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Hirzebruch signature operator for transitive Lie algebroids

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The aim of the paper is to construct Hirzebruch signature operator for transitive invariantly oriented Lie algebroids.

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1. Signature of Lie algebroids

1.1. *Definition of Lie algebroids, Atiyah sequence*

Lie algebroids appeared as infinitesimal objects of Lie groupoids, principal fibre bundles, vector bundles (Pradines, 1967), TC-foliations and non-closed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalents are known as Lie pseudo-algebras (Herz, 1953) called also further as Lie-Rinehart algebras (Huebschmann, 1990).

A *Lie algebroid* on a manifold M is a triple

$$A = (A, [\![\cdot,\cdot]\!], \#_A)$$

where A is a vector bundle on M, $(\operatorname{Sec} A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

$$\#_A : A \to TM$$

is a linear homomorphism (called the *anchor*) of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta, \quad f \in C^\infty(M),\ \xi,\eta \in \operatorname{Sec} A.$$

The anchor is bracket-preserving,[12]

$$\#_A \circ [\![\xi,\eta]\!] = [\#_A\circ\xi, \#_A\circ\eta].$$

A Lie algebroid is called *transitive* if $\#_A$ is an epimorphism.

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0,$$

$\boldsymbol{g} := \ker \#_A$. The fiber $\boldsymbol{g}_x$ of the bundle $\boldsymbol{g}$ in the point $x \in M$ is the Lie algebra with the commutator operation being

$$[v, w] = [\![\xi, \eta]\!](x), \quad \xi, \eta \in \operatorname{Sec} A, \quad \xi(x) = v, \eta(x) = w, \quad v, w \in \boldsymbol{g}_x.$$

The Lie algebra $\boldsymbol{g}_x$ is called the isotropy Lie algebra of A at $x \in M$. The vector bundle $\boldsymbol{g}$ is a Lie Algebra Bundle (LAB in short), called the *adjoint* of A, the fibres are isomorphic Lie algebras.

TM is a Lie algebroid with $id : TM \to TM$ as the anchor,

$\mathfrak{g}$ -finitely dimensional Lie algebra - is a Lie algebroid over a point $M = \{*\}$.

1.2. *Cohomology algebra, ellipticity of the complex of exterior derivatives* $\{d_A^k\}$

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \operatorname{Sec} \bigwedge A^*, \quad \text{- the space of cross-sections of } \bigwedge A^*$$

$$d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

$$\begin{aligned}(d_A\omega)(\xi_0, ..., \xi_k) = &\sum_{j=0}^{k} (-1)^j \ (\#_A \circ \xi_j)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \ \omega([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k),\end{aligned}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \operatorname{Sec} A$. The operators d_A^k satisfy

$$d_A(\omega \wedge \eta) = d_A\omega \wedge \eta + (-1)^k \omega \wedge d_A\eta,$$

so they are of first order and the symbol of d_A^k is equal to

$$S\left(d_A^k\right)_{(x,v)} : \bigwedge^k A_x^* \to \bigwedge^{k+1} A_x^*$$

$$S\left(d_A^k\right)_{(x,v)}(u) = (v \circ (\#_A)_x) \wedge u, \quad 0 \neq v \in T_x^* M.$$

In consequence the sequence of symbols

$$\bigwedge^{k} A_x^* \overset{S\left(d_A^k\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+1} A_x^* \overset{S\left(d_A^{k+1}\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+2} A_x^*$$

is exact if and only if A is transitive and then the complex $\left\{d_A^k\right\}$ is an elliptic complex.

The exterior derivative d_A introduces the cohomology algebra

$$\mathbf{H}\left(A\right) = \mathbf{H}\left(\Omega\left(A\right), d_A\right).$$

For the trivial Lie algebroid TM - the tangent bundle of the manifold M - the differential d_{TM} is the usual de-Rham differential d_M of differential forms on M whereas, for $L = \mathfrak{g}$ - a Lie algebra $\mathfrak{g}$ - the differential $d_{\mathfrak{g}}$ is the usual Chevalley-Eilenberg differential, $d_{\mathfrak{g}} = \delta_{\mathfrak{g}}$.

1.3. *Invariantly oriented Lie algebroids and signature*

The following theorem describes the class of transitive Lie algebroids (over compact oriented manifold) for which $\mathbf{H}^{\mathrm{top}}\left(A\right) \neq 0$.

