The Koszul homomorphism for a pair of Lie algebras in the theory of exotic characteristic classes of Lie algebroids

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Abstract

We examine fundamental properties of the universal exotic characteristic homomorphism in the category of Lie algebroids, introduced by the authors in [4]. The properties under study include: (a) functorial properties with respect to arbitrary morphisms of Lie algebroids, (b) homotopy properties, (c) relationships with the Koszul homomorphism for a pair of isotropy Lie algebras, (d) conditions under which the universal exotic characteristic homomorphism is a monomorphism.

1 Introduction

In [4] we constructed some exotic characteristic homomorphism

$$\Delta_{(A,B,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^\bullet(L)$$

for a triple (A, B, ∇), in which A is a regular Lie algebroid over a foliated manifold (M, F), B its regular subalgebroid on the same foliated manifold (M, F), $\boldsymbol{g}$ the kernel of the anchor of A, and $\nabla : L \to A$ a flat L-connection in A for an arbitrary Lie algebroid L over M. The domain of this homomorphism is the Lie algebroid analog to the relative cohomology algebra for a pair of Lie algebras defined in [5]. The exotic characteristic homomorphism generalizes some known secondary characteristic classes: for flat principal fibre bundles with a reduction (Kamber, Tondeur [13]), and two approaches to flat characteristic classes for Lie algebroids, the one for regular Lie algebroids due to Kubarski [19] and the one for representations of not necessarily regular Lie algebroids on vector bundles developed by Crainic ([6], [7]).

For $L = A$ and $\nabla = \mathrm{id}_A$ we obtain a new universal characteristic homomorphism $\Delta_{(A,B)\#}$, which is a factor of the characteristic homomorphism $\Delta_{(A,B,\nabla)\#}$ for each flat L-connection $\nabla : L \to A$, i.e.

$$\Delta_{(A,B,\nabla)\#} = \nabla^\# \circ \Delta_{(A,B)\#}. \tag{1.1}$$

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Clearly, no class from the kernel of $\Delta_{(A,B)\#}$ is an obstruction to the given flat connection $\nabla : L \to A$ being induced by a connection in B. For this reason, we raise the following question (usually in the theory of characteristic classes, see for example [13], we ask for the nontriviality of the characteristic homomorphisms):

- **Is the exotic universal characteristic homomorphism $\Delta_{(A,B)\#}$ a monomorphism?**

In sections 4 and 5, we give a positive answer under some assumptions.

The classical secondary (often also called exotic) characteristic homomorphism for a principal bundle with a given reduction measures the incompatibility of two geometric structures on the given principal bundle: its reduction and a flat connection. Namely, if the flat connection is a connection in a given reduction, the exotic characteristic homomorphism is trivial, i.e. it is the zero homomorphism in all positive degrees (cf. [13]). The exotic characteristic homomorphism $\Delta_{(A,B,\nabla)\#}$ for Lie algebroids has a similar meaning.

The classical secondary homomorphism for a given principal bundle P and its reduction P' has a stronger property: it is trivial if a given flat connection has values in any reduction homotopic to P'.

Chapter 3 concerns homotopy properties of the generalized exotic homomorphism $\Delta_{(A,B,\nabla)\#}$ for Lie algebroids. We examine the notion of homotopic Lie subalgebroids, introduced in [19] as a natural generalization of the notion of homotopic H-reductions of a principal bundle. We also show functorial properties of $\Delta_{(A,B,\nabla)\#}$ with respect to homomorphisms of Lie algebroids (not necessarily over the identity on the base manifold).

Chapter 4 studies $\Delta_{(A,B)\#}$ in two cases. First, for the trivial case of Lie algebroids over a point, i.e. for Lie algebras. This universal homomorphism is, in fact, equivalent (up to sign) to the well known "Koszul homomorphism" $\Delta_{(\mathfrak{g},\mathfrak{h})\#} : \mathsf{H}^\bullet(\mathfrak{g}/\mathfrak{h}) \to \mathsf{H}^\bullet(\mathfrak{g})$ for a pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$ [14], [11]. In [11] the injectivity of $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is considered and used to investigate the cohomology algebra of homogeneous manifolds G/H. Next, applying the Lie functor for principal fibre bundles gives a new universal homomorphism for the reduction of a principal fibre bundle. It is a factor of the standard secondary characteristic homomorphism $\Delta_{(P,P',\omega)\#}$ for any flat connection ω in P.

In Section 5, using functorial properties of the inclusion $\iota_x : (\boldsymbol{g}_x, \boldsymbol{h}_x) \to (A, B)$ over the map $\{x\} \hookrightarrow M$, where $\boldsymbol{g}_x, \boldsymbol{h}_x$ are the isotropy algebras of the Lie algebroids A and B at $x \in M$, we show a connection of $\Delta_{(A,B)\#}$ with the Koszul homomorphism for $(\boldsymbol{g}_x, \boldsymbol{h}_x)$. We find some conditions under which $\Delta_{(A,B)\#}$, for $B \subset A$, is a monomorphism. Our considerations show that the Koszul homomorphism plays an essential role in the study of exotic characteristic classes.

In the paper we suppose that the reader is familiar with Lie algebroids. For more about Lie algebroids and connections on them we refer to [21], [12], [9], [20], [3].

2 Construction of Exotic Characteristic Homomorphism

We now briefly explain the construction of the exotic characteristic homomorphism and the universal exotic characteristic homomorphism on Lie algebroids from [4].

Let $(A, [\![\cdot,\cdot]\!], \#_A)$ be a regular Lie algebroid over a foliated manifold (M,F), B its regular **subalgebroid** on the same foliated manifold (M,F), L a Lie algebroid over M and $\nabla : L \to A$ a **flat** L -connection in A. We call the triple

$$(A, B, \nabla)$$

an FS-*Lie algebroid.* Let $\lambda : F \to B$ be an arbitrary connection in B. Then $j \circ \lambda : F \to A$ is a connection in A. Let $\breve{\lambda} : A \to \boldsymbol{g}$ be its connection form. Summarizing, we have a flat L-connection $\nabla : L \to A$ in A and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{g} \arrow[r, hook, shift left, "\iota"] & A \arrow[l, shift left, "\breve{\lambda}"] \arrow[r, "\#_A"] & F \arrow[r] & 0 \\
0 \arrow[r] & \boldsymbol{h} \arrow[r] \arrow[u, hook] & B \arrow[u, hook, "j"'] \arrow[r, shift left, "\#_B"] & F \arrow[l, shift left, "\lambda"] \arrow[u, equal] \arrow[r] & 0.
\end{tikzcd}$$

