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Some exotic characteristic homomorphism for Lie algebroids

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The authors define some secondary characteristic homomorphism for the triple (A, B, ∇), in which $B \subset A$ is a pair of regular Lie algebroids over the same foliated manifold and $\nabla : L \to A$ is a homomorphism of Lie algebroids (i.e. a flat L-connection in A) where L is an arbitrary (not necessarily regular) Lie algebroid and show that characteristic classes from its image generalize known exotic characteristic classes for flat regular Lie algebroids (Kubarski) and flat principal fibre bundles with a reduction (Kamber, Tondeur). The generalization includes also the one given by Crainic for representations of Lie algebroids on vector bundles. For a pair of regular Lie algebroids $B \subset A$ and for the special case of the flat connection $\mathrm{id}_A : A \to A$ we obtain a characteristic homomorphism which is universal in the sense that it is a factor of any other one for an arbitrary flat L-connection $\nabla : L \to A$.

1. Introduction

From the very beginning the characteristic classes (primary and secondary) are global invariants of geometric structures on manifolds (more generally on principal fibre bundles on manifolds) determined mainly by connections, reductions of structure Lie groups, and so on, and having some important topological properties like homotopy independence, functoriality or rigidity. N. Teleman [23] showed in 1972 that the Chern–Weil homomorphism of any principal fibre bundle with a **connected** structural Lie group is an invariant of its infinitesimal object, i.e. of the Lie algebroid of this bundle. J. Kubarski [16] showed in 1991 that the condition of the connectedness of the structural Lie group is redundant, which means that primary characteristic homomorphisms (i.e. the Chern–Weil homomorphism) is really the "algebraic" notion belonging to the category of Lie algebroids. The crucial role was played by some generalization of the standard concepts of the representation of Lie groups (and Lie algebras) on vector spaces to the concept of the representation of principal fibre bundles and of Lie algebroids on vector bundles and comparing the spaces of suitable invariant cross-sections. It turns out (J. Kubarski [18,19]) that the same holds for the secondary (exotic) characteristic classes, in particular, for the characteristic classes of flat bundles. In [18] there was constructed a characteristic homomorphism for flat regular Lie algebroids equipped with some "reduction", i.e. with some Lie subalgebroid, generalizing this homomorphism for foliated principal fibre bundles given by F. Kamber and Ph. Tondeur [13–15]. Next, a different approaches to secondary classes by M. Crainic and R.L. Fernandes [4, 2003], [5,8], [6, 2005], appeared in the geometry of Lie algebroids (inspired, for example, by irregular Lie algebroids important in the Poisson geometry), for example, secondary characteristic classes for representations [4, 2003], characteristic classes up to homotopy [5] and intrinsic secondary characteristic classes [8], [6, 2005]. The last were lastly generalized by I. Vaisman [24].

The main goal of the paper is to build an exotic characteristic homomorphism in the category of Lie algebroids, which generalizes simultaneously the one given by Kubarski [18] and the one given by Crainic [4] and describes in the Lie

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algebroids language the classical case for foliated principal bundles. It is a characteristic homomorphism $\Delta_{(A,B,\nabla)\#}$ for the triple (A, B, ∇), where $B \subset A$ is a pair of regular Lie algebroids, both on the same regular foliated manifold (M, F), and $\nabla : L \to A$ is a flat L-connection in A (i.e. ∇ is a homomorphism of Lie algebroids), where L is an arbitrary (not necessarily regular) Lie algebroid on M. For a Lie algebroid of a principal fibre bundle, its reduction, and a usual flat connection, we obtain a homomorphism equivalent to the one by Kamber and Tondeur, mentioned above. Putting $L = A$ and the "trivial" flat connection $\nabla = \mathrm{id}_A : A \to A$, we obtain a quite new characteristic homomorphism previously unknown, even in the context of the principal fibre bundles. In fact, this homomorphism is obtained for a pair of Lie algebroids (A, B), $B \subset A$ and can be denoted by $\Delta_{(A,B)\#}$. It is in some sense a universal homomorphism. Namely, for any flat L-connection $\nabla : L \to A$ we have

$$\Delta_{(A,B,\nabla)\#} = \nabla^{\#} \circ \Delta_{(A,B)\#} \tag{1}$$

(we will say that $\Delta_{(A,B,\nabla)\#}$ is *factorized* by the universal characteristic homomorphism $\Delta_{(A,B)\#}$). The classes form the image of $\Delta_{(A,B)\#}$ (which belongs to the cohomology algebra $\mathsf{H}^{\bullet}(A)$) are called *universal* for the pair $B \subset A$.

In the context of the comparison with the Crainic classes (which concern [in our setting] the triple $(\mathrm{A}(\mathfrak{f}), \mathrm{A}(\mathfrak{f}, \{h\}), \nabla)$ where $\mathrm{A}(\mathfrak{f})$, $\mathrm{A}(\mathfrak{f}, \{h\})$ are Lie algebroids of a vector bundle $\mathfrak{f}$, its Riemann reduction $(\mathfrak{f}, \{h\})$, and $\nabla : L \to \mathrm{A}(\mathfrak{f})$ is an arbitrary flat L-connection in $\mathfrak{f}$, i.e. a representation of L on $\mathfrak{f}$), we present one—based on the Pfaffian—characteristic class for an even dimensional, oriented vector bundle not considered by Crainic. An example with such a nontrivial universal characteristic class is presented in Section 3.3.

2. The secondary (exotic) characteristic homomorphism for FS-Lie algebroids and the universal characteristic homomorphism

In this section we first define a characteristic homomorphism for the triple (A, B, ∇), where $B \subset A$ is a pair of regular Lie algebroids, both on the same regular foliated manifold (M, F), and $\nabla : L \to A$ is a flat L-connection in A, where L is an arbitrary Lie algebroid on a manifold M. Next we compare this homomorphism with characteristic homomorphisms for regular Lie algebroids and usual flat connections—given by Kubarski [18], and for principal fibre bundles—given by Kamber and Tondeur [15]. The comparison with the Crainic approach of secondary characteristic classes for representations will be given in the next section.

2.1. A few words about Lie algebroids

The notion of Lie algebroid (Pradines, 1967) had appeared as an infinitesimal object of Lie groupoids, principal bundles, vector bundles, TC-foliations, Poisson and Jacobi manifolds, etc. (for the historical approach see [20,21]). A *Lie algebroid* over a smooth manifold M is a triple $(L, [\![\cdot,\cdot]\!], \#_L)$ where L is a vector bundle over M, $(\Gamma(L), [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, $\#_L : L \to TM$ is a linear homomorphism of vector bundles such that $[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + \#_L(\xi)(f) \cdot \eta$ for all $f \in C^{\infty}(M)$, $\xi, \eta \in \Gamma(L)$. The anchor $\#_L$ of $(L, [\![\cdot,\cdot]\!], \#_L)$ is bracket-preserving, see [12] and [1]. The image $F := \mathrm{Im}\,\#_L \subset TM$ of the anchor $\#_L$ is an integrable (non-constant rank in general) distribution whose leaves form a Stefan foliation $\mathcal{F}$ of M [22,7]. We say also that A is a Lie algebroid over the foliated manifold (M, F). If the anchor $\#_L$ is of constant rank [an epimorphism], then the Lie algebroid L is called *regular* [*transitive*]. By a homomorphism of Lie algebroids $T : L \to A$ on a given manifold we mean a homomorphism of the underlying vector bundles which commutes with the anchors and preserves Lie bracket. The Lie algebroids of Lie groupoids, principal bundles, vector bundles and TC-foliations are transitive, but the Lie algebroids of general differential groupoids, Poisson manifolds, Jacobi manifolds etc. are rather nontransitive (and, in general, irregular). For details concerning Lie functors on the category of principal fibre bundles $P \rightsquigarrow \mathrm{A}(P)$ and vector bundles $\mathfrak{f} \rightsquigarrow \mathrm{A}(\mathfrak{f})$ see, for example, [16]. $\mathrm{A}(P) = TP/G$ is a Lie algebroid with the real Lie algebra $\Gamma(\mathrm{A}(P)) \cong \mathfrak{X}^r(P)$ and the anchor $\#_{\mathrm{A}(P)} : \mathrm{A}(P) \to TM$ determined by the projection π_* whereas $\mathrm{A}(\mathfrak{f})$ is the vector bundle whose global cross-sections form a Lie algebra of covariant derivative operators and the anchor $\#_{\mathrm{A}(\mathfrak{f})} : \mathrm{A}(\mathfrak{f}) \to TM$ is defined by the anchors of these operators. Together with a V-vector bundle $\mathfrak{f}$ (the vector space V is the typical fibre of $\mathfrak{f}$) we associate the $GL(V)$-principal fibre bundle of frames $\mathrm{L}(\mathfrak{f})$ and its Lie algebroid $\mathrm{A}(\mathrm{L}(\mathfrak{f}))$ which is canonically isomorphic to $\mathrm{A}(\mathfrak{f})$ [16]. For a regular Lie algebroid L we have the exact Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow L \xrightarrow{\#_L} F \to 0$ ($\boldsymbol{g} = \ker \#_L$), the fibre $\boldsymbol{g}_{|x}$ of $\boldsymbol{g}$ at x is a Lie algebra called the *isotropy Lie algebra at* x. Over any leaf of the foliation F the vector bundle $\boldsymbol{g}$ is a Lie algebra bundle. A splitting of this sequence $\nabla : F \to L$ (i.e. $\#_L \circ \nabla = \mathrm{id}_F$) is called a *connection* in L. If $L = \mathrm{A}(P)$ (here $F = TM$), then connections in L correspond one-to-one to usual connections in the principal fibre bundle P (i.e. a horizontal right-invariant distributions on P). We consider more general notion of L-connection in A (where L and A are arbitrary Lie algebroids on the same manifold) understanding as a linear homomorphism of vector bundles $\nabla : L \to A$ commuting with the anchors [1]. If A is a regular Lie algebroid with the adjoint LAB $\boldsymbol{g}$, then the curvature tensor $R^{\nabla} \in \Omega^2(L, \boldsymbol{g})$ is defined by $R^{\nabla}(\xi, \eta) = [\![\nabla\xi, \nabla\eta]\!] - \nabla([\![\xi, \eta]\!])$, $\xi, \eta \in \Gamma(L)$. Clearly, ∇ is a homomorphism of Lie algebroids if and only if ∇ is flat, i.e. if $R^{\nabla} = 0$. Any L-connection $\nabla : L \to A$ determines the standard operator $d^{\nabla} : \Omega(L; \boldsymbol{g}) \to \Omega(L; \boldsymbol{g})$ in the space of L-differential forms with values in $\boldsymbol{g}$ by the formula

