculation the signature $\operatorname{Sig}(W)$ using the Hirzebruch signature operator.

3.1. *Hodge Space*

In this subsection, the $*$-Hodge operator, the Hodge theorem, and the Hirzebruch signature operator are treated from an algebraic point of view.

Definition 3.1. By a *Hodge space* we mean the triple $(W, \langle\,,\rangle, (\,,))$, where W is a real vector space ($\dim W$ is finite or infinite), $\langle\,,\rangle$, $(\,,)\colon W \times W \to \mathbb{R}$ are 2-linear tensors such that

(1) $(\,,)$ is symmetric and positive definite (i.e., an inner product),

(2) there is a linear homomorphism $*_W\colon W \to W$, the so-called $*$-Hodge operator, such that

  (i) $\langle v, w\rangle = (v, *_W(w))$ for any $v \in V$,

  (ii) $*_W$ is an isometry with respect to $(\,,)$, i.e., $(v,w) = (*_W v, *_W w)$.

Clearly, the $*$-Hodge operator is uniquely defined (if it exists).

Two 2-tensors, $f\colon V \times V \to \mathbb{R}$ and $g\colon W \times W \to \mathbb{R}$, determine the tensor product $f \otimes g\,(V \otimes W) \times (V \otimes W) \to \mathbb{R}$, which is 2-linear.

Lemma 3.2 [3]. *The tensor $f \otimes g$ is symmetric and positive definite if both f and g are symmetric and positive definite (the dimensions of V and W can be infinite).*

The above consideration gives the following assertion.

Lemma 3.3. *If $(V, \langle\cdot,\cdot\rangle_V, (\cdot,\cdot)_V)$ and $(W, \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_W)$ are Hodge spaces, then their tensor product $(V \otimes W, \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_V \otimes (\cdot,\cdot)_W)$ is a Hodge space and $*_{V\otimes W} = *_V \otimes *_W$.*

3.2. *Finite-Dimensional Hodge Spaces. Examples*

Lemma 3.4. *Let $(W, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real vector space equipped with a 2-tensor $\langle\cdot,\cdot\rangle$. Then an inner product $(\cdot,\cdot)$ such that the system $(W, \langle\,,\rangle, (\,,))$ is a Hodge space exists if and only if there is a basis of W in which the matrix of $\langle\,,\rangle$ is orthogonal.*

Proof. Standard calculations.

It is an important observation that the calculation of the signature (in standard cases) using the idea of Hirzebruch operator is restricted to 2-tensors $\langle\cdot,\cdot\rangle$ (in fibers of some vector bundles) for which there is an auxiliary inner product $(\cdot,\,\cdot)$ such that the system $(W, \langle\,,\rangle, (\,,\,))$ is a Hodge space.

Let us now give examples of finite-dimensional Hodge spaces.

Example 3.5 [classical]. Let (V,G) be a real N-dimensional oriented Euclidean space with an inner product $G\colon V\times V\to\mathbb{R}$ and the volume tensor $\varepsilon=e_1\wedge\cdots\wedge e_N\in\wedge^N V$, (where $\{e_i\}_{i=1}^N$ is a positive ON basis of V). We identify $\wedge^N V=\mathbb{R}$ via the isomorphism $\rho\colon \wedge^N V\xrightarrow{\cong}\mathbb{R},\ \ s\cdot\varepsilon\longmapsto s$. We obtain the classical Hodge space $(\wedge V,\langle\,,\rangle,(\,,))$, where $\langle\cdot,\cdot\rangle\colon \wedge V\times\wedge V\to\mathbb{R}$, $\langle\cdot,\cdot\rangle^k:\wedge^k V\times\wedge^{N-k}V\to\wedge^N V=\mathbb{R}$, $\left\langle v^k,v^{N-k}\right\rangle=\rho\left(v^k\wedge v^{N-k}\right)$, $\langle\,,\rangle=0$ outside the pairs of degree $(k,N-k)$, and $(\cdot,\cdot)^k\colon \wedge^k V\times\wedge^k V\to\mathbb{R}$, $\left(v_1\wedge\ldots.\wedge v_k,w_1\wedge\cdots\wedge w_k\right)^k=\det\left[G\left(v_i,w_k\right)\right]$. The subspaces $\wedge^k V$, $k=0,1,\ldots,N$, are orthogonal (by definition).

The $*$-Hodge operator exists and is determined by an ON basis $\{e_i\}_{i=1}^N$ according to the formula $*\left(e_{i_1}\wedge\cdots\wedge e_{i_k}\right)=\varepsilon_{(j_1,\ldots,j_{n-k})}\cdot e_{j_1}\wedge\cdots\wedge e_{j_{n-k}}$, where $i_1<\cdots<i_k$ and $j_1<\cdots<j_{n-k}$, the sequence $(j_1,\ldots,j_{n-k})$ is complementary to $(i_1,\ldots,i_k)$, and $\varepsilon_{(j_1,\ldots,j_{n-k})}=\operatorname{sgn}\left(j_1,\ldots,j_{n-k},i_1,\ldots,i_k\right)$.

The above classical example is used

- for $V=T_xM$ or $V=T_x^*M$, where M is a Riemannian manifold,
- for $V=A_x$, where A is a TUIO-Lie algebroid over a Riemann manifold (see below).

Example 3.6 [Lusztig example [10], 1972]. Let $(\,,)_0:E\times E\to\mathbb{R}$ be a symmetric nondegenerate tensor (indefinite in general) on a finite-dimensional vector space E. Let G be an arbitrary positive inner product on E. Then there is exactly one direct product $E=E_+\oplus E_-$ which is ON with respect to both the inner products $(\,,)_0$ and G and such that $(\,,)_0$ is positive on E_+ and negative on E_-. Denote by $*_E$ the involution $*_E:E\to E$ such that $*_E|E_+=\mathrm{id}$, $*_E|E_-=-\,\mathrm{id}$. Then the quadratic form $(\,,)\colon E\times E\to\mathbb{R}$, $(v,w):=\left(v,*_E w\right)_0$, is symmetric and positive definite. The involution $*_E$ is an isometry, $(*_E v,*_E w)=\left(*_E v,*_E^2 w\right)_0=\left(*_E v,w\right)_0=\left(w,*_E v\right)_0=(w,v)=(v,w)$. Therefore, $(E,(\,,)_0,(\,,))$ is a Hodge space.

Example 3.7 [Gromov example [4], 1995]**.** Let $\langle\,,\rangle_0\colon E\times E\to\mathbb{R}$ be a skew-symmetric nondegenerate tensor on a finite-dimensional vector space E. There is an anti-involution τ on E, $\tau^2=-\,\mathrm{id}$ (i.e., a complex structure) such that $\langle\tau v,\tau w\rangle_0=\langle v,w\rangle_0$, $v,w\in E$, and $\langle v,\tau v\rangle_0>0$ for any $v\neq 0$. Namely, there is a basis of E for which the matrix of $\langle\,,\rangle_0$ is orthogonal and is of the form $\begin{bmatrix}0&\mathbf{1}\\-\mathbf{1}&0\end{bmatrix}$ and τ is given by the formula $\tau(v_i)=v_{n+i}$, $\tau(v_{n+i})=-v_i$. Then the tensor $(\,,):E\times E\to\mathbb{R}$, $(v,w):=\langle v,\tau w\rangle_0$, is symmetric and positive definite and $(\tau v,\tau w)=(v,w)$, i.e., τ preserves both the forms $\langle\,,\rangle_0$ and $(\,,)$. The system $(E,\langle\,,\rangle_0,(\,,))$ is a Hodge space since the operator $-\tau$ is the $*$-Hodge operator $\langle v,w\rangle_0=\left\langle v,-\tau^2 w\right\rangle_0=\langle v,\tau\left(-\tau w\right)\rangle_0=(v,-\tau w)$, and $-\tau$ is an isometry $(-\tau v,-\tau w)=(\tau v,\tau w)=(v,w)$.

Definition 3.8. By a Hodge vector bundle we mean a system $(\xi,\langle\,,\rangle,(\,,))$ consisting of a vector bundle ξ and of two smooth tensor fields $\langle\,,\rangle,(\,,):\xi\times\xi\to\mathbb{R}$ (sections of $(\xi\otimes\xi)^*$) such that, for each $x\in M$, the system $(\xi_x,\langle\,,\rangle_x,(\,,)_x)$ is a finite-dimensional Hodge space and the family of Hodge operators $*_x\colon\xi_x\to\xi_x$, $x\in M$, gives a smooth linear homomorphism of vector bundles, $*\colon\xi\to\xi$.