In the algebra $\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)$ of cross-sections of the bundle $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$ we distinguish the subalgebra $(\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ of invariant cross-sections with respect to the representation $\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}$ of the Lie algebroid B in the vector bundle $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$, associated to the adjoint representation $\mathrm{ad}_{B,\boldsymbol{h}} : B \to \mathrm{A}(\boldsymbol{g}/\boldsymbol{h})$, $\mathrm{ad}_{B,\boldsymbol{h}}(\xi)([\nu]) = [[\![\xi, \nu]\!]]$, $\xi \in \Gamma(B)$, $\nu \in \Gamma(\boldsymbol{g})$, and where $\mathrm{A}(\boldsymbol{g}/\boldsymbol{h})$ denotes the Lie algebroid of the vector bundle $\boldsymbol{g}/\boldsymbol{h}$. Recall that $\Psi \in (\Gamma(\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ if and only if

$$(\#_B \circ \xi)\langle \Psi, [\nu_1] \wedge \ldots \wedge [\nu_k]\rangle = \sum_{j=1}^{k} (-1)^{j-1} \langle \Psi, [[\![j \circ \xi, \nu_j]\!]] \wedge [\nu_1] \wedge \ldots \hat{\jmath} \ldots \wedge [\nu_k]\rangle$$

for all $\xi \in \Gamma(B)$ and $\nu_j \in \Gamma(\boldsymbol{g})$ (see [16]). In the space $(\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ of invariant cross-sections there exists a differential operator $\bar{\delta}$ defined by

$$\left\langle \bar{\delta}\Psi, [\nu_1] \wedge \ldots \wedge [\nu_k] \right\rangle = \sum_{i<j} (-1)^{i+j+1} \langle \Psi, [[\![\nu_i, \nu_j]\!]] \wedge [\nu_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [\nu_k]\rangle$$

(see [19]), and we obtain the cohomology algebra

$$\mathsf{H}^{\bullet}(\boldsymbol{g}, B) \equiv \mathsf{H}^{\bullet}\left(\left(\Gamma\left(\bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \right) \right)^{\Gamma(B)}, \bar{\delta} \right).$$

Denote by $\Omega(L)$ the algebra of differential forms on L. The cohomology space induced by the differential operator d_L in $\Omega(L)$ will be denoted by $\mathsf{H}^{\bullet}(L)$.

Notice that the homomorphism $\omega_{B,\nabla} : L \to \boldsymbol{g}/\boldsymbol{h}$, $\omega_{B,\nabla}(w) = [-(\breve{\lambda} \circ \nabla)(w)]$, does not depend on the choice of the auxiliary connection $\lambda : F \to B$, and $\omega_{B,\nabla} = 0$ if ∇ takes values in B.

Let us define a homomorphism of algebras

$$\Delta_{(A,B,\nabla)} : \left(\Gamma\left(\bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^* \right) \right)^{\Gamma(B)} \longrightarrow \Omega(L) \tag{2.1}$$

by

$$\left(\Delta_{(A,B,\nabla)}\Psi\right)_x (w_1 \wedge \ldots \wedge w_k) = \langle \Psi_x, \omega_{B,\nabla}(w_1) \wedge \ldots \wedge \omega_{B,\nabla}(w_k)\rangle, \quad w_i \in L_{|x}.$$

The homomorphism $\Delta_{(A,B,\nabla)}$ commutes with the differentials $\bar{\delta}$ and d_L (see [4]). In this way we obtain the cohomology homomorphism

$$\Delta_{(A,B,\nabla)\#} : \mathsf{H}^{\bullet}(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^{\bullet}(L).$$

In the case where $L = A$ and $\nabla = \mathrm{id}_A : A \to A$ is the identity map, we have the particular case of a homomorphism for the pair (A, B):

$$\Delta_{(A,B)} \equiv \Delta_{(A,B,\mathrm{id}_A)} : \Gamma(\bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*)^{\Gamma(B)} \longrightarrow \Omega(A),$$

$$\left(\Delta_{(A,B)}\Psi\right)_x (\upsilon_1 \wedge \ldots \wedge \upsilon_k) = \langle \Psi_x, [-\breve{\lambda}(\upsilon_1)] \wedge \ldots \wedge [-\breve{\lambda}(\upsilon_k)]\rangle, \qquad \upsilon_i \in A_{|x}.$$

The homomorphism $\Delta_{(A,B,\nabla)}$ can be written as the composition

$$\Delta_{(A,B,\nabla)}:\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^{*})\xrightarrow{\Delta_{(A,B)}}\Omega(A)\xrightarrow{\nabla^{*}}\Omega(L),$$

where ∇^{*} is the pullback of forms. For this reason, $\Delta_{(A,B)}$ induces the cohomology homomorphism

$$\Delta_{(A,B)\#}:\mathsf{H}^{\bullet}(\boldsymbol{g},B)\longrightarrow\mathsf{H}^{\bullet}(A),$$

which is a factor of $\Delta_{(A,B,\nabla)\#}$ for every flat L-connection $\nabla : L\to A$:

$$\Delta_{(A,B,\nabla)\#}:\mathsf{H}^{\bullet}(\boldsymbol{g},B)\xrightarrow{\Delta_{(A,B)\#}}\mathsf{H}^{\bullet}(A)\xrightarrow{\nabla^{\#}}\mathsf{H}^{\bullet}(L). \tag{2.2}$$

The map $\Delta_{(A,B,\nabla)\#}$ is called the *exotic characteristic homomorphism* of the FS-Lie algebroid (A,B,∇). We call elements of the subalgebra $\operatorname{Im}\Delta_{(A,B,\nabla)\#}\subset\mathsf{H}^{\bullet}(L)$ *exotic characteristic classes* of this algebroid. In particular, $\Delta_{(A,B)\#}=\Delta_{(A,B,\mathrm{id}_A)\#}$ is the characteristic homomorphism of the Lie subalgebroid $B\subset A$, which we call the *universal exotic characteristic homomorphism*; the characteristic classes from its image are called *universal characteristic classes* of the pair $B\subset A$.

The exotic characteristic homomorphism for FS-Lie algebroids generalizes the following known characteristic classes: for flat regular Lie algebroids (Kubarski), for flat principal fibre bundles with a reduction (Kamber, Tondeur) and for representations of Lie algebroids on vector bundles (Crainic).

1. For $L=F$ we obtain the case in which $\nabla : F\to A$ is a usual connection in A. In this way the exotic characteristic homomorphism is a generalization of one for a flat regular Lie algebroid given in [19] (see [4]).

2. For $L=TM$ and $A=TP/G$, $B=TP'/H$ (P' is an H-reduction of P) we obtain a case equivalent to the classical theory on principal fibre bundles [13] (see [4] and Section 4.2 below for more details).