$$\left(d^{\nabla}\Omega\right)(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} (-1)^{i-1} \left[\!\left[\nabla_{\xi_i}, \Omega(\xi_1, \ldots \hat{\imath} \ldots, \xi_n)\right]\!\right]_A + \sum_{i<j} (-1)^{i+j} \Omega\left([\![\xi_i, \xi_j]\!]_L, \xi_1, \ldots \hat{\imath} \ldots \hat{\jmath} \ldots, \xi_n\right). \tag{2}$$

The equality $d^\nabla d^\nabla \Omega = R^\nabla \wedge \Omega$ holds; in particular, d^∇ is a differential operator if ∇ is flat. For an arbitrary Lie algebroid L there is a derivative d_L in the space of real L-forms $\Gamma(\bigwedge L^*)$ giving the cohomology algebra $\mathsf{H}^\bullet(L)$. For more details about Lie algebroids we refer the reader, for example, to [21,20].

2.2. The construction of the secondary characteristic classes

Let us consider the triple (A, B, ∇), in which we have: a regular Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ on a foliated manifold (M, F), its regular Lie subalgebroid $B \subset A$, also on the same foliated manifold (M, F), and a flat L-connection $\nabla : L \to A$ in A for an arbitrary Lie algebroid L. We will call the triple

$$(A, B, \nabla)$$

an FS-*Lie algebroid*. The characteristic homomorphism for this triple constructed below measures the independence of these two geometric structures B and ∇ defined for A (in the sense that it is zero when ∇ takes values in B). In the diagram below $\lambda : F \to B$ is an arbitrary auxiliary connection in B. Then $j \circ \lambda : F \to A$ is a connection in A. Let $\breve{\lambda} : A \to \boldsymbol{g}$ be its connection form.

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{g} \arrow[r, hook, shift left, "i"] & A \arrow[l, shift left, "\breve{\lambda}"] \arrow[dr, "\#_A"] & L \arrow[l, "\nabla"'] \arrow[d, "\#_L"] \\
0 \arrow[r] & \boldsymbol{h} \arrow[r, hook] \arrow[u, hook] & B \arrow[u, hook, "j"] \arrow[r, shift left, "\#_B"] & F \supset F_1. \arrow[l, shift left, "\lambda"]
\end{tikzcd}$$

We define the homomorphism

$$\omega_{B,\nabla} : L \longrightarrow \boldsymbol{g}/\boldsymbol{h} \quad \text{by } \omega_{B,\nabla}(w) = \big[-(\breve{\lambda} \circ \nabla)(w)\big].$$

Observe that $\omega_{B,\nabla}$ does not depend on the choice of an auxiliary connection $\lambda : F \to A$ and $\omega_{B,\nabla} = 0$ if ∇ takes values in B. Let us define a homomorphism of algebras

$$\begin{aligned}
&\Delta_{(A,B,\nabla)} : \Gamma\Big(\bigwedge\nolimits^k (\boldsymbol{g}/\boldsymbol{h})^*\Big) \longrightarrow \Omega(L), \\
&(\Delta_{(A,B,\nabla)}\Psi)_x(w_1 \wedge \cdots \wedge w_k) = \big\langle \Psi_x, \omega_{B,\nabla}(w_1) \wedge \cdots \wedge \omega_{B,\nabla}(w_k)\big\rangle, \quad w_i \in L_{|x}.
\end{aligned} \tag{3}$$

In the special simple case $L = A$ and the flat connection $\nabla = \mathrm{id}_A : A \to A$ we have particular case of a homomorphism for the pair (A, B):

$$\begin{aligned}
&\Delta_{(A,B)} : \Gamma\Big(\bigwedge\nolimits^k (\boldsymbol{g}/\boldsymbol{h})^*\Big) \longrightarrow \Omega(A), \\
&(\Delta_{(A,B)}\Psi)_x(\upsilon_1 \wedge \cdots \wedge \upsilon_k) = \big\langle \Psi_x, \big[-\breve{\lambda}(\upsilon_1)\big] \wedge \cdots \wedge \big[-\breve{\lambda}(\upsilon_k)\big]\big\rangle, \quad \upsilon_i \in A_{|x}.
\end{aligned}$$

We assert that $\Delta_{(A,B,\nabla)}$ can be written as a superposition $\Delta_{(A,B,\nabla)} = \nabla^* \circ \Delta_{(A,B)}$,

$$\Delta_{(A,B,\nabla)} : \Gamma\Big(\bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\Big) \xrightarrow{\Delta_{(A,B)}} \Omega(A) \xrightarrow{\nabla^*} \Omega(L),$$

where ∇^* is the pullback of forms. In the algebra $\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)$ we distinguish the subalgebra $(\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ of invariant cross-sections with respect to the representation of the Lie algebroid B in the vector bundle $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$, associated to the adjoint one $\mathrm{ad}_{B,\boldsymbol{h}} : B \to \mathsf{A}(\boldsymbol{g}/\boldsymbol{h})$, $\mathrm{ad}_{B,\boldsymbol{h}}(\xi)([\nu]) = [[\![\xi, \nu]\!]]$, $\xi \in \Gamma(B)$, $\nu \in \Gamma(\boldsymbol{g})$, where $\mathsf{A}(\boldsymbol{g}/\boldsymbol{h})$ is the (transitive) Lie algebroid of $\boldsymbol{g}/\boldsymbol{h}$. Clearly, $\Psi \in (\Gamma(\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ if and only if $(\#_B \circ \xi)\langle \Psi, [\nu_1] \wedge \cdots \wedge [\nu_k]\rangle = \sum_{j=1}^k (-1)^{j-1} \langle \Psi, [[\![j \circ \xi, \nu_j]\!]] \wedge [\nu_1] \wedge \ldots \hat{\jmath} \ldots \wedge [\nu_k]\rangle$ for all $\xi \in \Gamma(B)$ and $\nu_j \in \Gamma(\boldsymbol{g})$ (see [16]). In the space $(\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ of invariant cross-sections there exists a differential $\bar{\delta}$ defined by the formula

$$\big\langle \bar{\delta}\Psi, [\nu_1] \wedge \cdots \wedge [\nu_k]\big\rangle = \sum_{i<j} (-1)^{i+j+1} \big\langle \Psi, [[\![\nu_i, \nu_j]\!]] \wedge [\nu_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [\nu_k]\big\rangle$$