Example 3.9 [important example]. Consider an arbitrary Riemannian oriented manifold M of dimension N and a Hodge vector bundle $(\xi,\langle\,,\rangle,(\,,))$. Then, for any point $x\in M$, we take the tensor product of Hodge spaces $\wedge T_x^*M\otimes\xi_x$. Assuming the compactness of M, we can define two 2-$\mathbb{R}$-linear tensors $((\alpha,\beta))$, $\langle\langle\alpha,\beta\rangle\rangle:\Omega\left(M;\xi\right)\times\Omega\left(M;\xi\right)\to\mathbb{R}$ by integrating over the Riemannian manifold,

$$((\alpha,\beta))=\int_M\left(\alpha_x,\beta_x\right)dM\quad\text{and}\quad\langle\langle\alpha,\beta\rangle\rangle=\int_M\langle\alpha_x,\beta_x\rangle dM=\int_M\alpha\wedge_\varphi\beta,$$

where $\varphi_x=\langle\cdot,\cdot\rangle_x^k\colon\wedge^k T_x^*M\otimes\xi_x\times\wedge^{N-k}T_x^*M\otimes\xi_x\to\bigwedge^N T_x^*M=\mathbb{R}$ stands for the wedge product with respect to the multiplication $\langle\,,\rangle_x$ of values. The 2-form $((\cdot,\cdot))$ is symmetric and positive definite, and the triple $\left(\Omega\left(M;W\right),\langle\langle\alpha,\beta\rangle\rangle,((\alpha,\beta))\right)$ is a Hodge space with the $*$-Hodge operator $\langle\langle\alpha,\beta\rangle\rangle=((\alpha,*\beta))$ defined pointwise, $\left(*\beta\right)_x=*_x\left(\beta_x\right)$. Indeed,

$$\langle\langle\alpha,\beta\rangle\rangle=\int_M\langle\alpha_x,\beta_x\rangle\,dM=\int_M\left(\alpha_x,*_x\beta_x\right)dM=((\alpha,*\beta)).$$

3.3. *Graded Differential Hodge Space*

Definition 3.10. By a *graded differential Hodge space* we mean a system $\left(W = \oplus_{k=0}^{N} W^k, \langle\cdot\,,\cdot\rangle, (\cdot\,,\cdot), d\right)$, where $(W, \langle\cdot\,,\cdot\rangle, (\cdot\,,\cdot))$ is a Hodge space (finite- or infinite-dimensional) and

(1) $\langle\cdot\,,\cdot\rangle^k : W^k \times W^{N-k} \to \mathbb{R}$ and $\langle\,,\rangle = 0$ outside the pairs $(k, N-k)$,
(2) W^k are orthogonal with respect to $(\cdot\,,\cdot)$,
(3) d is homogeneous of degree $+1$, i.e., $d\colon W^k \to W^{k+1}$, and $d^2 = 0$,
(4) $\langle dw, u\rangle = (-1)^{k+1}\langle w, du\rangle$ for $w \in W^k$.

Clearly,

a) the induced cohomology pairing $\langle\,,\rangle_{\mathbf{H}}^k : \mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$, $\langle [u],[w]\rangle_{\mathbf{H}}^k = \langle u, w\rangle^k$, is well defined,
b) $*\left[W^k\right] \subset W^{N-k}$, and $*\colon W^k \to W^{N-k}$ is an isomorphism.

Assume that W is a *graded differential Hodge space*. Let $d^*\colon W \to W$ be the adjoint operator with respect to $(\,,)$, i.e., such that $(d^*(w_1), w_2) = (w_1, d(w_2))$. Assume that d^* exists. It is easy to see that d^* is of degree -1, $d^*\colon W^{k+1} \to W^k$. Using standard calculations, we can show that the operator (the so-called *Laplacian*) $\Delta := (d + d^*)^2 = dd^* + d^*d$ is homogeneous of degree 0, i.e., $\Delta[W^r] \subset W^r$, it is selfadjoint, $(\Delta v, w) = (v, \Delta w)$, nonnegative, $(\Delta v, v) \geqslant 0$, and $\{v \in W;\ (\Delta v, v) = 0\} = \{v \in W;\ dv = 0 = d^*v\}$.

Definition 3.11. A vector $v \in W$ is said to be *harmonic* if $dv = 0$ and $d^*v = 0$ or, equivalently, if $v \perp (\Delta v)$. Write $\mathcal{H}(W) = \{v \in W;\ dv = 0,\ d^*v = 0\}$ and $\mathcal{H}^k(W) = \left\{v \in W^k;\ dv = 0,\ d^*v = 0\right\}$.

The harmonic vectors form the graded vector space $\mathcal{H}(W) = \oplus_{k=0}^{N}\mathcal{H}^k(W)$.

Lemma 3.12. $\mathcal{H}^k(W) = \ker\left\{d + d^* : W^k \to W\right\} = \ker\left\{\Delta^k : W^k \to W^k\right\}$, *i.e.,* $\mathcal{H}(W) = \ker\Delta = (\operatorname{Im}\Delta)^{\perp}$.

Proof. Standard calculations.

The kernel $\ker\Delta$ is the eigenspace of Δ corresponding to the zero eigenvalue.

If W is a Hilbert space and $Y \subset W$ is a closed subset, then W is the direct sum $Y \bigoplus Y^{\perp}$. For a Riemannian vector bundle ξ over a Riemannian manifold, the space $W = \mathrm{Sec}(\xi)$ is not a Hilbert space (because it is not complete). However, the following well-known and important theorem holds, see, e.g., [12].

Theorem 3.13. *Let ξ be a Riemannian vector bundle over a compact oriented Riemannian manifold M. If $\Delta\colon \mathrm{Sec}\,\xi \to \mathrm{Sec}\,\xi$ is a* selfadjoint nonnegative elliptic operator*, then* $\ker\Delta$ *is a finite-dimensional space and* $\mathrm{Sec}\,\xi = \operatorname{Im}\Delta \bigoplus \ker\Delta = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$.

Below, the symbol $\left(W = \oplus_{k=0}^{N} W^k, \langle\cdot\,,\cdot\rangle, (\cdot\,,\cdot), d\right)$ denotes an arbitrary graded differential Hodge space. The spaces $\ker\Delta^k$ and $\operatorname{Im} d^{k-1}$ are orthogonal; in particular, $\ker\Delta^k \cap \operatorname{Im} d^{k-1} = 0$. Therefore, the inclusion $\mathcal{H}^k(W) = \ker\Delta^k \hookrightarrow \ker d^k$ induces a monomorphism $\ker\Delta^k \rightarrowtail \mathbf{H}^k(W)$.

Let us now prove that, under the algebraic assumption $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$, this monomorphism is an isomorphism, i.e., every cohomology class contains (exactly one) harmonic vector.

Theorem 3.14. *If* $W = \operatorname{Im}\Delta \oplus (\operatorname{Im}\Delta)^{\perp}$, *then*

(1) $$W^k = \ker\Delta^k \oplus \operatorname{Im}\Delta^k,$$
(2) $$W^k = \ker\Delta^k \oplus \operatorname{Im} d^{k-1} \oplus \operatorname{Im}(d^*)^{k+1},$$
(3) $$\ker d^k = \ker\Delta^k \oplus \operatorname{Im} d^{k-1}.$$

In particular, if $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$, *then the inclusion* $\ker\Delta^k \hookrightarrow \ker d^k$ *induces an isomorphism* $\ker\Delta^k \hookrightarrow \ker d^k \to \ker d^k/\operatorname{Im} d^{k-1} = \mathbf{H}^k(W)$. *This means that every cohomology class contains exactly one nonzero harmonic vector.*

(4) *The equation $\Delta w = u$, for a given u, has a solution if and only if $u \in (\ker \Delta)^{\perp}$. Equivalently,* $\operatorname{Im} \Delta = (\ker \Delta)^{\perp}$.

Proof. (1) Evident.