3. Let $A=\mathcal{A}(\mathfrak{f})$ be the Lie algebroid of a vector bundle $\mathfrak{f}$ over a manifold M, $B=\mathcal{A}(\mathfrak{f},\{h\})\subset A$ its Riemannian reduction ([17]), L a Lie algebroid over M, and $\nabla : L\to\mathcal{A}(\mathfrak{f})$ an L-connection on $\mathfrak{f}$. Let $\Delta_{\#}$ denote the exotic characteristic homomorphism for the FS-Lie algebroid $(\mathcal{A}(\mathfrak{f}),\mathcal{A}(\mathfrak{f},\{h\}),\nabla)$. If the vector bundle $\mathfrak{f}$ is nonorientable, or orientable and of odd rank n, then the domain of $\Delta_{\#}$ is isomorphic to $\bigwedge(y_1,y_3,\ldots,y_{n'})$ where n' is the largest odd integer less than or equal to n and $y_{2k-1}\in\mathsf{H}^{4k-3}(\operatorname{End}\mathfrak{f},\mathcal{A}(\mathfrak{f},\{h\}))$ is represented by the multilinear trace form $\widetilde{y}_{2k-1}\in\Gamma(\bigwedge^{4k-3}(\operatorname{End}\mathfrak{f}/\operatorname{Sk}\mathfrak{f})^{*})$. Then the image of $\Delta_{\#}$ is generated by the Crainic classes $u_1(\mathfrak{f})$, $u_5(\mathfrak{f})$,…,$u_{4\left[\frac{n+3}{4}\right]-3}(\mathfrak{f})$ (for details see [6], [7], [4]). If $\mathfrak{f}$ is orientable of even rank $n=2m$ with a volume form v, the domain of $\Delta_{\#}$ is additionally generated by some class $y_{2m}\in\mathsf{H}^{2m}(\operatorname{End}\mathfrak{f},\mathcal{A}(\mathfrak{f},\{h,\mathrm{v}\}))$ represented by a form induced by the Pfaffian and where $\mathcal{A}(\mathfrak{f},\{h,\mathrm{v}\})$ is the Lie algebroid of the $SO(n,\mathbb{R})$-reduction $\mathcal{L}(\mathfrak{f},\{h,\mathrm{v}\})$ of the frame bundle $\mathcal{L}\mathfrak{f}$ of $\mathfrak{f}$ (see [4]). Then the algebra of exotic characteristic classes for $(\mathcal{A}(\mathfrak{f}),\mathcal{A}(\mathfrak{f},\{h,\mathrm{v}\}),\nabla)$ is generated by $u_1(\mathfrak{f})$, $u_5(\mathfrak{f})$,…,$u_{4\left[\frac{n+3}{4}\right]-3}(\mathfrak{f})$ and additionally by $\Delta_{\#}(y_{2m})$. In [4] we give an example of an FS-Lie algebroid where the Pfaffian induces a non-zero characteristic class.

4. The characteristic homomorphism $\Delta_{(A,B)\#}:\mathsf{H}^{\bullet}(\boldsymbol{g},B)\to\mathsf{H}^{\bullet}(A)$ depends only on the inclusion $i : B\hookrightarrow A$. This inclusion defines secondary characteristic classes $\mu_{2h-1}(i)\in\mathsf{H}^{4h-3}(B)$ in the sense of Vaisman [23]. The class $\mu_1(\#_E)$ of the anchor

$\#_E$ of a Lie algebroid E coincides with the modular class $\mathrm{mod}(E)$ of E. We point out (cf. [4], [6], [7], [8], [24]) that $\mathrm{mod}(E)$ can be expressed in terms of the exotic characteristic homomorphism in case where the basic connection $\nabla \equiv (\hat{\nabla}, \check{\nabla})$ is flat. For this, take $\Delta_{(A,B,\nabla)\#}$ for the Lie algebroid $A = \mathcal{A}(E \oplus T^*M)$ of the vector bundle $E \oplus T^*M$ and its Riemannian reduction B and the basic connection $\nabla = (\hat{\nabla}, \check{\nabla})$. Then, $\mathrm{mod}(E)$ and $\Delta_{(A,B,\nabla)\#}(y_1)$ are equal up to a constant. Notice that

$$\mu_1(i) = \mathrm{mod}(B) - i^{\#}(\mathrm{mod}(A)).$$

So, it can be expressed in terms of characteristic classes from the images of suitable exotic characteristic homomorphisms.

From (2.2) one can see that for a pair of regular Lie algebroids (A, B), $B \subset A$, both over a foliated manifold (M, F), and for an arbitrary element $\zeta \in \mathsf{H}^\bullet(\boldsymbol{g}, B)$ there exists a (universal) cohomology class $\Delta_{(A,B)\#}(\zeta) \in \mathsf{H}^\bullet(A)$ such that for any Lie algebroid L over M and a flat L-connection $\nabla : L \to A$ the equality

$$\Delta_{(A,B,\nabla)\#}(\zeta) = \nabla^{\#}\left(\Delta_{(A,B)\#}(\zeta)\right)$$

holds. Therefore, no element from the kernel of $\Delta_{(A,B)\#}$ can be used to compare the flat connection ∇ with a reduction $B \subset A$. Hence the following is of interest:

Problem 2.1 *Is the characteristic homomorphism $\Delta_{(A,B)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \to \mathsf{H}^\bullet(A)$ a monomorphism for a given $B \subset A$?*

The answer is yes in many cases, as we show below (see Section 5.2 for integrable transitive Lie algebroids and Section 5.3 for non-integrable transitive Lie algebroids).

3 Functoriality and Homotopy Properties

3.1 Functoriality

Let (A, B) and (A', B') be two pairs of regular Lie algebroids over (M, F) and (M', F'), respectively, where $B \subset A$, $B' \subset A'$, and let $H : A' \to A$ be a homomorphism of Lie algebroids over a mapping $f : (M', F') \to (M, F)$ of foliated manifolds such that $H[B'] \subset B$. We write $(H, f) : (A', B') \to (A, B)$. Let $H^{+\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \to \mathsf{H}^\bullet(\boldsymbol{g}', B')$ be the homomorphism of cohomology algebras induced by the pullback $H^{+*} : \Gamma(\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^*) \to \Gamma(\bigwedge^k(\boldsymbol{g}'/\boldsymbol{h}')^*)$ defined by

$$\left\langle H^{+*}(\Psi)_x, [\nu'_1] \wedge \ldots \wedge [\nu'_k] \right\rangle = \left\langle \Psi_{f(x)}, \left[H^+(\nu'_1)\right] \wedge \ldots \wedge \left[H^+(\nu'_k)\right] \right\rangle$$

where $\Psi \in \Gamma(\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^*)$, $x \in M$, $\nu'_1, \ldots, \nu'_k \in \boldsymbol{g}'_x$, and where $H^+ : \boldsymbol{g}' \to \boldsymbol{g}$ is the restriction of H to $\boldsymbol{g}'$ (see [19, Proposition 4.2]).

Theorem 3.1 (The functoriality of $\Delta_{(A,B)\#}$) *For a given pair of regular Lie algebroids (A, B), (A', B') and a homomorphism $(H, f) : (A', B') \to (A, B)$ we have the commutative diagram*

$$\begin{tikzcd}
\mathsf{H}^\bullet(\boldsymbol{g}, B) \arrow[r, "\Delta_{(A,B)\#}"] \arrow[d, "H^{+\#}"'] & \mathsf{H}^\bullet(A) \arrow[d, "H^{\#}"] \\
\mathsf{H}^\bullet(\boldsymbol{g}', B') \arrow[r, "\Delta_{(A',B')\#}"'] & \mathsf{H}^\bullet(A').
\end{tikzcd}$$

Proof. One can see that $H^{+} \circ \breve{\lambda}'(u') - \breve{\lambda}(Hu') \in \boldsymbol{h}$ for all $u' \in A'$, where λ and λ' are auxiliary connections in B and B', respectively. Applying this fact, it is sufficient to check the commutativity of the diagram on the level of forms. The calculations are left to the reader. ■

Definition 3.2 Let (A', B', ∇') and (A, B, ∇) be two FS-Lie algebroids on foliated manifolds (M', F') and (M, F), respectively, where $\nabla : L \to A$ and $\nabla' : L' \to A'$ are flat connections. By a *homomorphism*

$$H : (A', B', \nabla') \longrightarrow (A, B, \nabla)$$

over $f : (M', F') \to (M, F)$ we mean a pair (H, h) such that:

- $H : A' \to A$ is a homomorphism of regular Lie algebroids over f and $H[B'] \subset B$,
- $h : L' \to L$ is also a homomorphism of Lie algebroids over f,
- $\nabla \circ h = H \circ \nabla'$.