Theorem 1.1 (Ref. 5). *For each transitive Lie algebroid* $(A, [\![\cdot,\cdot]\!], \#_A)$ *with the Atiyah sequence*

$$0 \to \boldsymbol{g} \to A \overset{\#_A}{\longrightarrow} TM \to 0,$$

over a compact oriented manifold M the following conditions are equivalent ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, *i.e.* $\operatorname{rank} A = m + n$ *)*

- *(a)* $\mathbf{H}^{m+n}\left(A\right) \neq 0$,
- *(b)* $\mathbf{H}^{m+n}\left(A\right) \cong \mathbb{R}$ *and* $\mathbf{H}\left(A\right)$ *is an Poincaré algebra, i.e. the pairing* $\mathbf{H}^{j}\left(A\right) \times \mathbf{H}^{m+n-j}\left(A\right) \to \mathbf{H}^{m+n}\left(A\right) \cong \mathbb{R}$ *is nondegenerate,* $\mathbf{H}^{j}\left(A\right) \cong \left(\mathbf{H}^{m+n-j}\left(A\right)\right)^*$,
- *(c) there exists a global nonsingular cross-section* $\varepsilon \in \operatorname{Sec}\left(\bigwedge^n \boldsymbol{g}\right)$ *invariant with respect to the adjoint representation ad_A, that is, A is the so-called a TUIO-Lie algebroid, see Ref. 3, (shortly, A is invariantly oriented),*
- *(d) the vector bundle $\boldsymbol{g}$ is orientable and the modular class of A is trivial,* $\theta_A = 0$.

We recall the definition of the isomorphism $\mathbf{H}^{m+n}\left(A\right) \cong \mathbb{R}$ (for invariantly oriented transitive Lie algebroids). In Ref. 3 there is defined (for arbitrary transitive Lie algebroids) the so-called *fibre integral operator*

$$\int_A : \Omega^{\bullet}\left(A\right) \to \Omega_{dR}^{\bullet - n}\left(M\right)$$

by the formula

$$\left(\fint_A \omega^k\right)_x (w_1, ..., w_{k-n}) = (-1)^{kn}\, \omega_x(\varepsilon_x, \tilde{w}_1, ..., \tilde{w}_{k-n}), \quad \#_A(\tilde{w}_i) = w_i,$$

where $\varepsilon \in \mathrm{Sec}\,(\bigwedge^n \boldsymbol{g})$ is a nonsingular cross-section. The operator $\fint_A$ commutes with the differentials d_A and d_M if and only if ε is invariant. Then, the fibre integral gives a homomorphism in cohomology

$$\fint_A^{\#} : \mathbf{H}^\bullet(A) \to \mathbf{H}_{dR}^{\bullet-n}(M).$$

Assume in the sequel that a transitive Lie algebroid A over compact oriented manifold M is invariantly oriented and $\varepsilon \in \mathrm{Sec}\,(\bigwedge^n \boldsymbol{g})$ is an invariant cross-section. The scalar Poincaré product

$$\mathcal{P}_A^k : \mathbf{H}^k(A) \times \mathbf{H}^{m+n-k}(A) \to \mathbb{R},$$

$$([\omega], [\eta]) \longmapsto \int_A \omega \wedge \eta \quad \left(:= \int_M \left(\fint_A \omega \wedge \eta\right)\right)$$

is defined and is nondegenerated;[4] in consequence

$$\begin{aligned}\mathbf{H}^k(A) &\cong \mathbf{H}^{m+n-k}(A),\\ \mathbf{H}^{m+n}(A) &\cong \left(\mathbf{H}^0(M)\right)^* = \mathbb{R},\\ \dim \mathbf{H}(M) &< \infty,\end{aligned}$$

and we can consider an isomorphism

$$\mathbf{H}^{m+n}(A) \cong \mathbb{R}, \quad [\omega] \longmapsto \int_A \omega.$$

The pairing of A-differential forms

$$\begin{aligned}\langle\langle \cdot, \cdot \rangle\rangle^k &: \Omega^k(A) \times \Omega^{m+n-k}(A) \to \mathbb{R}\\ \langle\langle \omega, \eta \rangle\rangle^k &= \int_M \fint_A \omega \wedge \eta\end{aligned} \tag{1}$$

has the property

$$\langle\langle \omega, \eta \rangle\rangle^k = (-1)^{k(m+n-k)} \langle\langle \eta, \omega \rangle\rangle^{m+n-k}$$

and

$$\langle\langle d_A\omega, \eta \rangle\rangle = (-1)^{k+1} \langle\langle \omega, d_A\eta \rangle\rangle \quad \text{for} \quad \omega \in \Omega^k(A),\ \eta \in \Omega^{m+n-(k+1)}(A).$$

