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PRADINES-TYPE GROUPOIDS OVER FOLIATIONS; COHOMOLOGY, CONNECTIONS AND THE CHERN-WEIL HOMOMORPHISM

ABSTRACT We briefly introduce our concept of a Pradines-type groupoid over a foliation [7]. Examples of such groupoids can be found in the theory of foliations. Next, we define a cohomology module $H(A,\mathfrak{f})$ of the Lie algebroid A of a Pradines-type groupoid Φ over a foliation, with values in some vector bundle $\mathfrak{f}$, with respect to a given representation of Φ in $\mathfrak{f}$. It is shown that $H(A,\mathfrak{f})$ depends only on the derivative of this representation. Afterwards, the theory of connections in A and in is built. The last part – the main purpose of this paper – is devoted to defining the Chern-Weil homomorphism h_Φ of Φ and to proving its independence of the choice of connection. As an application of the introduced characteristic classes we give some generalization of the Bott Vanishing Theorem.

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2 Pradines-type groupoids over foliations and their Lie algebroids

There is well-known definition of a *differential groupoid* (see for example [10]) as groupoid

$$\Phi = (\Phi, \alpha, \beta, V, \cdot) \tag{1}$$

in which Φ and V are C^{∞}-manifolds, the mappings $\alpha, \beta : \Phi \to V$ (called a *source* and a *target*) are submersions, and $^{-1} : \Phi \to \Phi$, $h \mapsto h^{-1}$, $u : V \to \Phi$, $x \mapsto u_x$, (u_x – the unity over x) and $\cdot : \Phi * \Phi \to \Phi$, $(h, g) \mapsto h{\cdot}g$, ($\Phi * \Phi := \{(h, g)\,;\ \alpha h = \beta g\}$ is a proper submanifold of $\Phi \times \Phi$) are *smooth* (i.e. of the class C^{∞}).

A transitive differential groupoid is called a *Lie groupoid.* Each Lie groupoid is isomorphic to a Lie groupoid of Ehresmann PP^{-1} [1] for some principal fibre bundle P.

In the theory of foliations one can observe groupoids which do not possess any natural structures of differential groupoids (the spaces of these groupoids need not be manifolds).

Example 1 *The equivalence relation $R \subset V \times V$ determined by a foliation $\mathcal{F}$ of a manifold V ($x \sim y \Longleftrightarrow y \in L_x$, L_x – the leaf of $\mathcal{F}$ through x) is hardly ever regular [11].*

Example 2 *The subgroupoid $\Phi^{\mathcal{F}}$ of a Lie groupoid Φ, consisting of all elements of Φ for which the source and the target lie on some leaf of a given foliation $\mathcal{F}$ of V, is not – in general – a submanifold. This situation is a description (in a language of groupoids) of some important object consisting of a principal fibre bundle and a foliation on the base, studied, for example, by Kamber and Tondeur [5].*

It turns out that the spaces of the groupoids R and $\Phi^{\mathcal{F}}$, considered above, can be equipped with the structures of differential spaces in the sense of Sikorski in order to obtain smooth groupoids according to the definition below [7]. First, we recall that by a *differential space* (in the sense of Sikorski) (for brevity: d.s.) [12] we mean the pair (M, C) consisting of a set N and a nonempty family C of real functions on M, such that

(i) $\varphi(f_1(\cdot), ..., f_s(\cdot)) \in C$ for all $s \in \mathbb{N}$, $f_1, ..., f_s \in C$ and $\varphi \in C^{\infty}(\mathbb{R}^s)$,

(ii) $g : M \to \mathbb{R}$ belongs to C if, for each $x \in M$, there exists its neighbourhood $\tau \in \tau_C$ (τ_C – the weakest topology on M in which all functions from C are continuous) and a function $f \in C$ such that $f|U = g|U$.

For any d.s.'s (M, C) and (N, D), the mapping $f : M \to N$ is called *smooth* if $g \circ f \in C$ for each $g \in D$. If (M, C) is a d.s. and $A \subset M$ is any subset, then (A, C_A) is a d.s., too, where

$$g \in C_A \Longleftrightarrow \bigwedge_{x \in A} \bigvee_{x \in U \in \tau_C} \bigvee_{f \in C} (f|U \cap A = g|U \cap A)\,.$$

Returning to examples 1 and 2, we introduce on the sets R and $\Phi^{\mathcal{F}}$ the differential structures equal to $C^{\infty}(V \times V)_R$ and $C^{\infty}(\Phi)_{\Phi^{\mathcal{F}}}$, respectively. It is easy to see that all operations in these groupoids are then smooth (in the category of d.s.'s, of course).

Definition 3 *By a smooth groupoid [7] we mean groupoid (1) in which V is a C^{∞}-manifold, Φ is a d.s. and the mappings $\alpha, \beta, ^{-1}, u$ and $\cdot : \Phi * \Phi \to \Phi$ (where $\Phi * \Phi$ denotes the proper d.subsp. of $\Phi \times \Phi$) are smooth and, moreover, for each point $x \in V$ on the set $\alpha^{-1}(x)$, there exists a differential structure σ such that $\Phi_x := \left(\alpha^{-1}(x), \sigma\right)$ is a Hausdorff C^{∞}-manifold and*

(i) for each $h \in \alpha^{-1}(x)$, there exists its neighbourhood U open in the manifold Φ_x, such that $C_U = C^{\infty}(\Phi_x)_U$ where C is the differential structure of Φ,

(ii) for each locally arcwise connected topological space X and each continuous mapping $f : X \to \Phi$ such that $f[X] \subset \alpha^{-1}(x)$, the mapping $f : X \to \Phi_x$ is continuous, too.

The manifolds Φ_x, $x \in V$, are called *leaves* of the groupoid Φ.

The mapping

$$D_h : \Phi_{\beta h} \to \Phi_{\alpha h}, \quad g \longmapsto g \cdot h,$$

$h \in \Phi$, are diffeomorphisms.

With each smooth groupoid (1) we associate

(i) a differential subspace of the "*tangent bundle*" $(T\Phi, TC)$ [6] ($T\Phi = \bigsqcup_{h \in \Phi} T_h \Phi$, TC is the smallest of all differential structures containing the set $\{f \circ \pi;\ f \in C\} \cup \{df;\ f \in C\}$ where $\pi : T\Phi \to \Phi$ is the natural projection and $df : T\Phi \to \mathbb{R}$, $v \longmapsto v(f)$) equal to

$$\left(A(\Phi), (TC)_{A(\Phi)}\right)$$

where $A(\Phi) = \bigsqcup_{x \in V} T_{u_x} \Phi_x$,

(ii) a projection

$$p : A(\Phi) \to V, \quad p(v) = x \Leftrightarrow v \in T_{u_x} \Phi_x.$$

A smooth vector field X on Φ [12] is called *right-invariant* if (i) $X_h \in T_h(\Phi_{\alpha h})$, (ii) $(D_h)_{*g}(X_g) = X_{gh}$. The Lie bracket of right-invariant vector field is such a field, too. Each right-invariant vector field X determines a smooth section X_0 of the projection p by the formula $X_0(x) = X(u_x)$. Conversely:

Proposition 4 *For each smooth section $\eta : V \to A(\Phi)$ of p, there exists exactly one smooth right-invariant vector field η' on Φ such that $\eta'_{u_x} = \eta_x$, $x \in V$.*

Proof. Of course, $\eta'_h = (D_h)_{*u_{\beta h}}(\eta_{\beta h})$. To show the smoothness of η', we must prove that $\eta'(f) \in C$ for each $f \in C$. Let $f \in C$. For $h \in \Phi$, we have $\eta'_h(f) = \eta \circ \beta(h)(\Phi_{\beta h} \ni g \longmapsto f \circ (\cdot)(g,h))$. From the assumption about Φ we have $f \circ (\cdot) \in (C \times C)_{\Phi * \Phi}$. We fix $h_0 \in \Phi$ and find a neighbourhood $\Omega \in \tau_{C \times C}$ of $(u_{\beta h_0}, h_0)$ and a function $\tilde{f} \in C \times C$ such that $f \circ (\cdot) | \Omega \cap (\Phi * \Phi) = \tilde{f} | \Omega \cap (\Phi * \Phi)$. Thus, for h from some neighbourhood of h_0, we have $\eta'_h(f) = \eta \circ \beta(h)\left(\tilde{f} \circ (\cdot)\right)$. The function $h \longmapsto \eta \circ \beta(h)\left(\tilde{f}(\cdot, h)\right)$ belongs to C, which is not difficult to show. ■

For two sections ξ, η of p, we put

$$[\![\xi, \eta]\!] := \left([\xi', \eta']\right)_0.$$

Then the system $(\operatorname{Sec} A(\Phi), [\![\cdot,\cdot]\!])$ is a Lie algebra, where $\operatorname{Sec} A(\Phi)$ denotes the vector space of all global sections of p.

The mapping

$$\tilde{\beta}_* : A(\Phi) \to TV, \quad v \longmapsto \beta_*(v),$$

has the property: $\operatorname{Sec} \tilde{\beta}_* : \operatorname{Sec} A(\Phi) \to \mathfrak{X}(V)$ is a homomorphism of Lie algebras. Besides, the following equality

$$[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + \left(\tilde{\beta}_* \circ \xi\right)(f) \cdot \eta$$

holds for $\xi, \eta \in \operatorname{Sec} A(\Phi)$ and $f \in C^\infty(V)$. In general, the system

$$(A(\Phi), p, V) \tag{2}$$

is not a vector bundle for lack (among other things) of the equalities of dimensions of fibres of p.

In the case of a differential groupoid, $A(\Phi) \cong u^* T^\alpha \Phi$ is a vector bundle ($T^\alpha \Phi := \bigcup_h T_h \Phi_{\alpha h} \subset T\Phi$ is then equal to $\ker \alpha_*$) and $A(\Phi)$ is equal to the space of the so-called *Lie algebroid* of Φ defined by Pradines [8], [9]. There are smooth groupoids not being differential for which system (2) is a vector bundle. For example, the above-mentioned examples R and $\Phi^{\mathcal{F}}$ are such groupoids ($A(R) \cong T\mathcal{F}$, $A\left(\Phi^{\mathcal{F}}\right) \cong \tilde{\beta}_*^{-1}[T\mathcal{F}]$).

Definition 5 *By a groupoid of Pradines type [7] we mean smooth groupoid (1) for which system (2) is a vector bundle. The system* $\left(A(\Phi), [\![\cdot,\cdot]\!], \tilde{\beta}_*\right)$ *is then a Lie algebroid called a Lie algebroid of* Φ.

In the sequel, we shall be occupied with a groupoid Φ of Pradines type for which

(i) the family of abstract classes of the equivalence relation

$$R_\Phi := \left\{(x,y) \in V \times V;\ \bigvee_{h \in \Phi} (\alpha h = x, \quad \beta h = y)\right\}$$

is a foliation, say $\mathcal{F}$,

(ii) $\beta_x : \Phi_x \to L_x,\ h \longmapsto \beta h,\ x \in V$, are submersions (L_x is the leaf of $\mathcal{F}$ through x equipped the natural structure of an immerse submanifold of V).

This groupoid is called a *groupoid of Pradines type over the foliation* $\mathcal{F}$ [7]. (1) and (2) are examples of such groupoids.

Let Φ be a fixed groupoid of Pradines type over a foliation $\mathcal{F}$, and

$$A = (A, [\![\cdot,\cdot]\!], \gamma)$$

— its Lie algebroid. Then

(i) Φ_x is a principal fibre bundle with the projection β_x and the structural Lie group $G_x = \beta_x^{-1}(x)$,

(ii) $E := \operatorname{Im}\gamma$ is equal to $T\mathcal{F}$.

We put

$$\mathbf{g} = \ker\gamma.$$

$\mathbf{g}$ is a vector bundle whose each fibre $\mathbf{g}_{|x}$ possesses a natural structure of a Lie algebra (($[v,w] := [\![\xi,\eta]\!](x)$ for any $\xi,\eta \in \operatorname{Sec} A$ such that $\xi(x) = v$ and $\eta(x) = w$, $v, w \in \mathbf{g}_{|x}$). $\mathbf{g}_{|x}$ is called the *isotropy Lie algebra at* x and it is the Lie algebra of the Lie group G_x.