Clearly, $h^{\#} \circ \nabla^{\#} = \nabla'^{\#} \circ H^{\#}$. So, from (2.2) and Theorem 3.1 we obtain

Theorem 3.3 (*The functoriality of* $\Delta_{(A,B,\nabla)\#}$) *The following diagram commutes:*

$$\begin{tikzcd}
\mathsf{H}^{\bullet}(\boldsymbol{g}, B) \arrow[r, "\Delta_{(A,B,\nabla)\#}"] \arrow[d, "H^{+\#}"'] & \mathsf{H}^{\bullet}(L) \arrow[d, "h^{\#}"] \\
\mathsf{H}^{\bullet}(\boldsymbol{g}', B') \arrow[r, "\Delta_{(A',B',\nabla')\#}"'] & \mathsf{H}^{\bullet}(L').
\end{tikzcd}$$

3.2 Homotopy Invariance

We recall the definition of homotopy between homomorphisms of Lie algebroids.

Definition 3.4 [18] Let H_0, $H_1 : L' \to L$ be two homomorphisms of Lie algebroids. By a *homotopy joining* H_0 *to* H_1 we mean a homomorphism of Lie algebroids

$$H : T\mathbb{R} \times L' \longrightarrow L$$

such that $H(\theta_0, \cdot) = H_0$ and $H(\theta_1, \cdot) = H_1$, where θ_0 and θ_1 are null vectors tangent to $\mathbb{R}$ at 0 and 1, respectively. We say that H_0 and H_1 are *homotopic* and write $H_0 \sim H_1$. We say that $F : L' \to L$ is a *homotopy equivalence* if there is a homomorphism $G : L \to L'$ such that $G \circ F \sim \mathrm{id}_{L'}$ and $F \circ G \sim \mathrm{id}_L$.

The homotopy $H : T\mathbb{R} \times L' \to L$ determines a chain homotopy operator ([18], [2]) which implies that $H_0^{\#} = H_1^{\#} : \mathsf{H}^{\bullet}(L) \to \mathsf{H}^{\bullet}(L')$.

Definition 3.5 [19] Let B_0, $B_1 \subset A$ be two Lie subalgebroids (both over the same foliated manifold (M, F)). The Lie subalgebroids B_0 and B_1 are said to be *homotopic* if there exists a Lie subalgebroid $B \subset T\mathbb{R} \times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times F)$ such that for $t \in \{0, 1\}$,

$$\upsilon \in B_t \text{ if and only if } (\theta_t, \upsilon) \in B \tag{3.1}$$

for $\upsilon \in A$. A Lie subalgebroid $B \subset T\mathbb{R} \times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times F)$ satisfying (3.1) is called a *subalgebroid joining* B_0 *to* B_1.

See [19, Proposition 5.2] for a comparison of the homotopy of subbundles of a principal bundle and the homotopy of subalgebroids.

Let B_0, B_1 be two homotopic Lie subalgebroids over (M, F) and let $B \subset T\mathbb{R} \times A$ be a Lie subalgebroid of $T\mathbb{R} \times A$ joining B_0 to B_1, $t \in \{0, 1\}$. Consider the homomorphism of Lie algebroids $F_t^A : A \to T\mathbb{R} \times A$, $\upsilon_x \mapsto (\theta_t, \upsilon_x)$, over $f_t : M \to \mathbb{R} \times M$, $f_t(x) = (t, x)$. Then (3.1) yields $F_t^A[B_t] \subset B$. Applying the functoriality of $\Delta_{t\,\#} \equiv \Delta_{(A,B_t)\,\#}$ and $\Delta_{(A,B)\,\#}$ (see Theorem 3.1), we obtain the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{H}^\bullet(\boldsymbol{g}, B_0) \arrow[r, "\Delta_{0\#}"] \arrow[dd, bend right=60, "\alpha"'] & \mathsf{H}^\bullet(A) \\
\mathsf{H}^\bullet(0\times\boldsymbol{g}, B) \arrow[u, "{F_0}^{+\#}", "\simeq"'] \arrow[d, "{F_1}^{+\#}"', "\simeq"] \arrow[r, "\Delta_{(T\mathbb{R}\times A,B)\#}"'] & \mathsf{H}^\bullet(T\mathbb{R}\times A) \arrow[u, "{F_0}^{A\#}"'] \arrow[d, "{F_1}^{A\#}"] \\
\mathsf{H}^\bullet(\boldsymbol{g}, B_1) \arrow[r, "\Delta_{1\#}"'] & \mathsf{H}^\bullet(A)
\end{tikzcd}$$

where $F_t^{+\,\#} \equiv \left(F_t^A\right)^{+\,\#}$. In [19] it is shown that each $F_t^{+\,\#}$ is an isomorphism of algebras. We mention that in the proof of this fact we use Theorem 20.2 of [20] concerning invariant cross-sections over $\mathbb{R} \times M$.

For any flat L-connection $\nabla : L \to A$, the induced $T\mathbb{R} \times L$-connection $\mathrm{id}_{T\mathbb{R}} \times \nabla$ is also flat. Since F_t^A determines a homomorphism

$$(A, B_t, \nabla) \longrightarrow (T\mathbb{R}\times A, B, \mathrm{id}_{T\mathbb{R}} \times \nabla)$$

of FS-Lie algebroids over $f_t : M \to \mathbb{R} \times M$, we can complete the previous diagram to

$$\begin{tikzcd}
\mathsf{H}^\bullet(\boldsymbol{g}, B_0) \arrow[r, "\Delta_{0\#}"] \arrow[dd, bend right=60, "\alpha"'] & \mathsf{H}^\bullet(A) \arrow[r, "\nabla^\#"] & \mathsf{H}^\bullet(L) \\
\mathsf{H}^\bullet(0\times\boldsymbol{g}, B) \arrow[u, "{F_0}^{+\#}", "\simeq"'] \arrow[d, "{F_1}^{+\#}"', "\simeq"] \arrow[r, "\Delta_{(T\mathbb{R}\times A,B)\#}"'] & \mathsf{H}^\bullet(T\mathbb{R}\times A) \arrow[u, "{F_0}^{A\#}"'] \arrow[d, "{F_1}^{A\#}"] \arrow[r, "(\mathrm{id}\times\nabla)^\#"'] & \mathsf{H}^\bullet(T\mathbb{R}\times L) \arrow[u, "{F_0}^{L\#}"'] \arrow[d, "{F_1}^{L\#}"] \\
\mathsf{H}^\bullet(\boldsymbol{g}, B_1) \arrow[r, "\Delta_{1\#}"'] & \mathsf{H}^\bullet(A) \arrow[r, "\nabla^\#"'] & \mathsf{H}^\bullet(L)
\end{tikzcd}$$

Observe that the rows of the above diagram are characteristic homomorphisms of FS-Lie algebroids. Since F_0^L, $F_1^L : L \to T\mathbb{R} \times L$ are homotopic homomorphisms, we have $F_0^{L\,\#} = F_1^{L\,\#}$. To prove the homotopy independence of the exotic characteristic homomorphism (in the sense of Definition 3.5), it is sufficient to show that $F_0^{L\,\#}$, $F_1^{L\,\#}$ are isomorphisms.

