The signature operator for graded Hilbert subdifferential Hodge spaces and L_2-Hirzebruch operators for Lie algebroids

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Abstract

The purpose of the paper is to present the algebraic aspect of the Hodge theory and Hirzebruch signature operator originally given for smooth and L_2 manifolds and vector bundles. Finally, we give some aplications to transitive Lie algebroids.

Let $\langle \cdot | \cdot \rangle$ be a real pairing [Kronecker pairing, in the terminology of L. I. Nicolaescu [N]] on a graded differential (infinite dimensional) real vector space $\left(W = \bigoplus_{k=0}^{N=2n} W^k, d = \left(d^k\right) \right)$, where d is linear operator od degree 1 such that $\langle dw|u \rangle = (-1)^{|w|+1} \langle w|du \rangle$. The existence of an isomorphism $* : W \to W$ such that $(v|w) := \left\langle v| *^{-1} w \right\rangle$ is an inner product and $*$ is an isometry is very useful in examining the cohomology pairing $\mathbf{H}(W) \times \mathbf{H}(W) \to \mathbb{R}$, especially the signature [therefore the assumption N is even is necessary]. Signature is calculated via the index of a suitable Hirzebruch operator. The fundamental example is of course the Hodge theory for graded differential space of differential forms on a compact orientable Riemannian manifold. The second example is obtained for transitive Lie algebroids [K3]. The next two important examples (on the base of Lusztig example [L] and Gromov one [Gro]) concern the cohomology of a manifold with coefficients in flat symmetric or symplectic vector bundle [K-M-4]. The last two examples prove to be very important for Lie algebroids according to some spectral sequence argument [K-M-4]. In [K-M-4] we presented algebraic aspects of the Hirzebruch signature operator - and we obtain the general point of view on the above four examples defining and examining the so called Hodge spaces.

N.Teleman [T2] in 1983 defined and examined the signature operator on Lipschitz compact manifolds M for L_2-differential forms $L_2(M)$. The space $L_2(M)$ (in opposite to smooth differential forms) is Hilbert, but de-Rham derivative of such differential forms (determined in distributional manner) is not defined on the whole space $L_2(M)$ (but only on some dense subspace). The fundamental (in my opinion) observation by N.Teleman is that the Poincaré duality property - which is obtained via algebraic topology methods - is sufficient to obtain (in L_2-theory) the Hodge theory (i.e. Hodge isomorphism and the strong Hodge decomposition) and the calculation of the signature via the index of the suitable Hirzebruch operator (the convolution L_2-argument and the L_2-Poincaré Lemma is all what it is needed).

In our paper we provide an algebraic point of view on (abstract) graded Hilbert Hodge spaces with derivative defined on some dense subspace. The Poincaré duality (or even less - the weak nondegeneracy of the cohomology pairing) is sufficient to obtain a strong Hodge decomposition theorem and the Hodge isomorphism (Th. 2.2.6). Therefore the suitable (abstract) Hirzebruch operator has index equaling to the signature.

For non-Hilbert case W we can do Hilbert completion $\bar{W}$ and extend the derivative in a distributional sense. The diagram (3.1.2) joins two Hodge homomorphisms for W and $\bar{W}$. The conditions under which all four homomorphisms in this diagram are isomorphisms are given. Then the signature can be calculated via Hirzebruch operator for W and for $\bar{W}$ and we obtain the same number. As an application we examine the completion of the space of differential forms for our four fundamental examples (classical, Lie algebroid, Lusztig, Gromov). The mentioned above conditions giving all isomorphisms in Diagram (3.1.2) are fullfilled. Among them there is an L_2-convolution argument and the Weyl Lemma is used. Also we must check that $\mathbf{H}(W) = \mathbf{H}\left(\bar{W}\right)$ which generally follows from the so-called abstract Hodge theory of elliptic complexes ([L]) but for our four examples we can prove it without elliptic theory using the Mayer-Vietoris or spectral sequences arguments and sheaves argument.