Now, we put $\Phi_x^L := \{h \in \Phi;\ \beta h = x\},\ x \in V$. By means of the bijection $^{-1} : \Phi_x \to \Phi_x^L$ we define on Φ_x^L some structure of a C^∞-manifold. Then $L_h : \Phi_{\alpha h}^L \to \Phi_{\beta h}^L,\ g \longmapsto h \cdot g,\ h \in \Phi$, are diffeomorphisms; with their help *left-invariant* vector fields are defined. It is easy to see that

$$T_{u_x}\Phi_x \cap T_{u_x}\Phi_x^L = T_{u_x}G_x,$$

so each section $\xi \in \operatorname{Sec}\mathbf{g}$ extends not only to the right-invarint vector field ξ' but also to the left-invariant vector field ξ'_L (also smooth). ξ'_L is an α-field (i.e. is tangent to all manifolds Φ_x) and $\xi'_L|\Phi_x$ is a usual fundamental vector field on the principal fibre bundle Φ_x. The left-invariant vector field ξ'_L generated by a section $\xi \in \operatorname{Sec}\mathbf{g}$ is called *fundamental vector field on* Φ.

3 Cohomology of Pradines-type groupoids over foliations.

By an α-form of degree q on Φ with values in a vector bundle $(\mathfrak{f}, p, V)$ we mean an assignment Ψ of some covector $\Psi(h) \in \bigwedge^q (T_h^*(\Phi_{\alpha h})) \bigotimes \mathfrak{f}_{|\alpha h}$ to each element $h \in \Phi$. Ψ is called *smooth* if, for any smooth vector α-fields $X_1, ..., X_q$ on Φ, the mapping

$$\Phi \ni h \longmapsto \Psi(h)(X_{1h}, ..., X_{qh}) \in \mathfrak{f}$$

is smooth. If $\mathfrak{f}$ is the trivial bundle $\mathfrak{f} = V \times \mathbb{R}$, then Ψ is called an *α-form of degree q on* Φ. The set $\Omega^\alpha(\Phi, \mathfrak{f})$ of all smooth α-forms on Φ with values in $\mathfrak{f}$

constitutes a graded module over then ring C ($C =$ the differential structure of Φ); besides it is a left module over the algebra $\Omega^{\alpha}(\Phi)$ of smooth α-forms on Φ. Of course, for $\Psi \in \Omega^{\alpha}(\Phi, \mathfrak{f})$ and $x \in V$, we have

$$\Psi_{|x} := \iota_x^* \Psi \in \Omega\left(\Phi_x, \mathfrak{f}_{|x}\right)$$

where $\iota_x : \Phi_x \hookrightarrow \Phi$.

By a *representation* (in other words, a (*covariant*) *action) of* Φ *in a vector bundle* $\mathfrak{f}$ we mean an assignment T of some linear isomorphism $T(h) : \mathfrak{f}_{|\alpha h} \to \mathfrak{f}_{|\beta h}$ to each element $h \in \Phi$ in such a way that

(i) $T(g \cdot h) = T(g) \circ T(h)$,

(ii) $T(u_x) = \mathrm{id}_{\mathfrak{f}_{|x}}$,

(iii) the mapping $\tilde{T} : \Phi * \mathfrak{f}$, $(h, v) \mapsto T(h)(v)$, is smooth, where,

$$\Phi * \mathfrak{f} = \{(h, v) \in \Phi \times \mathfrak{f};\ \alpha h = pv\}$$

denotes the proper d.subsp. of $\Phi \times \mathfrak{f}$.

Example 6 *(i) the trivial representation* $T(h) = \mathrm{id}_{\mathbb{R}}$ *in the trivialbundle* $V \times \mathbb{R}$,

(ii) the adjoint representation Ad *in the vector bundle* $\mathbf{g}$ *of Lie algebras, defined by the formula:*

$$\mathrm{Ad}(h) = (\tau_h)_{*u_{\alpha h}} : \mathbf{g}_{|\alpha h} \to \mathbf{g}_{|\beta h}$$

where $\tau_h : G_{\alpha h} \to G_{\beta h}$, $a \longmapsto hah^{-1}$.

To give some representaion T is just the same as to give some (strong) smooth homomorphism of groupoids $T : \Phi \to \mathrm{GL}(\mathfrak{f})$ where $\mathrm{GL}(\mathfrak{f})$ denotes the Lie groupoid of all linear isomorphisms between fibres of $\mathfrak{f}$.

Let T be a fixed representation of Φ in $\mathfrak{f}$. $\Psi \in \Omega^{\alpha}(\Phi, \mathfrak{f})$ is called *equivariant with respect to* T if, for each $h \in \Phi$, the equality $(D_h)^* \left(\Psi_{|\alpha h}\right) = T\left(h^{-1}\right)_* \left(\Psi_{|\beta h}\right)$ holds. The graded vector space

$$\Omega_T^{\alpha}(\Phi, \mathfrak{f})$$

of all smooth α-forms on Φ with values in $\mathfrak{f}$ equivariant with respect to T is

(i) a graded module over the ring $C^{\infty}(V)$, with respect to the multiplication $f \bullet \Psi := f \circ \beta \cdot \Psi$,

(ii) a module over the algebra $\Omega_R^{\alpha}(\Phi)$ of all *right-invariant* α*-forms on* Φ, i.e. equivariant with respect to the trivial representation.

Each element of $\Omega(A, \mathfrak{f}) := \bigoplus^{q} \Omega^q(A, \mathfrak{f})$ where

$$\Omega^q(A, \mathfrak{f}) := \mathrm{Sec}\left(\bigwedge^{q} A^* \bigotimes \mathfrak{f}\right)$$

is called a *smooth form on the Lie algebroid A with values in $\mathfrak{f}$*, while, for the trivial bundle $\mathfrak{f} = V \times \mathbb{R}$, briefly: a *smooth form on the Lie algebroid A*. $\Omega(A, \mathfrak{f})$ is a graded module over $C^\infty(V)$ and a module over the algebra $\Omega(A)$ of all smooth forms on A.

Proposition 7 *The mapping*

$$\tau_T : \Omega_T^\alpha(\Phi, \mathfrak{f}) \to \Omega(A, \mathfrak{f}), \quad \tau_T(\Psi)(x) = \Psi(u_x), \quad x \in V,$$

is an isomorphism of graded $C^\infty(V)$-modules.

Proof. It is easy to see that

(i) $\tau_T(\Psi)$ is a smooth form on A with values in $\mathfrak{f}$,

(ii) τ_T is a monomorphism of graded $C^\infty(V)$-modules.

To prove that τ_T is epimorphic, we take any $\Theta \in \Omega^q(A, \mathfrak{f})$ and put

$$\Psi(h)(w_1, ..., w_q) = T\left(h^{-1}\right)\left(\Theta_{\beta h}, ..., (D_{h^{-1}})_{*h} w_i, ...\right), \quad h \in \Phi.$$

Then Ψ is a smooth equivariant α-form on Φ with values in $\mathfrak{f}$, such that $\tau_T \Psi = \Theta$. ■

The isomorphism from the above proposition for the trivial representation is denoted by τ_R. The formula

$$\tau_T(\psi \wedge \Psi) = \tau_R(\psi) \wedge \tau_T(\Psi)$$

holds for any $\psi \in \Omega_R^\alpha(\Phi)$ and $\Psi \in \Omega_T^\alpha(\Phi, \mathfrak{f})$; in particular, τ_R is an isomorphism of algebras.

Theorem 8 *Let X be any smooth vector vector α-field on Φ. There exists uniquely determined endomorphisms $\iota_X^{\alpha,\mathfrak{f}}$, $\Theta_X^{\alpha,\mathfrak{f}}$, $d^{\alpha,\mathfrak{f}}$ of the vector space $\Omega^\alpha(\Phi, \mathfrak{f})$ such that, for each $x \in V$, the following diagrams commutes:*

$$\begin{array}{ccc} \Omega^\alpha(\Phi, \mathfrak{f}) & \overset{\iota_X^{\alpha,\mathfrak{f}}\ \left(\Theta_X^{\alpha,\mathfrak{f}}, d^{\alpha,\mathfrak{f}}\right)}{\longrightarrow} & \Omega^\alpha(\Phi, \mathfrak{f}) \\ \downarrow \iota_x^* & & \downarrow \iota_x^* \\ \Omega\left(\Phi_x, \mathfrak{f}_{|x}\right) & \overset{\iota_{X|\Phi_x}\ \left(\Theta_{X|\Phi_x}, d\right)}{\longrightarrow} & \Omega\left(\Phi_x, \mathfrak{f}_{|x}\right) \end{array}$$

If X is, in addition, a right-invariant vector field, then the subspace $\Omega_T^\alpha(\Phi, \mathfrak{f})$ is stable with respect to all the three endomorphisms.

Proof. The uniqueness is evident. To prove the existence, we define the endomorphisms by the formulae (for a form Ψ of degree q):

$$\begin{aligned}
\left(\iota_X^{\alpha,\mathfrak{f}}\Psi\right)(X_1,...,X_{q-1}) &= \Psi(X,X_1,...,X_{q-1}), \qquad (3)\\
\left(\Theta_X^{\alpha,\mathfrak{f}}\Psi\right)(X_1,...,X_q) &= X(\Psi(X_1,...,X_q)) - \sum_{j=1}^{q}\Psi(X_1,...,[X,X_j],...,X_q)\\
\left(d^{\alpha,\mathfrak{f}}\Psi\right)(X_0,...,X_q) &= \sum_{j=0}^{q}(-1)^j X_j\left(\Psi\left(X_0,...\hat{X}_j...,X_q\right)\right)\\
&\quad + \sum_{i<j}(-1)^{i+j}\Psi\left([X_i,X_j],...\hat{X}_i...\hat{X}_j...\right)
\end{aligned}$$

where X_i are vector α-fields on Φ. The expresion $X(\Psi(X_1,...,X_q))$ has the following sense: it denotes the smooth function $\Phi \to \mathfrak{f}$ defined by $h \longmapsto X_h\left(\Psi_{|\alpha h}(X_1|\Phi_{\alpha h},...,X_q|\Phi_{\alpha h})\right)$. Let us notice that the homomorphisms so determined are C-linear skew-symmetric and possess values at each points. Besides, the diagrams above commute. ■

For the trivial vector bundle $\mathfrak{f} = V \times \mathbb{R}$, the index $\mathfrak{f}$ in the symbols of endomorphisms above (and below) is omitted.

Definition 9 *We take $\xi \in \operatorname{Sec} A$. We define endomorphisms*

$$\iota_\xi^{A,\mathfrak{f}},\ \Theta_\xi^{A,\mathfrak{f}},\ d^{A,\mathfrak{f}}$$

of the vector space $\Omega(A,\mathfrak{f})$ in such a way that the following diagrams commute:

$$\begin{array}{ccc}
\Omega(A,\mathfrak{f}) & \overset{\iota_\xi^{A,\mathfrak{f}}\quad\left(\Theta_\xi^{A,\mathfrak{f}},\ d^{A,\mathfrak{f}}\right)}{\longrightarrow} & \Omega(A,\mathfrak{f})\\
\cong\uparrow \tau_T & & \\
\Omega_T^\alpha(\Phi,\mathfrak{f}) & \overset{\iota_X^{\alpha,\mathfrak{f}}\quad\left(\Theta_X^{\alpha,\mathfrak{f}},d^{\alpha,\mathfrak{f}}\right)}{\longrightarrow} & \Omega_T^\alpha(\Phi,\mathfrak{f})
\end{array}$$

The fundamental properties of these endomorphisms are given below.