Let $t_o \in \mathbb{R}$. We shall see below that $F_{t_o}^L$ is a homotopy equivalence. Take the projection $\pi : T\mathbb{R} \times L \to L$ (over $\mathrm{pr}_2 : \mathbb{R} \times M \to M$). Of course, π is a homomorphism of Lie algebroids. Note that $F_{t_o}^L \circ \pi = \left(\hat{t}_o\right)_* \times \mathrm{id}_L$, where $\hat{t}_o : \mathbb{R} \to \mathbb{R}$ is defined by $t \mapsto t_o$. We take $\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(s, t) \mapsto t_o + s\,(t - t_o)$. Since the differential $f_* : TM \to TN$ of any smooth mapping $f : M \to N$ is a homomorphism of Lie algebroids [18], the map $\tau_* : T(\mathbb{R}\times\mathbb{R}) = T\mathbb{R}\times T\mathbb{R} \to T\mathbb{R}$ is a homomorphism of Lie algebroids. We put

$$\begin{aligned} &H : T\mathbb{R}\times (T\mathbb{R}\times L) \longrightarrow T\mathbb{R}\times L,\\ &H = \tau_*\times \mathrm{id}_L\,. \end{aligned}$$

Since

$$\begin{aligned} &H\left(\theta_0,\cdot,\cdot\right) = \tau\left(0,\cdot\right)_*\times \mathrm{id}_L = \left(\hat{t}_o\right)_*\times_L = F^L_{t_o}\circ\pi,\\ &H\left(\theta_1,\cdot,\cdot\right) = \tau\left(\cdot,1\right)_*\times \mathrm{id}_L = \mathrm{id}_{T\mathbb{R}\times L}, \end{aligned}$$

H is a homotopy joining $F^L_{t_o}\circ\pi$ to $\mathrm{id}_{T\mathbb{R}\times L}$, i.e. $F^L_{t_o}\circ\pi\sim \mathrm{id}_{T\mathbb{R}\times L}$. Evidently, $\pi\circ F^L_{t_o} = \mathrm{id}_L$. Therefore, each F^L_t is an isomorphism.

These facts lead us to the following result:

Theorem 3.6 (The Rigidity Theorem) *If B_0, $B_1\subset A$ are homotopic subalgebroids of A and $\nabla : L\to A$ is a flat L-connection in A, then the characteristic homomorphisms $\Delta_{(A,B_0,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g},B_0)\to \mathsf{H}^\bullet(L)$ and $\Delta_{(A,B_1,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g},B_1)\to \mathsf{H}^\bullet(L)$ are equivalent in the sense that there exists an isomorphism of algebras*

$$\alpha : \mathsf{H}^\bullet(\boldsymbol{g},B_0)\xrightarrow{\simeq}\mathsf{H}^\bullet(\boldsymbol{g},B_1)$$

such that

$$\Delta_{(A,B_1,\nabla)\#}\circ\alpha = \Delta_{(A,B_0,\nabla)\#}.$$

In particular, $\Delta_{(A,B_1)\#}\circ\alpha = \Delta_{(A,B_0)\#}$.

Corollary 3.7 *Let $\mathfrak{f}$ be a vector bundle, $\mathcal{A}(\mathfrak{f})$ its Lie algebroid, $B_0 = \mathcal{A}(\mathfrak{f},\{h_0\})$ and $B_1 = \mathcal{A}(\mathfrak{f},\{h_1\})$ Riemannian reductions of $\mathcal{A}(\mathfrak{f})$, corresponding to Riemannian metrics h_0, h_1, respectively. Then*

$$\Delta_{(\mathcal{A}(\mathfrak{f}),\mathcal{A}(\mathfrak{f},\{h_0\}))\#} = \Delta_{(\mathcal{A}(\mathfrak{f}),\mathcal{A}(\mathfrak{f},\{h_1\}))\#}. \tag{3.2}$$

Proof. The Lie subalgebroids $B_0 = \mathcal{A}(\mathfrak{f},\{h_0\})$ and $B_1 = \mathcal{A}(\mathfrak{f},\{h_1\})$ are homotopic [19, Theorem 5.5]. Therefore, according to the Rigidity Theorem 3.6, we obtain (3.2). ■

The last corollary shows that the characteristic homomorphism for the pair $\mathcal{A}(\mathfrak{f})$, $\mathcal{A}(\mathfrak{f},\{h\})$ is an intrinsic notion for $\mathcal{A}(\mathfrak{f})$ not depending on the metric h.

4 Particular Cases of the Universal Exotic Characteristic Homomorphism

4.1 The Koszul Homomorphism

In this section, we will consider the characteristic homomorphism $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ for a pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$, $\mathfrak{h}\subset\mathfrak{g}$, and give a class of such pairs for which $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is a monomorphism.

An arbitrary Lie algebra is a Lie algebroid over a point with the zero map as an anchor. Consider the homomorphism of pairs of Lie algebras $(\mathrm{id}_{\mathfrak{g}},0) : (\mathfrak{g},0)\to(\mathfrak{g},\mathfrak{h})$, $\mathfrak{h}\subset\mathfrak{g}$. By the definition of the universal exotic characteristic homomorphism, observe that

$$\Delta_{(\mathfrak{g},0)\#} : \mathsf{H}^\bullet(\mathfrak{g},0) = \mathsf{H}^\bullet(\mathfrak{g})\xrightarrow{(-\,\mathrm{id}_{\mathfrak{g}})_\#}\mathsf{H}^\bullet(\mathfrak{g}).$$

Now, the functoriality of $(\mathrm{id}_{\mathfrak{g}}, 0)$ described in Theorem 3.1 implies that

$$\Delta_{(\mathfrak{g},\mathfrak{h})\#} = \Delta_{(\mathfrak{g},0)\#} \circ (\mathrm{id}_{\mathfrak{g}})^{+\#} = (-\,\mathrm{id}_{\mathfrak{g}})^{+\#} : \mathsf{H}^{\bullet}(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathsf{H}^{\bullet}(\mathfrak{g}).$$

Let $(\bigwedge \mathfrak{g}^*)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}$ be the basic subalgebra of $\bigwedge \mathfrak{g}^*$, i.e. the subalgebra of invariant and horizontal elements of $\bigwedge \mathfrak{g}^*$ with respect to the Lie subalgebra $\mathfrak{h}$. Denote by k the inclusion from $(\bigwedge \mathfrak{g}^*)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}$ into $\bigwedge \mathfrak{g}^*$ (see [11, p. 412]). Moreover, consider the projection $s : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ and the map

$$(-s)^* : \left(\bigwedge (\mathfrak{g}/\mathfrak{h})^*\right)^{\mathfrak{h}} \longrightarrow \left(\bigwedge \mathfrak{g}^*\right)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}$$

given by

$$\left((-s)^* \Psi\right)(x_1 \wedge \ldots \wedge x_k) = \langle \Psi, (-s(x_1)) \wedge \ldots \wedge (-s(x_k))\rangle$$

for $\Psi \in (\bigwedge^k (\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$, $x_1, \ldots, x_k \in \mathfrak{g}$. One can see that $(-s)^*$ is an isomorphism of algebras and

$$\Delta_{(\mathfrak{g},\mathfrak{h})} = \mathrm{k} \circ (-s)^* .$$

Therefore, the exotic characteristic homomorphism $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ for the pair $(\mathfrak{g},\mathfrak{h})$ can be written as the composition

$$\Delta_{(\mathfrak{g},\mathfrak{h})\#} : \mathsf{H}^{\bullet}(\mathfrak{g},\mathfrak{h}) \xrightarrow[\cong]{(-s)^{\#}} \mathsf{H}^{\bullet}(\mathfrak{g}/\mathfrak{h}) \xrightarrow{\mathrm{k}^{\#}} \mathsf{H}^{\bullet}(\mathfrak{g}),$$

where $\mathsf{H}^{\bullet}(\mathfrak{g}/\mathfrak{h})$ denotes the cohomology algebra $\mathsf{H}^{\bullet}((\bigwedge \mathfrak{g}^*)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}, d_{\mathfrak{g}})$.