(see [18]) and we obtain the cohomology algebra

$$\mathsf{H}^\bullet(\boldsymbol{g}, B) := \mathsf{H}^\bullet\bigg(\Big(\Gamma\Big(\bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\Big)\Big)^{\Gamma(B)}, \bar{\delta}\bigg).$$

Theorem 1. *The homomorphism $\Delta_{(A,B,\nabla)}$ commutes with the differentials $\bar{\delta}$ and d_L, where d_L is the differential operator in $\Omega(L) = \Gamma(\bigwedge L^*)$.*

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Proof. Since the pullback of differential forms $\nabla^* : \Omega(L) \to \Omega(A)$ commutes with differentials d_L and d_A, it is sufficient to show the commutativity of $\Delta_{(A,B)}$ with differentials $\bar{\delta}$ and d_A. Let $\xi_0, \ldots, \xi_k \in \Gamma(A)$, and Ψ be an arbitrary invariant cross-section of the degree k. The curvature tensor $\Omega^{j\circ\lambda}$ of the connection $j \circ \lambda$ takes values in the bundle $\boldsymbol{h}$. Thus, we see that

$$\Omega^{j\circ\lambda}(\#_A \circ \xi_i, \#_A \circ \xi_j) = [\![\breve{\lambda} \circ \xi_i, \breve{\lambda} \circ \xi_j]\!] - [\![\xi_i, \breve{\lambda} \circ \xi_j]\!] + [\![\xi_j, \breve{\lambda} \circ \xi_i]\!] + \breve{\lambda} \circ [\![\xi_i, \xi_j]\!] \in \boldsymbol{h}.$$

Therefore, using invariance of Ψ, we have

$$\begin{aligned}(d_A \circ \Delta_{(A,B)})(\Psi)(\xi_0, \ldots, \xi_k) &= \sum_{i<j} (-1)^{i+j+1} \big\langle \Psi, [[\![\breve{\lambda} \circ \xi_i, \omega \circ \xi_j]\!]] \wedge [-\breve{\lambda} \circ \xi_0] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [-\breve{\lambda} \circ \xi_k] \big\rangle \\ &\quad - \sum_{i<j} (-1)^{i+j} \big\langle \Psi, \big[\Omega^{j\circ\lambda}(\#_A \circ \xi_i, \#_A \circ \xi_j)\big] \wedge [-\breve{\lambda} \circ \xi_0] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [-\breve{\lambda} \circ \xi_k] \big\rangle \\ &= \big\langle \bar{\delta}\Psi, [-\breve{\lambda} \circ \xi_0] \wedge \cdots \wedge [-\breve{\lambda} \circ \xi_k] \big\rangle \\ &= (\Delta_{(A,B)} \circ \bar{\delta})(\Psi)(\xi_0, \ldots, \xi_k). \qquad \square\end{aligned}$$

Corollary 2. *$\Delta_{(A,B)}$ and $\Delta_{(A,B,\nabla)}$ induce the cohomology homomorphisms*

$$\Delta_{(A,B)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^\bullet(A)$$

and

$$\Delta_{(A,B,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^\bullet(L).$$

The significance of $\Delta_{(A,B)\#}$ follows from the fact that $\Delta_{(A,B,\nabla)\#}$, for every flat L-connection $\nabla : L \to A$, is factorized by $\Delta_{(A,B)\#}$:

$$\Delta_{(A,B,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \xrightarrow{\Delta_{(A,B)\#}} \mathsf{H}^\bullet(A) \xrightarrow{\nabla^\#} \mathsf{H}^\bullet(L). \tag{4}$$

The map $\Delta_{(A,B,\nabla)\#}$ we will call the *characteristic homomorphism* of the FS-Lie algebroid (A, B, ∇). We call elements of a subalgebra $\operatorname{Im} \Delta_{(A,B,\nabla)\#} \subset \mathsf{H}^\bullet(L)$ the *secondary* (*exotic*) *characteristic classes* of this algebroid. In particular, $\Delta_{(A,B)\#} = \Delta_{(A,B,\mathrm{id}_A)\#}$ is the characteristic homomorphism of the Lie subalgebroid $B \subset A$. We define the last homomorphism as the *universal exotic characteristic homomorphism* and the characteristic classes from its image as the *universal characteristic classes* of the pair $B \subset A$.

2.3. The case for regular Lie algebroids and usual flat connections

Given a regular Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ over a regular, foliated manifold (M, F), consider two geometric structures:

- a flat connection $\omega : F \to A$,
- a Lie subalgebroid $j : B \hookrightarrow A$ over (M, F).

Let $\breve{\omega} : A \to \boldsymbol{g}$ be the connection form of ω. Let us consider an auxiliary connection $\lambda : F \to B$, its extension $j \circ \lambda$ to A and let $\breve{\lambda} : A \to \boldsymbol{g}$ be its connection form. Since $i \circ \breve{\omega} + \omega \circ \#_A = \mathrm{id}_A$, it follows that $i \circ \breve{\omega} \circ j \circ \lambda = -i \circ \breve{\lambda} \circ \omega$. Hence, we conclude that

$$\begin{aligned}(\Delta_{(A,B,\omega)}\Psi)_{|x}(w_1 \wedge \cdots \wedge w_k) &= \big\langle \Psi_x, \big[-(\breve{\lambda} \circ \omega)(w_1)\big] \wedge \cdots \wedge \big[-(\breve{\lambda} \circ \omega)(w_k)\big] \big\rangle \\ &= \big\langle \Psi_x, \big[\breve{\omega}_x(\tilde{w}_1)\big] \wedge \cdots \wedge \big[\breve{\omega}_x(\tilde{w}_k)\big] \big\rangle,\end{aligned}$$

where $\tilde{w}_i = \lambda(w_i)$. Since $\#_B(\tilde{w}_i) = w_i$,

$$\Delta_{(A,B,\omega)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^\bullet(F)$$

is the characteristic homomorphism for the regular flat Lie algebroid (A, B, ω), which was considered in [18].

2.4. The particular case: The classical Kamber–Tondeur homomorphism and universal characteristic homomorphism factorizing the classical one

Consider any G-principal fibre bundle P over a smooth manifold M, a flat connection $\omega \subset TP$ in P and a connected H-reduction $P' \subset P$, where $H \subset G$ is a closed Lie subgroup of G (we do not assume either connectedness or compactness of H). Applying the Lie functor for principal fibre bundles we can consider Lie algebroids $\mathsf{A}(P)$ and $\mathsf{A}(P')$ as well as the induced flat connection $\omega^A : TM \to \mathsf{A}(P)$ in the Lie algebroid $\mathsf{A}(P)$ and the secondary characteristic homomorphism

$$\Delta_{(\mathsf{A}(P),\mathsf{A}(P'),\omega^A)\#} : \mathsf{H}^\bullet\big(\boldsymbol{g}, \mathsf{A}(P')\big) \longrightarrow \mathsf{H}^\bullet_{dR}(M)$$

for the FS-Lie algebroid $(\mathrm{A}(P), \mathrm{A}(P'), \omega^A)$. This homomorphism is "equivalent" to the standard classical homomorphism on principal fibre bundles

$$\Delta_{(P,P',\omega)\#} : \mathsf{H}^\bullet(\mathfrak{g}, H) \longrightarrow \mathsf{H}^\bullet_{dR}(M)$$

given by F. Kamber and Ph. Tondeur [15], where $\mathsf{H}^\bullet(\mathfrak{g}, H)$—called the *relative Lie algebra cohomology of* $(\mathfrak{g}, H)$ (see [15,3])—is the cohomology space of the complex $(\bigwedge(\mathfrak{g}/\mathfrak{h})^{*H}, d^H)$ where $\bigwedge(\mathfrak{g}/\mathfrak{h})^{*H}$ is the space of invariant elements with respect to the adjoint representation of the Lie group H (see [15, 3.27]) and the differential d^H is defined by the formula:

$$\left\langle d^H(\psi), [v_1] \wedge \cdots \wedge [v_k]\right\rangle = \sum_{i<j} (-1)^{i+j} \left\langle \psi, [\![v_i, v_j]\!] \wedge [v_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [v_k]\right\rangle$$

for $\psi \in \bigwedge^k(\mathfrak{g}/\mathfrak{h})^{*H}$ and $v_i \in \mathfrak{g}$. The equivalence of these two characteristic homomorphisms lies in the fact that there exists an isomorphism of algebras $\kappa : \mathsf{H}^\bullet(\mathfrak{g}, H) \xrightarrow{\simeq} \mathsf{H}^\bullet(\boldsymbol{g}, \mathrm{A}(P'))$ such that

$$\Delta_{(\mathrm{A}(P),\mathrm{A}(P'),\omega^A)\#} \circ \kappa = \Delta_{(P,P',\omega)\#} \tag{5}$$

