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CHARACTERISTIC CLASSES OF SOME PRADINES-TYPE GROUPOIDS AND A GENERALIZATION OF THE BOTT VANISHING THEOREM

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ABSTRACT. This paper contains an application of characteristic classes of some Pradines-type groupoids over foliations, constructed by the author in [5]. Using these characteristic classes, we obtain a generalization of the Bott Vanishing Theorem to a flag $\{\mathfrak{F},\mathfrak{F}'\}$ of foliations $\mathfrak{F}$ and $\mathfrak{F}'$. The classical Bott Theorem follows from the above generalization if we put $\mathfrak{F}=\{V\}$.

Key words: the Bott Vanishing Theorem, the Chern-Weil homomorphism, Lie groupoid, Pradines-type groupoid over foliation, Lie algebroid.

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1. Pradines-type groupoids $\Phi^{\mathfrak{F}}$ and their Lie algebroids. There is a well-known definition of a Lie groupoid (see [8]) as a transitive groupoid

$$\Phi=(\Phi,\alpha,\beta,V,\cdot)$$

in which Φ and V are Hausdorff C^{∞}-manifolds, the mappings $\alpha,\beta:\Phi\longrightarrow V$ (called a source and a target) are submersions, and $^{-1}:\Phi\longrightarrow\Phi$, $u:V\longrightarrow\Phi$ and $\cdot:\Phi*\Phi\longrightarrow\Phi$ — defined by the formulae:

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${}^{-1}(h)=h^{-1}$, $u(x)=u_x$ (u_x - the unit over x), $\cdot(g,h)=g\cdot h$ ($\Phi*\Phi=\{(g,h)\in\Phi\times\Phi;\alpha g=\beta h\}$ is a proper submanifold of $\Phi\times\Phi$) — are of C^∞-class.

Any vector bundle F over V determines the Lie groupoid

$$\mathbf{GL}(F)=(\overline{GL(F)},\alpha,\beta,V,\cdot)$$

of all linear isomorphisms between fibres of F in which α, β and $\cdot$ are defined by $\alpha(h)=x$ and $\beta(h)=y$ iff $h:F_{|x}\xrightarrow{\approx}F_{|y}$, and $g\cdot h=g\circ h$ if $\alpha(g)=\beta(h)$.

Let Φ be any Lie groupoid over a manifold V and $\mathcal{F}$ - any foliation of V. Take a subgroupoid

$$\Phi^{\mathcal{F}}=(\Phi^{\mathcal{F}},\alpha^{\mathcal{F}},\beta^{\mathcal{F}},V,\cdot)$$

consisting of all elements of Φ such that the source and the target lie on the same leaf of $\mathcal{F}$. More precisely, $\Phi^{\mathcal{F}}=(\alpha,\beta)^{-1}[R]$ where $R\subset V\times V$ is the equivalence relation given by xRy iff $y\in L_x$. (L_x - the leaf of $\mathcal{F}$ through x). If $\mathcal{F}=\{V\}$, then $\Phi^{\mathcal{F}}=\Phi$. In general $\Phi^{\mathcal{F}}$ is not a submanifold of Φ. Denote by C the set of all real-valued functions defined on $\Phi^{\mathcal{F}}$ which can be locally extended to C^∞-functions on Φ (i.e. $C=C^\infty(\Phi)_{\Phi^{\mathcal{F}}}$, see [9]). C is a differential structure on $\Phi^{\mathcal{F}}$ and the pair $(\Phi^{\mathcal{F}},C)$ (further denoted briefly by $\Phi^{\mathcal{F}}$) is a <u>differential space</u> in the sense of R. Sikorski (see [9]). All operations in the groupoid $\Phi^{\mathcal{F}}$ are smooth in the category of differential spaces.

Because of the submersivity of $\alpha:\Phi\to V$, the set $\alpha^{-1}(x)$, $x\in V$, forms a proper C^∞-submanifold of Φ denoted by Φ_x. Φ_x constitutes a principal fibre bundle (for brevity p.f.b.) over V with the projection $\beta_x:\Phi_x\to V$, $h\mapsto\beta h$, the isotropy Lie group $G_x=\beta_x^{-1}(x)$ as the structural Lie group, and the action

$\Phi_x\times G_x\to\Phi_x$, $(h,a)\mapsto h\cdot a$.