Theorem 10 *For any forms $\psi \in \Omega^q(A)$, $\Psi \in \Omega(A,\mathfrak{f})$ and sections $\xi,\eta \in \operatorname{Sec} A$, the following formulas hold:*

(1⁰) $\iota_\xi^{A,\mathfrak{f}}(\psi\wedge\Psi) = \iota_\xi^A\psi\wedge\Psi + (-1)^q\psi\wedge\iota_\xi^{A,\mathfrak{f}}\Psi,$

(2⁰) $\Theta_\xi^{A,\mathfrak{f}}(\psi\wedge\Psi) = \Theta_\xi^A\psi\wedge\Psi + \psi\wedge\Theta_\xi^{A,\mathfrak{f}}\Psi,$

(3⁰) $d^{A,\mathfrak{f}}(\psi\wedge\Psi) = d^A\psi\wedge\Psi + (-1)^q\psi\wedge d^{A,\mathfrak{f}}\Psi,$

(4⁰) $\iota_{[\![\xi,\eta]\!]}^{A,\mathfrak{f}} = \Theta_\xi^A\circ\iota_\eta^{A,\mathfrak{f}} - ....$

(5⁰) $\Theta_{[\![\xi,\eta]\!]}^{A,\mathfrak{f}} = \Theta_\xi^{A,\mathfrak{f}}\circ\Theta_\eta^{A,\mathfrak{f}} - \Theta_\eta^{A,\mathfrak{f}}\circ\Theta_\xi^{A,\mathfrak{f}},$

(6⁰) $\Theta_{\xi}^{A,\mathfrak{f}} = \iota_{\xi}^{A,\mathfrak{f}} \circ d^{A,\mathfrak{f}} + d^{A,\mathfrak{f}} \circ \iota_{\xi}^{A,\mathfrak{f}}$,

(7⁰) $d^{A,\mathfrak{f}} \circ d^{A,\mathfrak{f}} = 0$,

(8⁰) $d^{A,\mathfrak{f}} \circ \Theta_{\xi}^{A,\mathfrak{f}} = \Theta_{\xi}^{A,\mathfrak{f}} \circ d^{A,\mathfrak{f}}$.

The endomorphisms $\iota_{\xi}^{A,\mathfrak{f}}$, $\Theta_{\xi}^{A,\mathfrak{f}}$, $d^{A,\mathfrak{f}}$ *are defined (on forms of degree* q *), by the following formulae, where* $T' : A \to A(\mathrm{GL}(\mathfrak{f}))$ *denotes the derivative of* T, *i.e. some homomorphism of Lie algebroids, while, for a section* $\sigma \in \mathrm{Sec}\,\mathfrak{f}$, $\tilde{\sigma}$ *denotes the mapping*

$$\tilde{\sigma} : \mathrm{GL}(\mathfrak{f}) \to \mathfrak{f}, \quad \mathfrak{h} \longmapsto \mathfrak{h}^{-1}(\sigma_{\beta h}), \tag{4}$$

(9⁰) $\left(\iota_{\xi}^{A,\mathfrak{f}}\Psi\right)(\xi_1, ..., \xi_{q-1}) = \Psi(\xi, \xi_1, ..., \xi_{q-1})$,

(10⁰) $\left(\Theta_{\xi}^{A,\mathfrak{f}}\Psi\right)(\xi_1, ..., \xi_q) = (T' \circ \xi)\left(\Psi(\xi_1, ..., \xi_q)^{\tilde{}}\right) - \sum_{j=1}^{q} \Psi(\xi_1, ..., [\![\xi, \xi_j]\!], ..., \xi_q)$,

(11⁰) $\left(d^{A,\mathfrak{f}}\Psi\right)(\xi_0, ..., \xi_q) = \sum_{j=0}^{q} (-1)^j (T' \circ \xi_j)\left(\Psi\left(\xi_0, ...\hat{\xi}_j ..., \xi_q\right)^{\tilde{}}\right)$
$+ \sum_{i<j} (-1)^{i+j} \Psi\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\xi}_i ...\hat{\xi}_j ..., \xi_q\right)$

where $\xi_i \in \mathrm{Sec}\,A$. *In particular, for the trivial representation,*

(12⁰) $\left(\Theta_{\xi}^{A}\psi\right)(\xi_1, ..., \xi_q) = (\gamma \circ \xi)(\psi(\xi_1, ..., \xi_q)) - \sum_{j=1}^{q} \psi(\xi_1, ..., [\![\xi, \xi_j]\!], ..., \xi_q)$,

(13⁰) $\left(d^{A}\psi\right)(\xi_0, ..., \xi_q) = \sum_{j=0}^{q} (-1)^j (\gamma \circ \xi_j)\left(\psi\left(\xi_0, ...\hat{\xi}_j ..., \xi_q\right)\right)$
$+ \sum_{i<j} (-1)^{i+j} \psi\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\xi}_i ...\hat{\xi}_j ..., \xi_q\right)$,

while, for the Lie algebroid A *equal to the tangent bundle* $E = T\mathcal{F}$,

(14⁰) $\left(\Theta_{\xi}^{E}\psi\right)(X_1, ..., X_q) = X(\psi(X_1, ..., X_q)) - \sum_{j=1}^{q} \psi(X_1, ..., [X, X_j], .., X_q)$,

(15⁰) $\left(d^{E}\psi\right)(X_0, ..., X_q) = \sum_{j=0}^{q} (-1)^j X_j\left(\psi\left(X_0, ...\hat{X}_j ..., X_q\right)\right)$
$+ \sum_{i<j} (-1)^{i+j} \psi\left([X_i, X_j], ...\hat{X}_i ...\hat{X}_j ..., X_q\right)$

for $X_i \in \mathrm{Sec}\,E$.

Proof. Formulae $(1^0)\div(8^0)$ are proved as follows: for example (1^0). First, we prove analogous formula for $\iota_X^{\alpha,\mathfrak{f}}$:

$(1^{0\prime})$ $\iota_X^{\alpha,\mathfrak{f}}(\psi \wedge \Psi) = \iota_X^{\alpha}(\psi \wedge \Psi) + (-1)^q \psi \wedge \iota_X^{\alpha,\mathfrak{f}}\Psi$
for $\psi \in \Omega^{\alpha,q}(\Phi)$, $\Psi \in \Omega^{\alpha}(\Phi, \mathfrak{f})$ and X − an α-field.

For the purpose, we show the equality, for any $x \in V$:

$$\iota_x^* \left(\iota_X^{\alpha,\mathfrak{f}} \left(\psi \wedge \Psi \right) \right) = \iota_x^* \left(\iota_X^{\alpha} \left(\psi \wedge \Psi \right) + (-1)^q \psi \wedge \iota_X^{\alpha,\mathfrak{f}} \Psi \right).$$

Next, in order to prove (1^0), we take any $\psi \in \Omega^q(A)$ and $\Psi \in \Omega(A,\mathfrak{f})$ as well as $\psi' \in \Omega_R^{\alpha}(\Phi)$ and $\Psi' \in \Omega_T^{\alpha}(\Phi,\mathfrak{f})$, such that $\tau_R(\psi') = \psi$ and $\tau_T(\Psi') = \Psi$. Then

$$\begin{aligned}
\iota_\xi^{A,\mathfrak{f}} \left(\psi \wedge \Psi \right) &= \iota_\xi^{A,\mathfrak{f}} \left(\tau_R(\psi') \wedge \tau_T(\Psi') \right) = \iota_\xi^{A,\mathfrak{f}} \left(\tau_T(\psi' \wedge \Psi') \right) \\
&= \tau_T \left(\iota_{\xi'}^{\alpha,\mathfrak{f}} \left(\psi' \wedge \Psi' \right) \right) = \tau_T \left(\iota_{\xi'}^{\alpha} \psi' \wedge \Psi' + (-1)^q \psi' \wedge \iota_{\xi'}^{\alpha,\mathfrak{f}} \Psi' \right) \\
&= \tau_R \left(\iota_{\xi'}^{\alpha} \psi' \right) \wedge \tau_T \Psi' + (-1)^q \tau_R \psi' \wedge \tau_T \left(\iota_{\xi'}^{\alpha,\mathfrak{f}} \Psi' \right) \\
&= \iota_\xi^{A,\mathfrak{f}} \psi \wedge \Psi + (-1)^q \psi \wedge \iota_\xi^{A,\mathfrak{f}} \Psi.
\end{aligned}$$

Formulae (2^0)÷(8^0) are proved analogously, while (9^0)÷(11^0) are proved by making successive use of formulae (10). E.g.:

$$\begin{aligned}
& \left(\Theta_\xi^{A,\mathfrak{f}} \Psi \right) (\xi_1, ..., \xi_q) \\
=\ & \Theta_\xi^{A,\mathfrak{f}} \left(\tau_T \Psi' \right) (\xi_1, ..., \xi_q) = \tau_T \left(\Theta_{\xi'}^{\alpha,\mathfrak{f}} \Psi' \right) (\xi_1, ..., \xi_q) \\
=\ & \left(\Theta_{\xi'}^{\alpha,\mathfrak{f}} \Psi' \right) (\xi_1', ..., \xi_q') \circ u \\
=\ & \left[\xi' \left(\Psi' \left(\xi_1', ..., \xi_q' \right) \right) - \sum_{j=1}^{q} \Psi' \left(\xi_1', ..., \left[\xi', \xi_j' \right], ..., \xi_q' \right) \right] \circ u \\
=\ & \xi \left(\Psi \left(\xi_1, ..., \xi_q \right)^{\tilde{}} \circ T \right) - \sum_{j=1}^{q} \Psi \left(\xi_1, ..., \left[\xi, \xi_j \right], ..., \xi_q \right) \\
=\ & (T' \circ \xi) \left(\Psi \left(\xi_1, ..., \xi_q \right)^{\tilde{}} \right) - \sum_{j=1}^{q} \Psi \left(\xi_1, ..., \left[\xi, \xi_j \right], ..., \xi_q \right).
\end{aligned}$$

■

Corollary 11 *Formulae (9^0)÷(11^0) states that $\iota_\xi^{A,\mathfrak{f}}$ depends only on A and $\mathfrak{f}$, while $\Theta_\xi^{A,\mathfrak{f}}$ and $d^{A,\mathfrak{f}}$ – on the derivative T' of T. In particular, the space $H(A,\mathfrak{f})$ of cohomology of the complex $\left(\Omega(A,\mathfrak{f}), d^{A,\mathfrak{f}}\right)$ depends only on T'. $H(A,\mathfrak{f})$ forms a graded module over the graded cohomology algebra of A, i.e. over the cohomology of the complex $\left(\Omega(A), d^A\right)$.*

Remark 12 *If the Lie algebroid A is equal to the trivial Lie algebroid $(TV, [\cdot,\cdot], \mathrm{id})$, then d^A stands for the usual exterior differentiation of smooth forms. If the manifold V is one-point, then any Lie algebroid is simply a Lie algebra. In this case, for any vector space F understood as a trivial bundle over this point, the differentiation $d^{A,F}$ is equal to the classical operator (see for example [3, Vol.III, p.211]).*

4 Connections

With the Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \gamma)$ we associate a short exact sequence of vector bundles (over the manifold V)

$$0 \to \mathbf{g} \overset{j}{\hookrightarrow} A \overset{\gamma}{\to} E \to 0$$

called an *Atiyah sequence assigned to the Lie algebroid* A (or a fundamental sequence assigned to A).

Definition 13 *By a connection in* A *we mean a splitting of the Atiyah-sequence for* A, *i.e. a morphism*

$$\lambda : E \to A$$

such that $\gamma \circ \lambda = \mathrm{id}_E$. *The corresponding subbundle* $\mathfrak{h} := \operatorname{Im} \lambda \subset A$ *is called horizontal, while the uniquely determined morphism* $\omega : A \to \mathbf{g}$ *such that* $\omega|\mathbf{g} = \mathrm{id}$ *and* $\omega|\mathfrak{h} = 0$ *– a connection form of* λ. *The morphism* $V := j \circ \omega : A \to A$ *is so-called connection homomorphism of* λ. *The isomorphism* $\lambda_* : \mathrm{Sec}\, E \xrightarrow{\cong} \mathrm{Sec}\, \mathfrak{h}$ *is called an isomorphism of horizontal lifting.*

The equality $\lambda_* ([X_1, X_2]) = H_* ([\![\lambda_* X, \lambda_* X_2]\!])$ holds, where $H := \mathrm{id} - V$.

With the groupoid Φ we associate another short exact sequence, this time, of the so-called vector bundles over the d.s. Φ, of the form

$$0 \to \mathbf{g}^\alpha \overset{j^\alpha}{\hookrightarrow} T^\alpha \Phi \overset{\gamma^\alpha}{\to} \beta^* E \to 0 \tag{5}$$

in which

(i) $T^\alpha \Phi$ is a (proper) differential subspace of $T\Phi$ with the set of points equal to $\bigsqcup_{h \in \Phi} T_h \Phi_{\alpha h}$,

(ii) $\gamma^\alpha (v) = (\pi^\alpha V, \beta_* v)$ where $\pi^\alpha : T^\alpha \Phi \to \Phi$ is the natural projection,

(iii) $\mathbf{g}^\alpha = \ker \gamma^\alpha$.

