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THE LIE GROUPOIDS AND LIE ALGEBROIDS IN DIFFERENTIAL GEOMETRY OF HIGHER ORDER

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1. Many problems of the classical global differential geometry can be considered from the unifying point of view of the theory of connections in principal fibre bundles. Differential geometry of higher order has been developing since the beginning of the 50's. It examines objects which locally depend on partial derivatives of higher order, for example: higher order connections, G-structures of higher order, pseudogroups. The first construction of higher order was the osculating spaces to a curve $f:(a,b)\to\mathbb{R}^3$. Many notions describing these objects have the form of differential operators, called now the Lie equations. Inquiries made by Ehresmann, Que, Spencer, Pradines, Kolař, Kumpera and others gave one view including all these problems, namely, the theory of the Lie groupoids and the Lie algebroids. The useful and indispensable technique is the jet theory. Notion of the Lie groupoid associated with a principal fibre bundle was introduced by Ehresmann in 1950.

2. The differentiable groupoids are from the formal point of view the generalization of the Lie groups. Many ideas and methods are derived from this fact. For example, the algebraization similar to that one of Lie groups is possible.

According to Ehresmann's idea every principal fibre bundle $P(M,G)$ determines a certain Lie groupoid consisting of the diffeomorphism of a fibre into another fibre. The algebraic structure in the groupoid is automatically determined by the composition of these diffeomorphisms. This groupoid is denoted by PP^{-1}. A Lie groupoid consisting of linear isomorphisms of a fibre into a fibre can be associated with any vector bundle E. This groupoid is denoted by $\pi(E)$. Ngo Van Que in 1967 formulated a precise abstract definition of the Lie groupoid.

Definition 1. *Lie groupoid is a collection* $\Phi=\left(\breve{\Phi},(\alpha,\beta),M,\cdot\right)$ *in which we have:*

a) *the space of the groupoid,* $\breve{\Phi}$*, it is a manifold of class* C^∞ *with countable basis,*
b) *the connected manifold of units,* M*,*
c) *the mapping "sources",* $\alpha:\breve{\Phi}\to M$*, and "target",* $\beta:\breve{\Phi}\to M$*, they are surmersions (surmersion=submersion&onto),*
d) *the partial multiplication,* $\cdot:D\to\breve{\Phi}$*, where* $D=\{(h,g);\beta(g)=\alpha(h)\}$ *is a submanifold of the manifold* $\breve{\Phi}\times\breve{\Phi}$*,*
e) *the algebraic structure is a structure of the groupoid in the sense of Ehresmann,*
f) *condition of transitivity that means:* $(\alpha,\beta):\breve{\Phi}\to M\times M$ *is surjective.*

The set $\Phi_{(x,x)}$ of the elements $h \in \breve{\Phi}$ such that $\alpha(h) = \beta(h) = x$ is called the isotropy group of the Lie groupoid Φ over x. It is a Lie group.

The unit of the Lie group $\Phi_{(x,x)}$ is called the unit of Lie groupoid over the point x. The Lie group is the Lie groupoid with one-element manifold of the units. For every $x \in M$ Lie groupoid $\Phi = \left(\breve{\Phi}, (\alpha, \beta), M, \cdot\right)$ determines a principal fibre bundle Φ_x in the following way: the set Φ_x consists of the element $h \in \breve{\Phi}$ such that $\alpha(h) = x$. Φ_x is a submanifold of M. The projection $\gamma : \Phi_x \to M$ is determined by the formula $\gamma = \beta | \Phi_x$, the Lie group $\Phi_{(x,x)}$ is taken and the action of this group on Φ_x is determined by the formula $\cdot (h, g) = h \cdot g$. One of the most important Lie groupoids is $\pi^k(M)$, the Lie groupoid of all invertible k-th order jets of manifold M. This groupoid determines a principal fibre bundles $L^k(M)$ of the k-order frames of manifold M.

a) G-structure on the manifold M is a subbundle of the bundle $L^1(M)$. It determines a Lie subgroupoid of the Lie groupoid $\pi^1(M)$. Similarly G-structure of higher order are defined.
b) Let Γ be a pseudogroupe of a local diffeomorphisms of the manifold M. Let $j^k\Gamma$ be a set of the k-order jets of elements of Γ. If a natural number k exists, for which the set $j^k\Gamma$ is an analytic Lie subgroupoid of the Lie groupoid $\pi^k(M)$ and Γ is the set of all solutions of $j^k\Gamma$, then Γ is called a Lie pseudogroup.

