Cyclic Čech-Hochschild bicomplex

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Abstract

In classical differential topology and geometry, the double Čech-de Rham complex for a good covering $\mathfrak{U}$ of M corresponds to the single de Rham complex. When passing to noncommutative geometry on smooth manifolds (i.e. to noncommutative geometry of the algebra $C^{\infty}(M)$ of smooth functions on M), the single Connes complex is the counterpart of the single de Rham complex. We can ask: which bicomplex corresponds to the single Connes complex in noncommutative geometry of smooth manifolds? We give an answer to this question defining the Cyclic Čech-Hochschild bicomplex for a good covering of a smooth manifold. We pose fundamental problems on the total homology of this bicomplex.

1 Čech-de Rham bicomplex for a good covering

In differential geometry and topology of smooth manifolds a fundamental role is played by

- the de Rham complex $(\Omega^{*}(M), d)$, $\dim M = n$,

$$\Omega^{0}(M) \xrightarrow{d} \ldots \xrightarrow{d^{q-1}} \Omega^{q}(M) \xrightarrow{d^{q}} \Omega^{q+1}(M) \xrightarrow{d^{q+1}} \ldots \xrightarrow{d} \Omega^{n}(M) \to 0$$

of differential forms and the de Rham differential operator $d = d_{dR}$. The cohomology of this complex is the de Rham cohomology of the manifold M,

$$\mathbf{H}^q_{DR}(M) = \ker d^q / \operatorname{Im} d^{q-1}.$$

To the single de Rham complex corresponds

- the Čech-de Rham bicomplex for a good covering $\mathfrak{U}$ of M,

$$\left(C^*\left(\mathfrak{U}, \Omega^*\right), \delta, \tilde{d}\right)$$

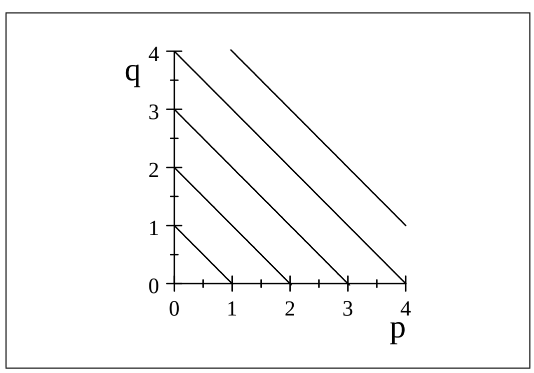
(over the first quadrant) where $C(\mathfrak{U}, \Omega) = \{C^p(\mathfrak{U}, \Omega^q)\}_{p \geq 0,\ q \geq 0}$, $\delta = \delta_{\check{C}}$ is the Čech differential and $\tilde{d} = (-1)^p d$ on $C^p(\mathfrak{U}, \Omega^*)$, $\delta\tilde{d} + \tilde{d}\delta = 0$,

$$\begin{array}{ccccccccc}
 & \uparrow & & & & & & & \\
\longrightarrow & C^{p-1}\left(\mathfrak{U}, \Omega^{q+2}\right) & \longrightarrow & & & & & & \\
 & \uparrow \tilde{d} & & \uparrow & & & & & \\
\longrightarrow & C^{p-1}\left(\mathfrak{U}, \Omega^{q+1}\right) & \xrightarrow{\delta} & C^{p}\left(\mathfrak{U}, \Omega^{q+1}\right) & \longrightarrow & & & & \\
 & \uparrow \tilde{d} & & \uparrow \tilde{d} & & \uparrow & & & \\
\longrightarrow & C^{p-1}\left(\mathfrak{U}, \Omega^{q+1}\right) & \xrightarrow{\delta} & C^{p}\left(\mathfrak{U}, \Omega^{q}\right) & \xrightarrow{\delta} & C^{p+1}\left(\mathfrak{U}, \Omega^{q}\right) & \longrightarrow & & \\
 & \uparrow & & \uparrow & & \uparrow \tilde{d} & & &
\end{array}$$

The total differential

$$D = \delta + \tilde{d} : C^{(r)} \to C^{(r+1)}$$

is homogeneous of degree $+1$ with respect to the total gradation $C^{(r)} = \bigoplus_{p+q=r} C^p(\mathfrak{U}, \Omega^q)$ along the lines $p + q = const.$



Total gradation along $p + q =$const

In spite of the fact that rows and columns of the double Čech-de Rham bicomplex are exact, the total cohomology of this bicomplex is on one hand isomorphic to the de Rham cohomology and on the other hand to the Čech cohomology:

$$\mathbf{H}_D^{(r)} = \mathbf{H}_{DR}^r (M) = \check{\mathbf{H}}^r (M) .$$

Therefore the characteristic classes of M (and more generally of any vector bundle or any G-principal fibre bundle) could be determined by locally defined cochain forms from the double Čech-de Rham bicomplex. See the results by Bott [B1], Bott-Tu [B-T, pp. 304-305] and Sharygin [Sh].