Let us explain that a *vector bundle over a d.s.* is defined identically as over a manifold (the property of local triviality is assumed). It is not difficult to see (basing on [7] that, for a groupoid Φ of Pradines type, $T^\alpha \Phi$ is a vector bundle over Φ.

We define, for a connection $\lambda : E \to A$, a mapping

$$\lambda^\alpha : \beta^* E \to T^\alpha \Phi, \quad (h, v) \longmapsto (D_h)_{* u_{\beta h}} \circ \lambda_{|\beta h} (v) .$$

λ^α is a strong homomorphism of vector bundles over Φ satisfying

(i) $\gamma^\alpha \circ \lambda^\alpha = \mathrm{id}_{\beta^* E}$,

(ii) $\lambda^\alpha_{|gh} = (D_h)_{*g} \circ \lambda^\alpha_{|g}$ where $\lambda^\alpha_{|h} : E_{|\beta h} \to T_h (\Phi_{\alpha h})$, $v \longmapsto \lambda^\alpha (h, v)$.

Conversely, for each smooth strong homomorphism $\mu : \beta^* E \to T^\alpha \Phi$ of vector bundles over Φ fulfilling (a) $\gamma^\alpha \circ \mu = \mathrm{id}_{\beta^* E}$, (b) $\mu_{|gh} = (D_h)_{*g} \circ \mu_{|g}$, there exists exactly one connection λ in A such that $\lambda^\alpha = \mu$.

Each homomorphism $\mu : \beta^* E \to T^\alpha \Phi$ fulfilling (a) and (b) is called a *connection in the groupoid* Φ. By a *connection form of* μ we mean the uniquely determined strong homomorphism $\zeta : T^\alpha \Phi \to \mathbf{g}^\alpha$ of vector bundles over Φ, for which $\zeta \circ j^\alpha = \mathrm{id}$ and $\zeta | \operatorname{Im} \mu = 0$. All connection forms are characterized by the properties

(i) $\zeta \circ j^\alpha = \mathrm{id}$,

(ii) $(D_h)_{*g} \circ \zeta_{|g} = \zeta_{|gh} \circ (D_h)_{*g}$.

The assignment $\lambda \longmapsto \lambda^\alpha$ establishes a bijection between connections in A and in Φ. One can verify that in the groupoid $\Phi^{\mathcal{F}}$ (example 2) where $\Phi \cong PP^{-1}$ (P – some principal fibre bundle) connections are in the 1-1 correspondence with partial connections in P [5] which project onto the tangent bundle to the foliation $\mathcal{F}$.

Proposition 14 *The mapping*

$$k : \alpha^* \mathbf{g} \to \mathbf{g}^\alpha, \quad (h, v) \longmapsto (A_h)_{*u_{\alpha h}} (v) ,$$

where $A_h : G_{\alpha h} \to \Phi_{\alpha h}$, $a \longmapsto ha$, *is a strong isomorphism of vevtor bundles over* Φ.

Proof. Since $k_{|h} : \mathbf{g}_{|\alpha h} \to \mathbf{g}^\alpha_{|h}$ is an isomorphism of vector spaces, it is sufficient to see the smoothness of k, but to prove this – the smoothness of the section $k \circ \xi$ of $\mathbf{g}^\alpha \subset T^\alpha \Phi \subset T\Phi$, where $\xi(h) = (h, \xi_{\alpha h})$, $h \in \Phi$, $\xi \in \operatorname{Sec} \mathbf{g}$. As $k \circ \xi = \xi'_L$ and the left-invariant vector field generated by ξ is smooth, $k \circ \xi$ is a smooth vector field. ■

Remark 15 *(a)* $A_h = L_h | G_x$, *so* $\xi'_L(h) = (A_h)_{*u_{\alpha h}} (\xi_{\alpha h})$ *for* $\xi \in \operatorname{Sec} \mathbf{g}$.

(b) Sequence (5) can be modified to the following diagram

$$\begin{array}{ccccccccccc}
 & & 0 & \to & \mathbf{g}^\alpha & \xrightarrow{j^\alpha} & T^\alpha \Phi & \xrightarrow{\gamma^\alpha} & \beta^* E & \to & 0 \\
 & & & {}^{k}\nearrow_{\cong} & & & & & & & \\
\mathbf{g} & \xleftarrow{\tilde{\alpha}} & \alpha^* \mathbf{g} & & \downarrow & & \downarrow & & \downarrow & & \\
\downarrow & & \downarrow & & & & & & & & \\
V & \xleftarrow{\alpha} & \Phi & = & \Phi & = & \Phi & = & \Phi & &
\end{array}$$

where $\tilde{\alpha}(h, v) = v$, *which is called a fundamental diagram for* Φ.

Let $\zeta : T^\alpha \Phi \to \mathbf{g}^\alpha$ be any connection form in Φ. Then the homomorphism

$$\zeta^\alpha := \tilde{\alpha} \circ k^{-1} \circ \zeta : T^\alpha \Phi \to \mathbf{g}$$

of bundles over $\alpha : \Phi \to V$ is called a connection α-form of ζ. This is a smooth α-form of degree 1 on Φ with values in the bundle $\mathbf{g}$. We show without difficulty the following

Proposition 16 *ζ^{α} has the properties:*

(a) $\iota^{\alpha,\mathbf{g}}_{\xi'_L}\zeta^{\alpha} = \xi \circ \alpha$ *(i.e.* $\zeta^{\alpha}_{|h}\left((A_h)_{*u_{\alpha h}} v\right) = v$ *),*

(b) $(D_h)^{*}\left(\zeta^{\alpha}_{|\alpha h}\right) = \left(\operatorname{Ad} h^{-1}\right)_{*}\left(\zeta^{\alpha}_{|\beta h}\right)$ *(i.e.* $\zeta^{\alpha} \in \Omega^{\alpha,1}_{\operatorname{Ad}}(\Phi, \mathbf{g})$ *).*

Conversely, for each homomorphism $\zeta^{\alpha} : T^{\alpha}\Phi \to \mathbf{g}$ of vector bundles over α, fulfilling (a) and (b) above, there exists exactly one connection form $\zeta : T^{\alpha}\Phi \to \mathbf{g}^{\alpha}$ such that $\zeta^{\alpha} := \tilde{\alpha} \circ k^{-1} \circ \zeta$.

We now take any connection form ω in the Lie algebroid A. ω determines some connection in a A which defines, in turn, some connection in Φ. The α-form of this last connection is given by the formula $\zeta^{\alpha}_{|h} = \left(\operatorname{Ad} h^{-1}\right) \circ \omega_{\beta h} \circ (D_{h^{-1}})_{*h}$. The restriction $\zeta^{\alpha}_{|x}$ of ζ^{α} to the manifold Φ_x is a usual connection form in the principal fibre bundle Φ_x. Besides $\tau_{\operatorname{Ad}}\zeta^{\alpha} = \omega$.

Now, we fix a connection $\lambda : E \to A$ in the Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \gamma)$ with a connection formω, a connection homomorphism V, and also some vector bundle $\mathfrak{f}$ and a representation T of Φ in $\mathfrak{f}$. A form $\Psi \in \Omega(A, \mathfrak{f})$ ($\psi \in \Omega(A)$) is called *horizontal* if $\iota^{A,\mathfrak{f}}_{\xi}\Psi = 0$ ($\iota^{A}_{\xi}\psi = 0$) for each $\xi \in \operatorname{Sec}\mathbf{g}$. All horizontal forms constitute a vector space $\Omega_i(A, \mathfrak{f})$ ($\Omega_i(A)$). Moreover, $\Omega_i(A)$ is an algebra and $\Omega_i(A, \mathfrak{f})$ – a submodule of the $\Omega_i(A)$-module $\Omega(A, \mathfrak{f})$. We define a *horizontal projection*

$$H^{A,\mathfrak{f}}_{*} : \Omega(A, \mathfrak{f}) \to \Omega(A, \mathfrak{f})$$

by the formula (for a form Ψ of degree q)

$$\left(H^{A,\mathfrak{f}}_{*}\Psi\right)(x; v_1, ..., v_q) = \Psi(x; Hv_1, ..., Hv_q)$$

where $H = \operatorname{id} - V$. For the trivial bundle $\mathfrak{f} = V \times \mathbb{R}$, the index $\mathfrak{f}$ is omitted. We show without difficulty that:

(i) $H^{A,\mathfrak{f}}_{*}$ is linear,

(ii) $H^{A,\mathfrak{f}}_{*}|\Omega_i(A, \mathfrak{f}) = \operatorname{id}$,

(iii) $\operatorname{Im} H^{A,\mathfrak{f}}_{*} = \Omega_i(A, \mathfrak{f})$,

(iv) $\left(H^{A,\mathfrak{f}}_{*}\right)^2 = H^{A,\mathfrak{f}}_{*}$,

(v) $H^{A,\mathfrak{f}}_{*}(\psi \wedge \Psi) = H^{A}_{*}\psi \wedge H^{A,\mathfrak{f}}_{*}\Psi$,

(vi) $H^{A,\mathbf{g}}_{*}\omega = 0$.

The endomorphism

$$\nabla^{A,\mathfrak{f}} := H^{A,\mathfrak{f}}_{*} \circ d^{A,\mathfrak{f}}$$

is called an *exterior covariant derivative in the Lie algebroid A (with values in $\mathfrak{f}$)* associated with the connection λ. For the trivial bundle $\mathfrak{f}$, the endomorphism $\nabla^{A,\mathfrak{f}}$ is denoted by ∇^{A}. It is easy to see the following properties of $\nabla^{A,\mathfrak{f}}$:

(i) $\nabla^{A,\mathfrak{f}}$ is linear,

(ii) $\operatorname{Im} \nabla^{A,\mathfrak{f}} \subset \Omega_i (A, \mathfrak{f})$,

(iii) $\nabla^{A,\mathfrak{f}} (\psi \wedge \Psi) = \nabla^A \psi \wedge H_*^{A,\mathfrak{f}} \Psi + (-1)^q H_*^A \psi \wedge \nabla^{A,\mathfrak{f}} \Psi$ for $\psi \in \Omega^q (A)$, $\Psi \in \Omega (A, \mathfrak{f})$.

The last property results from property (3^0) of $d^{A,\mathfrak{f}}$ (see Theorem 10).
Now, we define $\gamma_{\mathfrak{f}}^* \Theta \in \Omega^q (A, \mathfrak{f})$ for $\Theta \in \Omega^q (E, \mathfrak{f})$ by the formula

$$\left(\gamma_{\mathfrak{f}}^* \Theta\right) (x; v_1, ..., v_q) = \Theta (x; \gamma v_1, ..., \gamma v_q) .$$

Analogously, $\lambda_{\mathfrak{f}}^* \Psi \in \Omega^q (E, \mathfrak{f})$ for $\Psi \in \Omega^q (A, \mathfrak{f})$ by the formula

$$\left(\lambda_{\mathfrak{f}}^* \Psi\right) (x; w_1, ..., w_q) = \Psi (x; \lambda w_1, ..., \lambda w_q) .$$

It is easy to see that

(i) $\gamma_{\mathfrak{f}}^* \Theta \in \Omega_i (A, \mathfrak{f})$ for any form $\Theta \in \Omega (E, \mathfrak{f})$,

(ii) the mappings

$$\gamma_{\mathfrak{f}}^* : \Omega (E, \mathfrak{f}) \to \Omega_i (A, \mathfrak{f}), \quad \Theta \longmapsto \gamma_{\mathfrak{f}}^* \Theta,$$

and

$$\lambda_{\mathfrak{f}}^* : \Omega_i (A, \mathfrak{f}) \to \Omega (E, \mathfrak{f}), \quad \Psi \longmapsto \lambda_{\mathfrak{f}}^* \Psi,$$

are mutually inverse isomorphisms such that $(\theta \wedge \Theta) = \gamma^* \theta \wedge \gamma_{\mathfrak{f}}^* \Theta$ and $\lambda_{\mathfrak{f}}^* (\psi \wedge \Psi) = \lambda^* \psi \wedge \lambda_{\mathfrak{f}}^* \Psi$. Particularly, λ^* and γ^* are (defined for the trivial bundle $\mathfrak{f}$) isomorphisms of algebras.

Definition 17 *We define an endomorphism $\nabla^{\mathfrak{f}}$ of the vector space $\Omega (E, \mathfrak{f})$ as*

$$\nabla^{\mathfrak{f}} := \lambda_{\mathfrak{f}}^* \circ \nabla^{A,\mathfrak{f}} \circ \gamma_{\mathfrak{f}}^*$$

and call it an exterior covariant derivative in the bundle $\mathfrak{f}$ along leaves of the foliation $\mathcal{F}$ associated with the connection λ.