3. Miss Libermann in 1959 showed essential relations between the Lie groupoid $\pi^k(M)$ and a vector bundle $J^k(TM)$ of k-order jets of the vector field on the manifold M.

There is one-to-one relation between the cross-sections of the vector bundle $J^k(TM)$ and right-invariant vector fields on the Lie groupoid $\pi^k(M)$. The right-invariant vector fields are determined on every Lie groupoid. It is easy to state that a right-invariant vector field is uniquely determined by the values at the units.

4. The structures on the vector bundle $J^k(TM)$:

a) We take the vector space $C^\infty\left(J^k(TM)\right)$ consisting of all global sections of the bundle $J^k(TM)$. We take two sections ξ and η belonging to $C^\infty\left(J^k(TM)\right)$. These ξ and η determines a right-invariants vector fields ξ' and η' on the manifold $\pi^k(M)$. The bracket $[\xi', \eta']$ of this vector fields is a right-invariant vector field, too. This determines a global section of the vector bundle $J^k(TM)$, which is denoted by $[\![\xi, \eta]\!]$. We obtain the $\mathbb{R}$-Lie algebra structure in the space $C^\infty\left(J^k(TM)\right)$.
b) We define the morphism $\tilde{\beta}_* : J^k(TM) \to TM$ by the formula $\tilde{\beta}_*\left(j^k_x\theta\right) = \theta(x)$. Then
   (i) $\tilde{\beta}_*$ is an epimorphism,
   (ii) $C^\infty\left(\tilde{\beta}_*\right) : C^\infty\left(J^k(TM)\right) \to C^\infty(TM)$ is a $\mathbb{R}$-Lie algebra homomorphism. Which means that
$$\left[\tilde{\beta}_* \circ \xi, \tilde{\beta}_* \circ \eta\right] = \tilde{\beta}_* \circ [\![\xi, \eta]\!] \text{ for } \xi, \eta \in C^\infty\left(J^k(TM)\right),$$
   (iii) $[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + \left(\tilde{\beta}_* \circ \xi\right)(f) \cdot \eta$ for $\xi, \eta \in C^\infty\left(J^k(TM)\right)$ and $f \in C^\infty(M)$.

The object $\left(J^k(TM), [\![\cdot,\cdot]\!], \tilde{\beta}_*\right)$ is called the Lie algebroid of the Lie groupoid $\pi^k(M)$ (the notion was defined by J.Pradines).

5. J.Pradines constructed a similar object for an arbitrary Lie groupoid $\Phi = \left(\breve{\Phi}, (\alpha,\beta), M, \cdot\right)$.

Let us consider the vector bundle $i^*\left(T^\alpha\breve{\Phi}\right)$, it is first – the vector subbundle $T^\alpha\breve{\Phi} \subset T\breve{\Phi}$ of the tangent bundle $T\breve{\Phi}$ consisting of the α-vertical vectors and pull it back by imbedding $i : M \to \breve{\Phi}$. Algebraic structures can be introduced easier than in the case of $\pi^k(M)$.

a) a global section of the vector bundle $i^*\left(T^\alpha\breve{\Phi}\right)$ is an α-vertical vector field which is defined on the units 1_x, $x \in M$. The vector field is uniquely extended to globally defined right-invariant vector field on the $\breve{\Phi}$. We introduce the $\mathbb{R}$-Lie algebra structure on the vector space $C^\infty\left(i^*\left(T^\alpha\breve{\Phi}\right)\right)$ consisting of the global sections of the vector bundle $i^*\left(T^\alpha\breve{\Phi}\right)$. Let us consider two sections ξ,η and we take the bracket $[\xi',\eta']$, where ξ',η' are the right-invariant vector fields for which $\xi'(1_x) = \xi(x)$ and $\eta'(1_x) = \eta(x)$. We put $[\![\xi,\eta]\!] = [\xi',\eta'] | M$.
b) We define the morphism $\tilde{\beta}_* : i^*\left(T^\alpha\breve{\Phi}\right) \to TM$ by the formula $\tilde{\beta}_*(v) = \beta_*(v)$, where $\beta : \breve{\Phi} \to M$ is a mapping "target" in the Lie groupoid Φ.

We obtain the object $\left(i^*\left(T^\alpha\breve{\Phi}\right), [\![\cdot,\cdot]\!], \tilde{\beta}_*\right)$. It is a Lie algebroid of the Lie groupoid Φ.