For the leaf L_x of $\mathcal{F}$ through x, on the set

$$\Phi_x^{\mathcal{F}} := \beta_x^{-1}[L_x]$$

there exists exactly one C^∞-manifold structure such that if U is open in L_x and $L_{x|U}$ is a proper submanifold of V, then $\beta_x^{-1}[U]$ is open in $\Phi_x^{\mathcal{F}}$ and $\Phi^{\mathcal{F}}_{x|\beta_x^{-1}[U]}$ is a proper submanifold of Φ_x. Of course, $\Phi_x^{\mathcal{F}}$ is an immerse submanifold of Φ_x and $\beta_x^{\mathcal{F}}:\Phi_x^{\mathcal{F}} \longrightarrow L_x$, $h \longmapsto \beta h$, is a submersion. Besides, $\Phi_x^{\mathcal{F}}$ forms a p.f.b. over L_x analogously. For each $h \in \Phi^{\mathcal{F}}$, the mapping $D_h:\Phi^{\mathcal{F}}_{\beta h} \longrightarrow \Phi^{\mathcal{F}}_{\alpha h}$, $g \longmapsto g\cdot h$, is a diffeomorphism.

With the groupoid $\Phi^{\mathcal{F}}$ we associate a vector bundle

$$(A(\Phi^{\mathcal{F}}), p, V)$$

where $A(\Phi^{\mathcal{F}}) = \bigcup_{x\in V} T_{u_x}\Phi_x^{\mathcal{F}} \subset T\Phi$ and $p(v) = x$ iff $v \in T_{u_x}\Phi_x^{\mathcal{F}}$, $x \in V$. Moreover,

$$\tilde{\beta}^{\mathcal{F}}_*: A(\Phi^{\mathcal{F}}) \longrightarrow T\mathcal{F},\ v \longmapsto \beta_* v,$$

is an epimorphism. Therefore, it is not difficult to see that $\Phi^{\mathcal{F}}$ is a <u>Pradines-type groupoid over the foliation</u> $\mathcal{F}$ (see [4], [5]).

A smooth vector field X on the differential space $\Phi^{\mathcal{F}}$ (see [9]) is called <u>right-invariant</u> if

(a) $X_h \in T_h \Phi^{\mathcal{F}}_{\alpha h}$, $h \in \Phi^{\mathcal{F}}$,

(b) $(D_h)_{*g} X_g = X_{gh}$, $g,h \in \Phi^{\mathcal{F}}$, $\alpha g = \beta h$.

Each right-invariant vector field X on $\Phi^{\mathcal{F}}$ determines a C^∞-section $X_o: V \longrightarrow A(\Phi^{\mathcal{F}})$, $x \longmapsto X(u_x)$, of p. Conversely (see [5]),

<u>PROPOSITION 1</u>. For any C^∞-section $\xi: V \to A(\Phi^{\mathcal{F}})$ of p, there exists exactly one right-invariant vector field ξ' on $\Phi^{\mathcal{F}}$ such that $\xi'(u_x) = \xi(x)$, $x \in V$. The bracket $[\![\xi,\eta]\!] := [\xi',\eta']_o$ defines in the vector space Sec $A(\Phi^{\mathcal{F}})$ of all C^∞-sections of p a real Lie algebra structure.

PROPOSITION 2. The system

$$\mathcal{A}(\Phi^{\mathcal{F}}) = (A(\Phi^{\mathcal{F}}), [\![\cdot,\cdot]\!], \tilde{\beta}_*^{\mathcal{F}})$$

is a Lie algebroid (in the sense of J.Pradines [6], [7]).

With the Lie algebroid $\mathcal{A}(\Phi^{\mathcal{F}})$ we associate a short sequence of vector bundles over V

$$0 \longrightarrow \mathfrak{g} \overset{j}{\hookrightarrow} A(\Phi^{\mathcal{F}}) \xrightarrow{\gamma} T\mathcal{F} \longrightarrow 0$$

where γ denotes, for brevity, the mapping $\tilde{\beta}_*^{\mathcal{F}}$, and $\mathfrak{g} := \operatorname{Ker}\gamma$. Of course, $\mathfrak{g}$ is independent of the choice of $\mathcal{F}$. Each fibre $\mathfrak{g}_{|x}$, $x\in V$, is a Lie algebra with respect to the bracket $[v,w] := [\![\xi,\eta]\!](x)$ where $\xi,\eta\in \operatorname{Sec} A(\Phi^{\mathcal{F}})$ are such that $\xi(x)=v$, $\eta(x)=w$. Moreover, $\mathfrak{g}_{|x}$ is the Lie algebra of the isotropy Lie group G_x.