1 Graded differential Hodge spaces

All vector spaces will be over the field $\mathbb{R}$. A *pairing* between two vector spaces V and W is a bilinear map $B : V \times W \to \mathbb{R}$. The pairing B is called a *weakly nondegenerated*, i.e. the both null spaces are zero (i.e. if $B(v, \cdot) = 0$ then $v = 0$, and analogously for the second variable). The pairing B is called a *duality* (or *strongly nondegenerated*) if the adjunction morphism $V \to W^*$, $v \mapsto B(v, \cdot)$, associated to the pairing B is an isomorphism. If $B : V \times V \to \mathbb{R}$ is a duality then $V \cong V^*$ whence $\dim V < \infty$. If V and W are finitely dimensional and $B : V \times W \to \mathbb{R}$ is a weakly nondegenerated, then $\dim V = \dim W$ and B is a duality.

In [K-M-4] there is an abstract definition of a Hodge space which generalize classical examples for Riemannian manifolds, Lusztig [L] and Gromov [Gro] examples and one given for the theory of Lie algebroids [K3]. The main purpose of [K-M-4] is to define an abstract Hirzebruch operator for graded differential Hodge spaces. Here we present a review of this approach.

1.1 Hodge spaces

For details see [K-M-4]. Let W be a real vector space of an arbitrary dimension (finite or infinite).

Definition 1.1.1 *[K-M-4] By a* Hodge space *we mean the system*

$$(W, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *_W)$$

where $\langle\cdot|\cdot\rangle,\ (\cdot|\cdot) : W \times W \to \mathbb{R}$ *are* 2*-linear homomorphisms such that*

(1) $(\cdot|\cdot)$ *is symmetric and positive definite (i.e. it is an inner product),*

(2) $*_W : W \longrightarrow W$ *(called* $*$*-Hodge operator) is a linear mapping such that,*

(2a) $*_W$ *is an isometry with respect to* $(\cdot|\cdot)$,

(2b) $\langle v|w\rangle = (v|*_W(w))$ *for all* $v \in W$.

Clearly, the pairing $\langle\cdot|\cdot\rangle$ is weakly nondegenerated.
Remark: The unitary space $(W, (\cdot|\cdot))$ is not a Hilbert space in general. If it is Hilbert then we call it a Hilbert Hodge space. The $*$-Hodge operator $*_W$ fulfilling (2b) is uniquely determined (if exists) for given tensors $\langle\cdot|\cdot\rangle,\ (\cdot|\cdot)$.

Definition 1.1.2 *By the* tensor product of Hodge spaces $(V, \langle\cdot|\cdot\rangle_V, (\cdot|\cdot)_V, *_V)$ *and* $(W, \langle\cdot|\cdot\rangle_W, (\cdot|\cdot)_W, *_W)$ *we mean the following Hodge space*

$$(V \otimes W, \langle\cdot|\cdot\rangle_V \otimes \langle\cdot|\cdot\rangle_W, (\cdot|\cdot)_V \otimes (\cdot|\cdot)_W, *_V \otimes *_W)$$

(i.e. $*_{V\otimes W} = *_V \otimes *_W$ *).*

The tensor $(\cdot|\cdot)_V \otimes (\cdot|\cdot)_W$ is symmetric and positive definite (the dimensions of V and W can be infinite) according to [Gre].

Let $(W, \langle\cdot|\cdot\rangle)$ be a finite dimensional real vector space equipped with a 2-linear tensor $\langle\cdot|\cdot\rangle : W \times W \to \mathbb{R}$. Then there exist an inner product $(\cdot|\cdot)$ and an operator $*_W$ such that the system $(W, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *_W)$ is a Hodge space if and only if there exists a basis of W in which the matrix of $\langle\cdot|\cdot\rangle$ is orthogonal.

The inner product and the $*$-Hodge operator play an auxiliary role in the study of properties of the pairing $\langle\cdot|\cdot\rangle$.

Now we give some examples of finite dimensional Hodge spaces.