(2) Since $\operatorname{Im} \Delta^k \subset \operatorname{Im} d^{k-1} + \operatorname{Im} d^{*(k+1)}$ ($\Delta u = d\,(d^* u) + d^*\,(du) \in \operatorname{Im} d^{k-1} + \operatorname{Im} d^{*(k+1)}$), it follows that $W^k = \ker \Delta^k \bigoplus \operatorname{Im} \Delta^k = \ker \Delta^k + \operatorname{Im} d^{k-1} + \operatorname{Im}(d^*)^{k+1}$. Thus, we must only prove (which is easy) that these three subspaces are ON.

(3) Since $\ker \Delta^k$ and $\operatorname{Im} d^{k-1}$ are ON and $\ker \Delta^k + \operatorname{Im} d^{k-1} \subset \ker d^k$, we must show only that $\ker \Delta^k + \operatorname{Im} d^{k-1} \subset \ker d^k$. Let $u^k \in \ker d^k$. Expand u^k according to (2), $u^k = w_1 + dw_2 + d^* w_3$, $\Delta w_1 = 0$. In particular, $dw_1 = 0$ and $0 = dd^* w_3$. Since $(d^* w_3, d^* w_3) = (w_3, dd^* w_3) = 0$, we have $d^* w_3 = 0$ and $u^k = w_1 + dw_2 \in \ker \Delta^k + \operatorname{Im} d^{k-1}$.

(4) Assume that the equation $\Delta w = u$ has a solution w for a given u. Then $(u, v) = (\Delta w, v) = (w, \Delta v) = (w, 0) = 0$ for any $v \in \ker \Delta$. Therefore, $u \in (\ker \Delta)^{\perp}$. To show the converse, take $u \in (\ker \Delta)^{\perp}$ and assume that $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$. Representing u in this direct sum as $u = u_1 + u_2$, where $u_1 \in \operatorname{Im} \Delta$ and $u_2 \in (\operatorname{Im} \Delta)^{\perp} = \ker \Delta$, we have $u \in (\ker \Delta)^{\perp}$ and $u_2 \in \ker \Delta$. Therefore, if $u_1 = \Delta h$, then $0 = (u, u_2) = (\Delta h, u_2) + (u_2, u_2) = (h, \Delta u_2) + (u_2, u_2) = (u_2, u_2)$, which gives $u_2 = 0$, i.e., $u = u_1 \in \operatorname{Im} \Delta$ and $u = \Delta h$.

Remark 3.15. The above fact (4) means that the condition $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$ is sufficient to show the equality $\operatorname{Im} \Delta = (\ker \Delta)^{\perp}$. We can ask: Is this condition necessary?

Let us now try to formulate a condition assuring the existence of the adjoint operator d^* in a graded differential Hodge space.

Theorem 3.16. *Let $\left(W = \bigoplus_{k=0}^{N} W^k, \langle\cdot,\cdot\rangle, (\cdot,\cdot), d\right)$ be a graded differential Hodge space. Let $\varepsilon\colon \{0, 1, \dots, N\} \to \{-1, 1\}$ be an arbitrary function such that $\varepsilon_k = \varepsilon_{N-k}$ for any k. Assume that $\langle\,,\rangle$ is ε-anticommutative, i.e., $\langle v^k, v^{N-k}\rangle = \varepsilon_k \langle v^{N-k}, v^k\rangle$ for $v^k \in W^k$, $v^{N-k} \in W^{N-k}$ [note that, if $\langle\,,\rangle^{k,N-k}$ is nontrivial, then $\varepsilon_k \varepsilon_{N-k} = +1$, i.e., $\varepsilon_k = \varepsilon_{N-k}$]. Then the following assertions hold.*

(a) *$**\,(w^k) = \varepsilon_k \cdot w^k$; in particular, $*^{-1}(u^k) = \varepsilon_k \cdot *(u^k)$ and $(*^{N-k})^{-1}(u^k) = \varepsilon_k \cdot *^k(u^k)$.*

(b) *The adjoint operator d^* exists and is given by $d^*(w^k) = \varepsilon_k (-1)^k * d * (w^k)$, $w^k \in W^k$, where $*$ stands for the $*$-Hodge operator on W.*

(c) $(*d^*)(w^k) = \varepsilon_k (-1)^k \varepsilon_{N-k+1} (d*)(w^k)$, $(d^**)(w^k) = (-1)^{N-k} \varepsilon_k \varepsilon_{N-k} (*d) w^k = (-1)^{N-k} (*d) w^k$.

(d) *If $\varepsilon_{k-1} = \varepsilon_{k+1}$, then $*\Delta = \pm \Delta *$ (to be precise, $*\Delta w^k = \varepsilon_{k-1} \varepsilon_k (-1)^{N+1} \Delta * w^k$), and we then conclude that $*\left[\mathcal{H}^k(W)\right] \subset \mathcal{H}^{N-k}(W)$, and $*\colon \mathcal{H}^k(W) \to \mathcal{H}^{N-k}(W)$ is an isomorphism.*

Proof. (a) Immediate calculations.

(b) The form $(\,,)$ is symmetric and positive definite (i.e., this is an inner product). Since the tensor $(\,,)$ is an inner product, it is sufficient to prove that the operator $\tilde{d}(w^k) = \varepsilon_k (-1)^k * d * (w^k)$, $w^k \in W^k$, is adjoint to d. Take $w^{k-1} \in W^{k-1}$. Since $*d*(w^k) \in W^{k-1}$, it follows that, by (a), $\left(w^{k-1}, \tilde{d}(w^k)\right) = \left(w^{k-1}, \varepsilon_k (-1)^k * d * (w^k)\right) = \left(w^{k-1}, \varepsilon_k (-1)^k * d \varepsilon_k *^{-1}(w^k)\right) = (-1)^k \left(w^{k-1}, *d *^{-1}(w^k)\right) = (-1)^k \left\langle w^{k-1}, d *^{-1}(w^k)\right\rangle$, which is equal [note that $\langle dw^k, u\rangle = (-1)^{k+1} \langle w^k, du\rangle$] to the expression $\left\langle dw^{k-1}, *^{-1}(w^k)\right\rangle = \left(dw^{k-1}, **^{-1}(w^k)\right) = \left(dw^{k-1}, w^k\right)$.

(c) Easy calculations.

(d) The manipulations $\Delta * (w^k) = (-1)^{N-k} d * dw^k + \varepsilon_{k-1} (-1)^{N-k+1} * d * d * \left(w^k\right)$ and $*\Delta(w^k) = \varepsilon_{k+1} \varepsilon_k (-1)^{N+1} \left((-1)^{N-k} d * dw^k + \varepsilon_{k+1} (-1)^{N-k+1} * d * d * (w^k)\right)$ imply the relation $*\Delta w^k = \varepsilon_{k-1} \varepsilon_k (-1)^{N+1} \Delta * w^k$ for $\varepsilon_{k-1} = \varepsilon_{k+1}$.

The above assertion (d) yields the following theorem.

Theorem 3.17 [duality theorem]. *If $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$, then $\mathbf{H}^k(W) \simeq \mathbf{H}^{N-k}(W)$.*

The composition $\mathbf{H}^k(W) \underset{\cong}{\longleftarrow} \mathcal{H}^k(W) \underset{\cong}{\xrightarrow{*}} \mathcal{H}^{N-k}(W) \underset{\cong}{\longrightarrow} \mathbf{H}^{N-k}(W)$, $[v] \longmapsto [*v]$, is an isomorphism given by the above formula for harmonic vectors only!