Definition 2. *[Transitive] Lie algebroid is a collection* $(E, [\![\cdot,\cdot]\!], \gamma)$ *in which*

a) *E is a vector bundle over the manifold M,*
b) *$[\![\cdot,\cdot]\!] : C^\infty(E) \times C^\infty(E) \to C^\infty(E)$ and $(C^\infty(E), [\![\cdot,\cdot]\!])$ is a $\mathbb{R}$-Lie algebra,*
c) *$\gamma : E \to TM$ is an epimorphism such that $C^\infty(\gamma) : C^\infty(E) \to C^\infty(TM)$ is $\mathbb{R}$-Lie algebra homomorphism,*
d) *if $\sigma,\tau \in C^\infty(E)$ and $f \in C^\infty(M)$, then*

$$[\![\sigma, f\cdot\tau]\!] = f\cdot[\![\sigma,\tau]\!] + (\gamma\circ\sigma)(f)\cdot\tau.$$

6. Let us consider Φ a Lie subgroupoid of the Lie groupoid $\pi^k(M)$. It determines a certain subbundle of the principal fibre bundle $L^k(M)$. We take the linear isomorphism $\lambda_x : J^k(TM)_{|x} \to T_{1_x}\left(\pi^k(M)_x\right)$ defined by Libermann by the formula $\lambda_x\left(j^k_x\theta\right) = \theta^k(1_x)$ where θ^k is k-th prolongation of θ. We put $E_x = (\lambda_x)^{-1}\left[T_{1_x}\left(\breve{\Phi}\right)\right]$, we obtain the vector subbundle E of $J^k(TM)$. The right-invariants vector field on the groupoid $\pi^k(M)$ tangent to submanifold $\breve{\Phi}$ determines a global section of the bundle E and conversely. The bundle E has the following properties:

(a) $\tilde{\beta}_*|E : E \to TM$ is an epimorphism,
b) $[\![\xi,\eta]\!] \in C^\infty(E)$ for $\xi,\eta \in C^\infty(E)$, that is $C^\infty(E)$ is a Lie subalgebra of the Lie algebra $C^\infty\left(J^k(TM)\right)$.

Conversely, if $E \subset J^k(TM)$ is a vector subbundle which has properties a) and b) then there exists exactly one the Lie subgroupoid of the Lie groupoid $\pi^k(M)$ such that it determines the subbundle E. The vector subbundle $E \subset J^k(TM)$ for which the property b) hold is called a Lie equation. Their theory is a well developed owing to work of Malgrange, Kumpera, Spencer, Goldschmidt, Que and others.

Example 1. *(1) Let A be m-dimensional vector subbundle of the tangent bundle TM and let us assume $n = \dim M$. It determines a G-structure consisting of frames $(v_1, ..., v_m, v_{m+1}, ..., v_n) \in L(M)_x$ for which $(v_1, ..., v_m)$ is the basis A_x, $x \in M$. This G-structure determines a certain Lie subgroupoid of the Lie groupoid $\pi^1(M)$. This Lie subgroupoid determines the vector subbundle $E \subset J^1(TM)$ for which*

$$E = \left\{ j^1_x \theta;\ (\mathcal{L}_\theta \xi)(x) \in A_{|x},\ \xi, \theta \in C^\infty(TM),\ x \in M \right\}.$$

(2) Let q be a metric tensor on the manifold M and P be a corresponding to it $O(n)$-structure. Thus

$$E = \left\{ j^1_x \theta;\ (\mathcal{L}_\theta q)(x) = 0,\ \theta \in C^\infty(TM) \right\}.$$

7. Let us consider a connection Γ in principal fibre bundle $L(M)$. Γ determines a covariant derivative, ∇, in the vector bundle TM. ∇ determines the differential operator of the order one by the formula

$$\nabla(\sigma) = (\tau \longmapsto \nabla_\tau \sigma), \quad \text{for } \tau, \sigma \in C^\infty(TM),$$

so ∇ determines a linear morphism

$$\delta : J^1(TM) \longrightarrow \mathrm{Hom}(TM, TM).$$

It is a splitting of the exact sequence

$$0 \longrightarrow \mathrm{Hom}(TM, TM) \underset{\delta}{\overset{\longrightarrow}{\longleftarrow}} J^1(TM) \longrightarrow TM \longrightarrow 0.$$

δ determines a splitting $C : TM \to J^1(TM)$, which is an interpretation of connection in terms of jets. C is a homomorphism of the Lie algebroid if and only if a covariant derivative ∇ has a curvature 0.

8. The Lie groupoid Φ determines the second very important object, namely, the groupoid β-admissible α-sections $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$. It consists of such local sections $\sigma : M_{|U} \to \breve{\Phi}$ of the surmersion $\breve{\Phi} \xrightarrow{\alpha} M$ for which $U' = \beta \circ \sigma[U] \subset M$ is an open set and $\beta \circ \sigma : M_{|U} \to M_{|U'}$ is a diffeomorphism.