Example 1.1.3 (Classical) *Let* (V, G) *be a real* N*-dimensional oriented vector space with an inner product* $G : V \times V \to \mathbb{R}$. *We identify* $\bigwedge^N V = \mathbb{R}$ *via arbitrary positive ON-base* $\{e_i\}_{i=1}^N$ *of* V. *We have the classical Hodge space*

$$\left(\bigwedge V = \bigoplus_{r=0}^{N} \bigwedge^r V, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *\right)$$

where for $v^r \in \bigwedge^r V$

$$\langle v^r | v^s \rangle = \begin{cases} v^r \wedge v^s \in \bigwedge^N V = \mathbb{R}, & \textit{if} \;\; r+s = N, \\ 0 & \textit{if} \;\; r+s \neq N, \end{cases} ,$$

whereas for $v_i, w_i \in V$

$$(v_1 \wedge .... \wedge v_r | w_1 \wedge ... \wedge w_r) = \det \left[G\left(v_i, w_k\right) \right].$$

The subspaces $\bigwedge^r V$ *are orthogonal and*

$$* \left(e_{i_1} \wedge ... \wedge e_{i_r}\right) = \varepsilon_{(j_1,...,j_{N-r})} \cdot e_{j_1} \wedge ... \wedge e_{j_{N-r}},$$

where the sequence $j_1 < ... < j_{N-r}$ *is complementary to* $i_1 < ... < i_r$ *and* $\varepsilon_{(j_1,...,j_{N-r})} = \operatorname{sgn}\left(j_1, ..., j_{N-r}, i_1, ..., i_r\right)$ *(it is a small modification with respect to the classical theory on Riemann manifold).*

Example 1.1.4 (Lusztig example, 1972) *[L] Let* $\langle \cdot | \cdot \rangle : E \times E \to \mathbb{R}$ *be a symmetric (indefinite in general) nondegenerated scalar product on a finite dimensional vector space* E. *Let* G *be an arbitrary inner product in* E *(i.e. symmetric positive). Then there exists exactly one direct sum decomposition* $E = E_+ \oplus E_-$ *which is ON with respect to the both scalar products* $\langle \cdot | \cdot \rangle$ *and* G *and such that* $\langle \cdot | \cdot \rangle$ *is positive definite on* E_+ *and negative definite on* E_-. *We denote by* $*_E$ *the involution* $*_E : E \to E$ *such that*

$$*_E | E_+ = id, \qquad *_E | E_- = -id.$$

Then the quadratic form

$$(\cdot|\cdot) : E \times E \longrightarrow \mathbb{R}, \qquad (v,w) \longmapsto (v|w) := \langle v | *_E w \rangle$$

is symmetric and positive definite. The involution $*_E$ *is an isometry, therefore the system*

$$\left(E, \langle \cdot | \cdot \rangle, (\cdot|\cdot,), *_E\right)$$

is a finite dimensional Hodge-space.

Example 1.1.5 (Gromov example, 1995) *[Gro] Let* $\langle \cdot | \cdot \rangle : E \times E \to \mathbb{R}$, *be a symplectic form on a finite dimensional vector space* E *[i.e. skew-symmetric and nondegenerated]. There exists an anti-involution* τ *in* E, $\tau^2 = -id$ *(i.e. a complex structure) such that*

$$\langle \tau v | \tau w \rangle = \langle v | w \rangle, \quad v, w \in E,$$

$$\langle v | \tau v \rangle > 0 \quad \textit{for all} \quad v \neq 0.$$

Then the tensor

$$(\cdot|\cdot) : E \times E \longrightarrow \mathbb{R} \qquad (v,w) \longmapsto (v|w) := \langle v | \tau w \rangle$$

is symmetric and positive definite and $(\tau v | \tau w) = (v|w)$. *The system*

$$\left(E, \langle \cdot | \cdot \rangle, (\cdot|\cdot), -\tau\right)$$

is a finite dimensional Hodge-space since $-\tau$ *is an isometry and* $\langle v|w \rangle = (v| - \tau w)$.