(see [18, Theorem 6.1]). Therefore, the obtained algebras of characteristic classes are identical. We recall that the isomorphism κ on the level of cochains is defined via the isomorphism $\tilde{\kappa} : (\bigwedge(\mathfrak{g}/\mathfrak{h})^*)^H \to (\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ given by $\tilde{\kappa}(\psi)(x) = \mathrm{Ad}^\wedge_{P',\boldsymbol{g}}(z)(\psi)$, $z \in P'_{|x}$, where the representation $\mathrm{Ad}^\wedge_{P',\boldsymbol{g}}$ of P' on $\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^\star$ is induced by $\mathrm{Ad}_{P',\boldsymbol{g}} : P' \to L(\boldsymbol{g}/\boldsymbol{h})$, $z \mapsto [\hat{z}]$, and $\hat{z} : \mathfrak{g} \xrightarrow{\cong} \boldsymbol{g}_{|x}$, $v \mapsto [A_{z\star v}]$ $(A_z : G \to P,\ a \mapsto za)$. The fact that $\tilde{\kappa}$ is an isomorphism is obtained by Proposition 5.5.3 from [16] (just here the assumption that P' is connected is needed). We recall briefly the definition of the homomorphism $\Delta_{(P,P',\omega^A)\#}$ and reasoning giving (5). Let $\breve{\omega} : TP \to \mathfrak{g}$ denote the connection form of ω. There exists a homomorphism of G-DG-algebras $\breve{\omega}_\wedge : \bigwedge \mathfrak{g}^* \to \Omega(P)$ (in view of the flatness of ω) induced by the algebraic connection $\breve{\omega} : \mathfrak{g}^* \to \Omega(P)$, $\alpha \mapsto \alpha\breve{\omega} = \langle \alpha, \breve{\omega}\rangle$. The homomorphism $\breve{\omega}_\wedge$ is given by the formula $\breve{\omega}_\wedge(\phi^k)_z(v_1, \ldots, v_k) = \langle \phi; \omega_z(v_1) \wedge \cdots \wedge \omega_z(v_k)\rangle$ and can be restricted to H-basic elements $\breve{\omega}_H : (\bigwedge \mathfrak{g}^*)_H \to \Omega(P)_H$. According to the isomorphisms $(\bigwedge \mathfrak{g}^*)_H \cong \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H}$ and $\Omega(P)_H \cong \Omega(P/H)$ it gives a DG-homomorphism of algebras $\breve{\omega}_H : \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H} \to \Omega(P/H)$. Composing it with $s^* : \Omega(P/H) \to \Omega(M)$, where $s : M \to P/H$ is the cross-section determined by the H-reduction P', we obtain a homomorphism of DG-algebras $\Delta_{P,P',\omega} : \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H} \xrightarrow{\breve{\omega}_H} \Omega(P/H) \xrightarrow{s^*} \Omega(M)$. Passing to cohomology we obtain $\Delta_{(P,P',\omega)\#}$. Because of the algebraic formula for $\breve{\omega}_\wedge$ we see that this homomorphism on the level of forms is given by

$$\left(\Delta_{P,P',\omega}(\psi)\right)_x (w_1 \wedge \cdots \wedge w_k) = \left\langle \psi, \left[\breve{\omega}_z(\tilde{w}_1)\right] \wedge \cdots \wedge \left[\breve{\omega}_z(\tilde{w}_k)\right]\right\rangle,$$

where $z \in P'_{|x}$, $w_i \in T_xM$, $\tilde{w}_i \in T_zP'$, $\pi'_*\tilde{w}_i = w_i$, with $\pi' : P' \to M$. Therefore, the equality (5) holds (for details see [18, Theorem 6.1]).

Using the universal *exotic characteristic homomorphism* $\Delta_{(\mathrm{A}(P),\mathrm{A}(P'))\#}$ for the pair of transitive Lie algebroids $(\mathrm{A}(P), \mathrm{A}(P'))$, $\mathrm{A}(P') \subset \mathrm{A}(P)$, we can define the *universal exotic characteristic homomorphism*

$$\Delta_{(P,P')\#} := \Delta_{(\mathrm{A}(P),\mathrm{A}(P'))\#} \circ \kappa : \mathsf{H}^\bullet(\mathfrak{g}, H) \longrightarrow \mathsf{H}^\bullet\big(\mathrm{A}(P)\big) \longrightarrow \mathsf{H}^{r\bullet}_{dR}(P)$$

for the reduction of a principal fibre bundle $P' \subset P$ (where $\mathsf{H}^{r\bullet}_{dR}(P)$ is the space of cohomology of right-invariant differential forms on P; we recall that $\mathsf{H}^{r\bullet}_{dR}(P) := \mathsf{H}^\bullet(\Omega^r(P)) \simeq \mathsf{H}^\bullet_{dR}(P)$ if G is compact and connected).

Theorem 3. *The homomorphism $\Delta_{(P,P')\#}$ on the level of differential forms is given by the following formula*:

$$(\Delta_{P,P'}\psi)_z(w_1 \wedge \cdots \wedge w_k) = \left\langle \psi, \left[-\lambda_z(w_1)\right] \wedge \cdots \wedge \left[-\lambda_z(w_k)\right]\right\rangle,$$

where λ is the form of any connection on P extending an arbitrary connection on P'.

The commutativity of the diagram

$$\begin{tikzcd}
\mathsf{H}^\bullet(\mathfrak{g}, H) \arrow[r, "\Delta_{(P,P')\#}"] \arrow[d, "\kappa"', "\cong"] & \mathsf{H}^{r\bullet}_{dR}(P) \arrow[r] \arrow[d, equal] \arrow[dr, "\omega^\#"'] & \mathsf{H}^\bullet_{dR}(P) \\
\mathsf{H}^\bullet(\boldsymbol{g}, \mathrm{A}(P')) \arrow[r, "\Delta_{(\mathrm{A}(P),\mathrm{A}(P'))\#}"'] & \mathsf{H}^\bullet(\mathrm{A}(P)) \arrow[r, "\omega^{A\#}"'] & \mathsf{H}^\bullet_{dR}(M)
\end{tikzcd}$$

where $\omega^\#$ on the level of right-invariant differential forms $\Omega^r(P)$ is given as the pullback of differential forms:

$$\omega^* : \Omega^r(P) \longrightarrow \Omega(M), \quad \omega^*(\phi)_x(u_1 \wedge \cdots \wedge u_k) = \phi_z(\tilde{u}_1 \wedge \cdots \wedge \tilde{u}_k),$$

where $z \in P_{|x}$, $\tilde{u}_i$ is the ω-horizontal lift of u_i, yields the following

Theorem 4. *The homomorphism $\Delta_{(P,P',\omega)\#}$ is factorized by $\Delta_{(P,P')\#}$, i.e. the diagram below commutes*

$$\begin{tikzcd} & \mathsf{H}^{r\bullet}_{dR}(P) \arrow[dr, "\omega^{\#}"] & \\ \mathsf{H}^{\bullet}(\mathfrak{g},H) \arrow[ur, "\Delta_{(P,P')\#}"] \arrow[rr, "\Delta_{(P,P',\omega)\#}"'] & & \mathsf{H}^{\bullet}_{dR}(M). \end{tikzcd}$$

From the above we get a corollary on the existing of the new universal exotic characteristic homomorphism for a G-principal fibre bundle P and its H-reduction P'.