Theorem 18 *(a)* $\nabla^{\mathfrak{f}} = \lambda_{\mathfrak{f}}^* \circ d^{A,\mathfrak{f}} \circ \gamma_{\mathfrak{f}}^*$,

(b) for the trivial bundle $\mathfrak{f} = V \times \mathbb{R}$, the equality $\nabla^{\mathfrak{f}} = d^E$ holds, i.e. $d^E = \lambda^* \circ d^A \circ \gamma^*$,

(c) $\nabla^{\mathfrak{f}} (\theta \wedge \Theta) = d^E \theta \wedge \Theta + (-1)^q \theta \wedge \nabla^{\mathfrak{f}} \Theta$ *for* $\theta \in \Omega^q (E)$, $\Theta \in \Omega (E, \mathfrak{f})$,

(d) $\left(\nabla^{\mathfrak{f}} \Theta\right) (X_0, ..., X_q) = \sum_{j=0}^q (-1)^j \nabla_{X_j}^{\mathfrak{f}} \left(\Theta \left(X_0, ...\tilde{X}_j ..., X_q\right)\right)$

$+ \sum_{i<j} (-1)^{i+j} \Theta \left([X_i, X_j], ...\tilde{X}_i ... \tilde{X}_j ..., X_q\right)$,

(e) $\nabla^{\mathfrak{f}}$ *restricted to* $\operatorname{Sec}\mathfrak{f}$*, i.e.* $\nabla^{\mathfrak{f}}:\operatorname{Sec}\mathfrak{f}\rightarrow{}^{1}(E,\mathfrak{f})$*, is defined by the formula* $\nabla^{\mathfrak{f}}_{X}(\sigma)=(T'\circ\lambda_{*}X)(\tilde{\sigma})$ *for* $\sigma\in\operatorname{Sec}\mathfrak{f}$ *and* $X\in\operatorname{Sec}E$ *(for* $\tilde{\sigma}$*, see 4), and has the properties:*

(i) $\nabla^{\mathfrak{f}}$ *is linear,*

(ii) $\nabla^{\mathfrak{f}}_{fX}\sigma=f\nabla^{\mathfrak{f}}_{X}\sigma,$

(iii) $\nabla^{\mathfrak{f}}_{X}(f\sigma)=X(f)\,\sigma+f\nabla^{\mathfrak{f}}_{X}\sigma$ *for* $f\in C^{\infty}(V)$, $\sigma\in\operatorname{Sec}\mathfrak{f}$.

Proof. (a) follows from the equality $\lambda_{\mathfrak{f}}^{*}H_{*}^{A,\mathfrak{f}}\Psi=\lambda_{\mathfrak{f}}^{*}\Psi$ for any $\Psi\in\Omega(A,\mathfrak{f})$, while (b) – from the suitable properties (mentioned above) of $\lambda_{\mathfrak{f}}^{*}$ i $\gamma_{\mathfrak{f}}^{*}$ as well as from property (3^0) of $d^{A,\mathfrak{f}}$. (b) is shown by a direct calculation with the use of formulae (13^0) and (15^0), (c) follows from (3^0), as to (d): by (11^0), we have

$$\begin{aligned}
&\left(\nabla^{\mathfrak{f}}\Theta\right)(X_0,...,X_q)\\
=\;&\lambda_{\mathfrak{f}}^{*}\circ d^{A,\mathfrak{f}}\circ\gamma_{\mathfrak{f}}^{*}\Theta(X_0,...,X_q)\\
=\;&\sum_{j=0}^{q}(-1)^{j}(T'\circ\lambda_{*}X_j)\left(\gamma_{\mathfrak{f}}^{*}\Theta(\lambda_{*}X_0,...\hat{\jmath}...,\lambda_{*}X_q)\right)\\
&+\sum_{i<j}(-1)^{i+j}\left(\gamma_{\mathfrak{f}}^{*}\Theta\right)([\![\lambda_{*}X_i,\lambda_{*}X_j]\!],...\hat{\imath}...\hat{\jmath}...)\\
=\;&\sum_{j=0}^{q}(-1)^{j}\nabla^{\mathfrak{f}}_{X_j}(\Theta(X_0,...\hat{\jmath}...,X_q))+\sum_{i<j}(-1)^{i+j}\Theta([X_i,X_j],...\hat{\imath}...\hat{\jmath}...).
\end{aligned}$$

(e) is easy to see. ∎

Remark 19 $\nabla^{\mathfrak{f}}$ *restricted to any leaf of the foliation* $\mathcal{F}$*, i.e.* $\nabla^{\mathfrak{f}}:\operatorname{Sec}\left(\mathfrak{f}_{|L}\right)\rightarrow\Omega^{1}\left(TL,\mathfrak{f}_{|L}\right)$*, is a usual covariant derivative. Operators having the above property appeared in the work by Kamber and Tondeur [5] as partial connections in a vector bundle.*

By a *curvature form of* λ we mean the form

$$\Omega:=\nabla^{A,\mathbf{g}}\omega\in\Omega^{2}(A,\mathbf{g}).$$

This form has the following properties:

(i) $\Omega\in\Omega^{2}_{i}(A,\mathbf{g})$,

(ii) $\Omega(\xi_1,\xi_2)=-\omega([\![H_{*}\xi_1,H_{*}\xi_2]\!])$ for $\xi_j\in\operatorname{Sec}A$.

Indeed, (i) follows from property (iii) of the horizontal projection $H_{*}^{A,\mathfrak{f}}$, while (b) from the calculation:

$$\begin{aligned}
&\Omega(\xi_1,\xi_2)\\
=\;&\left(\nabla^{A,\mathbf{g}}\omega\right)(\xi_1,\xi_2)=\left(d^{A,\mathbf{g}}\omega\right)(H_{*}\xi_1,H_{*}\xi_2)\\
&\overset{(11^0)}{=}(\operatorname{ad}\circ H_{*}\xi_1)\left(\omega(H_{*}\xi_2)^{\tilde{}}\right)-(\operatorname{ad}\circ H_{*}\xi_2)\left(\omega(H_{*}\xi_1)^{\tilde{}}\right)-\omega([\![H_{*}\xi_1,H_{*}\xi_2]\!])\\
=\;&-\omega([\![H_{*}\xi_1,H_{*}\xi_2]\!])
\end{aligned}$$

where ad denotes the derivative of the adjoint representation Ad.

by a *curvature base-form of* λ we mean the form

$$\Omega_B = \lambda_{\mathbf{g}}^{*}\Omega \in \Omega^2(E, \mathbf{g}).$$

This form has the properties:

(i) $\Omega_B(X_1, X_2) = -\omega([\![\lambda_* X_1, \lambda_* X_2]\!])$,

(ii) $[\![\lambda_* X_1, \lambda_* X_2]\!] = \underbrace{\lambda_*[X_1, X_2]}_{\text{horizontal part}} \underbrace{-\Omega_B(X_1, X_2)}_{\text{vertical part}}$,

(iii) $\Omega = 0 \iff \Omega_B = 0$,

(iv) $\Omega_B = 0$ iff the Lie bracket of two horizontal vector fields (i.e. sections of $\mathfrak{h} = \operatorname{Im}\lambda$) is such a field.

It remains to examine two classical equations:

(a) the structure equation of Maurer-Cartan

$$\Omega = d^{A,\mathbf{g}}\omega + \frac{1}{2}[\omega, \omega],$$

(b) the Bianchi identity

$$\nabla^{A,\mathbf{g}}\Omega = 0 \quad (\text{also } \nabla^{\mathbf{g}}\Omega_B = 0). \tag{6}$$

In equation (a), we take the connection μ in Φ, determined by λ. Let ζ^α be its connection α-form. The classical Maurer-Cartan equation for the connection $\zeta^\alpha_{|x}$ in the principal fibre bundle Φ_x has the form

$$d\left(\zeta^\alpha_{|x}\right) + \frac{1}{2}\left[\zeta^\alpha_{|x}, \zeta^\alpha_{|x}\right] = H(x)_* \, d\left(\zeta^\alpha_{|x}\right)$$

where $H(x)_*$ denotes here the horizontal projection in Φ_x associated with $\zeta^\alpha_{|x}$. Let us denote by V^α the connection homomorphism of μ, i.e.

$$V^\alpha : T^\alpha\Phi \to T^\alpha\Phi, \quad v \mapsto \zeta(v),$$

where ζ is a connection form of μ, and next, define the horizontal projection

$$H_*^{\alpha,\mathbf{g}} : \Omega^\alpha(\Phi, \mathbf{g}) \to \Omega^\alpha(\Phi, \mathbf{g})$$

by the formula

$$(H_*^{\alpha,\mathbf{g}}\Psi)(h; v_1, ..., v_q) = \Psi(h; H^\alpha v_1, ..., H^\alpha v_q)$$

where $H^\alpha = \mathrm{id} - V^\alpha$. Of course,

$$H(x)_* = (H_*^{\alpha,\mathbf{g}})_{|x}$$

and both the horizontal projections $H_*^{\alpha,\mathbf{g}}$ and $H_*^{A,\mathbf{g}}$ commute with τ_{Ad}. Defining $[\zeta^\alpha,\zeta^\alpha]$ analogously as $[\omega,\omega]$, we get

$$\left(d^{\alpha,\mathbf{g}}\zeta^\alpha+\frac{1}{2}[\zeta^\alpha,\zeta^\alpha]\right)_x=d\left(\zeta^\alpha_{|x}\right)+\frac{1}{2}\left[\zeta^\alpha_{|x},\zeta^\alpha_{|x}\right]=H(x)_*\,d\left(\zeta^\alpha_{|x}\right)=\left(H_*^{\alpha,\mathbf{g}}d^{\alpha,\mathbf{g}}\zeta^\alpha\right)_{|x},$$

so

$$d^{\alpha,\mathbf{g}}\zeta^\alpha+\frac{1}{2}[\zeta^\alpha,\zeta^\alpha]=H_*^{\alpha,\mathbf{g}}d^{\alpha,\mathbf{g}}\zeta^\alpha$$

which further gives

$$\begin{aligned} d^{A,\mathbf{g}}\omega+\frac{1}{2}[\omega,\omega] &= d^{A,\mathbf{g}}\tau_{\mathrm{Ad}}\zeta^\alpha+\frac{1}{2}[\tau_{\mathrm{Ad}}\zeta^\alpha,\tau_{\mathrm{Ad}}\zeta^\alpha] \\ &= \tau_{\mathrm{Ad}}\left(d^{\alpha,\mathbf{g}}\zeta^\alpha\frac{1}{2}[\zeta^\alpha,\zeta^\alpha]\right)=\tau_{\mathrm{Ad}}H_*^{\alpha,\mathbf{g}}d^{\alpha,\mathbf{g}}\zeta^\alpha \\ &= H_*^{A,\mathbf{g}}d^{A,\mathbf{g}}\omega=\nabla^{A,\mathbf{g}}\omega=\Omega. \end{aligned}$$

The Bianchi identity easily follows from the Maurer-Cartan equation.

The form $\Omega^\alpha:=H_*^{\alpha,\mathbf{g}}d^{\alpha,\mathbf{g}}\zeta^\alpha$ is called a *curvature α-form of the connection* μ in Φ. It is the so-called *basic form,* i.e. equivariant and horizontal at the same time, where the *horizontality* of a form $\Psi\in\Omega^\alpha(\Phi,\mathfrak{f})$ states that $\iota_X^{\alpha,\mathfrak{f}}\Psi=0$ for each vertical vector X (i.e. each section X of the bundle $\mathbf{g}^a$). The space of all basic forms is denoted by $\Omega_B^\alpha(\Phi,\mathfrak{f})$. $\Omega_B^\alpha(\Phi)$ (for the trivial bundle $\mathfrak{f}$) forms an algebra. The isomorphism τ_T restricts to the isomorphism $\tau_{T,i}:\Omega_B^\alpha(\Phi,\mathfrak{f})\to\Omega_i(A,\mathfrak{f})$, moreover, $\tau_{R,i}:\Omega_B^\alpha(\Phi)\to\Omega_i(A)$ is, of course, an isomorphism of algebras. Besides, $\tau_{\mathrm{Ad}}\Omega^\alpha=\Omega$.