We restrict the positive inner product $(\cdot\,,\cdot)\colon W^k \times W^k \to \mathbb{R}$ to the space of harmonic vectors $(\cdot\,,\cdot)_{\mathcal{H}}\colon \mathcal{H}^k(W) \times \mathcal{H}^k(W) \to \mathbb{R}$, the tensor $\langle\cdot\,,\cdot\rangle\colon W^k \times W^{N-k} \to \mathbb{R}$ to harmonic vectors as well, $\mathcal{B}^k = \langle\cdot\,,\cdot\rangle_{\mathcal{H}} : \mathcal{H}^k(M) \times \mathcal{H}^{N-k}(M) \to \mathbb{R}$. Properties of the $*$-Hodge operator imply the commutative diagram

$$\begin{array}{rl} \langle\cdot\,,\cdot\rangle_{\mathcal{H}} : & \mathcal{H}^k(M) \times \mathcal{H}^{N-k}(M) \longrightarrow \mathbb{R} \\ & \qquad\qquad \Big\downarrow \mathrm{id} \times * \\ (\cdot\,,\cdot)_{\mathcal{H}} : & \quad \mathcal{H}^k(W) \times \mathcal{H}^k(W) \longrightarrow \mathbb{R}. \end{array}$$

3.4. *Signature and the Hirzebruch Operator*

Let $\left(W = \bigoplus_{k=0}^N W^k, \langle\cdot\,,\cdot\rangle, (\cdot\,,\cdot), d\right)$ be a graded differential Hodge space. Let $\varepsilon : \{0, 1, \ldots, N\} \to \{-1, 1\}$ be an arbitrary function such that $\varepsilon_k = \varepsilon_{N-k}$ for any k. Assume the ε-anticommutativity of $\langle\,,\rangle$. From the point of view of signature, we must consider an even N, $N = 2n$, and take $\varepsilon_n = +1$. Then $\langle\cdot\,,\cdot\rangle^n\colon W^n \times W^n \to \mathbb{R}$ and $\langle\cdot\,,\cdot\rangle^n_{\mathcal{H}} = \mathcal{B}^n\colon \mathcal{H}^n(W) \times \mathcal{H}^n(W) \to \mathbb{R}$ are symmetric and nondegenerate. Therefore, in cohomology, the tensor $\langle\cdot\,,\cdot\rangle^n_{\mathbf{H}}\colon \mathbf{H}^n\,(W) \times \mathbf{H}^n(W) \to \mathbb{R}$ is also symmetric, and it is an extension of $\langle\cdot\,,\cdot\rangle^n_{\mathcal{H}}$.

Definition 3.18. If $\dim \mathbf{H}^n(W) < \infty$, then we define the signature of W as the signature of $\langle\cdot\,,\cdot\rangle^n_{\mathbf{H}}$. $\mathrm{Sig}(W) := \mathrm{Sig}\,\langle\cdot\,,\cdot\rangle^n_{\mathbf{H}}$.

Remark 3.19. Under the assumption $W = \operatorname{Im}\Delta \oplus (\operatorname{Im}\Delta)^{\perp}$, we have $\mathcal{H}^n(W) \cong \mathbf{H}^n(W)$ and $\mathcal{B}^n = \langle\cdot\,,\cdot\rangle^n_{\mathbf{H}}$. Therefore, if $\dim \mathbf{H}^n(W) < \infty$, then $\mathrm{Sig}(W) = \mathrm{Sig}\,\langle\cdot\,,\cdot\rangle^n_{\mathbf{H}} = \mathrm{Sig}\,\mathcal{B}^n$ because $\langle\cdot\,,\cdot\rangle_{\mathcal{H}} = \langle\cdot\,,\cdot\rangle^n_{\mathbf{H}}$ under the identification $\mathbf{H}^k(W) = \mathcal{H}^k(W)$.

Remark 3.20. The condition $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$ does not imply the relation $\dim\ker\Delta < \infty$. Indeed, if $d = 0$, then $d^* = 0$, $\Delta = 0$, and $\ker\Delta = W$. Therefore, $W = 0 \oplus 0^{\perp}$, and $\dim\mathcal{H} = \dim\mathbf{H} = \dim W$ can be arbitrary.

In the construction of the Hirzebruch signature operator, the fundamental role is played by an operator (a small modification of the $*$-Hodge operator)

$$\tau\colon W \to W, \qquad \tau^k\colon W^k \to W^{N-k}, \qquad \tau^k(w) = \tilde{\varepsilon}_k \cdot *w, \quad \tilde{\varepsilon}_k \in \{-1, 1\}\,, \tag{3}$$

such that (i) $\tau \circ \tau = \mathrm{id}$, (ii) $d^* = -\tau \circ d \circ \tau$, and (iii) $\tau^n = *$, i.e., $\tilde{\varepsilon}_n = 1$ ($n = N/2$).

Let us verify the existence of τ and prove its uniqueness (assuming that $d^k \neq 0$). (If an odd N is considered, then one must admit complex $\tilde{\varepsilon}_k \in \mathbb{C}$ with the absolute value 1).

Theorem 3.21. *If $N = 2n$ and*

$$\varepsilon_k = (-1)^n(-1)^{k(N-k)} = (-1)^n(-1)^k, \tag{4}$$

then there is an operator τ satisfying (i)–(iii) *and given by $\tau^k(w^k) = (-1)^{k(k+1)/2}(-1)^{n(n+1)/2} \cdot *(w^k)$. Conversely, if $d^k \neq 0$ for all $k = 0, 1, \ldots, N-1$ and if τ exists, then ε_k is given by* (4). *The function ε satisfies the relation $\varepsilon_{k-1} = \varepsilon_{k+1}$.*

Proof. Easy calculations show that, for an arbitrary positive integer N, even or odd, the operator τ, which is defined by (3),

(a) satisfies condition (i) if and only if

$$\tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k} = \varepsilon_k, \qquad k \in \{0, 1, \ldots, N\}\,, \tag{5}$$

(b) and satisfies condition (ii) if and only if

$$\tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k+1} = (-1)^{k+1}\varepsilon_k, \qquad k \geqslant 1. \tag{6}$$

Let us now prove that, for a sequence $\varepsilon_k \in \{-1,1\}$, there is a complex sequence $\tilde{\varepsilon}_k \in \mathbb{C}$ satisfying (5) and (6) if and only if ε_k is given by the formula

$$\varepsilon_k = (-1)^{k(N-k)}(-1)^{N(N-1)/2}\,(\tilde{\varepsilon}_0)^2 \tag{7}$$

for some $\tilde{\varepsilon}_0 \in \{1,-1,i,-i\}$. Every value $\tilde{\varepsilon}_0 \in \{1,-1,i,-i\}$ defines a sequence $\tilde{\varepsilon}_k$ uniquely by the rule $\tilde{\varepsilon}_k = (-1)^{\frac{2N-k-1}{2}k}\tilde{\varepsilon}_0$.

Let us assume first that, for ε_k, there is an $\tilde{\varepsilon}_k \in \mathbb{C}$ satisfying (5) and (6). Substituting (5) into (6), we obtain (for $k = 1,2,\dots,N$) $\tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k+1} = (-1)^{k+1}\tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k}$, $\tilde{\varepsilon}_{N-k+1} = (-1)^{k+1}\tilde{\varepsilon}_{N-k}$ ($k \rightsquigarrow N-k+1$), and $\tilde{\varepsilon}_k = (-1)^{N-k}\tilde{\varepsilon}_{k-1}$. It follows that $\tilde{\varepsilon}_k = (-1)^{(N-k+N-1)k/2}\tilde{\varepsilon}_0$. Next $\varepsilon_0 \overset{(5)}{=} \tilde{\varepsilon}_0\tilde{\varepsilon}_N = (-1)^{N(N-1)/2}(\tilde{\varepsilon}_0)^2$ and $\varepsilon_k = \tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k} = (-1)^{k(N-k)}(-1)^{N(N-1)/2}(\tilde{\varepsilon}_0)^2$. Since $\varepsilon_k \in \{-1,1\}$, we have $\tilde{\varepsilon}_0 \in \{1,-1,i,-i\}$.

Conversely, let ε_k satisfy (7) and let $\tilde{\varepsilon}_k = (-1)^{(N-k+N-1)k/2}\tilde{\varepsilon}_0$. It can readily be seen that the condition $\tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k} = \varepsilon_k$ holds, and $\tilde{\varepsilon}_k\tilde{\varepsilon}_{N-k+1} = (-1)^{k+1}\varepsilon_k$. Adding k and noticing that $(-1)^{N(N-1)/2}\,(\tilde{\varepsilon}_0)^2 \in \{-1,1\}$, we see that (for $d^k \neq 0$) there are only two possibilities for ε_k for which there is a suitable τ, namely, $\varepsilon_k = (-1)^{k(N-k)}$ and $\varepsilon_k = -(-1)^{k(N-k)}$. If $N = 2n$ and $\varepsilon_n = +1$, then $\varepsilon_k = (-1)^{k(N-k)}(-1)^{N(N-1)/2}\,(\tilde{\varepsilon}_0)^2 = (-1)^k(-1)^{N(N-1)/2}(\tilde{\varepsilon}_0)^2$, $1 = \varepsilon_n = (\tilde{\varepsilon}_0)^2$. Therefore, $\tilde{\varepsilon}_0 \in \{-1,1\}$ and $\varepsilon_k = (-1)^k(-1)^{N(N-1)/2} = (-1)^k(-1)^n$, which yields two possible τ given by $\tilde{\varepsilon}_k = (-1)^{(2N-k-1)k/2}\tilde{\varepsilon}_0 = (-1)^{k(k+1)/2}\tilde{\varepsilon}_0$ and $\tilde{\varepsilon}_0 \in \{-1,1\}$. Finally, we must take $\tilde{\varepsilon}_0$ in such a way that $\tilde{\varepsilon}_n = +1$, and thus $\tilde{\varepsilon}_0 = (-1)^{n(n+1)/2}$. Therefore, $\tilde{\varepsilon}_k = (-1)^{k(k+1)/2}(-1)^{n(n+1)/2}$.