Corollary 5. *If G is a compact, connected Lie group and P' is a connected H-reduction in a G-principal bundle P, $H \subset G$, then there exists a homomorphism of algebras*

$$\Delta_{(P,P')\#} : \mathsf{H}^{\bullet}(\mathfrak{g},H) \longrightarrow \mathsf{H}^{\bullet}_{dR}(P)$$

(*called a universal exotic characteristic homomorphism for the pair $P' \subset P$*) *such that for arbitrary flat connection ω in P, the characteristic homomorphism $\Delta_{(P,P',\omega)\#} : \mathsf{H}^{\bullet}(\mathfrak{g},H) \to \mathsf{H}^{\bullet}_{dR}(M)$ is factorized by $\Delta_{(P,P')\#}$, i.e. the following diagram is commutative*

$$\begin{tikzcd} & \mathsf{H}^{\bullet}_{dR}(P) \arrow[dr, "\omega^{\#}"] & \\ \mathsf{H}^{\bullet}(\mathfrak{g},H) \arrow[ur, "\Delta_{(P,P')\#}"] \arrow[rr, "\Delta_{(P,P',\omega)\#}"'] & & \mathsf{H}^{\bullet}_{dR}(M). \end{tikzcd}$$

3. Comparison with the Crainic classes

3.1. The Crainic approach

We briefly explain the Crainic approach to characteristic classes of a representation [4]. Primarily we notice that arbitrary representation $\nabla_{\xi}\nu$ of an arbitrary Lie algebroid L (not necessarily regular) in a vector bundle f can be described by a homomorphism of Lie algebroids $\nabla : L \to \mathsf{A}(\mathfrak{f})$ (i.e. a flat L-connection in $\mathsf{A}(\mathfrak{f})$). The Crainic classes of ∇ live in the cohomology algebra $\mathsf{H}^{\bullet}(L)$ of the Lie algebroid L. In the simplest case of the trivial vector bundle $\mathfrak{f} = M \times V$ ($\dim V = n$) they are constructed as follows: For a frame $(e_1, \ldots, e_n)$ of $\mathfrak{f}$ we introduce a matrix $\omega = [\omega^i_j] \in M_{n\times n}(\Omega(L))$ of 1-forms on L such that $\nabla_{\xi} e_j = \sum_i \omega^i_j(\xi) \cdot e_i$ for all $\xi \in \Gamma(L)$. Clearly, $\operatorname{tr}(\omega) = \operatorname{tr}(\tilde{\omega})$, where $\tilde{\omega} = \frac{\omega + \omega^T}{2}$ is the symmetrization of ω, and the flatness condition implies that for all natural numbers k,

$$\operatorname{tr}\big(\tilde{\omega}^{2k-1}\big) \tag{6}$$

are closed forms on L. Their cohomology classes are independent of the choice of frames. These classes vanish if ∇ is a Riemannian connection with respect to some Riemannian metric h in $\mathfrak{f}$. A Riemannian connection is a connection in a Riemannian reduction $\mathrm{L}(\mathfrak{f}, \{h\})$ of the principal fibre bundle of frames $\mathrm{L}(\mathfrak{f})$. For an arbitrary vector bundle $\mathfrak{f}$ Crainic uses a local construction (a suitable cocycle) and the Čech double complex $\check{C}^*(\mathcal{U}, C^*(L))$ together with the Mayer–Vietoris argument. For $L = TM$ the usual exotic characteristic classes of flat vector bundles $\mathfrak{f}$ are obtained. An explicit formula for an arbitrary L-flat real vector bundle $(\mathfrak{f}, \nabla)$ is based on the observation that in a local orthonormal frame $(e_1, \ldots, e_n)$ of $(\mathfrak{f}, \{h\})$ the symmetrization $\tilde{\omega}$ of ω is equal to the matrix of the symmetric-values form $\omega(\mathfrak{f}, h) = \frac{1}{2}(\nabla - \nabla^h)$, where ∇^h is the adjoint L-connection induced by the metric h. The adjoint connection ∇^h remains flat. The classes will be given as in (6), with $\tilde{\omega}$ replaced by $\omega(\mathfrak{f}, h)$. One explicit formula up to a constant uses the Chern–Simons transgression differential forms cs_k for suitable two connections and is given by (see [6])

$$u_{2k-1}(\mathfrak{f}) = \big[u_{2k-1}(\mathfrak{f}, \nabla)\big] \in \mathsf{H}^{2k-1}(L),$$

where

$$u_{2k-1}(\mathfrak{f}, \nabla) = (-1)^{\frac{k+1}{2}} \mathrm{cs}_k\big(\nabla, \nabla^h\big) \tag{7}$$

and k is an odd natural ($u_{2k-1}(\mathfrak{f})$ is trivial if k is even). We recall that

$$\mathrm{cs}_k\big(\nabla, \nabla^h\big) = \int_0^1 \mathrm{ch}_k\big(\nabla^{\mathrm{aff}}\big) \in \Omega^{2k-1}(L),$$

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$$\left(\int_0^1 \mathrm{ch}_k\left(\nabla^{\mathrm{aff}}\right)\right)_{\xi_1,\ldots,\xi_{2k-1}} = \int_0^1 \mathrm{ch}_k\left(\nabla^{\mathrm{aff}}\right)_{\frac{\partial}{\partial t},\xi_1,\ldots,\xi_{2k-1}}\Big|_{(t,\bullet)}\,dt, \quad \xi_i \in \Gamma(L),$$

where $\nabla^{\mathrm{aff}} = (1-t)\cdot\tilde{\nabla} + t\cdot\tilde{\nabla}^h : T\mathbb{R}\times A \to \mathrm{A}(\mathrm{pr}_2^*\mathfrak{f})$ is the affine combination of the connections $\tilde{\nabla}$ and $\tilde{\nabla}^h$ whereas $\mathrm{ch}_k(\nabla) = \mathrm{Tr}((R^{\nabla})^k)$. The connection $\tilde{\nabla} : T\mathbb{R}\times L \to A(\mathrm{pr}_2^*\mathfrak{f})$, $\tilde{\nabla}_{(v_t,\xi_x)}(\nu\circ\mathrm{pr}_2) = \nabla_{\xi_x}(\nu)$, is the lifting of the connection ∇ through the projection $\mathrm{pr}_2 : \mathbb{R}\times M \to M$. If ∇ is flat, then $\tilde{\nabla}$ is flat, too.

3.2. The secondary characteristic homomorphism for Riemannian reductions

Let $(\mathfrak{f},\{h\})$ denote a vector bundle of the rank n over a manifold M with a Riemannian metric h. The metric h yields [17] the Lie subalgebroid $B = \mathrm{A}(\mathfrak{f},\{h\})$ of the algebroid $\mathrm{A}(\mathfrak{f})$ of the vector bundle $\mathfrak{f}$ and the reduction $\mathrm{L}(\mathfrak{f},\{h\})$ of the frames bundle $\mathrm{L}\mathfrak{f}$ of $\mathfrak{f}$; $(u:\mathbb{R}^n \to \mathfrak{f}_{|x}) \in \mathrm{L}(\mathfrak{f},\{h\})$ if and only if u is an isometry. Taking the canonical isomorphism $\Phi_{\mathfrak{f}} : \mathrm{A}(\mathrm{L}\mathfrak{f}) \to \mathrm{A}(\mathfrak{f})$ of Lie algebroids [16] we have $\mathrm{A}(\mathfrak{f},\{h\}) = \Phi_{\mathfrak{f}}[\mathrm{A}(\mathrm{L}(\mathfrak{f},\{h\}))]$. We observe that $\alpha \in \Gamma(\mathrm{A}(\mathfrak{f}))$ belongs to $\Gamma(\mathrm{A}(\mathfrak{f},\{h\}))$ if and only if for any cross-sections $\nu,\mu \in \Gamma(\mathfrak{f})$ the formula $h(\alpha(\nu),\mu) = (\#\alpha)(h(\nu,\mu)) - h(\nu,\alpha(\mu))$ holds. The Atiyah sequences for $\mathrm{A}(\mathfrak{f})$ and $\mathrm{A}(\mathfrak{f},\{h\})$ are

$$0 \longrightarrow \mathrm{End}(\mathfrak{f}) \xrightarrow{i} \mathrm{A}(\mathfrak{f}) \xrightarrow{\pi} TM \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Sk}(\mathfrak{f}) \longrightarrow \mathrm{A}\big(\mathfrak{f},\{h\}\big) \longrightarrow TM \longrightarrow 0,$$

where $\mathrm{Sk}(\mathfrak{f}) \subset \mathrm{End}(\mathfrak{f})$ is the vector subbundle of h-skew symmetric endomorphisms. Let L be a Lie algebroid over M and $\nabla : L \to \mathrm{A}(\mathfrak{f})$ any flat L-connection in $\mathfrak{f}$. Let us consider FS-Lie algebroids $((\mathrm{A}(\mathfrak{f}),\mathrm{A}(\mathfrak{f},\{h\})),\nabla)$ and $(\mathrm{A}(\mathfrak{f}),\mathrm{A}(\mathfrak{f},\{h\}),\mathrm{id}_{A(\mathfrak{f})})$ and theirs secondary characteristic homomorphisms denote, for shortness, by

$$\Delta_\# : \mathrm{H}^\bullet\big(\mathrm{End}\,\mathfrak{f},\mathrm{A}\big(\mathfrak{f},\{h\}\big)\big) \longrightarrow \mathrm{H}^\bullet(L),$$
$$\Delta_{o\#} : \mathrm{H}^\bullet\big(\mathrm{End}\,\mathfrak{f},\mathrm{A}\big(\mathfrak{f},\{h\}\big)\big) \longrightarrow \mathrm{H}^\bullet\big(\mathrm{A}(\mathfrak{f})\big),$$