5 The Chern-Weil homomorphism of groupoids of Pradines-type over foliations

Let $\mathfrak{f}_1,...,\mathfrak{f}_k,\mathfrak{f}$ be any vector bundles over V. For a smooth k-linear homomorphism

$$\Gamma:\mathfrak{f}_1\times...\times\mathfrak{f}_k\to\mathfrak{f}$$

of vector bundles, we define

(i) for forms $\Psi_i\in\Omega^{\alpha,q_i}(\Phi,\mathfrak{f}_i)$, $i\le k$, the form

$$\Gamma_*^\alpha(\Psi_1,,,,\Psi_k)\in\Omega^{\alpha,q}(\Phi,\mathfrak{f}),\quad q=\sum q_i,$$

by the formula

$$\begin{aligned} &\Gamma_*^\alpha(\Psi_1,,,,\Psi_k)(h;v_1,...,v_q) \\ =\ &\frac{1}{q_1!\cdot...\cdot q_k!}\sum_\sigma \operatorname{sgn}\sigma\cdot\Gamma_{|\alpha h}\left(\Psi_1\left(h;v_{\alpha(1)},...\right),...,\Psi_k\left(h;...v_{\sigma(q)}\right)\right). \end{aligned}$$

Of course

$$\Gamma_*^\alpha(\Psi_1,,,,\Psi_k)_{|x}=\left(\Gamma_{|x}\right)\left(\Psi_{1|x},,,,\Psi_{k|x}\right).$$

(ii) for forms $\Psi_i \in \Omega^{\alpha,q_i}(A,\mathfrak{f}_i)$, $i \leq k$, the form

$$\Gamma_*^A(\Psi_1,,,,\Psi_k) \in \Omega^q(A,\mathfrak{f})$$

– by the analogous formula; in particular

$$\Gamma_*^E(\Theta_1,...,\Theta_k) \in \Omega^q(E,\mathfrak{f})$$

is defined for $\Theta_i \in \Omega^{q_i}(E,\mathfrak{f}_i)$.

It is easy to see that the following formulae ($\tilde{\tau}_\mathfrak{f} : \Omega^\alpha(\Phi,\mathfrak{f}) \to \Omega(A,\mathfrak{f})$, $\tilde{\lambda}_\mathfrak{f}^* : \Omega(A,\mathfrak{f}) \to \Omega(E,\mathfrak{f})$ denote here the mappings $\Psi \longmapsto \Psi(u_x)$ and $\Psi \longmapsto \lambda_\mathfrak{f}^*\Psi$, respectively) hold:

(i) $\Gamma_*^A(\tilde{\tau}_{\mathfrak{f}_1} \times ... \times \tilde{\tau}_{\mathfrak{f}_k})(\Psi_1,,,,\Psi_k) = \tilde{\tau}_\mathfrak{f}(\Gamma_*^\alpha(\Psi_1,,,,\Psi_k))$,

(ii) $\Gamma_*^E \circ \left(\tilde{\lambda}_{\mathfrak{f}_1}^* \times ... \times \tilde{\lambda}_{\mathfrak{f}_k}^*\right)(\Psi_1,,,,\Psi_k) = \tilde{\lambda}_\mathfrak{f}^*\left(\Gamma_*^A(\Psi_1,,,,\Psi_k)\right)$.

Besides,

(a) $\iota_X^{\alpha,\mathfrak{f}}(\Gamma_*^\alpha(\Psi_1,,,,\Psi_k)) = \sum_i (-1)^{q_1+...+q_{i-1}} \Gamma_*^\alpha\left(\Psi_1,,,,\iota_X^{\alpha,\mathfrak{f}_i}\Psi_i,...,\Psi_k\right)$ for any α-field X,

(b) $\iota_\xi^{A,\mathfrak{f}}\left(\Gamma_*^A(\Psi_1,,,,\Psi_k)\right) = \sum_i (-1)^{q_1+...+q_{i-1}} \Gamma_*^A\left(\Psi_1,,,,\iota_\xi^{A,\mathfrak{f}_i}\Psi_i,...,\Psi_k\right)$ for any $\xi \in \operatorname{Sec}\mathbf{g}$,

(c) $d^{\alpha,\mathfrak{f}}(\Gamma_*^\alpha(\Psi_1,,,,\Psi_k)) = \sum_i (-1)^{q_1+...+q_{i-1}} \Gamma_*^\alpha\left(\Psi_1,,,,d^{\alpha,\mathfrak{f}_i}\Psi_i,...,\Psi_k\right)$.

Formulae (a) and (c) can be proved by the method "for each point x on the manifold Φ_x ", used bedore, while (b) follows from (a) and the equality $\iota_\xi^{A,\mathfrak{f}} \circ \tilde{\tau}_\mathfrak{f} = \tilde{\tau}_\mathfrak{f} \circ \iota_{\xi'}^{\alpha,\mathfrak{f}}$.

Assume that $T_1,...,T_k,T$ are representations of Φ in the bundles $\mathfrak{f}_1,...,\mathfrak{f}_k,\mathfrak{f}$, respectively. A k-linear homomorphism $\Gamma : \mathfrak{f}_1 \times ... \times \mathfrak{f}_k \to \mathfrak{f}$ is called $(T_1,...,T_k;T)$-invariant if, for each $h \in \Phi$, the diagram

$$\begin{array}{ccc} \mathfrak{f}_{1|x} \times ... \times \mathfrak{f}_{k|x} & \xrightarrow{\Gamma_{|x}} & \mathfrak{f}_{|x} \\ T_1(h) \times ... \times T_k(h) \downarrow & & \downarrow T(h) \\ \mathfrak{f}_{1|y} \times ... \times \mathfrak{f}_{k|y} & \xrightarrow{\Gamma_{|y}} & \mathfrak{f}_{|y} \end{array}$$

commutes, where $x = \alpha h$, $y = \beta h$. All invariant sections of the bundle $\bigotimes^k \mathfrak{f}_i^* \bigotimes \mathfrak{f}$ (considered as k-linear homomorphisms) are denoted by

$$\left(\operatorname{Sec} \bigotimes^k \mathfrak{f}_i^* \bigotimes \mathfrak{f}\right)_I.$$

We notice that

(i) the value $\Gamma_{|x}$ of an invariant section Γ is an invariant element with respect to induced representations of the Lie group G_x in the vector spaces $\mathfrak{f}_{1|x}, ..., \mathfrak{f}_{k|x}, \mathfrak{f}_{|x}$,

(ii) for an invariant section Γ, knowing the value $\Gamma_{|x}$, one can calculate the value $\Gamma_{|y}$ for each $y \in L_x$ (L_x – the leaf of $\mathcal{F}$ through x).

Denote by $\left(\bigotimes^k \mathfrak{f}^*_{i|x} \bigotimes \mathfrak{f}_{|x}\right)_I$ the space of invariant homomorphisms $\mathfrak{f}_{1|x} \times ... \times \mathfrak{f}_{k|x} \longrightarrow \mathfrak{f}_{|x}$ (invariant with respect to the above-mentioned representation of G_x) and take the "bundle" $\left(\bigotimes^k \mathfrak{f}^*_i \bigotimes \mathfrak{f}\right)_I := \bigcup_{x\in V} \left(\left(\bigotimes^k \mathfrak{f}^*_{i|x} \bigotimes \mathfrak{f}_{|x}\right)_I\right)$ (with the differential structure induced from $\bigotimes^k \mathfrak{f}^*_i \bigotimes \mathfrak{f}$). This "bundle" is (i) a usual trivial vector bundle over each leaf of $\mathcal{F}$, while (ii) invariant homomorphisms are some of its sections.

For the groupoid $\Phi^{\mathcal{F}}$ (Example 2), each element of this "bundle" is a value of a certain invariant homomorphism. More exactly, the bundle $\left(\bigotimes^k \mathfrak{f}^*_i \bigotimes \mathfrak{f}\right)_I$ possesses then a global, canonical teleparallelism and each invariant homomorphism has the form $\sum_i f^i \cdot \Gamma_i$ for some smooth functions f^i constant along the leaves of $\mathcal{F}$ and some homomorphisms Γ_i "constant" with respect to this telepallelism.

A representation $T : \Phi \to \mathrm{GL}(\mathfrak{f})$ defines the 2-linear $(\mathrm{Ad}; T)$-invariant homomorphism $T_i : \mathbf{g} \times \mathfrak{f} \to \mathfrak{f}$, $(v, w) \longmapsto T(x)'(v)(w)$, where $T(x) : G_x \to \mathrm{GL}(\mathfrak{f}_{|x})$ denotes the induced representation and $T(x)'$ – its derivative. In particular, for the adjoint representation $\mathrm{Ad} : \Phi \to \mathrm{GL}(\mathbf{g})$, we have the $(\mathrm{Ad}, \mathrm{Ad})$-invariant homomorphism $[\cdot,\cdot] = \mathrm{Ad}_I : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$, $(k, l) \longmapsto [k, l]$.

Let $\Gamma : \mathfrak{f}_1 \times ... \times \mathfrak{f}_k \to \mathfrak{f}$ be an invariant homomorphism. Then

(i) for $\Psi_j \in \Omega^\alpha_{T_j}(\Phi, \mathfrak{f}_j)$, $j \le k$, we have $\Gamma^\alpha_*(\Psi_1, ..., \Psi_k) \in \Omega^\alpha_T(\Phi, \mathfrak{f})$,

(ii) for $\Psi_j \in \Omega^\alpha_i(A, \mathfrak{f}_j)$, $j \le k$, $-\Gamma^A_*(\Psi_1, ..., \Psi_k) \in \Omega_i(A, \mathfrak{f})$,

(iii) the formula

$$d^{A,\mathfrak{f}}\Gamma^A_*(\Psi_1, ..., \Psi_k) = \sum_j (-1)^{q_1+...+q_{j-1}} \Gamma^A_*\left(\Psi_1, ..., d^{A,\mathfrak{f}_j}\Psi_j, ..., \Psi_k\right)$$

holds for $\Psi_j \in \Omega^{q_j}(A, \mathfrak{f}_j)$.

Furthemore

$$\nabla^{\mathfrak{f}}\left(\Gamma^E_*(\Theta_1, ..., \Theta_k)\right) = \sum_j (-1)^{q_1+...+q_{j-1}} \Gamma^E_*\left(\Theta_1, ..., \nabla^{\mathfrak{f}_j}\Theta_j, ..., \Theta_k\right)$$

for $\Theta_j \in \Omega(E, \mathfrak{f}_j)$; in particular, for the trivial bundle $\mathfrak{f}$, we have

$$d^E\left(\Gamma^E_*(\Theta_1, ..., \Theta_k)\right) = \sum_j (-1)^{q_1+...+q_{j-1}} \Gamma^E_*\left(\Theta_1, ..., \nabla^{\mathfrak{f}_j}\Theta_j, ..., \Theta_k\right) \qquad (7)$$

for $\Theta_j \in \Omega(E, \mathfrak{f}_j)$.

For a k-linear $(\mathrm{Ad}, ..., \mathrm{Ad})$-invariant homomorphism $\Gamma : \mathbf{g} \times ... \times \mathbf{g} \to \mathbb{R}$ we put

(i) $\beta^\alpha \Gamma := \Gamma_*^A (\Omega^\alpha, ..., \Omega^\alpha) \in \Omega^{\alpha, 2k}(\Phi)$,

(ii) $\beta^A \Gamma := \Gamma_*^A (\Omega, ..., \Omega) \in \Omega^{2k}(A)$,

(iii) $\beta^E \Gamma := \Gamma_*^E (\Omega_B, ..., \Omega_B) \in \Omega^{2k}(E)$,

where Ω^α, Ω, Ω_B are the curvature α-form, the curvature form and the curvature base-form of a given connection, respectively.