Example 3.22 [classical]. For differential forms, the ε-anticommutativity is defined by $\varepsilon_k = (-1)^{k(N-k)}$. Then the operator τ such that $\tau\circ\tau = \mathrm{id}$ and $d^* = -\tau\circ d\circ\tau$, exists, and we must take $(-1)^{N(N-1)/2}\,(\tilde{\varepsilon}_0)^2 = 1$, i.e., $(\tilde{\varepsilon}_0)^2 = (-1)^{N(N-1)/2}$. The operator τ is real if and only if $(-1)^{N(N-1)/2} = +1$, which is equivalent to $N = 4k$ or $N = 4k+1$. Note that $\tilde{\varepsilon}_0$ is then given by

$$\tilde{\varepsilon}_0 = \begin{cases} \pm 1 & \text{for } N = 4k \text{ or } N = 4k+1, \\ \pm i & \text{for } N = 4k+2 \text{ or } N = 4k+3. \end{cases}$$

We assume below that $N = 2n$ and $\varepsilon_k = (-1)^n(-1)^k$ and take the suitable operator τ. Denote by $W_\pm = \{w \in W;\ \tau w = \pm w\}$ the sets of eigenspaces corresponding to the eigenvalues $+1$ and -1 of τ. Note that $(d+d^*)\,[W_+] \subset W_-$.

Definition 3.23. The operator $D_+ = d + d^* \colon W_+ \to W_-$ is referred to as the *Hirzebruch signature operator.*

Remark 3.24. If $\dim\mathcal{H} < \infty$, then the index $\operatorname{Ind} D_+ := \dim_{\mathbb{R}}\ker(D_+) - \dim_{\mathbb{R}}\ker\left(D_+^*\right)$ is well defined (the dimensions are finite), $\ker(D_+) = W_+\cap\mathcal{H}(W)$, and analogously for the adjoint operator $(D_+)^* = D_- \colon W_- \to W_+$, $\ker(D_-) = W_-\cap\mathcal{H}(W)$.

Theorem 3.25 [Hirzebruch signature theorem]. *If* $\dim\mathcal{H} < \infty$, *then* $\operatorname{Ind} D_+ = \operatorname{Sig}(\mathcal{B}^n \colon \mathcal{H}^n(W) \times\mathcal{H}^n(W) \to \mathbb{R})$. *If, additionally,* $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^\perp$, *then* $\operatorname{Ind} D_+ = \operatorname{Sig} W$.

Proof. For the subspace $V \subset W$ stable under τ, $\tau\,[V] \subset V$, we write $V_+ = \{v \in V;\ \tau v = v\}$, and similarly $V_- = \{v \in V;\ \tau v = -v\}$. The mapping $\langle\cdot\,,\cdot\rangle^n_{\mathcal{H}} = \mathcal{B}^n \colon \mathcal{H}^n(W) \times \mathcal{H}^n(W) \to \mathbb{R}$ is nondegenerate. It can readily be seen that

(a) $\mathcal{H}^n(W) = V_1 \bigoplus V_2$ for $\mathcal{H}^n_+(W) = V_1 = \{\alpha \in \mathcal{H}^n(W);\ *\alpha = \alpha\}$ and $\mathcal{H}^n_-(W) = V_2 = \{\alpha \in \mathcal{H}^n(W);\ *\alpha = -\alpha\}$.

(b) The subspaces $\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)$ are τ-stable and, for $s = 0,1,\dots,n-1$, the mapping $\varphi_\pm \colon \mathcal{H}^s(W) \to \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_\pm$, $X \longmapsto \frac{1}{2}(X \pm \tau X)$, is an isomorphism of real spaces.

(c) The subspaces $W^s + W^{2n-s}$ are τ-invariant. Therefore, $W_\pm = \oplus_{s=0}^{n-1}\left(W^s + W^{2n-s}\right)_\pm \oplus W^n_\pm$, which implies $\ker D_+ = W_+ \cap \ker(d + d^* : W \to W) = W_+\cap\mathcal{H}(W) = \oplus_{s=0}^{n-1}\left(W^s + W^{2n-s}\right)_+ \oplus W^n_+ \bigcap \oplus_{s=0}^{n-1}\left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right) \oplus \mathcal{H}^n\,(W) = \oplus_{s=0}^{n-1}\left(\mathcal{H}^s(W) + \mathcal{H}^{4k-s}(M)\right)_+ \oplus \mathcal{H}^n(W)_+$, and thus

$(W_{\pm}^n \cap \mathcal{H}^n = \mathcal{H}_{\pm}^n$ since $\tau^n = *^n)$ $\dim\ker D_+ - \dim\ker D_+^* = \sum_{s=0}^{n-1} \dim_{\mathbb{R}} \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_+ + \dim_{\mathbb{R}} \mathcal{H}_+^n(W) - \sum_{s=0}^{n-1} \dim_{\mathbb{R}} \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_- - \dim_{\mathbb{R}} \mathcal{H}_-^n(W) = \dim_{\mathbb{R}} \mathcal{H}_+^n(W) - \dim_{\mathbb{R}} \mathcal{H}_-^n(W) = \operatorname{Sig}(\mathcal{B}^n)$.

4. FOUR FUNDAMENTAL EXAMPLES AND THEIR GENERAL SETTINGS

4.1. *Four Fundamental Examples*

In the previous section, we have described a general algebraic approach to the Hirzebruch signature operator. Thanks to it, the following four fundamental examples can be understood as special cases of the general setting. Here are the four examples of the spaces with grading and differential operator $\left(W = \oplus_{k=0}^N W^k, d\right)$ in which M is a connected compact oriented Riemannian manifold and W^k is equal to (1) $(\Omega^k(M),\, d_{dR})$ for $N = 4p$ [the *classical example*], (2) $(\Omega^k(A),\, d_A)$ for $N = m + n = 4p$, where A is a TUIO-Lie algebroid [the *Lie algebroid example*], (3) $(\Omega^k(M;E),\, d_\nabla)$, where $(E, (\,,)_0)$ is a flat vector bundle, $(\,,)_0$-symmetric nondegenerate parallel, for $N = 4p$ [the *Lusztig example*], (4) $(\Omega^k(M;E),\, d_\nabla)$, where $(E, \langle\,,\rangle_0)$ is a flat vector bundle, $\langle\,,\rangle_0$-symplectic parallel, for $N = 4p+2$ [the *Gromov example*]. In the above, in all cases, the sequences of differentials $\left\{d_{dR}^k\right\}$, $\left\{d_A^k\right\}$, $\left\{d_\nabla^k\right\}$ are elliptic complexes, $\dim \mathbf{H}^k(W) < \infty$, and the pairing $\mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$ is defined, which is symmetric in the middle degree $N/2$. Its signature, $\operatorname{Sig}(W)$, is defined to be the signature of W.

4.2. *General Setting of the Above Four Examples*

Let us give some applications of the above algebraic theory to vector bundles over manifolds. Other applications (to more general objects than manifolds) are probably available; see the last section.

Consider a graded vector bundle $\xi = \bigoplus_{k=0}^N \xi^k$ of Hodge spaces over a connected compact oriented Riemannian manifold M, $\left(\xi = \oplus_{k=0}^N \xi^k, \langle\,,\rangle, (\,,), d\right)$, where

1) $\langle\,,\rangle, (\,,)$ are fields of smooth 2-tensors in ξ such that $(\xi_x, \langle\,,\rangle_x, (\,,)_x)$ is a Hodge space, $x \in M$, with a $*$-Hodge operator $*_x\colon \xi_x \to \xi_x$; it is assumed that $\langle v, w\rangle = 0$ if $v \in \xi^r$, $w \in \xi^s$, and $r + s \neq N$, and that subbundles ξ^k are orthogonal with respect to $(\,,)$,
2) the ε-anticommutativity axiom holds, $\left\langle v^k, v^{N-k}\right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k\right\rangle$, where $\varepsilon_k \in \{-1, +1\}$.