respectively, and take into the consideration the isomorphism

$$\kappa : \mathrm{H}^\bullet\big(\mathfrak{gl}(n,\mathbb{R}),O(n)\big) \xrightarrow{\cong} \mathrm{H}^\bullet\big(\mathrm{End}\,\mathfrak{f},\mathrm{A}\big(\mathfrak{f},\{h\}\big)\big)$$

of algebras described in Section 2.4. If the vector bundle $\mathfrak{f}$ is nonorientable, then

$$\mathrm{H}^\bullet\big(\mathrm{End}\,\mathfrak{f},\mathrm{A}\big(\mathfrak{f},\{h\}\big)\big) \overset{\kappa}{\cong} \mathrm{H}^\bullet\big(\mathfrak{gl}(n,\mathbb{R}),O(n)\big) \cong \bigwedge(y_1,y_3,\ldots,y_{n'}),$$

where n' is the largest odd integer $\leqslant n$ ($n' = 2[\frac{n+1}{2}]-1$) and $y_{2k-1} \in \mathrm{H}^{4k-3}(\mathrm{End}\,\mathfrak{f},\mathrm{A}(\mathfrak{f},\{h\}))$ for $k \in \{1,2,\ldots,[\frac{n+1}{2}]\}$ are represented by the multilinear trace forms $\widetilde{y}_{2k-1} \in \Gamma(\bigwedge^{4k-3}(\mathrm{End}\,\mathfrak{f}/\,\mathrm{Sk}\,\mathfrak{f})^*)$, see [15, 6.31, p. 142].

In the case of an oriented vector bundle $\mathfrak{f}$ with a volume form v, the metric h and v induce an $SO(n,\mathbb{R})$-reduction $\mathrm{L}(\mathfrak{f},\{h,\mathrm{v}\})$ of the frames bundle $\mathrm{L}\mathfrak{f}$ of $\mathfrak{f}$; $(u:\mathbb{R}^n \to \mathfrak{f}_{|x}) \in \mathrm{L}(\mathfrak{f},\{h,\mathrm{v}\})$ if and only if u is an isometry keeping the orientation. Clearly, $\mathrm{A}(\mathfrak{f},\{h,\mathrm{v}\}) = \mathrm{A}(\mathfrak{f},\{h\})$, and hence $\mathrm{H}^\bullet(\mathrm{End}\,\mathfrak{f},\mathrm{A}(\mathfrak{f},\{h\})) \cong \mathrm{H}^\bullet(\mathrm{End}\,\mathfrak{f},\mathrm{A}(\mathfrak{f},\{h,\mathrm{v}\}))$. If $\mathfrak{f}$ is orientable and odd rank (see [9]),

$$\mathrm{H}^\bullet\big(\mathfrak{gl}(n,\mathbb{R}),SO(n)\big) \cong \mathrm{H}^\bullet\big(\mathfrak{gl}(n,\mathbb{R}),O(n)\big).$$

If the vector bundle $\mathfrak{f}$ is orientable of even rank $n = 2m$, then

$$\mathrm{H}^\bullet\big(\mathrm{End}\,\mathfrak{f},\mathrm{A}\big(\mathfrak{f},\{h,\mathrm{v}\}\big)\big) \cong \mathrm{H}^\bullet\big(\mathfrak{gl}(2m,\mathbb{R}),SO(2m)\big) \cong \bigwedge(y_1,y_3,\ldots,y_{2m-1},y_{2m}),$$

where $y_{2k-1} \in \mathrm{H}^{4k-3}(\mathrm{End}\,\mathfrak{f},\mathrm{A}(\mathfrak{f},\{h,\mathrm{v}\}))$ are defined as above and

$$y_{2m} \in \mathrm{H}^{2m}\big(\mathfrak{gl}(n,\mathbb{R}),SO(n)\big) \cong \mathrm{H}^{2m}\big(\mathrm{End}\,\mathfrak{f},\mathrm{A}\big(\mathfrak{f},\{h,\mathrm{v}\}\big)\big)$$

is represented by

$$\widetilde{y}_{2m} \in \Gamma\Big(\bigwedge^{2m}(\mathrm{End}\,\mathfrak{f}/\,\mathrm{Sk}\,\mathfrak{f})^*\Big),$$
$$\widetilde{y}_{2m}\big([A_1],\ldots,[A_{2m}]\big) = d(z_{2m-1})(\widetilde{A}_1,\ldots,\widetilde{A}_{2m}), \tag{8}$$

$A_1,\ldots,A_{2m} \in \Gamma(\mathrm{End}\,\mathfrak{f})$, and where $\widetilde{A}_j$ denotes the symmetrization of A_j, d is the usual differential on the algebra $\bigwedge(\mathrm{End}\,\mathfrak{f})^*$ and $z_{2m-1} \in \Gamma(\bigwedge^{2m-1}(\mathrm{End}\,\mathfrak{f})^*)$ is given by

$$z_{2m-1}(A_1,\ldots,A_{2m-1}) = c(m)\sum_{\sigma\in S_{2m-1}} \mathrm{sgn}\,\sigma\,\big(e,\alpha A_{\sigma_1}\wedge\alpha[A_{\sigma_2},A_{\sigma_3}]\wedge\cdots\wedge\alpha[A_{\sigma_{2m-2}},A_{\sigma_{2m-1}}]\big),$$

where $c(m)=\frac{(-1)^{m-1}(m-1)!}{2^{m-1}(2m-1)!}\in\mathbb{R}$, e is a non-zero cross-section of $\bigwedge^{2m}\mathfrak{f}$ and $\alpha:\operatorname{End}\mathfrak{f}\to\bigwedge^{2}\mathfrak{f}$ is given by $(\alpha(A),\nu\wedge\mu)=\frac{1}{2}((A\nu,\mu)-(\nu,A\mu))$, $A\in\Gamma(\operatorname{End}\mathfrak{f})$, $\nu,\mu\in\Gamma(\mathfrak{f})$. We add that z_{2m-1} is the image of the Pfaffian for a pair $(\mathfrak{f},e)$ by the Cartan map for $\operatorname{End}\mathfrak{f}$ (for the Cartan map see for example [10, Ch. VI, 6.7, 6.8]).

We shall show that $\Delta_\#(y_{2j-1})$ is (up to a constant) equal to the Crainic class $u_{4j-3}(\mathfrak{f})$ for all $j\in\{1,2,\dots,[\frac{n+1}{2}]\}$. Let $\nabla_0,\nabla_1:L\to \mathrm{A}(\mathfrak{f})$ be arbitrary two L-connections in $\mathfrak{f}$ and let $\nabla^{\mathrm{aff}}=(1-t)\tilde{\nabla}_0+t\tilde{\nabla}_1:T\mathbb{R}\times L\to A(\mathrm{pr}_2^*\mathfrak{f})$ be their affine combination. Observe

$$R^{\nabla_1}=R^{\nabla_0}+d^{\nabla_0}\theta+[\theta,\theta], \tag{9}$$

where

$$\theta=\nabla_1-\nabla_0\in\Omega^1(L;\operatorname{End}\mathfrak{f}) \tag{10}$$

and $[\theta,\theta]=\theta^2=\theta\wedge\theta\in\Omega^2(L;\operatorname{End}\mathfrak{f})$, $[\theta,\theta](\xi,\eta)=[\theta(\xi),\theta(\eta)]$. The 1-form θ we can lifted to $\tilde{\theta}\in\Omega^1(T\mathbb{R}\times L;\operatorname{End}\mathfrak{f})$ putting $\tilde{\theta}_{(v_t,\xi_x)}=\theta_{\xi_x}$. The cross-section $(0,\xi)$ of $T\mathbb{R}\times L$ will be denoted by ξ and $(\frac{\partial}{\partial t},0)$ by $\frac{\partial}{\partial t}$. Observe that $\nabla^{\mathrm{aff}}=\tilde{\nabla}_0+\Xi$, where $\Xi_{|(t,x)}=t\cdot\tilde{\theta}_x$ and $(d^{\tilde{\nabla}_1}\tilde{\theta})_{\xi,\eta}(\nu\circ\mathrm{pr}_2)=(d^{\nabla_1}\theta)_{\xi,\eta}(\nu)\circ\mathrm{pr}_2$ for any $\xi,\eta\in\Gamma(L)$, $x\in M$, $t\in\mathbb{R}$. The affine combination ∇^{aff} of flat connections cannot be flat even if ∇_0 is flat: by (9) for flat ∇_0 we have