The Bott, Bott-Tu and Sharygin approach consists in producing local formulas in the Čech-de Rham bicomplex by using transition functions $g_{ij} : U_i \cap U_j \to G$ (forming a cocycle, i.e. $g_{ij}g_{jk} = g_{ik}$) with values in a suitable Lie group G. The transition functions g_{ij} are obtained from a system of local trivializations of a suitable principal fibre bundle.

All such formulas for characteristic classes obtained for chains defined for a good covering are called *local formulas for characteristic classes.*

And what is the corresponding object in noncommutative geometry?

The counterpart of the de Rham complex is the **Connes complex** for an associative algebra A with unit over the field $K = \mathbb{R}$ or $\mathbb{C}$, which lies at the foundations of noncommutative geometry [C1], [C2], [C-M].

2 Single Connes complex

Roughly speaking, the (abstract) *Connes complex* consists of cyclic Hochschild chains and bar differentials

$$\left(C_* (A) / \operatorname{Im} (1 - T_*) , [b_*]\right) \cong \left(C_*^\lambda (A) = \ker (1 - T_*) , b'_*\right)$$

where $C_q (A) = \bigotimes^{q+1} (A)$ which determines the cyclic homology $\mathbf{H}_*^\lambda (A)$ [L]. For a review of abstract Hochschild and Cyclic Homology see [K-T], [T].

Now we turn to noncommutative geometry on smooth manifolds, i.e. to the algebra $A = C^\infty (M)$ of smooth functions on M.

The relationship between the above defined abstract Hochschild theory of an arbitrary algebra A and the "smooth theory" for manifolds uses the tensor power $\widehat{\bigotimes}^{q+1} C^\infty (M)$ in the category of topological linear spaces, following the introduction of the Fréchet topology in the algebra of smooth functions $C^{\infty}(M)$ (see for example [E]). We have the following identification of $\widehat{\bigotimes}^{q+1} C^{\infty}(M)$ with the space of smooth functions on the cartesian product M^{q+1}:

$$\widehat{\bigotimes}^{q+1} C^{\infty}(M) = C_q(C^{\infty}(M)) := \{\phi^q : M^{q+1} \to \mathbb{R}\}.$$

$$f_0 \otimes ... \otimes f_q(x_0, ..., x_q) = f_0(x_0) \cdot ... \cdot f_q(x_q).$$

The above allows us to apply all the cyclic homology differentials like b, b', B etc. to Hochschild q-chains $C_q(C^{\infty}(M))$. The *bar differential*

$$b'_q : C_q(C^{\infty}(M)) \to C_{q-1}(C^{\infty}(M))$$

is defined by the formula

$$\left(b'_q(\phi^q)\right)(x_0, ..., x_{q-1}) = \sum_{i=0}^{q-1} (-1)^i \phi^q(x_0, ..., x_i, x_i, ..., x_{q-1}).$$

We introduce the *graded cyclic permutation*

$$T_q : C_q(C^{\infty}(M)) \to C_q(C^{\infty}(M)),$$

$$T_q(\phi^q)(x_0, ..., x_q) = (-1)^q \phi^q(x_1, ..., x_q, x_0).$$

Definition 2.1. A q-chain ϕ^q is called *cyclic* if $T_q(\phi^q) = \phi^q$, i.e.

$$\phi^q \text{ is cyclic} \quad \equiv \quad \phi^q \in \ker(1 - T_q).$$

The space of cyclic q-chains is denoted by $C_q^{\lambda}(C^{\infty}(M))$.