By integration over M, introduce 2-linear tensors $\langle\langle,\rangle\rangle, ((,)) : \operatorname{Sec}(\xi) \times \operatorname{Sec}(\xi) \to \mathbb{R}$, $\langle\langle \alpha, \beta\rangle\rangle := \int_M \langle \alpha_x, \beta_x\rangle dM$ and $((\alpha, \beta)) := \int_M (\alpha_x, \beta_x) dM$. Then $((,))$ is a positive definite inner product on $\operatorname{Sec}(\xi)$, the $*$-Hodge operator is an isometry $((\alpha,\beta)) = ((*\alpha, *\beta))$, and $\langle\langle \alpha, \beta\rangle\rangle = ((\alpha, *\beta))$.

3) d is a differential on $\operatorname{Sec}(\xi)$, $d^2 = 0$, of degree $+1$, $d^k\colon \operatorname{Sec}\left(\xi^k\right) \to \operatorname{Sec}\left(\xi^{k+1}\right)$, such that, by definition,

3a) d^k are differential operators of first order,

3b) $\langle\langle dw, u\rangle\rangle = (-1)^{k+1} \langle\langle w, du\rangle\rangle$ for $w \in \operatorname{Sec}\left(\xi^k\right)$ and $u \in \operatorname{Sec}(\xi)$.

Therefore, $\left(\operatorname{Sec}(\xi) = \oplus_{k=0}^N \operatorname{Sec}(\xi)^k, \langle\langle\cdot,\cdot\rangle\rangle, ((\cdot,\cdot)), d\right)$ is a graded differential Hodge space. Then the adjoint operator $d^*\colon \operatorname{Sec}(\xi) \to \operatorname{Sec}(\xi)$, $((\alpha, d^*\beta)) = ((d\alpha, \beta))$, exists and $d^*\left(\alpha^k\right) = \varepsilon_k (-1)^k * d * \left(\alpha^k\right)$.

Theorem 4.1. *If $\left\{d^k\right\}$ is an elliptic complex, then the Laplacian Δ is a selfadjoint, nonnegative, and elliptic operator. Consequently,* $\operatorname{Sec}\xi = \operatorname{Im}\Delta \oplus (\operatorname{Im}\Delta)^\perp$, $\mathcal{H}(\operatorname{Sec}\xi) \cong \mathbf{H}(\operatorname{Sec}\xi, d)$, *and* $\dim \mathcal{H}(\operatorname{Sec}\xi) < \infty$. *If $N = 2n$ and $\varepsilon_k = (-1)^n(-1)^k$, then we obtain the Hirzubruch operator $D_+ = d + d^*\colon \operatorname{Sec}\xi_+ \to \operatorname{Sec}\xi_-$ and the relation* $\operatorname{Sig}\langle\langle,\rangle\rangle_{\mathbf{H}}^n = \operatorname{Ind} D_+$.

The ellipticity of Δ follows from [16, Remark 6.34]. The fact that the symbol $\sigma\left(D^*\right)_{(x,v)}$ of the adjoint operator of a first-order operator $D : \operatorname{Sec}\xi \to \operatorname{Sec}\eta$ is equal to $-\sigma(D)_{(x,v)}^*$ is well known and can readily be verified. Indeed, the symbol $\sigma(D)_{(x,v)}\colon \xi_x \to \eta_x$ is a linear mapping such that $D(fW)_x = \sigma(D)_{(x,(df)_x)}(W_x) + f(x)D(W)_x$ for $f \in C^\infty(M)$ and $W \in \operatorname{Sec}\xi$. Let D^* be the adjoint operator for D, i.e., $((D^*(V), W)) = ((V, D(W)))$ for $W \in \operatorname{Sec}\xi$ and $V \in \operatorname{Sec}\eta$. To prove that $\sigma\left(D^*\right)_{(x,v)} = -\sigma(D)_{(x,v)}^*$, we can note that, for f, W, and V as above,

$$\big((x \longmapsto -\sigma(D)^*_{(x,(df)_x)}(V_x) + f(x)D^*(V)_x, W)\big) = \int_M \big(-\sigma(D)^*_{(x,(df)_x)}(V_x) + f(x)D^*(V)_x, W_x\big)$$
$$= \int_M \big(-V_x, \sigma(D)_{(x,(df)_x)}(W_x)\big) + \int_M (V_x, D(fW)_x) = \int_M (V_x, f(x)D(W)_x) = ((fV, D(W)))$$
$$= ((D^*(fV), W)).$$

5. APPLICATIONS TO LIE ALGEBROIDS

In all four above examples, the complexes of differentials $\{d^k\}$, $\{d^k_A\}$, and $\{d^k_\nabla\}$ are elliptic because the sequences of symbols are exact.

Let us describe the four fundamental examples of graded differential Hodge space. The fundamental idea is as follows. We have a 2-tensor $\langle\, ,\rangle$ and intend to find a positive definite scalar tensor $(\, ,\,)$ for which the $*$-Hodge operator exists and is an isometry.

Example 5.1 [standard]. Let M be a compact oriented Riemannian manifold, $\dim M = 4p$.

(a) $W^k = \Omega^k(M) = \operatorname{Sec}\left(\wedge^k T^*M\right)$.

(b) $\langle\langle,\rangle\rangle^k \colon W^k \times W^{N-k} \to \mathbb{R}$, $(\alpha,\beta) \longmapsto \int_M \alpha\wedge\beta$,

$$\langle\langle\alpha,\beta\rangle\rangle^k = \int_M \alpha\wedge\beta = (-1)^{k(N-k)}\int_M \beta\wedge\alpha = \underbrace{(-1)^k}_{\varepsilon_k}\langle\langle\beta,\alpha\rangle\rangle^{N-k}.$$

In the middle degree $2p$, the tensor $\langle\langle,\rangle\rangle$ is symmetric.

(c) $d\colon W^k \to W^{k+1}$ is the differentiation of differential forms.

(d) $\langle\langle d\alpha,\beta\rangle\rangle = (-1)^{k+1}\langle\langle\alpha, d\beta\rangle\rangle$ for $\alpha\in W^k$, $\beta \in W^{N-(k+1)}$ (this holds since $\int_M d\,(\alpha\wedge\beta) = 0$).

With respect to the standardly defined inner product on $\wedge T^*_xM$, we have a finite-dimensional Hodge space $\left(\wedge T^*_xM, \langle\, ,\rangle_x, (\, ,)_x\right)$. By integrating over the Riemannian manifold M, we obtain 2-linear tensors $\langle\langle,\rangle\rangle$, $((,)) : \Omega(M)\times\Omega(M) \to \mathbb{R}$,

$$\langle\langle\alpha,\beta\rangle\rangle = \int_M \langle\alpha,\beta\rangle\, dM = \int_M \alpha\wedge\beta, \quad ((\alpha,\beta)) = \int_M (\alpha,\beta)\, dM,$$

and the relation $\langle\langle\alpha,\beta\rangle\rangle = ((\alpha, *\beta))$ holds, thus producing a graded Hodge space with a differential operator, $(\Omega(M), \langle\langle,\rangle\rangle, ((\, ,)), d)$. The signature $\operatorname{Sig} M = \operatorname{Sig}\langle\langle,\rangle\rangle^{2p}_{\mathbf{H}}$ can be calculated as the index of the Hirzebruch operator $D_+ = d_{dR} + d^*_{dR} : \Omega(M)_+ \to \Omega(M)_-$ (d^*_{dR} is the adjoint operator to d_{dR} with respect to the inner product $((\, ,))$).

Example 5.2 [5]. Let A be a transitive Lie algebroid over a compact oriented manifold M and let $\operatorname{rank} A = N = 4p = m+n$, $m = \dim M$, and $n = \dim \mathbf{g}_{|x}$. Assume that A is invariantly oriented by a volume tensor $\varepsilon \in \operatorname{Sec}\left(\wedge^n \mathbf{g}\right)$, which is invariant with respect to the adjoint representation Ad_A.