$$R^{\nabla^{\mathrm{aff}}}=d^{\tilde{\nabla}_0}\Xi+[\Xi,\Xi]. \tag{11}$$

Lemma 6. *The curvature tensor $R^{\nabla^{\mathrm{aff}}}$ of the affine combination ∇^{aff} of two flat L-connections ∇_0, ∇_1 has the following properties*

$$\left(R^{\nabla^{\mathrm{aff}}}\right)_{\frac{\partial}{\partial t},\xi}(\nu\circ\mathrm{pr}_2)=\theta_\xi(\nu), \tag{12}$$

$$\left(R^{\nabla^{\mathrm{aff}}}\right)_{\xi,\eta}(\nu\circ\mathrm{pr}_2)_{|(t,\cdot)}=\left(t^2-t\right)\cdot(\theta\wedge\theta)_{\xi,\eta}(\nu), \tag{13}$$

$$\left(\left(R^{\nabla^{\mathrm{aff}}}\right)^k_{\frac{\partial}{\partial t},\xi_1,\dots,\xi_{2k-1}}\right)_{|(t,\cdot)}=k\cdot t^{k-1}\cdot(t-1)^{k-1}\cdot\theta^{2k-1}_{\xi_1,\dots,\xi_{2k-1}}. \tag{14}$$

Proof. Formula (12) is clear. To see (13), we need only to observe (because of (11), (9) and the flatness of ∇_0 and ∇_1) that

$$\left(d^{\tilde{\nabla}_0}\Xi\right)_{\xi,\eta}(\nu\circ\mathrm{pr}_2)_{|(t,\cdot)}=t\cdot R^{\nabla_1}_{\xi,\eta}(\nu)-t\cdot(\theta\wedge\theta)_{\xi,\eta}(\nu)=-t\cdot(\theta\wedge\theta)_{\xi,\eta}(\nu),$$
$$[\Xi,\Xi]_{\xi,\eta}(\nu\circ\mathrm{pr}_2)_{|(t,\cdot)}=t^2\cdot(\theta\wedge\theta)_{\xi,\eta}(\nu).$$

Formula (14) can be proved by induction with respect to k. Indeed, from (12) we have the step $k=1$. Let $n\in\mathbb{N}$. Assume that (14) holds for all $k\leqslant n$. From this, (13), (12) and the associativity of the algebra $(\operatorname{End}\mathfrak{f},\circ)$ we get

$$\begin{aligned}
\left(\left(R^{\nabla^{\mathrm{aff}}}\right)^{n+1}_{\frac{\partial}{\partial t},\xi_1,\dots,\xi_{2n+1}}\right)_{|(t,\cdot)}&=\left(\left(\left(R^{\nabla^{\mathrm{aff}}}\right)^n\wedge R^{\nabla^{\mathrm{aff}}}\right)_{\frac{\partial}{\partial t},\xi_1,\dots,\xi_{2n+1}}\right)_{|(t,\cdot)}\\
&=\sum_{\sigma\in S(2n-1,2)}\operatorname{sgn}\sigma\cdot\left(\left(R^{\nabla^{\mathrm{aff}}}\right)^n_{\frac{\partial}{\partial t},\xi_{\sigma_1},\dots,\xi_{\sigma_{2n-1}}}\circ R^{\nabla^{\mathrm{aff}}}_{\xi_{\sigma_{2n}},\xi_{\sigma_{2n+1}}}\right)_{|(t,\cdot)}\\
&\quad+\sum_{\sigma\in S(2n,1)}\operatorname{sgn}\sigma\cdot\left(\left(R^{\nabla^{\mathrm{aff}}}\right)^n_{\xi_{\sigma_1},\dots,\xi_{\sigma_{2n}}}\circ R^{\nabla^{\mathrm{aff}}}_{\frac{\partial}{\partial t},\xi_{\sigma_{2n+1}}}\right)_{|(t,\cdot)}\\
&=\sum_{\sigma\in S(2n-1,2)}\operatorname{sgn}\sigma\cdot\left(n\cdot t^n(t-1)^n\cdot\theta^{2n-1}_{\xi_{\sigma_1},\dots,\xi_{\sigma_{2n-1}}}\circ(\theta\wedge\theta)_{\xi_{\sigma_{2n}},\xi_{\sigma_{2n+1}}}\right)\\
&\quad+\sum_{\sigma\in S(2n,1)}\operatorname{sgn}\sigma\cdot\left(t^n(t-1)^n\cdot\theta^{2n}_{(a_{\sigma_1},\dots,a_{\sigma_{2n}})}\circ\theta_{a_{\sigma_{2n+1}}}\right)\\
&=nt^n(t-1)^n\cdot\theta^{2n+1}_{\xi_1,\dots,\xi_{2n+1}}+t^n(t-1)^n\cdot\theta^{2n+1}_{\xi_1,\dots,\xi_{2n+1}}\\
&=(n+1)t^{(n+1)-1}(t-1)^{(n+1)-1}\cdot\theta^{2(n+1)-1}_{\xi_1,\dots,\xi_{2n+1}}.\qquad\square
\end{aligned}$$

From the above we have the following theorem.

Theorem 7. *The Chern–Simons transgression differential form $cs_k(\nabla_0,\nabla_1)$ for two flat L-connections ∇_0, ∇_1, is equal to*

$$\mathrm{cs}_k(\nabla_0,\nabla_1)=(-1)^{k+1}\frac{k!\cdot(k-1)!}{(2k-1)!}\cdot\operatorname{tr}\theta^{2k-1}, \tag{15}$$

where θ is defined by (10)*.*

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The above formula is well known in the classical cases on principal fibre bundles (see, for example, the papers by Chern and Simons [2], Heitsch and Lawson [11, 1974]).

Let $\widetilde{\ }: \mathrm{End}\,\mathfrak{f} \to \mathrm{End}\,\mathfrak{f}$, $v \mapsto \widetilde{v} := \frac{1}{2}(v + v^*)$ denote the symmetrization. Let us consider $\mathrm{id}_{\mathrm{A}(\mathfrak{f})}$ as an $\mathrm{A}(\mathfrak{f})$-connection in $\mathfrak{f}$ and take its adjoint $\mathrm{id}^h_{\mathrm{A}(\mathfrak{f})}$ induced by the metric h. Let $\lambda : TM \to \mathrm{A}(\mathfrak{f})$ be any h-Riemannian connection (i.e. a connection such that $\operatorname{Im}\lambda \subset \mathrm{A}(\mathfrak{f}, h)$, or equivalently $\lambda^h = \lambda$) and $\breve{\lambda} : \mathrm{A}(\mathfrak{f}) \to \mathrm{End}\,\mathfrak{f}$ be its connection form. Since $i \circ \breve{\lambda} + \lambda \circ \pi = \mathrm{id}_{\mathrm{A}(\mathfrak{f})}$, for any cross-section α of $\mathrm{A}(\mathfrak{f})$ we have

$$-\widetilde{\breve{\lambda}(\alpha)} = \frac{1}{2}\big(\mathrm{id}^h_{A(\mathfrak{f})} - \mathrm{id}_{\mathrm{A}(\mathfrak{f})}\big)(\alpha). \tag{16}$$

Using (16), (15) and (7) we get that

$$\Delta_o(\widetilde{y}_{2k-1}) = (-1)^k \cdot 2^{3-4k} \cdot \frac{(4k-3)!}{(2k-1)! \cdot (2k-2)!} \cdot u_{4k-3}(\mathfrak{f}, \mathrm{id}_{A(\mathfrak{f})}).$$

Since $\mathrm{cs}_k(\nabla, \nabla^h) = \nabla^*(\mathrm{cs}_k(\mathrm{id}_{A(\mathfrak{f})}, \mathrm{id}^h_{A(\mathfrak{f})}))$ and $u_{4k-3}(\mathfrak{f}, \nabla) = \nabla^\# u_{4k-3}(\mathfrak{f}, \mathrm{id}_{A(\mathfrak{f})})$,

$$\Delta_\#(y_{2k-1}) = \big[\nabla^* \Delta_o(\widetilde{y}_{2k-1})\big] = \frac{(-1)^k \cdot (4k-3)!}{2^{4k-3} \cdot (2k-1)! \cdot (2k-2)!} \cdot u_{4k-3}(\mathfrak{f}).$$

From the above formulae we can explain the relation between the characteristic homomorphism $\Delta_\# : \mathsf{H}^\bullet(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\})) \to \mathsf{H}^\bullet(L)$ of $(A(\mathfrak{f}), A(\mathfrak{f}, \{h\}), \nabla)$ and the family of the Crainic classes $\{u_{4k-3}(\mathfrak{f})\}$.