The space of cyclic chains is stable under the bar differential:

$$b'\left[C_q^{\lambda}(C^{\infty}(M))\right] \subset C_{q-1}^{\lambda}(C^{\infty}(M)).$$

Definition 2.2. The complex

$$\left(C_*^{\lambda}(C^{\infty}(M)), b'\right)$$

is called the *cyclic complex* or *Connes complex* of the manifold M. Its k-th homology group is denoted by $\mathbf{H}_k^{\lambda}(C^{\infty}(M))$,

$$\mathbf{H}_k^{\lambda}(C^{\infty}(M)) := \mathbf{H}_k\left(C_*^{\lambda}(C^{\infty}(M)), b'\right).$$

The homology of this complex was given by Connes (for compact manifolds) and by Teleman (for paracompact manifolds). See also [L-Q].

Theorem 2.3 (Connes-Teleman theorem)**.** *For a paracompact manifold M we have*

$$\mathbf{H}_0^\lambda (C^\infty (M)) = \Omega^0 (M) = C^\infty (M),$$

and for $i \geq 1$,

$$\mathbf{H}_{2i}^\lambda (C^\infty (M))$$
$$= \mathbf{H}^0 (M) \oplus \mathbf{H}^2 (M) \oplus \mathbf{H}^4 (M) \oplus ... \oplus \mathbf{H}^{2i-2} (M) \oplus \Omega^{2i} (M) / d \left[\Omega^{2i-1} (M)\right],$$

$$\mathbf{H}_{2i+1}^\lambda (C^\infty (M))$$
$$= \mathbf{H}^1 (M) \oplus \mathbf{H}^3 (M) \oplus \mathbf{H}^5 (M) \oplus ... \oplus \mathbf{H}^{2i-1} (M) \oplus \Omega^{2i+1} (M) / d \left[\Omega^{2i} (M)\right].$$

In particular if $\dim M$ *is even,* $\dim M = 2m$*, then*

$$\begin{aligned} \mathbf{H}_{2m+2}^\lambda (C^\infty (M)) &= \mathbf{H}_{2m+4}^\lambda (C^\infty (M)) = ... \\ &= \mathbf{H}^0 (M) \oplus \mathbf{H}^2 (M) \oplus \mathbf{H}^4 (M) \oplus ... \oplus \mathbf{H}^{2m} (M), \end{aligned}$$

while if $\dim M$ *is odd,* $\dim M = 2m+1$*, then*

$$\begin{aligned} \mathbf{H}_{2m+2}^\lambda (C^\infty (M)) &= \mathbf{H}_{2m+4}^\lambda (C^\infty (M)) = ... \\ &= \mathbf{H}^0 (M) \oplus \mathbf{H}^2 (M) \oplus \mathbf{H}^4 (M) \oplus ... \oplus \mathbf{H}^{2m} (M). \end{aligned}$$

Therefore the whole algebra of characteristic classes (Chern character) is sitting in one homology space, $\mathbf{H}_{2m+2}^\lambda (C^\infty (M))$.

Conclusion 2.4. *If $U \subset M$ is a contractible open subset ($U \cong \mathbb{R}^m$), then $\mathbf{H}^k (U) = 0$ for $k > 0$, $\mathbf{H}^0 (U) = \mathbb{R}$, therefore*

$$\begin{aligned} \mathbf{H}_{2i}^\lambda (C^\infty (U)) &= \mathbb{R} \oplus \left(\Omega^{2i} (U) / d \left[\Omega^{2i-1} (U)\right]\right) \\ &= \mathbb{R} \oplus \left(\Omega^{2i} (U) / \ker d^{2i}\right) = \mathbb{R} \oplus \operatorname{Im} d^{2i}, \quad i \geq 1, \\ \mathbf{H}_0^\lambda (C^\infty (U)) &= C^\infty (U), \\ \mathbf{H}_{2i-1}^\lambda (C^\infty (U)) &= \left(\Omega^{2i-1} (U) / \ker d_{2i-1}\right) = \operatorname{Im} d_{2i-1}. \end{aligned}$$

3 Čech-Hochschild bicomplex for a good covering

In classical differential topology and geometry, the single de Rham complex corresponds to the double Čech-de Rham complex for a good covering $\mathfrak{U}$ of M. And what is the case in noncommutative geometry of smooth manifolds?

We give an answer to this question. A double complex counterpart in noncommutative geometry of smooth manifolds has been defined by N.Teleman.