If $\Phi = GL(F)$, then $\mathfrak{g}$ is canonically isomorphic to $\operatorname{Hom}(F;F)$.

2. CONNECTIONS IN $\mathcal{A}(\Phi^{\mathcal{F}})$.

By a connection in $\mathcal{A}(\Phi^{\mathcal{F}})$ (see [5]) we mean any mapping

$$\lambda : T\mathcal{F} \longrightarrow A(\Phi^{\mathcal{F}})$$

such that $\gamma\circ\lambda = \mathrm{id}_{T\mathcal{F}}$.

PROPOSITION 3. Connections in $\mathcal{A}(\Phi^{\mathcal{F}})$ are in one-to-one correspondence with partial connections in the p.f.b. Φ_x projectable onto $T\mathcal{F}$ (for definition of a partial connection see [3]).

Proof. A connection λ determines a partial connection H^λ in Φ_x by the formula $H^\lambda_h := \operatorname{Im}((D_h)_{*u_{\beta h}} \circ \lambda_{|\beta h})$. The correspondence $\lambda \longmapsto H^\lambda$ is the sought-for bijection. q.e.d.

For a connection λ in $\mathcal{A}(\Phi^{\mathcal{F}})$, the uniquely determined morphism

$$\omega : A(\Phi^{\mathcal{F}}) \longrightarrow \mathfrak{g}$$

fulfilling $\omega|\mathfrak{g} = \mathrm{id}$ and $\omega|\operatorname{Im}\lambda = 0$ is called a connection form of λ.

Let F be any vector bundle over V. By a <u>C^∞-form of degree q on $T\mathfrak{F}$ with values in F</u> we shall mean each C^∞-section of the bundle

$$\Lambda^q(T\mathfrak{F})^* \otimes F.$$

The set

$$\Omega(T\mathfrak{F};F)$$

of all such forms is a graded module over $C^\infty(V)$. Moreover, it has a structure of a module over the algebra

$$\Omega(T\mathfrak{F};\mathbb{R})$$

of all real-valued C^∞-forms on $T\mathfrak{F}$.

By a <u>curvature base-form</u> (or a <u>curvature tensor</u>) of a connection λ we shall mean the form

$$\Omega_B \in \Omega^2(T\mathfrak{F};\mathfrak{g})$$

defined by the formula $\Omega_B(X,Y) = -\omega([\![\lambda\circ X,\lambda\circ Y]\!])$, $X,Y\in \mathrm{Sec}\,T\mathfrak{F}$.

<u>PROPOSITION 4.</u> $[\![\lambda\circ X,\lambda\circ Y]\!] = \lambda\circ[X,Y] - \Omega_B(X,Y)$.

<u>3. The Chern-Weil homomorphism for $\Phi^{\mathfrak{F}}$.</u> The groupoid $\Phi^{\mathfrak{F}}$ acts on the bundle $\mathfrak{g}$ by the <u>adjoint representation $Ad^{\mathfrak{F}}$</u> defined by

$$Ad^{\mathfrak{F}}(h) = (\tau_h)_{*u_x} : \mathfrak{g}_{|x} \xrightarrow{\approx} \mathfrak{g}_{|y}, \quad h\in\Phi^{\mathfrak{F}},$$

where $\tau_h : G_x \longrightarrow G_y$, $a \longmapsto hah^{-1}$, $x=\alpha h$, $y=\beta h$.

Let $\overset{k}{\vee}\mathfrak{g}^*$ be the k-symmetric power of $\mathfrak{g}^*$. Denote by $(Ad^{\mathfrak{F}})^{\vee}$ the action of $\Phi^{\mathfrak{F}}$ on $\overset{k}{\vee}\mathfrak{g}^*$ induced by $Ad^{\mathfrak{F}}$. A section

$$\Gamma\in \mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}^*$$

is called <u>$Ad^{\mathfrak{F}}$-invariant</u> if $(Ad^{\mathfrak{F}})^{\vee}(h)(\Gamma_{\alpha h}) = \Gamma_{\beta h}$ for each $h\in\Phi^{\mathfrak{F}}$. The set of all $Ad^{\mathfrak{F}}$-invariant sections of the bundle $\overset{k}{\vee}\mathfrak{g}^*$ is denoted by

$$(\mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}^*)_I^{\mathfrak{F}}.$$

Of course, $\bigoplus^k(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I^{\mathcal{F}}$ forms an algebra.