Example 4.1 Let $(\mathfrak{g},\mathfrak{h})$ be a reductive pair of Lie algebras ($\mathfrak{h} \subset \mathfrak{g}$), and $s : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ the canonical projection. Theorems IX and X from [11, sections 10.18, 10.19] imply that $\mathrm{k}^{\#}$ is injective if and only if $\mathsf{H}^{\bullet}(\mathfrak{g}/\mathfrak{h})$ is generated by 1 and odd-degree elements. Therefore, because $(-s)^{\#}$ is an isomorphism of algebras, it follows that $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is injective if and only if $\mathsf{H}^{\bullet}(\mathfrak{g},\mathfrak{h})$ is generated by 1 and odd-degree elements. In a wide class of pairs of Lie algebras $(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{h}$ is reductive in $\mathfrak{g}$, the homomorphism $\mathrm{k}^{\#}$ is injective if and only if $\mathfrak{h}$ is noncohomologous to zero (briefly: n.c.z.) in $\mathfrak{g}$ (i.e. if the homomorphism of algebras $\mathsf{H}^{\bullet}(\mathfrak{g}) \to \mathsf{H}^{\bullet}(\mathfrak{h})$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is surjective). Tables I, II and III at the end of Section XI of [11] contain many n.c.z. pairs, e.g.: $(\mathfrak{gl}(n), \mathfrak{so}(n))$ for odd n, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(k,\mathbb{C}))$ for $k < n$, $(\mathfrak{so}(2m+1), \mathfrak{so}(2k+1))$, $(\mathfrak{so}(2m), \mathfrak{so}(2k+1))$ for $k < m$.

In view of the above, the examples below show that the exotic characteristic homomorphism for the reductive pair $(\mathrm{End}(V), \mathrm{Sk}(V))$ of Lie algebras is a monomorphism for any odd-dimensional vector space V and is not a monomorphism for even-dimensional ones.

Example 4.2 (The pair of Lie algebras $(\mathrm{End}(V), \mathrm{Sk}(V))$) (a) Let V be a vector space of odd dimension, $\dim V = 2m-1$. Then

$$\mathsf{H}^{\bullet}(\mathrm{End}(V), \mathrm{Sk}(V)) \cong \mathsf{H}^{\bullet}(\mathfrak{gl}(2m-1,\mathbb{R}), O(2m-1)) \cong \bigwedge(y_1, y_3, \ldots, y_{2m-1}),$$

where $y_{2k-1} \in \mathsf{H}^{4k-3}(\mathrm{End}(V), \mathrm{Sk}(V))$ are represented by the multilinear trace forms ([10], [13]). We conclude from the previous example that $\Delta_{(\mathrm{End}(V),\mathrm{Sk}(V))\#}$ is injective.
(b) In the case where V is an even-dimensional vector space, we have [10]

$$\mathsf{H}^{\bullet}(\mathrm{End}(V), \mathrm{Sk}(V)) \cong \mathsf{H}^{\bullet}(\mathfrak{gl}(2m,\mathbb{R}), SO(2m)) \cong \bigwedge(y_1, y_3, \ldots, y_{2m-1}, y_{2m}),$$

where $2m = \dim V$, y_{2k-1} are as above and $y_{2m} \in \mathsf{H}^{2m}(\mathrm{End}(V), \mathrm{Sk}(V))$ is a nonzero class determined by the Pfaffian. For details concerning the elements y_{2m} see [4]. Example 4.1 shows that if $\dim V$ is even, the homomorphism $\Delta_{(\mathrm{End}(V),\mathrm{Sk}(V))\#}$ is not a monomorphism.

Example 4.3 Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie algebras and $\mathfrak{g}\oplus\mathfrak{h}$ their direct product. The characteristic homomorphism of the pair $(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ is a monomorphism. It is equal to

$$\Delta_{(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\#} : \mathsf{H}^\bullet(\mathfrak{g}) \longrightarrow \mathsf{H}^\bullet(\mathfrak{g})\otimes\mathsf{H}^\bullet(\mathfrak{h}),\quad \Delta_{\#(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})}([\Phi]) = [(-1)^{|\Phi|}\cdot\Phi]\otimes 1.$$

4.2 The Exotic Universal Characteristic Homomorphism of Principal Fibre Subbundles

We recall briefly a connection between the secondary characteristic homomorphism

$$\Delta_{(P,P',\omega)\#} : \mathsf{H}^\bullet(\mathfrak{g},H) \longrightarrow \mathsf{H}^\bullet_{dR}(M)$$

for flat principal bundles [13] and the exotic characteristic homomorphism

$$\Delta_{(\mathrm{A}(P),\mathrm{A}(P'),\omega)\#} : \mathsf{H}^\bullet(\boldsymbol{g},\mathrm{A}(P')) \longrightarrow \mathsf{H}^\bullet_{dR}(M)$$

for the induced pair of Lie algebroids $\mathrm{A}(P)$, $\mathrm{A}(P')$ and a suitable flat connection in $\mathrm{A}(P)$ determined by ω (cf. [4]) and denoted here also by ω. Namely, there exists an isomorphism of algebras $\kappa : \mathsf{H}^\bullet(\mathfrak{g},H) \xrightarrow{\simeq} \mathsf{H}^\bullet(\boldsymbol{g},\mathrm{A}(P'))$ such that $\Delta_{(\mathrm{A}(P),\mathrm{A}(P'),\omega)\#}\circ\kappa = \Delta_{(P,P',\omega)\#}$ (see [19, Theorem 6.1]). This leads to a new universal characteristic homomorphism

$$\Delta_{(P,P')\#} = \Delta_{(\mathrm{A}(P),\mathrm{A}(P'))\#}\circ\kappa : \mathsf{H}^\bullet(\mathfrak{g},H) \longrightarrow \mathsf{H}^{r\bullet}_{dR}(P)$$

where $\mathsf{H}^{r\bullet}_{dR}(P)$ denotes the cohomology space of right-invariant forms in P. Then $\Delta_{(P,P',\omega)\#}$ can be described as the composition

$$\Delta_{(P,P',\omega)\#} : \mathsf{H}^\bullet(\mathfrak{g},H) \xrightarrow{\Delta_{(P,P')\#}} \mathsf{H}^{r\bullet}_{dR}(P) = \mathsf{H}^\bullet(\mathrm{A}((P)) \xrightarrow{\omega^\#} \mathsf{H}^\bullet_{dR}(M)$$

In particular, if G is a compact, connected Lie group and P' is a connected H-reduction in a G-principal bundle P, then $\mathsf{H}^{r\bullet}_{dR}(P) = \mathsf{H}^\bullet_{dR}(P)$ and $\Delta_{(P,P')\#}$ acts from $\mathsf{H}^\bullet(\mathfrak{g},H)$ into $\mathsf{H}^\bullet(P)$.