If

$$m + n = 4p$$

then

$$\mathcal{P}_A^{2p} : \mathbf{H}^{2p}(A) \times \mathbf{H}^{2p}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and is called the signature of A, and is denoted by

$$\mathrm{Sign}\,(A)\,.$$

The problem is:[4]

- to calculate the signature Sign (A) and give some conditions to the equality Sign $(A) = 0$. There are examples for which Sign $(A) \neq 0$ (this is announced in Ref. 6).

2. $*$- operator and exterior coderivative d_A^*

2.1. *Associated scalar product and $*$-Hodge operator*

Consider

- any Riemannian tensor G_1 in the vector bundle $\boldsymbol{g} = \ker \#_A$ for which ε is the volume tensor (such a tensor exists).
- any Riemannian tensor G_2 on M.

Next, taking an arbitrary connection $\lambda : TM \to A$ in the Lie algebroid A i.e. a splitting of the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \underset{\lambda}{\overset{\#_A}{\rightarrow}} TM \longrightarrow 0,$$

and the horizontal space

$$H = \operatorname{Im}\lambda,$$
$$A = \boldsymbol{g} \bigoplus H$$

we define a Riemannian tensor G (called *scalar product associated* to ε) on $A = \boldsymbol{g} \bigoplus H$ such that $\boldsymbol{g}$ and H are orthogonal, on $\boldsymbol{g}$ we have G_1 but on H we have the pullback $\lambda^* G_2$. The vector bundle A is oriented (since $\boldsymbol{g}$ and M are oriented).

At each point $x \in M$ we consider the scalar product G_x on $A_{|x}$ and the pairing of tensors

$$\langle , \rangle_x^k : \bigwedge^k A_x^* \times \bigwedge^{m+n-k} A_x^* \to \bigwedge^{m+n} A_x^* \overset{\rho_x}{\to} \mathbb{R}$$

where ρ_x is defined via the volume form for G_x.

We can notice that ρ_x is equal to the composition

$$\begin{array}{lcc}
\bigwedge^{m+n} A_x^* & \xrightarrow{\rho_x} & \mathbb{R} \\
\downarrow (-1)^{(m+n)n}\, i_{\varepsilon_x} \searrow \fint_{A_p} & & \nearrow \rho_{G_2 x} \\
\bigwedge^{m} A_x^* & \xrightarrow{\rho_{\lambda x}} \bigwedge^{m} T_x^* M &
\end{array}$$

$$i_{\varepsilon_x}\omega_x\left((v_1,...,v_{k-n})\right) = \omega_x\left(\varepsilon_x, v_1, ..., v_{k-n}\right),$$
$$\rho_{\lambda x}\left(\omega_x\right)\left(w_1,...,w_{k-n}\right) = \omega_x\left(\varepsilon_x, \lambda_x\left(w_1\right),...,\lambda_x\left(w_{k-n}\right)\right).$$

Standardly, we can extend the scalar product G_x in A_x to a scalar product $(\cdot,\cdot)_x$ in $\bigwedge A_x^*$. There exists exactly one the so-called $*$-Hodge operator $*_x : \bigwedge^k A_x^* \to \bigwedge^{m+n-k} A_x^*$ such that

$$\langle \alpha_x, \beta_x \rangle = \left(\alpha_x, *_x\beta_x\right),$$

and it is given by

$$*_x\left(e_{i_1}^* \wedge ... \wedge e_{i_k}^*\right) = \operatorname{sgn}\left(j_1,...,j_{m+n-k}, i_1,..,i_k\right) e_{j_1}^* \wedge ... \wedge e_{j_{m+n-k}}^*$$

where $1 \leq i_1 < ... < i_k \leq m+n$, $1 \leq j_1 < ... < j_{m+n-k} \leq m+n$ and $\{i_1,...,i_k\} \cap \{j_1,...,j_{m+n-k}\} = \varnothing$ (for ON positive frame e_i in A_x). Using all points $x \in M$ we obtain two $C^\infty(M)$-linear 2-tensors

$$\langle , \rangle, (,) : \Omega(A) \times \Omega(A) \to C^\infty(M)$$

defined as above point by point. Integrating along M we get $\mathbb{R}$-linear 2-tensors

$$\langle\langle , \rangle\rangle, ((,)) : \Omega(A) \times \Omega(A) \to \mathbb{R}.$$