The set of the unit of the groupoid $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$ is a topology of manifold M. The unit 1_U over the point $U \in \mathrm{Top}\, M$ is a mapping $1_U = \left(M_{|U} \ni x \mapsto 1_x \in \breve{\Phi} \right)$. Let $\Gamma_\alpha(M, \Phi)$ be a isotropy group of the groupoid $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$ over the unit M. The partial multiplication is defined by the following way: if $\sigma : M_{|U} \to \breve{\Phi}$, $\tau : M_{|U'} \to \breve{\Phi}$ belonging to $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$ and $\beta \circ \sigma[U] = U'$, then $\tau \cdot \sigma : M_{|U} \to \breve{\Phi}$ is defined by the formula

$$(\tau \cdot \sigma)_x = \tau_{\beta \circ \sigma(x)} \cdot \sigma_x, \quad x \in U.$$

The importance of the notion of the groupoid $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$ is based on the fact that there is one-to-one correspondence between a local sections of the bundle $i^*\left(T^a \breve{\Phi}\right)$ and a local smooth 1-parameter subgroups of the groupoid $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$.

Let us explain it on the example of global complete sections of the bundle $i^*\left(T^a \breve{\Phi}\right)$. Let $\xi : M \to i^*\left(T^a \breve{\Phi}\right)$ be a global complete section of the bundle

$i^*\left(T^a\breve{\Phi}\right)$. A global section ξ of the bundle $i^*\left(T^a\breve{\Phi}\right)$ is complete if the corresponding right-invariant vector field ξ' on the groupoid Φ is complete. Let $\varphi_t : \breve{\Phi} \to \breve{\Phi}$ be a global one-parameter subgroup of diffeomorphisms, which generates the vector field ξ'. We put $\operatorname{Exp} t\xi = \varphi_t \circ i : M \to \breve{\Phi}$.

$\operatorname{Exp} t\xi$ is a β-admissible α-section. It is an element of the group $\Gamma_\alpha(M,\Phi)$. It has the following properties:

(i) $\mathbb{R} \ni t \mapsto \operatorname{Exp} t\xi \in \Gamma_\alpha(M,\Phi)$ is a homomorphism of the additive group $\mathbb{R}$ into the group $\Gamma_\alpha(M,\Phi)$.

(ii) $M \times \mathbb{R} \ni (x,t) \mapsto (\operatorname{Exp} t\xi)(x) \in \breve{\Phi}$ is a smooth mapping.

Conversely, every group homomorphism $\Xi : \mathbb{R} \to \Gamma_\alpha(M,\Phi)$ such that the mapping $M \times \mathbb{R} \ni (x,t) \mapsto \Xi(t)(x) \in \breve{\Phi}$ is smooth is determined by exactly one global complete section of the bundle $i^*\left(T^a\breve{\Phi}\right)$. Then Ξ is called a global smooth one parameter subgroup of the groupoid $\Gamma_{\alpha,\mathrm{loc}}(M,\Phi)$.

9. Let $C_0^\infty\left(i^*\left(T^a\breve{\Phi}\right)\right)$ be a set of global section with compact support. Every such section is complete. The exponential mapping on the Lie groupoid is the mapping

$$\operatorname{Exp}_\Phi = \left(C_0^\infty\left(i^*\left(T^a\breve{\Phi}\right)\right) \ni \xi \longmapsto \operatorname{Exp} 1\xi \in \Gamma_\alpha(M,\Phi)\right).$$

This mapping was defined by Kumpera in 1971. The following theorem can be proved:

Theorem 1. *Let* $\xi_1, ..., \xi_m \in C_0^\infty\left(i^*\left(T^a\breve{\Phi}\right)\right)$ *be sections such that they are a basis of bundle* $i^*\left(T^a\breve{\Phi}\right)$ *over an open set* U *contained in* M*. Then, for every point* $x_0 \in U$*, there exist: open neighbourhood* $U_m \subset \mathbb{R}^m$ *containing* 0 *and open neighbourhood* $U' \subset U$ *containing* x_0 *such that the mapping*

$$\overline{\operatorname{Exp}} = \left(U_m \times U' \ni \left(a^1, .., a^m, x\right) \longmapsto \left(\operatorname{Exp}\sum_{i=1}^m a^i \cdot \xi_i\right)(x) \in \breve{\Phi}\right)$$

is a diffeomorphism onto an open set in $\breve{\Phi}$*.*

10. Using this theorem it can be proved that

Theorem 2. *A Lie subgroupoid* Ψ *of the Lie groupoid* Φ *is a topological subspace if and only if* $\breve{\Psi}$ *is a closed set.*

Besides, the following theorem holds.