During his visit to the Institute of Mathematics of the Lodz University of Technology in December 2009 at the invitation of the author, N. Teleman suggested the definition of the *Cyclic Čech-Hochschild bicomplex* for a good covering of a manifold, which corresponds to the single Connes complex, and posed the problem of computing its total homology in connection with the problem of computing characteristic classes.

Definition 3.1 (Čech-Hochschild bicomplex)**.** By the *Čech-Hochschild bicomplex* for a good covering $\mathfrak{U}$ of M we mean a bicomplex (over the first quadrant) of vector spaces

$$\left(\left\{\check{C}H^{\lambda}_{p,q}\left(\mathfrak{U},\mathbb{R}\right)\right\}_{p\geq 0,\ q\geq 0},\tilde{\delta},b'\right)$$

where

(a)

$$\check{C}H_{p,q}\left(\mathfrak{U},\mathbb{R}\right)=\bigoplus\nolimits_{(i_0,\ldots,i_p)\in I^{p+1}}C_{[i_0,\ldots,i_p],q}\left(\mathfrak{U},\mathbb{R}\right),$$
$$C_{[i_0,\ldots,i_p],q}\left(\mathfrak{U},\mathbb{R}\right)=\left\{\phi^q_{i_0\ldots i_p}:\left(U_{i_0\ldots i_p}\right)^{q+1}\to\mathbb{R}\right\},$$

(b) b' is the bar differential for $p\geq 0, q\geq 1$,

$$b'_{p,q}:\check{C}H_{p,q}\left(\mathfrak{U},V\right)\to\check{C}H_{p,q-1}\left(\mathfrak{U},V\right),$$
$$\left(b'_{p,q}\left(\phi^q_p\right)\right)_{i_0,\ldots,i_p}\left(x_0,\ldots,x_{q-1}\right)=\sum\nolimits_{i=0}^{q-1}\left(-1\right)^i\phi^q_{i_0\ldots i_p}\left(x_0,\ldots,x_i,x_i,\ldots,x_{q-1}\right).$$

(c) $\tilde{\delta}$ is a sign modification of the Čech differential for $p,q\geq 0$,

$$\delta^{p,q}:\check{C}H_{p,q}\left(\mathfrak{U},V\right)\to\check{C}H_{p+1,q}\left(\mathfrak{U},V\right),$$
$$\delta^{p,q}\left(\phi^q\right)_{i_0,\ldots,i_{p+1}}=\sum\nolimits_{r=0}^{p+1}\left(-1\right)^r\phi^q_{i_0,\ldots,\widehat{i_r},\ldots,i_{p+1}},$$
$$\tilde{\delta}^{p,q}=\left(-1\right)^q\delta^{p,q}.$$

It is a standard calculation that $\tilde{\delta}b' + b'\tilde{\delta} = 0$. We modify the sign of δ, not of b', because in the standard CC-bicomplex in cyclic homology we have "$-b'$" everywhere, and technically the CC-bicomplex is used in this theory.

We take the graded cyclic permutation

$$\begin{gathered} T_{p,q} : \check{C}H_{p,q}\left(\mathfrak{U}, \mathbb{R}\right) \to \check{C}H_{p,q}\left(\mathfrak{U}, \mathbb{R}\right), \\ T_{p,q}\left(\phi^{q}\right)_{i_0,\ldots,i_p}\left(x_0, \ldots, x_q\right) = (-1)^{q}\, \phi^{q}_{i_0 \ldots i_p}\left(x_1, \ldots, x_q, x_0\right), \end{gathered}$$

and the *cyclic functions*

$$\check{C}H^{\lambda}_{p,q}\left(\mathfrak{U}, \mathbb{R}\right) = \left\{\phi \in \check{C}H_{p,q}\left(\mathfrak{U}, \mathbb{R}\right);\ T_{p,q}\left(\phi\right) = \phi\right\} = \ker\left(1 - T_q\right).$$

The spaces of cyclic chains are stable under the bar and Čech differentials,

$$b'\left[\check{C}H^{\lambda}_{p,q}\left(\mathfrak{U}, \mathbb{R}\right)\right] \subset \check{C}H^{\lambda}_{p,q-1}\left(\mathfrak{U}, \mathbb{R}\right), \quad \delta^{p,q}\left[\check{C}H^{\lambda}_{p,q}\left(\mathfrak{U}, \mathbb{R}\right)\right] \subset \check{C}H^{\lambda}_{p+1,q}\left(\mathfrak{U}, \mathbb{R}\right).$$

In this way we introduce a subcomplex $(\check{C}H^{\lambda}\left(\mathfrak{U}, \mathbb{R}\right), \tilde{\delta}, b')$, $\check{C}H^{\lambda} = \ker\left(1 - T\right)$, of the Čech-Hochschild bicomplex.