The first $\langle\langle , \rangle\rangle$ is, clearly, given by (1)

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \langle \omega, \eta \rangle = \int_M \fint_A \omega \wedge \eta.$$

The $*$-Hodge operator $* : \Omega(A) \to \Omega(A)$ is defined point by point

$$*(\omega)(x) = *_x\left(\omega_x\right)$$

and we have

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta)).$$

2.2. *Exterior coderivative*

We define exterior coderivative $d_A^* : \Omega^k(M) \to \Omega^{m+n-k}(A)$ by the formula

$$d_A^*(\omega) = (-1)^{k(m+n-k)}(-1)^k * d_A * (\omega), \quad \omega \in \Omega^k(A),$$

where $*$ is the $*$-Hodge operator in $\Omega(A)$. We have

(a)

$$* * (\omega) = (-1)^{k(m+n-k)} \cdot \omega, \quad \omega \in \Omega^k(A), \tag{2}$$

(b)

$$((d^*(\omega), \eta)) = ((\omega, d_A(\eta))),$$

i.e. d_A^* is adjoint to d_A with respect to the scalar product $((,))$ in $\Omega(A)$.

3. Laplacian and harmonic differential forms

It enables us to introduce the Laplacian

$$\Delta_A = (d_A + d_A^*)^2 = d_A d_A^* + d_A^* d_A.$$

Clearly

$$\ker \Delta_A = (\operatorname{Im} \Delta_A)^{\perp}$$

(with the respect to the scalar product $((,))$).

Proposition 3.1. *The Laplacian Δ_A is elliptic, self-adjoint and nonnegative operator. In consequence*

$$\Omega(A) = \ker \Delta_A \bigoplus \operatorname{Im} \Delta_A. \tag{3}$$

Proof. The first property follows from the ellipticity of the complex $\{d_A^k\}$ (namely: the symbol of the adjoint operator d_A^{*k} is equal to the minus of the adjoint symbol of d_A^k and to prove the ellipticity of the Laplacian we use Remark 6.34 from Ref. 8), the next two properties are trivial consequence of the definition. The last property (3) can be proved in the same way as the Theorem 55 from Ref. 7 using extension $\operatorname{Sec} \bigwedge A^*$ to the Hilbert Sobolev spaces $H_s(\bigwedge A^*)$, extension Δ to continuous operator $\Delta_s : H_s(\bigwedge A^*) \to H_{s-2}(\bigwedge A^*)$ and the fact that $\ker \Delta_s$ consists only of smooth sections, $\ker \Delta_s = \ker \Delta$. $\square$

A A-differential form $\omega \in \Omega(A)$ is called *harmonic* if $d_A\omega = 0$ i $d_A^*\omega = 0$. Denote the space of harmonic A-differential forms by $\mathcal{H}(A)$ and harmonic k- A-differential forms by $\mathcal{H}^k(A)$. $\mathcal{H}(A)$ is a graded vector space

$$\mathcal{H}(A) = \bigoplus_{k=0}^{m+n} \mathcal{H}^k(A)$$

and

$$\mathcal{H}(A) = \ker \Delta_A.$$

is the eigenspace of the operator Δ corresponding to the zero value of the eigenvalue.

Simple calculations assert that $\ker \Delta^k$ and $\operatorname{Im} d^{k-1}$ are orthogonal, therefore the inclusion

$$\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$$

induce a monomorphism

$$\ker \Delta^k \rightarrowtail \mathbf{H}^k(W). \tag{4}$$

Since $\operatorname{Im} \Delta^k \subset Imd_A^{k-1} + \operatorname{Im} d_A^{*(k+1)}$, the inclusion (3) yields $\Omega^k(A) = \ker \Delta_A^k + \operatorname{Im} d_A^{k-1} + \operatorname{Im} d_A^{*(k+1)}$ and easily we can notice that these three subspaces are orthogonal. Therefore

$$\Omega^k(A) = \ker \Delta_A^k \bigoplus \operatorname{Im} d_A^{k-1} \bigoplus \operatorname{Im} d_A^{*(k+1)}$$

and

$$\ker d^k = \ker \Delta^k \bigoplus \operatorname{Im} d^{k-1}$$

which implies the Hodge Theorem for Lie algebroids:

Corollary 3.1. *The monomorphism (4) is an isomorphism*

$$\mathcal{H}^k(W) \cong \ker d^k / \operatorname{Im} d^{k-1} = \mathbf{H}^k(W).$$

It means that in each cohomology class $\alpha \in \mathbf{H}^k(A)$ there is exactly one harmonic A-differential form $\omega \in \mathcal{H}^k(W)$.