Theorem 3. *If* A *is the Lie algebroid of the Lie groupoid* Φ*, then every Lie subalgebroid* $A' \subset A$ *determines exactly one connected Lie subgroupoid of the Lie groupoid* Φ*.*

Using the properties of the exponential mapping the following theorems can be proved:

Theorem 4. *If* Ψ *is a Lie subgroupoid of the Lie groupoid* Φ *then*

$$C_0^\infty\left(i^*\left(T^a\breve{\Psi}\right)\right) = \left\{\xi \in C_0^\infty\left(i^*\left(T^a\breve{\Phi}\right)\right);\ (\operatorname{Exp} t\xi)(x) \in \breve{\Psi} \text{ for } (x,t) \in M \times \mathbb{R}\right\}.$$

The following theorem was proved in a quite complicated way by N.V.Que in 1969.

Theorem 5. *If A and A' are the Lie algebroids of trivial Lie groupoids Φ and Φ', then every homomorphism $\sigma : A \to A'$ of these algebroids determines local homomorphism of the Lie groupoids $F : \Phi_{|U} \to \Phi'$, where U is some open set containing all units.*

Using the above theorems one can prove a stronger theorem for any Lie groupoids (non necessary trivial). The proof is simpler.

Let $\gamma : A \to A'$ be a homomorphism of the Lie algebroids and let $A = i^* \left(T^a \breve{\Phi}\right)$, $A' = i^* \left(T^\alpha \breve{\Phi}'\right)$. We put

$$\lambda = \{(v, \gamma(v)) \in A \boxplus A';\ v \in A\}$$

where $A \boxplus A'$ is the Whitney product of the Lie algebroids A and A'. Let ε be a connected Lie subgroupoid of the Lie groupoid $\Phi \boxplus \Phi'$ (where $\Phi \boxplus \Phi'$ is the Whitney product of the Lie groupoids) such that its Lie algebroid is equal to λ. The projection $\omega_1 : \Phi \boxplus \Phi' \to \Phi$ is a homomorphism of the Lie groupoids. We put $\omega_1' = \omega_1 | \varepsilon$. If $v \in i^* \left(T^a \breve{\Phi}\right)_{|x}$ then $\omega_1' (v, \gamma(v)) = v$, so

$$(\omega_1' | \varepsilon_x)_{*1_x} : T_{1_x} (\breve{\varepsilon}_x) \longrightarrow T_{1_x} \left(\breve{\Phi}_x\right)$$

is a linear isomorphism.

It is easy to see that

$$(\omega_1')_{*1_x} : T_{1_x} (\breve{\varepsilon}) \longrightarrow T_{1_x} \left(\breve{\Phi}\right)$$

is a linear isomorphism. Therefore ω_1' is a diffeomorphism in a neighbourhood of every unit 1_x, $x \in M$. Consequently, there exists a neighbourhood $\Theta \subset \breve{\varepsilon}$ which contains all units and such that $\omega_1' : \Theta \to \breve{\Phi}$ is a diffeomorphism onto an open set $\Omega \subset \breve{\Phi}$. ω_1' is a local isomorphism of the Lie groupoids. We put $F = \omega_2 \circ (\omega_1')^{-1} : \Omega \to \breve{\Phi}'$. It is the required local isomorphism of the Lie groupoids.

11. In the Lie group theory is known Yamabe's theorem. It is interesting whether the analogous theorem in the Lie groupoid theory holds. Namely:

Conjecture 1. *Let Φ be a Lie groupoid and Ψ its subgroupoid (in an algebraical sense). Then the subset $H \subset \breve{\Phi}$ consisting of the elements $h \in \breve{\Phi}$ for which there exists a global smooth 1-parameter subgroup Ξ of the groupoid $\Gamma_{\alpha,\mathrm{loc}}(M, \Phi)$ for which $\Xi(t)(x) \in \breve{\Psi}$ for $(x, t) \in M \times \mathbb{R}$ and $\Xi(t_0)(x_0) = h$ for certain $x_0 \in M$, $t_0 \in \mathbb{R}$, is a Lie subgroupoid.*

It seems that the technique necessary to examine this hypothesis will be a suitable version of Campbel-Hausdorff's formula.

In the Lie algebra theory is known the theorem of Ado: Every Lie algebra admits the linear exact and finitely dimensional representation. It is interesting whether the analogous theorem in the Lie algebroid theory holds.

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