Example 1.1.6 (Trivial Hodge space) *Let $(E, (\cdot|\cdot))$ be a real p-dimensional Euclidean space (a vector space with an inner product $(\cdot|\cdot)$). Then the system $(E, \langle\cdot|\cdot\rangle = (\cdot|\cdot), (\cdot|\cdot), * = Id)$ is called a trivial Hodge space.*

Infinite dimensional Hodge structures can be given on the space of cross-sections of a vector bundle.

Definition 1.1.7 *[K-M-4] By the* Hodge vector bundle *(or a* vector bundle of finitely dimensional Hodge spaces*) we mean a system*

$$(\xi, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *_\xi)$$

consisting of a Riemann vector bundle $(\xi, (\cdot|\cdot))$ and smooth tensor field $\langle\cdot|\cdot\rangle \in \mathrm{Sec}\left(\otimes^2\xi\right)^*$ and linear isomorphism $*_\xi : \xi \to \xi$ such that for each $x \in M$ the system

$$\left(\xi_x, \langle\cdot|\cdot\rangle_x, (\cdot|\cdot)_x, *_{\xi_x}\right)$$

is a finitely dimensional Hodge space. If $*_\xi = Id$, i.e. $\langle\cdot|\cdot\rangle = (\cdot|\cdot)$ then ξ is called trivial Hodge vector bundle.

Remark 1.1.8 In all considered four examples below (see Subsection 1.4.1 below) the Hodge vector bundle is a Σ-bundle (in the sense of [G-H-V, Ch. VIII]). We recall that it is equivalent to the existence of a covariant derivative in the vector bundle ξ such that all three tensors $\langle\cdot|\cdot\rangle, (\cdot|\cdot), *_\xi$ are parallel. Therefore with a Σ-Hodge vector bundle we can associate a suitable structure Lie group, a principal fibre bundle (also a Lie groupoid and a Lie algebroid), and the characteristic Chern-Weil homomorphism [G-H-V, Ch. VIII].

Lemma 1.1.9 *If $(\xi, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *_\xi)$ is a Hodge vector bundle over a compact oriented Riemannian manifold M, then the system*

$$(\mathrm{Sec}\,\xi, \langle\!\langle\cdot|\cdot\rangle\!\rangle, (\!(\cdot|\cdot)\!), *)$$

where $\langle\!\langle\cdot|\cdot\rangle\!\rangle, (\!(\cdot|\cdot)\!) : \mathrm{Sec}\,\xi \times \mathrm{Sec}\,\xi \to \mathbb{R}$ are pairings defined via the integral operator

$$\langle\!\langle\alpha|\beta\rangle\!\rangle = \int_M \langle\alpha_x|\beta_x\rangle\, dM, \qquad (\!(\alpha|\beta)\!) = \int_M (\alpha_x|\beta_x)\, dM,$$

is a Hodge space (infinite dimensional if $\dim M > 0$) induced by the Hodge vector bundle ξ. If M is not compact, then the system as above for the subspace $\mathrm{Sec}_c\,\xi$ of cross-sections with compact supports forms a Hodge space as well. For arbitrary M (compact or not) passing to L_2-theory we obtain a Hodge space on L_2-cross-sections $L_2(\xi)$.

Example 1.1.10 *The following are examples of Hodge vector bundles:*

- *$\xi = \bigwedge T^*M$ for a Riemannian manifold M; we use the classical Hodge space $\bigwedge T_x^*M$ for the cotangent space T_x^*M.*

- *Lusztig example of a vector bundle ξ with nondegenerated indefinite symmetric 2-linear tensor field $\langle\cdot|\cdot\rangle$ [and equipped with a flat covariant derivative ∇ such that the tensor $\langle\cdot|\cdot\rangle$ is parallel],*
- *Gromov example of a vector bundle ξ with a symplectic form $\langle\cdot|\cdot\rangle$ [and equipped with a flat covariant derivative such that $\langle\cdot|\cdot\rangle$ is parallel],*
- *Tensor product $\bigwedge T^*M\otimes\xi$ of a Riemann manifold M with an arbitrary Hodge vector bundle ξ. Particularly, this concerns a trivial Hodge vector bundle ξ. (Remark: with this structure we really have to deal - in the category of Lipschitz manifolds - in the second and third part of the paper by Teleman [T4]).*
- *$\xi=\bigwedge A^*$ for an invariantly oriented transitive Lie algebroid A, see [K3].*

The Lusztig and Gromov examples of Hodge vector spaces are very important for the calculation of the signature of transitive Lie algebroids, thanks to some spectral sequence argument [K-M-4], see Subsection (1.4.1).