Let $\varepsilon_k = (-1)^{k(n+m-k)}$. Simple calculations yields the equality

$$*\Delta_A\omega = \varepsilon_{k-1}\varepsilon_k (-1)^{n+m+1} \Delta_A * \omega, \quad \omega \in \Omega^k(A),$$

therefore

$$*\left[\mathcal{H}^k(W)\right] \subset \mathcal{H}^{m+n-k}(W),$$

and (thanks (2))

$$* : \mathcal{H}^k(W) \to \mathcal{H}^{m+n-k}(W)$$

is an isomorphism. In consequence we obtain (independently on Ref. 4) the Duality Theorem

$$\mathbf{H}^{k}(A) \simeq \mathbf{H}^{m+n-k}(A).$$

We restrict the scalar product $((\cdot,\cdot)) : \Omega^{k}(A) \times \Omega^{k}(A) \to \mathbb{R}$ to the space of harmonic A-differential forms

$$((\cdot,\cdot)) : \mathcal{H}^{k}(A) \times \mathcal{H}^{k}(A) \to \mathbb{R},$$

and we restrict the tensor $\langle\langle\cdot,\cdot\rangle\rangle : \Omega^{k}(A) \times \Omega^{m+n-k}(A) \to \mathbb{R}$ to harmonic A-differential forms

$$\mathcal{B}^{k} = \langle\langle\cdot,\cdot\rangle\rangle : \mathcal{H}^{k}(A) \times \mathcal{H}^{m+n-k}(A) \to \mathbb{R}.$$

Using the isomorphism $\mathcal{H}^{k}(A) \cong \mathbf{H}^{k}(A)$ we see that

$$\mathcal{B}^{k} = \mathcal{P}_{A}^{k},$$

therefore, if $m+n=4p$ then

$$\mathrm{Sign}\ (A) = \mathrm{Sign}\,\mathcal{B}^{2p}.$$

4. Hirzebruch signature operator

Assume $m+n = 4p$. Considering the direct sum $\mathcal{H}^{2p}(A) = \mathcal{H}_{+}^{2p}(A) \bigoplus \mathcal{H}_{-}^{2p}(A)$, where

$$\mathcal{H}_{\pm}^{2p}(W) = \left\{\omega \in \mathcal{H}^{2p}(W);\ *\omega = \pm\omega\right\},$$

and noticing that $\mathcal{B}^{2p}$ is positive on $\mathcal{H}_{+}^{2p}(A)$, and is negative on $\mathcal{H}_{-}^{2p}(A)$ we see that

$$\mathrm{Sign}\ \left(\mathcal{B}^{2p}\right) = \dim_{\mathbb{R}} \mathcal{H}_{+}^{2p}(W) - \dim_{\mathbb{R}} \mathcal{H}_{-}^{2p}(W).$$

To construction the Hirzebruch signature operator the fundamental role is played by an auxiliary operator

$$\tau : \Omega(A) \to \Omega(A)$$

defined by

$$\tau^{k}\left(\omega^{k}\right) = \tilde{\varepsilon}_{k} * \left(\omega^{k}\right), \quad \tilde{\varepsilon}_{k} \in \{-1,+1\}, \quad \omega^{k} \in \Omega^{k}(A),$$

fulfilling the properties

i) $\tau \circ \tau = Id$,
ii) $d_{A}^{*} = -\tau \circ d_{A} \circ \tau$,
iii) $\tau^{2p} = *$.

Lemma 4.1. *The operator τ fulfills axioms i)-iii) if and only if* $\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} (-1)^p$.

Proof. Easy calculations. □

We put

$$\Omega(A)_+ = \{\omega \in \Omega(A);\ \tau\omega = +\omega\},$$
$$\Omega(A)_- = \{\omega \in \Omega(A);\ \tau\omega = -\omega\},$$

The spaces $\Omega(A)_+$ and $\Omega(A)_-$ are eigenspaces of τ corresponding to the eigenvalues $+1$ i -1 and are spaces of cross-sections of suitable vector bundles.