(a) $W^k = \Omega^k(A) = \operatorname{Sec}\left(\bigwedge^k A^*\right)$.

(b)
$$\langle\langle,\rangle\rangle^k : W^k\times W^{N-k} \to \mathbb{R}, \quad (\alpha,\beta) \longmapsto \int_M\int_A \alpha\wedge\beta,$$
$$\langle\langle\alpha,\beta\rangle\rangle^k = \int_M\int_A \alpha\wedge\beta = (-1)^{k(N-k)}\int_M\int_A \beta\wedge\alpha = \underbrace{(-1)^k}_{\varepsilon_k}\langle\langle\beta,\alpha\rangle\rangle^{N-k}.$$

This tensor is symmetric in the middle degree $2p$.

(c) $d_A\colon W^k \to W^{k+1}$ is the differentiation of A-differential forms.

(d) $\langle\langle d_A\alpha,\beta\rangle\rangle = (-1)^{k+1}\langle\langle\alpha, d_A\beta\rangle\rangle$ for $\alpha\in W^k$, $\beta\in W^{N-(k+1)}$.

There is an inner product $((\, ,))$ in $W = \Omega(A)$ such that $(\Omega(A), \langle\langle,\rangle\rangle, ((\, ,)))$ is a graded Hodge space with a differential. Indeed [6], let G' be an arbitrary Riemannian tensor in $\mathbf{g} = \ker \#_A$. Then the volume tensor $\varepsilon_{G'}$ of G' is equal to $f\cdot\varepsilon$ for some smooth function $f>0$. The tensor $G := f^{-\frac{2}{n}}G'$ is a Riemannian tensor in $\mathbf{g}$ for which ε is the volume tensor. Let G_2 be any Riemannian tensor on M. Taking an arbitrary connection $\lambda\colon TM \to A$ on A and the horizontal space $H = \operatorname{Im}\lambda \subset A$, we obtain $A = \mathbf{g}\bigoplus H$. Define a Riemannian tensor G on $A = \mathbf{g}\bigoplus H$ such that $\mathbf{g}$ and H are orthogonal. On $\mathbf{g}$, we have G_1, and on H, the pullback λ^*G_2. The vector bundle A is oriented (since $\mathbf{g}$ and M are oriented). At each point $x \in M$, we consider the above inner product G_x on $A_{|x}$ and the multiplication of tensors $\langle\, ,\rangle_x^k\colon \wedge^k A_x^* \times \wedge^{N-k} A_x^* \longrightarrow \wedge^N A_x^* \xrightarrow{\rho_x} \mathbb{R}$, where ρ_x is defined using the volume form for G_x.

We can note that $\rho_x = \rho_{G_2 x} \circ \oint_{A_x}$. The inner product G_x in A_x can be extended to an inner product on $\wedge A_x^*$, and we obtain the classical finite-dimensional Hodge space $\left(\bigwedge A_x^*, \langle\, ,\rangle_x, (\, ,)_x\right)$ and two $C^\infty(M)$-tensors $\langle\, ,\rangle, (\, ,) : \Omega(A) \times \Omega(A) \to C^\infty(M)$ defined as above, pointwise. Integrating over M, we obtain a graded Hodge space with differential operator $(\Omega(M), \langle\langle, \rangle\rangle, ((\, ,)), d_A)$. The tensor $\langle\langle, \rangle\rangle$ induces a 2-tensor in cohomology, $\langle\langle, \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$, which is symmetric in the middle degree $\langle\langle, \rangle\rangle_{\mathbf{H}}^{2p} : \mathbf{H}^{2p}(M) \times \mathbf{H}^{2p}(M) \to \mathbb{R}$. The dimension $\dim \mathbf{H}(A)$ is finite (Kubarski and Mishchenko [7], 2003). Therefore, the signature of $\langle\langle, \rangle\rangle_{\mathbf{H}}^{2p}$ can be calculated as the index of the Hirzebruch operator $D_+ = d_A + d_A^* : \Omega(A)_+ \to \Omega(A)_-$, where d_A^* stands for the adjoint to d_A with respect to the inner product $((\, ,))$.

Our previous considerations concerning the signature of a Lie algebroid using the *Hochschild-Serre spectral sequence* of A permit us to calculate the signature of a Lie algebroid by using another Hirzebruch operator, following the Lusztig and Gromov examples.

Example 5.3 [Lusztig [10] (1971), Gromov [4] (1995). Signature for flat bundles]. Let M be a compact oriented $N = 4p$-dimensional manifold and let $E \to M$ be a flat bundle equipped with a flat covariant derivative ∇ and a nondegenerate indefinite symmetric tensor $G_0 = (\, ,\,)_0\colon E \times E \to M \times \mathbb{R}$, $(\, ,)_{0x}\colon E_x \times E_x \to \mathbb{R}$, constant for ∇, i.e., satisfying the relation $\partial_X (\sigma, \eta)_0 = (\nabla_X \sigma, \eta)_0 + (\sigma, \nabla_X \eta)_0$. Take

- $W^k = \Omega^k(M; E)$,
- the differential operator $d_\nabla\colon W^k \to W^{k+1}$ defined standardly by using ∇.

Since $\nabla G_0 = 0$, it follows that $d(\alpha \wedge_{G_0} \beta) = d_\nabla \alpha \wedge_{G_0} \beta + (-1)^{|\alpha|} (\alpha \wedge_{G_0} d_\nabla \beta)$, and therefore, if $|\alpha| + |\beta| = N-1$, then $\int_M (d_\nabla \alpha) \wedge_{G_0} \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge_{G_0} (d_\nabla \beta)$. Define the duality $\langle\langle \cdot, \cdot \rangle\rangle^k\colon W^k \times W^{N-k} \to \mathbb{R}$ by the rule $\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge_{G_0} \beta$; one can see that the condition $\langle\langle d_\nabla \alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$ is satisfied. Since G_0 is symmetric, we have

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge_{G_0} \beta = \int_M (-1)^{k(N-k)} \beta \wedge_{G_0} \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}.$$

This tensor is symmetric in the middle degree. There is an inner product $((\, ,))$ on W^k for which the $*$-Hodge operator for $(W, \langle\langle, \rangle\rangle, ((\, ,)))$ is an isometry. Indeed ([10, 4]), choose some positive-definite inner product $(\, ,)'$ on E. Then we take a unique splitting $E = E_+ \oplus E_-$ of the vector bundle E which is both $(\, ,\,)_0$- and $(\, ,\,)'$-orthogonal and such that $(\, ,\,)_0$ is positive definite on E_+ and $(\, ,)_0$ is negative definite on E_-. Denote by τ the involution $\tau\colon E \to E$ ($\tau^2 = \mathrm{id}$) such that $\tau|E_+ = \mathrm{id}$ and $\tau|E_- = -\mathrm{id}$. Then the quadratic form $(v, w) = (v, \tau w)_0$ is symmetric and positive definite, and $(E_x, (\, ,)_{0x}, (\, ,)_x)$ is a Hodge space.

In each fiber $\wedge T_x^* M \otimes E_x$, we introduce the tensor product of the classical Hodge space $\wedge T_x^* M$ and the above one in E_x. Pointwise, we obtain tensors $\langle\, ,\rangle,\ (\, ,) : \Omega(M; E) \times \Omega(M; E) \to C^\infty(M)$, $*\colon \Omega(M; E) \to \Omega(M; E)$, such that $\langle \alpha, \beta \rangle = (\alpha, *\beta)$ and, integrating over M, we obtain a Hodge space $(\Omega(M; E), \langle\langle, \rangle\rangle, ((\, ,)))$,

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle dM = \int_M \alpha \wedge_{G_0} \beta, \qquad ((\alpha, \beta)) = \int_M (\alpha, \beta) dM,$$

with $*\colon \Omega(M) \to \Omega(M)$, $*(\alpha)(x) = *_x(\alpha_x)$, and $\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$. Let d_∇^* be the adjoint operator to d_∇ with respect to $((\, ,))$. The tensor $\langle\langle, \rangle\rangle$ induces a 2-tensor in cohomology, $\langle\langle, \rangle\rangle_{\mathbf{H}}\colon \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$, which is symmetric in the middle degree $\langle\langle, \rangle\rangle_{\mathbf{H}}^{2p}\colon \mathbf{H}^{2p}(M; E) \times \mathbf{H}^{2p}(M; E) \to \mathbb{R}$, and its signature is the index of the Hirzebruch operator $D_+ = d_\nabla + d_\nabla^* : \Omega(M; E)_+ \to \Omega(M; E)_-$.