Definition 3.2 (**Cyclic Čech-Hochschild bicomplex**)**.** By the *Cyclic Čech-Hochschild bicomplex* we mean the bicomplex

$$(\check{C}H^{\lambda}\left(\mathfrak{U}, \mathbb{R}\right), \tilde{\delta}, b')$$

$$\begin{array}{ccccccc}
 & & & & & & \downarrow & \\
 & & & & & \xrightarrow{\tilde{\delta}} & \check{C}H^{\lambda}_{p+2,q+1}\left(\mathfrak{U}, \mathbb{R}\right) & \longrightarrow \\
 & & & & \downarrow b' & & \downarrow b' & \\
 & & & \xrightarrow{\tilde{\delta}} & \check{C}H^{\lambda}_{p,q}\left(\mathfrak{U}, \mathbb{R}\right) & \xrightarrow{\tilde{\delta}} & \check{C}H^{\lambda}_{p+1,q}\left(\mathfrak{U}, \mathbb{R}\right) & \longrightarrow \\
 & \downarrow b' & & & \downarrow b' & & \downarrow b' & \\
\longrightarrow & \check{C}H^{\lambda}_{p-1,q-1}\left(\mathfrak{U}, \mathbb{R}\right) & \xrightarrow{\tilde{\delta}} & & \check{C}H^{\lambda}_{p,q-1}\left(\mathfrak{U}, \mathbb{R}\right) & \xrightarrow{\tilde{\delta}} & \check{C}H^{\lambda}_{p+1,q-1}\left(\mathfrak{U}, \mathbb{R}\right) & \longrightarrow \\
 & \downarrow & & & \downarrow & & \downarrow &
\end{array}$$

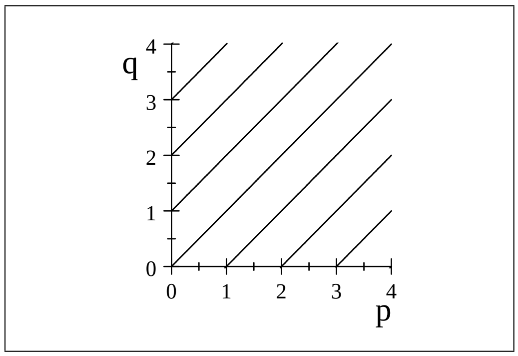
The total differential

$$D = \tilde{\delta} + b' : C^{[r]} \to C^{[r+1]}$$

is homogeneous of degree +1 with respect to the total gradation

$$C^{[r]} = \prod_{p-q=r} \check{C}H^{\lambda}_{p,q}(\mathfrak{U}, \mathbb{R})$$

along the lines $p - q = const$



Total gradation along the lines $p - q = const$,

(we observe that on each line $p - q = const$ we have infinitely many vector spaces and we must use the cartesian product of vector spaces, and not the direct sum; see [K, Th. 3.2]). Since $D^2 = 0$ we can define the total homology

$$\mathbf{H}^{[r]}\left(\check{C}H^{\lambda}(\mathfrak{U}, \mathbb{R})\right).$$

FIRST MAIN PROBLEM:

(1) Calculate the total homology of the Cyclic Čech-Hochschild bicomplex.

We observe that the homology of the columns of the Cyclic Čech-Hochschild bicomplex is the homology of the suitable Connes complex for contractible sets $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$, therefore the above mentioned Connes-Teleman theorem on homology of the complex $C^{\lambda}_{*}\left(C^{\infty}\left(U_{i_0 \ldots i_p}\right)\right)$ is very useful.

It turns out that the total homology of the cyclic Čech-Hochschild bicomplex $(\check{C}H^{\lambda}_{p,.q}, \tilde{\delta}, b')$ is very rich [K], so it is reasonable to look for representations of characteristic classes in the homology of this bicomplex.

SECOND MAIN PROBLEM:

(2) Find a Cyclic Čech-Hochschild chain representing characteristic classes.

This problem will be discussed elsewhere.

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