If $\mathcal{F}=\{V\}$, then the letter $\mathcal{F}$ in the symbols $\mathrm{Ad}^{\mathcal{F}}$, $(\mathrm{Ad}^{\mathcal{F}})^{\vee}$ and $(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I^{\mathcal{F}}$ will be omitted.

We have $(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I\subset(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I^{\mathcal{F}}$.

PROPOSITION 5. Each $\mathrm{Ad}^{\mathcal{F}}$-invariant section of $\overset{k}{\vee}\mathfrak{g}^*$ is equal to $\sum_i f^i\Gamma_i$ for some C^{∞}-functions f^i constant along the leaves of $\mathcal{F}$ and for some Ad-invariant sections Γ_i.

PROPOSITION 6. The algebra $\bigoplus^k(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I$ of all Ad-invariant sections is canonically isomorphic to the algebra $(\vee\mathfrak{g}_{|x}^*)_I$ of all invariant polynomials on $\mathfrak{g}_{|x}$ with respect to the adjoint representation of G_x on $\mathfrak{g}_{|x}$, $x\in V$. This isomorphism is built with the help of the family of isomorphisms $\mathrm{Ad}(h)$, $h\in\Phi_x$.

Let $\lambda: T\mathcal{F}\longrightarrow A(\Phi^{\mathcal{F}})$ be any connection in $\mathcal{A}(\Phi^{\mathcal{F}})$ and $\Omega_B\in\Omega^2(T\mathcal{F};\mathfrak{g})$ - its curvature base-form. We define the following homomorphism of algebras

$$\gamma^{\mathcal{F}}:\bigoplus^k(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I^{\mathcal{F}}\longrightarrow\Omega(T\mathcal{F};\mathbb{R}),\quad \Gamma\longmapsto\Gamma_*(\Omega_B,\dots,\Omega_B),$$

where $\Gamma\in\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*$ is treated as a symmetric k-linear homomorphism $\mathfrak{g}\times\dots\times\mathfrak{g}\longrightarrow\mathbb{R}$ via the isomorphism $\overset{k}{\vee}\mathfrak{g}^*\cong\mathcal{L}_s^k(\mathfrak{g};\mathbb{R})$,

$$t_1\vee\dots\vee t_k\longmapsto((v_1,\dots,v_k)\longmapsto\frac{1}{k!}\sum_{\sigma}t_{\sigma(1)}(v_1)\cdot\ldots\cdot t_{\sigma(k)}(v_k))$$

and $\Gamma_*(\Omega_B,\dots,\Omega_B)\in\Omega^{2k}(T\mathcal{F};\mathbb{R})$ is defined by the formula

$$\Gamma_*(\Omega_B,\dots,\Omega_B)(x;v_1,\dots,v_{2k})$$
$$=\frac{1}{2^k}\sum_{\sigma}\operatorname{sgn}\sigma\,\Gamma_x(\Omega_B(x;v_{\sigma(1)},v_{\sigma(2)}),\dots,\Omega_B(x;v_{\sigma(2k-1)},v_{\sigma(2k)})).$$

Now, we define a differential operator

$$d^{T\mathcal{F}}:\Omega(T\mathcal{F};\mathbb{R})\longrightarrow\Omega(T\mathcal{F};\mathbb{R})$$

by (for a form Θ of degree q)

$$d^{T\mathcal{F}}\Theta(X_0,\dots,X_q)=\sum_{j=0}^{q}(-1)^j X_j(\Theta(X_0,\dots,\hat{X}_j,\dots,X_q))$$
$$+\sum_{i<j}(-1)^{i+j}\Theta([X_i,X_j],\dots,\hat{X}_i,\dots,\hat{X}_j,\dots).$$

Let $H(T\mathcal{F};\mathbb{R})$ be the cohomology algebra of the complex

$$(\Omega(T\mathcal{F};\mathbb{R}),d^{T\mathcal{F}}).$$

We have the following two theorems which are particular cases of general theorems on the theory of cohomology of Pradines--type groupoids over foliations (see [6]).