It is easy to show that

$$\beta^\alpha \Gamma \in \Omega_B^{\alpha, 2k}(\Phi) \quad \text{and} \quad \beta^A \Gamma \in \Omega_i^{2k}(A).$$

We define in an evident manner) the mappings β^α, β^A, β^E from the space $\bigoplus^k \left(\left(\mathrm{Sec} \bigotimes^k \mathbf{g}^* \right)_I \right)$ into $\Omega_B^\alpha(\Phi)$, $\Omega_i(A)$ and $\Omega(E)$, respectively, and notice the following equations

$$\tau_R \circ \beta^\alpha = \beta^A \quad \text{and} \quad \lambda^* \circ \beta^A = \beta^E.$$

The space $\bigoplus^k \left(\mathrm{Sec} \bigotimes^k \mathbf{g}^* \right)$ of all sections is an algebra (in the natural manner), while the subspace $\bigoplus^k \left(\mathrm{Sec} \bigotimes^k \mathbf{g}^* \right)_I$ of invariant sections is, of course, its subalgebra. β^α is a homomorphism of algebras, whence β^A and β^E, too (the formula $\beta^\alpha (\Gamma_1 \cdot \Gamma_2) = \beta^\alpha \Gamma_1 \wedge \beta^\alpha \Gamma_2$ follows from the fact that it holds "for each point x on the manifold Φ_x "). We define a smooth homomorphism

$$\pi_S^k : \bigotimes^k \mathbf{g}^* \to \bigvee^k \mathbf{g}^*, \quad t_1 \otimes ... \otimes t_k \longmapsto t_1 \vee ... \vee t_k$$

of vector bundles.

- We identify $\bigotimes^k \mathbf{g}^* \cong \mathcal{L}^k(\mathbf{g}, \mathbb{R})$ via the isomorphism

$$t_1 \otimes ... \otimes t_k \longmapsto ((v_1, ..., v_k) \longmapsto t_1(v_1) \cdot ... \cdot t_k(v_k))$$

while $\bigvee^k \mathbf{g}^* \cong \mathcal{L}_s^k(\mathbf{g}, \mathbb{R})$ via –

$$t_1 \vee ... \vee t_k \longmapsto \left((v_1, ..., v_k) \longmapsto \frac{1}{k!} \sum_\sigma t_{\sigma(1)}(v_1) \cdot ... \cdot t_{\sigma(k)}(v_k) \right),$$

therefore the embedding

$$\bigvee^k \mathbf{g}^* \cong \mathcal{L}_s^k(\mathbf{g}, \mathbb{R}) \subset \mathcal{L}^k(\mathbf{g}, \mathbb{R}) \cong \bigotimes^k \mathbf{g}^*$$

is defined by the formula

$$t_1 \vee \ldots \vee t_k \longmapsto \frac{1}{k!} \sum_{\sigma} t_{\sigma(1)} \cdot \ldots \cdot t_{\sigma(k)}.$$

Further, we treat $\overset{k}{\bigvee} \mathbf{g}^*$ as a subspace of $\overset{k}{\bigotimes} \mathbf{g}^*$ (of course, with its own algebra structure). With such an interpretation,

$$\pi_S^k | \overset{k}{\bigvee} \mathbf{g}^* = \mathrm{id}\,.$$

We understand $\mathrm{Sec} \overset{k}{\bigvee} \mathbf{g}^* \subset \mathrm{Sec} \overset{k}{\bigotimes} \mathbf{g}^*$ analogously. γ^α, γ^A, γ^E are defined as restrictions of β^α, β^A, β^E to the subspace $\overset{k}{\bigoplus} \left(\mathrm{Sec} \overset{k}{\bigvee} \mathbf{g}^* \right)_I$. To prove the equation $\gamma^\alpha \circ \mathrm{Sec}\, \pi_S^k =$ and the fact that γ^α is a homomorphism of algebras, it is sufficient to show

(i) the commutativity of the diagram

$$\begin{array}{ccc} \overset{k}{\bigotimes} \mathbf{g}^*_{|x} & & \\ \pi^k_{S|x} \downarrow & \searrow^{\beta^\alpha(x)} & \\ \overset{k}{\bigvee} \mathbf{g}^*_{|x} & \overset{\gamma^\alpha(x)}{\longrightarrow} & \Omega\left(\Phi_x\right), \end{array}$$

where $\beta^\alpha(x)$ and $\gamma^\alpha(x)$ are defined by $\delta \longmapsto \delta_* \left(\Omega^\alpha_{|x}, \ldots, \Omega^\alpha_{|x} \right)$,

(ii) the fact that $\gamma^\alpha(x)$ is a homomorphism of algebras.

But it follows from the suitable properties of the commutative algebra $\mathrm{Im}\, \beta^\alpha(x)$ [2]. The above implies

$$\gamma^A(\Gamma) = \Gamma^A_*(\Omega, \ldots, \Omega) \quad \text{ and } \quad \gamma^E \Gamma = \Gamma^E_*(\Omega_B, \ldots, \Omega_B)$$

and the commutativity of the fundamental diagram

$$\begin{array}{ccccccc} \overset{k}{\bigoplus} \left(\mathrm{Sec} \overset{k}{\bigvee} \mathbf{g}^* \right)_I & \overset{\gamma^\alpha}{\longrightarrow} & \Omega^\alpha_B(\Phi) & \overset{d^\alpha}{\longrightarrow} & \Omega^\alpha_R(\Phi) & \overset{H^\alpha_*}{\longrightarrow} & \Omega^\alpha_B(\Phi) \\ & \searrow^{\gamma^A} & \cong\downarrow \tau_{R,i} & & \cong\downarrow \tau_R & & \downarrow \tau_{R,i} \\ & & \Omega_i(A) & \overset{d^A}{\longrightarrow} & \Omega(A) & \overset{H^A_*}{\longrightarrow} & \Omega_i(A) \\ & {}^{\gamma^E} \searrow & \cong\downarrow \lambda^* & & \downarrow \lambda^* & \cong \swarrow \lambda^* & \\ & & \Omega(E) & & \Omega(E) & & \end{array}$$

Theorem 20 $d^E \circ \gamma^E = 0$.

Proof. It is an immediate consequence of (7) and the Bianchi identity (6) (in brackets). ■

Definition 21 *The superposition*

$$h_\Phi : \bigoplus^k \left(\operatorname{Sec} \bigvee^k \mathbf{g}^*\right)_I \xrightarrow{\gamma^E} Z(E) \to H(E)$$

is called the Chern-Weil homomorphism of Φ*. The image of* h_Φ *is a graded subalgebra of* $H(E)$ *called the Pontryagin algebra of* Φ *and denoted*

$$\operatorname{Pont}(\Phi).$$

Remark 22 *Proceeding in the same way, we may build the Chern-Weil homomorphism* $h_\Phi^{\mathfrak{f}} : \bigoplus^k \left(\operatorname{Sec} \bigvee^k \mathbf{g}^* \bigotimes \mathfrak{f}\right)_I \to H^\nabla(E, \mathfrak{f})$ *with values in any vector bundle* $\mathfrak{f}$*, with respect to any representation* $T : \Phi \to \mathrm{GL}(\mathfrak{f})$*, where* $H^\nabla(E, \mathfrak{f})$ *is the space of Vaisman cohomology of the false complex* $\left(\Omega(E, \mathfrak{f}), \nabla^{\mathfrak{f}}\right)$*. For* $\mathfrak{f} = \mathbf{g}$*,* $T = \mathrm{Ad}$ *and* $\Gamma = \mathrm{id}_{\mathbf{g}} : \mathbf{g} \to \mathbf{g}$*, we get the universal Halperin-Lehman characteristic class of curvature (see [4]).*

Theorem 23 *The Chern-Weil homomorphism* h_Φ *is independent of the choice of connection.*

Lemma 24 *Let* Φ *and* Φ' *be any groupoid of Pradines type over foliations* $\mathcal{F}$ *and* $\mathcal{F}'$ *of manifolds* V *and* V'*, while* $A = (A, [\![\cdot,\cdot]\!], \gamma)$ *and* $A = (A', [\![\cdot,\cdot]\!]', \gamma')$ *their Lie algebroids. If* $F : \Phi \to \Phi'$ *is any smooth homomorphism of groupoids over* $f : V \to V'$ *(i.e.* $\alpha' \circ F = f \circ \alpha$*,* $\beta \circ F = f \circ \beta$ *), and* $\omega : A \to \mathbf{g}$ *and* $\omega' : A' \to \mathbf{g}'$ *are any connection forms in* A *and* A'*, respectively, for which the diagram*

$$\begin{array}{ccc} \mathbf{g} & \xleftarrow{\omega} & A \\ \tilde{F}_*^0 \downarrow & & \downarrow \\ \mathbf{g}' & \xleftarrow{\omega'} & A \end{array}$$

commutes (where $\tilde{F}_*$ *and* $\tilde{F}_*^0$ *denote the suitable restrictions of* $F_* : T\Phi \to T\Phi'$*), then the Chern-Weil homomorphism* h_Φ *and* $h_{\Phi'}$*, built by using the forms* ω *and* ω'*, give the commuting diagram*

$$\begin{array}{ccc} \bigoplus^k \left(\operatorname{Sec} \bigvee^k \mathbf{g}'^*\right)_I & \xrightarrow{h_{\Phi'}} & H(E') \\ \operatorname{Sec}\left(\tilde{F}_*^0\right)^\vee \downarrow & & \downarrow f^\# \\ \bigoplus^k \left(\operatorname{Sec} \bigvee^k \mathbf{g}^*\right)_I & \xrightarrow{h_\Phi} & H(E) \end{array}$$

where $E := T\mathcal{F}$ *and* $E' := T\mathcal{F}'$*.*

Proof of the lemma. First, we notice (by the meyhod "for each point x on the bundles Φ_x and $\Phi'_{f(x)}$ ") that, for the curvature forms Ω and Ω' associated with ω and ω', the following diagram

$$\begin{array}{ccc} \mathbf{g}_{|x} & \overset{\Omega_{|x}}{\longleftarrow} & A_{|x}\times A_{|x} \\ \tilde{F}^{0}_{*|x}\downarrow & & \downarrow \tilde{F}_{*|x}\times \tilde{F}_{*|x} \\ \mathbf{g}'_{|x} & \overset{\Omega'_{|f(x)}}{\longleftarrow} & A_{|f(x)}\times A_{|f(x)} \end{array}$$

commutes. Next, we show that, for the corresponding curvature base-forms $\Omega_B \in \Omega^2(E,\mathbf{g})$ and $\Omega'_B \in \Omega^2(E',\mathbf{g})$ the diagram

$$\begin{array}{ccc} \mathbf{g}_{|x} & \overset{\Omega_{B|x}}{\longleftarrow} & E_{|x}\times E_{|x} \\ \tilde{F}^{0}_{*|x}\downarrow & & \downarrow \tilde{f}_{*|x}\times \tilde{f}_{*|x} \\ \mathbf{g}'_{|x} & \overset{\Omega'_{B|f(x)}}{\longleftarrow} & E'_{|f(x)}\times E'_{|f(x)} \end{array}$$

commutes ($\tilde{f}_* : E \to E'$ denotes here the differential of f restricted to E). Using this diagram, we can easily prove that the diagram below also commutes:

$$\begin{array}{ccc} \Omega(E') & \overset{(\tilde{f}_*)^*}{\longrightarrow} & \Omega(E) \\ \Gamma \longmapsto \Gamma_*(\Omega'_B,...,\Omega'_B)\uparrow & & \uparrow\ \Gamma \longmapsto \Gamma_*(\Omega_B,...,\Omega_B) \\ \overset{k}{\bigoplus}\left(\mathrm{Sec}\overset{k}{\bigvee}\mathbf{g}'^*\right) & \overset{\mathrm{Sec}\left(\tilde{F}^0_*\right)}{\longrightarrow} & \overset{k}{\bigoplus}\left(\mathrm{Sec}\overset{k}{\bigvee}\mathbf{g}^*\right) \end{array}$$

To end the proof, it is sufficient to show that

$$d^E \circ \left(\tilde{f}_*\right)^* = \left(\tilde{f}_*\right)^* \circ d^{E'},$$

which implies the possibility of defining $f^* : H(E) \to H(E')$.