5 When the Universal Exotic Characteristic Homomorphism for a Pair of Transitive Lie Algebroids is a Monomorphism

5.1 The Koszul Homomorphism Versus the Universal Exotic Characteristic Homomorphism

Consider a pair (A,B), $B\subset A$, of transitive Lie algebroids on a manifold M, $x\in M$, and a pair of adjoint Lie algebras $(\boldsymbol{g}_x,\boldsymbol{h}_x)$. Clearly, the inclusion $\iota_x : (\boldsymbol{g}_x,\boldsymbol{h}_x)\to(A,B)$ is a homomorphism of pairs of Lie algebroids over $\{x\}\hookrightarrow M$. Theorem 3.1 gives rise to the commutative diagram

$$\begin{tikzcd}
\mathsf{H}^\bullet(\boldsymbol{g},B) \arrow[r, "\Delta_{(A,B)\#}"] \arrow[d, "\iota_x^{+\#}"'] & \mathsf{H}^\bullet(A) \arrow[d, "\iota_x^{\#}"] \\
\mathsf{H}^\bullet(\boldsymbol{g}_x,\boldsymbol{h}_x) \arrow[r, "\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}"'] & \mathsf{H}^\bullet(\boldsymbol{g}_x).
\end{tikzcd} \tag{5.1}$$

Obviously, if the left and bottom homomorphisms are monomorphisms, then so is $\Delta_{(A,B)\#}$. The homomorphism $\iota_x^{+\#}$ is a monomorphism if each invariant element $v \in (\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x^*))^{\mathfrak{h}_x}$ can be extended to a global invariant cross-section of the vector bundle $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$. Consequently, we obtain the following theorem linking the Koszul homomorphism with exotic characteristic classes:

Theorem 5.1 *Let (A, B) be a pair of transitive Lie algebroids over a manifold M, $B \subset A$, $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ be a pair of adjoint Lie algebras at $x \in M$, and suppose each element of $(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\mathfrak{h}_x}$ extends to an invariant cross-section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$. If the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}$ for the pair $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ is a monomorphism, then $\Delta_{(A,B)\#}$ is a monomorphism.*

Remark 5.2 Theorem 6.5.15 of [21] implies that an element of $(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\mathfrak{h}_x}$, $x \in M$, can be extended to an invariant cross-section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$ if and only if it is invariant with respect to the $\pi_1(M)$-action on $(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\mathfrak{h}_x}$ via the holonomy morphism of the flat B-connection $\nabla^{\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}} = \overline{\mathrm{ad}^{\wedge}_{B,\,\boldsymbol{h}}} \circ \gamma$ in the bundle

$$\left(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*\right)^{\boldsymbol{h}} = \bigsqcup_{x\in M}\left(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*\right)^{\mathfrak{h}_x}$$

where $\gamma : TM \to B$ is any TM-connection in B and $\overline{\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}}$ is the representation of B in $(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)^{\boldsymbol{h}}$ induced by $\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}$.

In the next two sections we give examples of pairs $B \subset A$ satisfying the assumptions of the last theorem, among which are non-integrable Lie algebroids.

5.2 The Case of Integrable Lie Algebroids

Take the Lie algebroid $\mathrm{A}(P)$ of some principal G-bundle P and its subalgebroid $\mathrm{A}(P')$ for some reduction P' of P with connected structural Lie group $H \subset G$. Notice that, on account of Theorem 1.1 in [15], for any transitive Lie subalgebroid $B \subset \mathrm{A}(P)$ there exists a connected reduction P' of P having B as its Lie algebroid, i.e. $B = \mathrm{A}(P')$. However, in general, the structural Lie group of P' need not be connected.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. The representation $\mathrm{ad}_{B,\boldsymbol{h}}$ is integrable, because it is the differential of the representation $\mathrm{Ad}_{P',\boldsymbol{h}} : P' \to L(\boldsymbol{g}/\boldsymbol{h})$ of the principal fibre bundle P' defined by $z \mapsto [\hat{z}]$ (see [19, p. 218]). We recall that for each $z \in P'$, the isomorphism $\hat{z} : \mathfrak{g} \to \boldsymbol{g}_x$, $v \mapsto [(A_z)_{*e} v]$ $(A_z : G \to P,\ a \mapsto za)$ maps $\mathfrak{h}$ onto $\boldsymbol{h}_x$ (see [16, Sect. 5.1]) and determines an isomorphism $[\hat{z}] : \mathfrak{g}/\mathfrak{h} \to \boldsymbol{g}_x/\boldsymbol{h}_x$. Therefore (see also [16, Prop. 5.5.2-3]), we have a natural isomorphism

$$\kappa : \left(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*\right)^H \xrightarrow{\cong} \left(\Gamma\left(\bigwedge(\boldsymbol{g}/\boldsymbol{h}^*)\right)\right)^{\Gamma(B)}.$$

By the connectedness of H, $(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^H = (\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\mathfrak{h}_x}$, which implies that $\iota_x^{+\#}$ is an isomorphism. Thus we see that the assumptions of Theorem 5.1 hold for any pair of integrable Lie algebroids A and B $(B \subset A)$, i.e. if A is a Lie algebroid of some principal bundle P and B is its Lie subalgebroid of some reduction of P.

Now, Theorem 5.1 implies the following result:

Theorem 5.3 *Let A be a Lie algebroid of some principal bundle $P\,(M,G)$, $B=\mathrm{A}(P')$ its Lie subalgebroid for some reduction P' of P, $(\boldsymbol{g}_x,\boldsymbol{h}_x)$ be a pair of adjoint Lie algebras at $x\in M$. If the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}$ for the pair $(\boldsymbol{g}_x,\boldsymbol{h}_x)$ is a monomorphism for any $x\in M$, then the universal exotic characteristic homomorphism $\Delta_{(A,B)\#}$ is a monomorphism.*

Remark 5.4 The example in [4, Sect. 3.3] of a nontrivial universal characteristic class determined by the Pfaffian shows that there exists a pair of transitive and integrable Lie algebroids (A,B) for which the left arrow in the diagram (5.1) is an isomorphism, the bottom one is not a monomorphism, but the top one is a monomorphism.

5.3 The Case of Non-integrable Lie Algebroids of Some TC-foliations

We start with the following general theorem on Lie algebroids with abelian isotropy Lie algebras.

Theorem 5.5 *Let (A,B) with $B\subset A$ be a pair of transitive Lie algebroids over a manifold M, for which the kernels $\boldsymbol{g},\boldsymbol{h}$ of the anchors are abelian Lie algebra bundles and let $x\in M$. If each element of $\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*$ can be extended to an invariant cross-section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$, then the universal exotic characteristic homomorphism $\Delta_{(A,B)\#}$ is a monomorphism.*

Proof. Let $x\in M$. Since the homomorphism $\mathsf{H}^\bullet(\boldsymbol{g}_x)=\bigwedge(\boldsymbol{g}_x)^*\to\bigwedge(\boldsymbol{h}_x)^*=\mathsf{H}^\bullet(\boldsymbol{h}_x)$ induced by the inclusion $\boldsymbol{g}_x\hookrightarrow\boldsymbol{h}_x$ is surjective and $(\boldsymbol{g}_x,\boldsymbol{h}_x)$ is a reductive pair of Lie algebras, the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}$ is injective (see Example 4.1). Since the isotropy algebras $\boldsymbol{g}$ and $\boldsymbol{h}$ are abelian, every $v\in\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*$ is invariant, i.e. $\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*=(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\boldsymbol{h}_x}$. Theorem 5.1 now shows that if each element of $\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*$ can be extended to an invariant cross-section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$, then $\Delta_{(A,B)\#}$ is a monomorphism. ■

We find non-integrable Lie algebroids satisfying the assumptions of the last theorem. Our examples are Lie algebroids $A(G,H)$ of transversely complete foliations (TC-foliations for short) [16].