Example 5.4 [Gromov (1995) [4]. Signature for a symplectic bundle]. Let M be a compact oriented manifold M of dimension $\dim M = N = 4p+2$ and let $E \to M$ be a symplectic vector bundle (i.e., equipped with a flat covariant derivative ∇ and parallel symplectic structure [i.e., skew symmetric and nondegenerate] $S = \langle\,,\rangle \colon E \times E \to M \times \mathbb{R}$ and $\nabla S = 0$). Take

- $W^k = \Omega^k(M;E)$,
- and let $d_\nabla \colon W^k \to W^{k+1}$ be the differential operator defined by ∇.

The condition $\int_M (d_\nabla \alpha) \wedge_S \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge_S d_\nabla \beta$ holds for $|\alpha| + |\beta| = N-1$.

The form $\langle\langle \cdot , \cdot \rangle\rangle^k \colon W^k \times W^{N-k} \to \mathbb{R}$ is defined by $\langle\langle \alpha, \beta\rangle\rangle^k = \int_M \alpha \wedge_S \beta$, and the relation $\langle\langle d_\nabla \alpha, \beta\rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta\rangle\rangle$ holds. Since S is skew symmetric, we have $\alpha \wedge_S \beta = -(-1)^{k(N-k)} \beta \wedge_S \alpha$ and

$$\langle\langle \alpha, \beta\rangle\rangle^k = \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha\rangle\rangle^{N-k}.$$

There is an inner product $((\,,))$ on W^k for which $(W, \langle\langle,\rangle\rangle, ((\,,)))$ is a Hodge space. Namely [15, p. 56], there exists an anti-involution τ on E, $\tau^2 = -\tau$ (i.e., a complex structure), such that

- $\langle \tau v, \tau w\rangle = \langle v, w\rangle$, $v, w \in E_x$,
- $\langle v, \tau v\rangle > 0$ for all $v \neq 0$.

Then the tensor $(v,w) := \langle v, \tau w\rangle$ is symmetric, positive definite, and $(\tau v, \tau w) = (v,w)$, i.e., τ preserves both the forms $\langle\,,\rangle$ and $(\,,)$. The operator $-\tau$ is the $*$-Hodge operator on $(E_x, \langle\,,\rangle_x, (\,,)_x)$. Hence, the system $(E_x, \langle\,,\rangle_x, (\,,)_x)$ is a Hodge space.

At each point $x \in M$, we take the tensor product $\bigwedge T_x^* M \bigotimes E_x$ of the classical Hodge space $\bigwedge T_x^* M$ and the above E_x. Similarly to the above example, we obtain a graded Hodge space $(\Omega(M;E), \langle\langle,\rangle\rangle, ((\,,)), d)$ with a differential (where the $*$-Hodge operator is defined pointwise by $* \colon \Omega(M) \to \Omega(M)$, $*(\alpha)(x) = *_x(\alpha_x)$). Passing to the cohomology, we obtain $\langle\langle,\rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$,

$$\langle\langle \alpha, \beta\rangle\rangle^k = \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha\rangle\rangle^{N-k},$$

which is symmetric in the middle degree $2p+1$ (because $\langle\,,\rangle$ is skew symmetric),

$$\langle\langle\,,\rangle\rangle_{\mathbf{H}}^{2p+1} \colon \mathbf{H}^{2p+1}(M;E) \times \mathbf{H}^{2p+1}(M;E) \to \mathbb{R},$$

and $\langle\langle \alpha, \beta\rangle\rangle^{2p+1} = -(-1)^{2p+1} \langle\langle \beta, \alpha\rangle\rangle^{2p+1} = \langle\langle \beta, \alpha\rangle\rangle^{2p+1}$. We can calculate the signature of the form $\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p+1}$ as the index of the Hirzebruch operator $D_+ = d_\nabla + d_\nabla^* : \Omega(M;E)_+ \to \Omega(M;E)_-$.

Example 5.5. As a consequence, for a TUIO-Lie algebroid A over a compact oriented manifold M for which $m = \dim M$, $n = \operatorname{rank} \mathbf{g} = \dim \mathbf{g}_x$, and under the assumption that $\mathbf{H}^{m+n}(A) \neq 0$ and $m+n = 4p$, we have two Hirzebruch signature operators.

(I) The first one is $D_+ = d_A + d_A^* : \Omega(A)_+ \to \Omega(A)_-$, where d_A^* is the adjoint to d_A with respect to the inner product $((\alpha,\beta)) = \int_M (\alpha,\beta)$ defined in Example 5.2 above, and $W_\pm = \{\alpha \in \Omega(A);\ \tau\alpha = \pm\alpha\}$ for $\tau(\alpha^k) = (-1)^{k(k+1)/2}(-1)^p \cdot *(\alpha^k)$.

(II) To introduce the other one, we use the equality $\operatorname{Sig} \mathbf{H}(A) = \operatorname{Sig} E_2$ for the second term E_2, $E_2^{p,q} = \mathbf{H}^p_{\nabla^q}(M; \mathbf{H}^q(\mathbf{g}))$, of the Hochschild–Serre spectral sequence. The flat covariant derivative ∇^q in the cohomology vector bundle $\mathbf{H}^q(\mathbf{g})$ depends on the structure of the Lie algebroid A.

Let $m+n = 4p$. The signature $\operatorname{Sign} E_2$ is equal to the signature of the quadratic form $E_2^{2p} \times E_2^{2p} \to E_2^{m+n} = \mathbb{R}$, and,

a) if n is odd, then $\operatorname{Sig} E_2 = 0$;

b) if n is even, then $\operatorname{Sig} E_2 = \operatorname{Sig}\big(E_2^{m/2,n/2} \times E_2^{m/2,n/2} \to E_2^{m+n} = E_2^{m,n} = \mathbb{R}\big)$, where $E_2^{m/2,n/2} = \mathbf{H}^{m/2}_{\nabla^{n/2}}\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big)$. Consider the form $\langle\langle,\rangle\rangle \colon \mathbf{H}_{\nabla^{n/2}}\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big) \times \mathbf{H}_{\nabla^{n/2}}\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big) \to \mathbb{R}$,

$\langle\langle , \rangle\rangle^k : \mathbf{H}^k_{\nabla^{n/2}}\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big) \times \mathbf{H}^{m-k}_{\nabla^{n/2}}\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big) \to \mathbf{H}^m_{\nabla^n}\big(M; \mathbf{H}^n(\mathbf{g})\big) = \mathbb{R}$, which is symmetric in the middle degree $k = m/2$. The signature of this form is equal to the signature of A. For $k = n$, the bundle $\mathbf{H}^n(\mathbf{g})$ is trivial, $\mathbf{H}^n(\mathbf{g}) \cong M \times \mathbb{R}$, the connection ∇^n is equal to ∂, and the multiplication of values is taken with respect to the pairing $\langle , \rangle : \mathbf{H}^{n/2}(\mathbf{g}) \times \mathbf{H}^{n/2}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$.

We have $m/2 + n/2 = 2p$. Consider two different cases.

(a) Let $m/2$ and $n/2$ be even. Then the form $\mathbf{H}^{n/2}(\mathbf{g}) \times \mathbf{H}^{n/2}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$ is symmetric, and we can use Example 5.3 (of Lusztig type) to obtain the Hirzebruch signature operator $D_+ = d_{\nabla^{n/2}} + d^*_{\nabla^{n/2}} : \Omega_+\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big) \to \Omega_-\big(M; \mathbf{H}^{n/2}(\mathbf{g})\big)$.

(b) Let $m/2$ and $n/2$ be odd. Then the form $\mathbf{H}^{n/2}(\mathbf{g}) \times \mathbf{H}^{n/2}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$ is symplectic, and we can use Example 5.4 (of Gromov type) to obtain the Hirzebruch signature operator $D_+ = d_{\nabla^{n/2}} + d^*_{\nabla^{n/2}}$.

For each of the cases, the index of D_+ is equal to the signature of A. Therefore, the Atiyah–Singer formula for the index can be used.

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