Using theorem 18(b) and the relationship between d^A and d^α, one can reduce this equality to the commutativity of the usual operations of differentiation and pull-back of differential forms on the manifolds Φ_x and $\Phi'_{f(x)}$. ■

Proof of theorem 23. We consider the Pradines-type groupoid $\breve{\Phi} = \Phi \times \mathbb{R}^2$ (in which $\breve{\alpha}(h,x,y) = (\alpha h, x)$, $\beta(h,x,y) = (\beta h, y)$). $\breve{\Phi}$ is over the foliation $\mathcal{F}\times\mathbb{R} := \{L \times \mathbb{R};\ L \in \mathcal{F}\}$. The sequence

$$0 \to \mathbf{g}\times 0 \hookrightarrow A \times T\mathbb{R} \overset{\gamma\times \mathrm{id}}{\longrightarrow} E \times T\mathbb{R} \to 0$$

is the Atiyah sequence associated with the Lie algebroid $A \times T\mathbb{R}$ (i.e. with the Lie algebroid of $\breve{\Phi}$). The homomorphism $\widetilde{\mathrm{pr}}_1 : \Phi \times \mathbb{R}^2 \to \Phi$ of groupoids defines some homomorphisms (over $\mathrm{pr}_1 : V \times \mathbb{R} \to V$) of vector bundles:

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{g}\times 0 & \hookrightarrow & A\times T\mathbb{R} & \overset{\gamma\times\mathrm{id}}{\longrightarrow} & E\times T\mathbb{R} & \to & 0 \\ & & \downarrow & & \downarrow (\widetilde{\mathrm{pr}}_1)_*^{\sim} & & \downarrow & & \\ 0 & \to & \mathbf{g} & \hookrightarrow & A & \longrightarrow & E & \to & 0 \end{array}$$

A connection form ω in A determines a connection form $\tilde{\omega}=\omega\times 0 : A\times T\mathbb{R}\rightarrow\mathbf{g}\times 0$ in the Lie algebroid $A\times T\mathbb{R}$, for which the following diagram commutes:

$$\begin{array}{ccc} \mathbf{g}\times 0 & \overset{\tilde{\omega}}{\longleftarrow} & A\times T\mathbb{R} \\ \downarrow & & \downarrow \\ \mathbf{g} & \overset{\omega}{\longleftarrow} & A \end{array}$$

Now, we take two connection forms $\omega_i : A\rightarrow\mathbf{g}$, $i=0,1$, and the connection forms $\tilde{\omega}_i$ in $A\times T\mathbb{R}$, corresponding to them. These last together define a certain connection form $\tilde{\omega} : A\times T\mathbb{R}\rightarrow\mathbf{g}\times 0$ by the formula:

$$\tilde{\omega}_{|(x,t)}\left(v,w\right)=\left(\omega_{0|x}\left(v\right)\cdot\left(1-t\right)+\omega_{1|x}\left(v\right)\cdot t,0\right).$$

We now consider the homomorphism $F_{\nu} : \Phi\rightarrow\Phi\times\mathbb{R}^{2}$, $h\longmapsto\left(h,\left(\nu,\nu\right)\right)$, $\nu=0,1$, of groupoids over $i_{\nu} : V\rightarrow V\times\mathbb{R}$, $x\longmapsto\left(x,\nu\right)$. Then we get the commuting diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbf{g} & \overset{\omega_{\nu}}{\overleftarrow{\hookrightarrow}} & A & \rightarrow & E & \rightarrow & 0 \\ & & \left(\tilde{F}_{\nu}\right)_{*}^{0}\downarrow & & \left(\tilde{F}_{\nu}\right)_{*}\downarrow & & \downarrow & & \leftarrow \text{hom.'s over } i_{\nu} \\ 0 & \rightarrow & \mathbf{g}\times 0 & \overset{\omega}{\overleftarrow{\hookrightarrow}} & A\times T\mathbb{R} & \rightarrow & E\times T\mathbb{R} & \rightarrow & 0 \end{array}$$

According to lemma 24, we get the diagram

$$\begin{array}{ccc} \bigoplus^{k}\left(\operatorname{Sec}\bigvee^{k}\left(\mathbf{g}\times 0\right)^{*}\right)_{I} & \overset{h_{\Phi\times\mathbb{R}^{2}}}{\longrightarrow} & H\left(E\times T\mathbb{R}\right) \\ \operatorname{Sec}\left(\tilde{F}_{\nu}\right)_{*}^{0\vee}\downarrow & & \downarrow i_{\nu}^{\#} \\ \bigoplus^{k}\left(\operatorname{Sec}\bigvee^{k}\mathbf{g}^{*}\right)_{I} & \overset{h_{\Phi}}{\longrightarrow} & H\left(E\right) \end{array}$$

To notice the equality

$$i_{0}^{\#}=i_{1}^{\#}$$

will be the next step of the proof.

Lemma 25 *Let V and V' be any manifolds with arbitrary foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. If $f,g : V\rightarrow V'$ are any smooth mappings and $H : V\times\mathbb{R}\rightarrow V'$ is an homotopy between them, such that, for each leaf L of $\mathcal{F}$ and for $t\in\mathbb{R}$, the set $H\left(\cdot,t\right)\left[L\right]$ is contained in some leaf of $\mathcal{F}'$, then*

$$f^{*}=g^{*} : H\left(E\right)\rightarrow H\left(E'\right)$$

where $E=T\mathcal{F}$ and $E'=T\mathcal{F}'$.

Proof of the lemma. We define some cochain homotopy operator

$$h : \Omega^{q}\left(E'\right)\rightarrow\Omega^{q-1}\left(E\right),$$

$q = 0, 1, 2, ..,$ by the formula

$$h(\Theta)(x; v_1, ..., v_{q-1}) = \int_0^1 (h^*\Theta)_{|(x,t)} \left(v_1, ..., v_{q-1}, \frac{\partial}{\partial t} \right) dt$$

for $\Theta \in \Omega^q(E')$. The correctness of this definiyion follows from the fact that

$$H_{*(x,t)} \left[E_{|x} \times T_t \mathbb{R} \right] \subset E'_{|H(x,t)},$$

which is a consequence of the assumptions. The condition

$$f^* - g^* = h \circ D^{E'} + d^E \circ h$$

can be checked in a standard way. ■

Continuation of the proof of the theorem.. Applying lemma 24 to the homotopy $H := \mathrm{id}_{V \times \mathbb{R}}$, we get the equality $i_0^{\#} = i_1^{\#}$.

Finally, we consider the homomorphism

$$\mathbf{g} \times 0 \xrightarrow{p_1} \mathbf{g}, \quad (v, 0) \longmapsto v,$$

over $\mathrm{pr}_1 : V \times \mathbb{R} \to V$. Of course, $p_1 \circ \left(\tilde{F}_\nu \right)_*^o = \mathrm{id}_{V \times \mathbb{R}}$, so

$$\mathrm{id} = \left(\mathrm{Sec} \bigvee^k \mathbf{g}^* \overset{\mathrm{Sec}(p_1)^\vee}{\longrightarrow} \mathrm{Sec} \bigvee^k (\mathbf{g} \times 0)^* \overset{\mathrm{Sec}\left(\tilde{F}_\nu\right)_*^{o\vee}}{\longrightarrow} \mathrm{Sec} \bigvee^k \mathbf{g}^* \right).$$

Thus, considering the diagram

$$\begin{array}{ccc} \mathrm{Sec} \bigvee^k (\mathbf{g} \times 0)^* & \overset{h_{\Phi \times \mathbb{R}^2}}{\longrightarrow} & H(E \times T\mathbb{R}) \\ \mathrm{Sec}(p_1)^\vee \uparrow\downarrow \mathrm{Sec} \left(\tilde{F}_\nu \right)_*^{o\vee} & & \downarrow i_0^{\#} = i_1^{\#} \\ \mathrm{Sec} \bigvee^k \mathbf{g}^* & \overset{h_\Phi}{\longrightarrow} & H(E) \end{array}$$

we obtain

$$h_\Phi = \left(h_\Phi \circ \mathrm{Sec} \left(\tilde{F}_\nu \right)_*^{o\vee} \right) \circ \mathrm{Sec}(p_1)^\vee = i_\nu^{\#} \circ h_{\Phi \times \mathbb{R}^2} \circ \mathrm{Sec}(p_1)^\vee.$$

The right-hand side of this equality is the same for both connections ω_0 and ω_1, which proves the independence of h_Φ of the choice of connection. ■ ■

Remark 26 *The equivalence of the Chern-Weil homomorphism h_Φ of the Lie groupoid of Ehresmann $\Phi = PP^{-1}$ determined by a principal fibre bundle P, with the Chern-Weil homomorphism h_P of P (see, for example, [3, Vol II]) follows from the commutativity of the diagram*

$$\begin{array}{ccccccc} \ulcorner & \left(\bigvee^k \tilde{\mathfrak{g}}^* \right)_I & & \overset{\Xi}{\longrightarrow} & & \left(\mathrm{Sec} \bigvee^k \mathbf{g}^* \right)_I & \urcorner \\ | & \gamma^P \downarrow & & & & \downarrow \gamma^A & | \\ \gamma_B | & \Omega_B(P) & & \overset{\varkappa}{\longleftarrow} & & \Omega_i(A) & |\gamma^{TV} \\ | & & \nwarrow^{\pi^*} & & \gamma^* \nearrow\swarrow \lambda^* & & | \\ \llcorner & - - - - \longrightarrow & & \Omega(V) & & \longleftarrow - - - - - - & \lrcorner \end{array}$$

in which

(i) $\tilde{\mathfrak{g}}$ *is the Lie algebra of structural Lie group* G *of* P,

(ii) $\mathbf{g}$ *is the bundle of Lie algebras of the Lie algebroid* A *of* Φ.

(iii) $\Xi(\Gamma)_{|x} = \Gamma \circ \left((H_z)_*^{-1} \times ... \times (H_z)_*^{-1} \right)$ *where* $(H_z)_* : \tilde{\mathfrak{g}} \to \mathbf{g}_{|x}$ *is the derivative of the homomorphism of Lie groups* $H_z : G \to G_x,\ a \longmapsto [z, za]$, $z \in P_{|x}$,

(iv) $\varkappa(\Psi)(z; v_1, ..., v_q) = \Psi(\pi z; (\varphi_z)_{*z} v_q)$ *where* $\varphi_z : P \overset{\cong}{\longrightarrow} \left(PP^{-1}\right)_x,\ t \longmapsto [z, t],\ x = \pi z$.

Remark 27 *Let* Φ *be any Pradines-type groupoid over a foliation* $\mathcal{F}$. *We take* $L \in \mathcal{F}$ *and* $x \in L$. *Then the Chern-Weil homomorphism* h_Φ *of* Φ *and* h_{Φ_x} *of the principal fibre bundle* Φ_x *are connected by the commuting diagram*

$$\begin{array}{ccc} \overset{k}{\bigoplus} \left(\operatorname{Sec} \overset{k}{\bigvee} \mathbf{g}^* \right)_I & \overset{h_\Phi}{\longrightarrow} & H(E) \\ \Gamma \mapsto \Gamma_{|x} \ \downarrow & & \downarrow [\Theta] \mapsto [\Theta|L] \\ \overset{k}{\bigoplus} \left(\operatorname{Sec} \overset{k}{\bigvee} \mathbf{g}^*_{|x} \right)_I & \overset{h_{\Phi_x}}{\longrightarrow} & H(TL) . \end{array}$$

Remark 28 *For the groupoid* $\Phi^{\mathcal{F}}$ *(from example 2) in which* $\Phi \cong PP^{-1}$, *the Chern-Weil homomorphism* h_P, $h_{\Phi^{\mathcal{F}}}$, *and* h_{Φ_x} *are connected by the commuting diagram (*$\tilde{\mathfrak{g}}$ *is the Lie algebra of structural Lie group* G *of* P *):*

$$\begin{array}{cccccc} \ulcorner & ---- & \left(\bigvee \tilde{\mathfrak{g}}^*\right)_I & \overset{h_P}{\longrightarrow} & H(TV) \\ | & & \downarrow & & \downarrow \\ \cong | & & \overset{k}{\bigoplus} \left(\operatorname{Sec} \overset{k}{\bigvee} \mathbf{g}^* \right)_I & \overset{h_{\Phi^{\mathcal{F}}}}{\longrightarrow} & H(E) \\ | & & \downarrow & & \downarrow \\ \llcorner & --- \longrightarrow & \left(\bigvee \mathbf{g}^*_{|x}\right)_I & \overset{h_{\Phi_x}}{\longrightarrow} & H(TL) \end{array}$$

As an application of the introduced characteristic classes we have the following theorem (see [13]):

Theorem 29 (Some generalization of the Bott Vanishing Theorem) *Let* $\{\mathcal{F}, \mathcal{F}'\}$ *be a flag of foliation on a manifold* V; *suppose that*

$$T\mathcal{F} = \mathcal{T}\mathcal{F}' \bigoplus \mathfrak{f},$$

$q = \operatorname{rank} \mathfrak{f}$, *then*

$$\operatorname{Pont}^k \left(\operatorname{GL}(\mathfrak{f})^{\mathcal{F}} \right) = 0,$$

for $k > 2q$.

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