Theorem 8. *Let $\mathfrak{f}$ be a real vector bundle over a manifold M and*

$$\Delta_\# : \mathsf{H}^\bullet\big(\mathrm{End}\,\mathfrak{f}, A\big(\mathfrak{f}, \{h\}\big)\big) \longrightarrow \mathsf{H}^\bullet(L)$$

the secondary characteristic homomorphism corresponding to the FS-Lie algebroid $(A(\mathfrak{f}), A(\mathfrak{f}, \{h\}), \nabla)$, where $\nabla : L \to A(\mathfrak{f})$ is a flat L-connection in $A(\mathfrak{f})$.

(a) *If the vector bundle $\mathfrak{f}$ is nonorientable or orientable and of odd rank n, then the image of $\Delta_\#$ is generated by $u_1(\mathfrak{f}), u_5(\mathfrak{f}), \ldots, u_{4[\frac{n+3}{4}]-3}(\mathfrak{f})$.*

(b) *If the vector bundle $\mathfrak{f}$ is orientable and of even rank $n = 2m$, then the image of $\Delta_\#$ is generated by $u_1(\mathfrak{f}), u_5(\mathfrak{f}), \ldots, u_{4[\frac{n+3}{4}]-3}(\mathfrak{f})$ and additionally by $\Delta_\#(y_{2m})$, where y_{2m} is given in (8).*

3.3. Example of a nontrivial universal characteristic class determined by the Pfaffian

Let M be an oriented, connected manifold, $\dim M \geqslant 1$, and $\mathfrak{g} = \mathrm{End}(\mathbb{R}^2)$. Given a transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ over M, where $A = TM \oplus \mathrm{End}(\mathbb{R}^2) \cong A(M \times \mathbb{R}^2)$ and $\#_A = \mathrm{pr}_1$ is the projection on the first factor, and

$$\big[\!\big[(X_1, \sigma_1), (X_2, \sigma_2)\big]\!\big] = \big([X_1, X_2], X_1(\sigma_2) - X_2(\sigma_1) + [\sigma_1, \sigma_2]\big)$$

for all $X_1, X_2 \in \mathfrak{X}(M)$, $\sigma_1, \sigma_2 \in C^\infty(M; \mathrm{End}(\mathbb{R}^2))$, we have the Atiyah sequence $0 \to M \times \mathrm{End}(\mathbb{R}^2) \cong \mathrm{End}(M \times \mathbb{R}^2) \xhookrightarrow{i} A \xrightarrow{\mathrm{pr}_1} TM \to 0$. Let $B \subset A$ be the Riemannian reduction of A, i.e. $B = TM \oplus \mathrm{Sk}(\mathbb{R}^2)$ is a transitive subalgebroid of A. Observe that in the domain of the universal characteristic homomorphism $\Delta_{o\#} : \mathsf{H}^\bullet(M \times \mathfrak{g}, B) \to \mathsf{H}^\bullet(A)$ is $[\widetilde{y}_2] \in \mathsf{H}^2(M \times \mathfrak{g}, B)$, where $\widetilde{y}_2([\sigma_1], [\sigma_2]) = \mathrm{Pf}([\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!])$ for all $\sigma_1, \sigma_2 \in \Gamma(\ker \#_A) \cong C^\infty(M; \mathfrak{g})$. $\Delta_{o\#}([\widetilde{y}_2]) \in \mathsf{H}^2(A)$ is represented by $\Delta_o(\widetilde{y}_2) \in \Omega^1(A)$ given by

$$\Delta_o(\widetilde{y}_2)\big((X_1, \sigma_1), (X_2, \sigma_2)\big) = \mathrm{Pf}\big([\widetilde{\sigma}_1, \widetilde{\sigma}_2]\big).$$

Theorem 9. $\Delta_{o\#}(\widetilde{y}_2) \neq 0$.

Proof. Suppose that $\Delta_o(\widetilde{y}_2)$ is exact. Let $\Delta_o(\widetilde{y}_2) = d_A(\zeta)$ for some $\zeta \in \Omega^1(A)$. Thus we get that for all $(X_1, \sigma_1), (X_2, \sigma_2) \in \mathfrak{X}(M) \times C^\infty(M; \mathfrak{g})$, $\mathrm{Pf}([\widetilde{\sigma}_1, \widetilde{\sigma}_2])$ is equal to

$$X_1\big(\zeta(X_2, \sigma_2)\big) - X_2\big(\zeta(X_1, \sigma_1)\big) + \zeta\big([X_1, X_2], X_1(\sigma_2) - X_2(\sigma_1) + [\sigma_1, \sigma_2]\big).$$

Observe that $\zeta = 1 \otimes \zeta_1 + \zeta_2 \otimes 1$ for some $\zeta_1 \in \Gamma(M \times \mathfrak{g}^*)$ and $\zeta_2 \in \Omega^1(M)$. For this reason, for $\sigma_1 = \sigma_2 = 0$, we obtain that $d_{dR}(\zeta_2) = 0$. Moreover, for $X_1 = 0$ and $\sigma_2 = 0$ we have

$$X_2\big(\zeta_1(\sigma_1)\big) = -\zeta_1\big(X_2(\sigma_1)\big) \tag{17}$$

for all $X_2 \in \mathfrak{X}(M)$, $\sigma_1 \in C^\infty(M; \mathfrak{g})$. Let $\{E_1, E_2, E_3, E_4\}$ be a base of $\mathfrak{g}$. Fix $X \in \mathfrak{X}(M)$, $\sigma \in C^\infty(M; \mathfrak{g})$, and let $\zeta_1 = \sum_j \zeta_1^j E_j$ for some $\zeta_1^j \in C^\infty(M)$. Note that $X(\zeta_1(\sigma)) = \zeta_1(X(\sigma)) + X(\zeta_1)(\sigma)$. Combining this with (17) we deduce that

$$2\zeta_1\big(X(\sigma)\big) + X(\zeta_1)(\sigma) = 0. \tag{18}$$

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Taking in (18) constant functions $\sigma_j = 1 \cdot E_j \in C^\infty(M;\mathfrak{g})$, $j \in \{1,2,3,4\}$, we see that $X(\zeta_1) = 0$ for all $X \in \mathfrak{X}(M)$. It follows that ζ_1 is a constant function. Let $\widetilde{\sigma}_j = \sigma^j E_j \in C^\infty(M;\mathfrak{g})$ for some non-constant functions $\sigma^j \in C^\infty(M;\mathfrak{g})$. (18) now implies $X(\sigma^j)\zeta_1^j = 0$ for all $X \in \mathfrak{X}(M)$ and $j \in \{1,2,3,4\}$. Hence $\zeta_1^j = 0$. Since $\zeta_1 = 0$ and $d_{dR}(\zeta_2) = 0$, $\mathrm{Pf}([\widetilde{\sigma}_1, \widetilde{\sigma}_2]) = 0$ for all $\sigma_1, \sigma_2 \in C^\infty(M;\mathfrak{g})$. On the second hand $\mathrm{Pf}([\widetilde{\sigma}_1, \widetilde{\sigma}_2])$ is not a zero function for all σ_1, σ_2. Indeed, let $\{e_1, e_2\}$ be a base of $\mathbb{R}^2$ and $\{e^{*1}, e^{*2}\}$ the associated dual base of $(\mathbb{R}^2)^*$. Let $E_1, E_2, E_3 \in \mathrm{Sym}(\mathbb{R}^2) \subset \mathrm{End}(\mathbb{R}^2)$, $E_4 \in \mathrm{Sk}(\mathbb{R}^2) \subset \mathrm{End}(\mathbb{R}^2)$ be defined by $E_1(x) = \langle e^{*1}, x\rangle e_1$, $E_2(x) = \langle e^{*2}, x\rangle e_2$, $E_3(x) = \langle e^{*1}, x\rangle e_2 + \langle e^{*2}, x\rangle e_1$, $E_4(x) = \langle e^{*1}, x\rangle e_2 - \langle e^{*2}, x\rangle e_1$. Observe that $\mathrm{Pf}([E_1, E_3]) = \mathrm{Pf}(E_4) = 1 \neq 0$. Thus $\Delta_{o\#}([\widetilde{y}_2]) \in \mathsf{H}^2(A)$ is a nontrivial secondary characteristic class for $(TM \oplus \mathrm{End}(\mathbb{R}^2), TM \oplus \mathrm{Sk}(\mathbb{R}^2), \mathrm{id})$ of even rank. □

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