We notice that

$$(d_A + d_A^*)\left[\Omega(A)_+\right] \subset \Omega(A)_- .$$

Definition 4.1. The operator

$$(D_A)_+ = d_A + d_A^* : \Omega(A)_+ \to \Omega(A)_-$$

is called the *Hirzebruch operator* (or the *signature operator*) for the Lie algebroid A.

Clearly

$$(D_A)_+^* = d_A + d_A^* : \Omega(A)_- \to \Omega(A)_+ .$$

Theorem 4.1.

$$\operatorname{Sign} A = \operatorname{Ind}\,(D_A)_+ = \dim_{\mathbb{R}} \ker\left((D_A)_+\right) - \dim_{\mathbb{R}} \ker\left((D_A)_+^*\right).$$

Proof. It is sufficient to prove that $\operatorname{Ind}\,(D_A)_+ = \operatorname{Sign}\left(\mathcal{B}^{2p}\right)$. The proof is analogous to the classical case.[9] Firstly, we notice that subspaces $\mathcal{H}^s(A) + \mathcal{H}^{m+n-s}(A)$ are τ-stable and for $s = 0, 1, ..., 2p-1$

$$\begin{aligned} \varphi_\pm : \mathcal{H}^s(A) &\to \left(\mathcal{H}^s(A) + \mathcal{H}^{m+n-s}(A)\right)_\pm \\ \omega &\longmapsto \frac{1}{2}(\omega \pm \tau\omega) \end{aligned}$$

is an isomorphism of real vector spaces. Secondly $\Omega^{2p}(A)_\pm \cap \mathcal{H}^{2p}(A) = \mathcal{H}^{2p}_\pm(A)$, the space $\Omega^s(A) + \Omega^{m+n-s}(A)$ is τ-stable and

$$\Omega(A) = \bigoplus_{s=0}^{2p-1} \left(\Omega^s(A) + \Omega^{m+n-s}(A)\right) \bigoplus \Omega^{2p}(A),$$

therefore

$$\Omega\left(A\right)_{+}=\bigoplus\nolimits_{s=0}^{2p-1}\left(\Omega^{s}\left(A\right)+\Omega^{m+n-s}\left(A\right)\right)_{+}\bigoplus\Omega^{2p}\left(A\right)_{+}.$$

Thirdly, (a)

$$\begin{aligned}
&\ker\left(D_{A}\right)_{+}\\
&=\Omega\left(A\right)_{+}\cap\ker\left(d_{A}+d_{A}^{*}:\Omega\left(A\right)\to\Omega\left(A\right)\right)\\
&=\Omega\left(A\right)_{+}\cap\mathcal{H}\left(A\right)\\
&=\bigoplus\nolimits_{s=0}^{2p-1}\left(\Omega^{s}\left(A\right)+\Omega^{m+n-s}\left(A\right)\right)_{+}\bigoplus\Omega^{2p}\left(A\right)_{+}\\
&\cap\bigoplus\nolimits_{s=0}^{2p-1}\left(\mathcal{H}^{s}\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)\oplus\mathcal{H}^{n}\left(A\right)\\
&=\bigoplus\nolimits_{s=0}^{2p-1}\left(\mathcal{H}^{s}\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)_{+}\bigoplus\mathcal{H}^{2p}\left(A\right)_{+}
\end{aligned}$$

(b)

$$\begin{aligned}
&\dim\ker\left(D_{A}\right)_{+}-\dim\ker\left(D_{A}\right)_{+}^{*}\\
&=\sum_{s=0}^{2p-1}\dim_{\mathbb{R}}\left(\mathcal{H}^{s}\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)_{+}+\dim_{\mathbb{R}}\mathcal{H}_{+}^{2p}\left(A\right)\\
&-\sum_{s=0}^{2p-1}\dim_{\mathbb{R}}\left(\mathcal{H}^{s}\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)_{-}-\dim_{\mathbb{R}}\mathcal{H}_{-}^{2p}\left(A\right)\\
&=\dim_{\mathbb{R}}\mathcal{H}_{+}^{2p}\left(A\right)-\dim_{\mathbb{R}}\mathcal{H}_{-}^{2p}\left(A\right)\\
&=\operatorname{Sign}\left(\mathcal{B}^{n}\right).
\end{aligned}$$

□

Thanks the above Theorem, we can use the Atiyah-Singer formula for calculating the signature of A.

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