<u>THEOREM 1.</u> $d^{T\mathcal{F}}\circ\gamma^{\mathcal{F}}=0.$

This theorem allows us to define the following homomorphism of algebras

$$h_{\Phi^{\mathcal{F}}}:\bigoplus^k(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I^{\mathcal{F}}\longrightarrow H(T\mathcal{F};\mathbb{R}),\quad \Gamma\longmapsto[\gamma^{\mathcal{F}}(\Gamma)].$$

<u>THEOREM 2.</u> $h_{\Phi^{\mathcal{F}}}$ is independent of the choice of connection.

<u>DEFINITION.</u> We shall call the homomorphism $h_{\Phi^{\mathcal{F}}}$ the <u>Chern--Weil homomorphism for $\Phi^{\mathcal{F}}$</u>. Its image $\mathrm{Im}\,h_{\Phi^{\mathcal{F}}}$ is called the <u>Pontryagin algebra for $\Phi^{\mathcal{F}}$</u> and denoted by

$$\mathrm{Pont}(\Phi^{\mathcal{F}}).$$

<u>REMARK.</u> If $\mathcal{F}=\{V\}$, then the superposition

$$(\vee\mathfrak{g}_{|x}^{*})_I\cong\bigoplus^k(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}^*)_I\longrightarrow H(TV;\mathbb{R})=H_{dR}(V)$$

is the classical Chern-Weil homomorphism for the p.f.b. Φ_x.

<u>4. A generalization of the Bott Vanishing Theorem.</u> As an application of the characteristic classes described above we give

<u>THEOREM 3.</u> (<u>A generalization of the Bott Vanishing Theorem</u>)

Let $\{\mathcal{F},\mathcal{F}'\}$ be a flag of foliations $\mathcal{F}$ and $\mathcal{F}'$ on V. If

$$T\mathcal{F}=T\mathcal{F}'\oplus F$$

then $\mathrm{Pont}^k(\mathbf{GL}(F)^{\mathcal{F}})=0$ for $k>2\cdot\mathrm{rank}\,F$.

Proof. A connection λ in $GL(F)^{\mathcal{F}}$ defines (see [5]) an operator

(1) $$\nabla^{\lambda}: \mathrm{Sec}\, F \longrightarrow \Omega^1(T\mathcal{F}; F)$$

by the formula

(2) $$\nabla^{\lambda}_{v}\sigma = (\lambda v)(\tilde{\sigma}_x), \quad v \in (T\mathcal{F})_{|x}, \ \sigma \in \mathrm{Sec}\, F,$$

where $\tilde{\sigma}_x := \tilde{\sigma} | GL(F)_x : GL(F)_x \longrightarrow F_{|x}$ and $\tilde{\sigma} : GL(F) \longrightarrow F$, $h \mapsto h^{-1}(\sigma_{\beta h})$

∇^{λ} is linear and fulfils the conditions (a) $\nabla^{\lambda}_{fX}\sigma = f\nabla^{\lambda}_{X}\sigma$, (b) $\nabla^{\lambda}_{X} f\sigma = X(f)\sigma + f\nabla^{\lambda}_{X}\sigma$ where $X \in \mathrm{Sec}\, T\mathcal{F}$, $\sigma \in \mathrm{Sec}\, F$, $f \in C^{\infty}(V)$.

Any linear operator (1) such that (a) and (b) hold is called a covariant derivative in $(F, \mathcal{F})$ (or, after [3], a partial connection in F with respect to $\mathcal{F}$).

LEMMA 1. The correspondence $\lambda \longmapsto \nabla^{\lambda}$ establishes a bijection between connections in $\mathcal{A}(GL(F)^{\mathcal{F}})$ and covariant derivatives in $(F, \mathcal{F})$.

Proof of lemma 1. It is sufficient to show that

(i) for $v \in (T\mathcal{F})_{|x}$, $x \in V$, a vector $\lambda v \in A(GL(F)^{\mathcal{F}})_{|x}$ satisfying $\Upsilon(\lambda v) = v$ is, by (2), uniquely determined,

(ii) the mapping $\lambda : T\mathcal{F} \longrightarrow A(GL(F)^{\mathcal{F}})$, $v \longmapsto \lambda v$, is of the C^{∞}--class.

(i) and (ii) easily follow from local calculations.

By a curvature tensor of a covariant derivative ∇ in $(F, \mathcal{F})$ we mean a tensor

$$R \in \Omega^2(T\mathcal{F}; \mathrm{Hom}(F; F))$$

defined by the formula $R_{X,Y}\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma$ for $X, Y \in \mathrm{Sec}\, T\mathcal{F}$, $\sigma \in \mathrm{Sec}\, F$.