Take a Lie group G and its connected nonclosed Lie subgroup H. Let $A(G,H)$ be a Lie algebroid of a TC-foliation. Let us recall that $A(G,H)$ is the Lie algebroid of left cosets of H in G (see [1], [16]). Notice that such Lie algebroids are non-integrable if and only if the TC-foliation is not developable (see [1]) — which occurs in the case where G is connected and simply connected.

These TC-foliations play an essential role in the theory of Riemannian foliations (see [22]). Lie algebroids $A(G,H)$ are cohomologically quite complicated. There are examples of non-integrable Lie algebroids $A(G,H)$ for which the Chern-Weil homomorphism is nontrivial [16].

Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Moreover, let $\overline{H}$ denote the closure of H. Using trivializations of $TG=G\times\mathfrak{g}$ given by left-invariant vector fields one can check that $A(G,H)$ is a vector bundle over $G/\overline{H}$ which is the quotient space $(G\times(\mathfrak{g}/\mathfrak{h}))_{\overline{H}}$ with respect to the right action of $\overline{H}$ on G and the adjoint action of $\overline{H}$ on $\mathfrak{g}/\mathfrak{h}$. Next, recall that there exists an isomorphism

$$c:l(G,H)\xrightarrow{\cong}\Gamma(A(G,H))$$

of the module $l(G,H)$ of transverse fields onto the module of global cross-sections of $A(G,H)$. Notice that c is also an isomorphism of real Lie algebras.

Let $\mathfrak{s}$ denote the Lie algebra of $\overline{H}$. Every right-invariant vector field $\overline{Y}_w$ generated by $w \in \mathfrak{g}$ and every left-invariant vector field $\overline{X}_w$ generated by $w \in \mathfrak{s}$ is a transverse field [16]. Moreover, the Lie algebra bundle $\boldsymbol{g}$ associated with $A(G,H)$ is a trivial vector bundle of abelian Lie algebras with the trivialization $G/\overline{H} \times \mathfrak{s}/\mathfrak{h} \to \boldsymbol{g}$ defined by $(x,[w]) \mapsto (c(\overline{X}_w))(x)$.

Let H_1 and H_2 be Lie subgroups of G such that

$$H_1 \subsetneq H_2 \subsetneq T \subsetneq G \quad \text{where} \quad T = \overline{H_1} = \overline{H_2}.$$

Denote by $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{t}$ the Lie algebras of H_1, H_2 and T, respectively. Denote

$$A = A(G,H_1).$$

Consider the Lie subalgebroid

$$B = A(G,H_2).$$

The Lie algebra bundles of A and B are $\boldsymbol{g} = G/T \times \mathfrak{t}/\mathfrak{h}_1$ and $\boldsymbol{h} = G/T \times \mathfrak{t}/\mathfrak{h}_2$, respectively. Take an element

$$\sigma_o \in \bigwedge^k (\boldsymbol{g}_x/\boldsymbol{h}_x)^* \cong \bigwedge^k ((\mathfrak{t}/\mathfrak{h}_1)\,/\,(\mathfrak{t}/\mathfrak{h}_2))^*$$

and the constant cross-section Π of the vector bundle $\bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$ which is equal to σ_o at all points of G/T.

We will show that Π is invariant. Let $\nu_1,\ldots,\nu_k \in \Gamma(\boldsymbol{g}) = \Gamma(G/T \times \mathfrak{t}/\mathfrak{h}_1)$ and $\eta \in \Gamma(B)$. Every ν_i $(i \in \{1,\ldots,k\})$ can be written in the form $\sum_{j_i} f_i^{j_i} \cdot c(\overline{X}_{w_{j_i}})$ where $f_i^{j_i} \in C^\infty(G/T)$, $w_{j_i} \in \mathfrak{t}$. We see that locally $\eta = \sum_j h^j \cdot c(\overline{Y}_{g_j})$ for some $h^j \in C^\infty(G/T)$ and $g_j \in \mathfrak{g}$. Denote by $\#_2$ the anchor of the Lie algebroid $A(G,H_2)$. Then

$$[\![\eta,\nu_r]\!] = \sum\nolimits_{j,j_r} h^j \cdot \left(\#_2 \circ c\left(\overline{Y}_{g_j}\right)\right)\left(f_r^{j_r}\right) c\left(\overline{X}_{w_{j_r}}\right) \tag{5.2}$$

(see [16, p. 56]). Since Π is a constant cross-section of $\bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$, since vector fields are derivations of the algebra of smooth functions, and since (5.2) holds, we deduce that

$$\begin{aligned}
&(\#_2 \circ \eta)\,\langle \Pi, [\nu_1]\wedge\ldots\wedge[\nu_k]\rangle \\
&= \sum_j \sum_{j_1,\ldots,j_k} h^j \left(\#_2 \circ c\left(\overline{Y}_{g_j}\right)\right) \left(f_1^{j_1}\cdots f_k^{j_k} \left\langle \Pi, \left[c\left(\overline{X}_{w_{j_1}}\right)\right]\wedge\ldots\wedge\left[c\left(\overline{X}_{w_{j_k}}\right)\right]\right\rangle\right) \\
&= \sum_{j,r} \sum_{j_1,\ldots,j_k} h^j f_1^{j_1}\cdots\left(\#_2 \circ c\left(\overline{Y}_{g_j}\right)\right)\left(f_r^{j_r}\right)\cdots f_k^{j_k} \left\langle \Pi, \left[c\left(\overline{X}_{w_{j_1}}\right)\right]\wedge\ldots\wedge\left[c\left(\overline{X}_{w_{j_k}}\right)\right]\right\rangle \\
&= \sum_r (-1)^{r-1} \left\langle \Pi, \left[\sum\nolimits_{j,j_r} h^j\left(\#_2 \circ c\left(\overline{Y}_{g_j}\right)\right)\left(f_r^{j_r}\right) c\left(\overline{X}_{w_{j_r}}\right)\right] \wedge [\nu_1]\wedge\ldots\widehat{r}\ldots\wedge[\nu_k]\right\rangle \\
&= \sum_r (-1)^{r-1} \left\langle \Pi, [\![\eta,\nu_r]\!]\wedge[\nu_1]\wedge\ldots\widehat{r}\ldots\wedge[\nu_k]\right\rangle,
\end{aligned}$$

which proves that Π is an invariant cross-section of $\bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$.

Since every element of $\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*$ can be extended to an invariant cross-section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$, Theorem 5.5 now shows the following

Theorem 5.6 *Let G be a Lie group, and H_1 and H_2 be its Lie subgroups such that $H_1 \subsetneq H_2 \subsetneq \overline{H_1} = \overline{H_2} \subsetneq G$. Then the universal exotic characteristic homomorphism for the pair of Lie algebroids $A(G,H_2) \subset A(G,H_1)$ is a monomorphism. Additionally, if G is connected and simply connected, the algebroids $A(G,H_2)$ and $A(G,H_1)$ are non-integrable.*

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