LEMMA 2. The curvature tensor R of ∇^{λ} is equal to the curvature base-form Ω_B of λ.

Proof of lemma 2. By proposition 4, we have

$$R_{X,Y}\sigma = \lambda X((\lambda Y)(\tilde{\sigma})^{\sim}) - \lambda Y((\lambda X)(\tilde{\sigma})^{\sim}) - (\lambda[X,Y])(\tilde{\sigma})$$
$$= ([\![\lambda X,\lambda Y]\!] - \lambda[X,Y])(\tilde{\sigma}) = -\Omega_B(X,Y)(\tilde{\sigma}) = \Omega_B(X,Y)(\sigma).$$

Continuing the proof of theorem 3, we construct (analogously to Bott [1]) a covariant derivative in $(F,\mathcal{F})$ whose curvature tensor R has the property $R_{X,Y}=0$ for all $X,Y\in \operatorname{Sec} T\mathcal{F}'$. For the purpose, take any covariant derivative $\bar{\nabla}$ in $(F,\mathcal{F})$. For $X\in \operatorname{Sec} T\mathcal{F} = \operatorname{Sec} T\mathcal{F}'\oplus \operatorname{Sec} F$, write $X=X_{\mathcal{F}'}+X_F$ where $X_{\mathcal{F}'}\in \operatorname{Sec} T\mathcal{F}'$ and $X_F\in \operatorname{Sec} F$.

Then define $\nabla_X\sigma = \pi[X_{\mathcal{F}'},\sigma] + \bar{\nabla}_{X_F}\sigma$ for $X\in \operatorname{Sec} T\mathcal{F}$, $\sigma\in \operatorname{Sec} F$, where $\pi: T\mathcal{F}'\oplus F \longrightarrow F$ is the projection onto the second factor. It is not difficult to see that this formula defines a covariant derivative in $(F,\mathcal{F})$ which fulfils therequirement condition. By lemma 1, there exists a connection λ in $\mathcal{A}(GL(F)^{\mathcal{F}})$ such that $\nabla^{\lambda}=\nabla$. By lemma 2, the curvature base-form Ω_B of λ has the property $\Omega_B(X,Y)=0$ for all $X,Y\in \operatorname{Sec} T\mathcal{F}'$.

Using the decomposition $(T\mathcal{F})_{|x} = (T\mathcal{F}')_{|x}\oplus F_{|x}$, we see that $\gamma^{\mathcal{F}}(\Gamma)=0$ for $\Gamma\in(\operatorname{Sec}\bigvee^{k}\mathfrak{g}^{*})^{\mathcal{F}}_{I}$ such that $k>\operatorname{rank}F$. q.e.d.

REMARK. Let $\Phi = GL(F)$ and let F be as in theorem 3. By remark 33 from [5], we have that the Chern-Weil homomorphisms h_{Φ_x}, $h_{\Phi_x^{\mathcal{F}}}$ of p.f.b.'s Φ_x and $\Phi_x^{\mathcal{F}}$, respectively, and $h_{\Phi^{\mathcal{F}}}$ of the Pradines-type groupoid $\Phi^{\mathcal{F}}$ are connected by the commuting diagram

$$\begin{tikzcd}
(\bigvee^{k}(\mathfrak{g}_{|x})^{*})_I \arrow[r, "h_{\Phi_x}"] \arrow[d] \arrow[dd, bend right=60, "id"'] & H^{2k}_{dR}(V) \arrow[d] \\
(\operatorname{Sec}\bigvee^{k}\mathfrak{g}^{*})^{\mathcal{F}}_{I} \arrow[r, "h_{\Phi^{\mathcal{F}}}"] \arrow[d] & H^{2k}(T\mathcal{F};\mathbb{R}) \arrow[d] \\
(\bigvee^{k}(\mathfrak{g}_{|x})^{*})_I \arrow[r, "h_{\Phi_x^{\mathcal{F}}}"] & H^{2k}_{dR}(L_x)\,.
\end{tikzcd}$$

For $k>\operatorname{rank}F$, the bottom row is zero by the classical Bott Vanishing Theorem applied to the foliation $\mathcal{F}'|L_x$ of L_x, $x\in V$, which - of course - also follows from the vanishing of the middle row.

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