1.2 Graded differential Hodge spaces

To define a signature we introduce a gradation and a derivative to the definition of Hodge space. We are interested only in the case when the top degree is even (because then the middle degree can be considered).

Definition 1.2.1 *[K-M-4] By a* graded anticommutative differential Hodge space of even degree $N=2n$ *we mean a system*

$$\left(W=\bigoplus\nolimits_{r=0}^{N=2n}W^r,\left\langle\cdot|\cdot\right\rangle,\left(\cdot|\cdot\right),*,d\right)$$

where $(W,\left\langle\cdot|\cdot\right\rangle,\left(\cdot|\cdot\right),*)$ is a Hodge space (finitely or infinitely dimensional) and

(1) $\left\langle\cdot|\cdot\right\rangle|W^k\times W^r=0$ if $k+r\neq N$,

(2) W^r are orthogonal with respect to $\left(\cdot|\cdot\right)$,

(3) the operator d, called derivative, is homogeneous of degree $+1$, i.e. $d:W^r\to W^{r+1}$, and $d^2=0$,

(4) $\left\langle dw^r|u^{N-r-1}\right\rangle=(-1)^{r+1}\left\langle w^r|du^{N-r-1}\right\rangle$ for $w^r\in W^r$, $u^{N-r-1}\in W^{N-r-1}$,

(5) the tensor $\left\langle\cdot|\cdot\right\rangle$ is anticommutative in the sense

$$\left\langle v^r|v^{N-r}\right\rangle=\varepsilon_r\left\langle v^{N-r}|v^r\right\rangle,$$

where

$$\varepsilon_r:=(-1)^n(-1)^{r(N-r)}=(-1)^n(-1)^r.$$

Denote $\left\langle\cdot|\cdot\right\rangle^r:=\left\langle\cdot|\cdot\right\rangle|W^r\times W^{N-r}$. Clearly,

a) $\varepsilon_n = +1$ so the tensor $\langle\cdot|\cdot\rangle^n$ on W^n is symmetric,

b) $*[W^r] \subset W^{N-r}$, and $\quad * : W^r \to W^{N-r} \quad$ is an isomorphism,

c) the induced cohomology pairing

$$\langle\cdot|\cdot\rangle^r_{\mathbf{H}} : \mathbf{H}^r(W)\times\mathbf{H}^{N-r}(W) \longrightarrow \mathbb{R}, \quad ([u],[v]) \longmapsto \langle [u] \mid [w]\rangle^r_{\mathbf{H}} := \langle u|w\rangle^r,$$

is correctly defined,

d) the condition (5) above enables us to consider the cohomology pairing in the middle degree

$$\langle\cdot|\cdot\rangle^n_{\mathbf{H}} : \mathbf{H}^n(W) \times \mathbf{H}^n(W) \longrightarrow \mathbb{R}$$

which is symmetric, therefore if $\dim \mathbf{H}^n(W) < \infty$ then we can consider the signature of the quadratic form $\langle\cdot|\cdot\rangle^n_{\mathbf{H}}$.

Definition 1.2.2 *If* $\dim \mathbf{H}^n(W) < \infty$ *we define the* signature *of* W *as the signature of* $\langle\cdot|\cdot\rangle^n_{\mathbf{H}}$

$$\mathrm{Sig}(W) := \mathrm{Sig}\,\langle\cdot|\cdot\rangle^n_{\mathbf{H}}.$$

The above condition (4) in the classical geometrical examples on an N-dimensional compact manifold M without boundary follows from the Stokes theorem.

Remark 1.2.3 The finiteness of the dimension of the cohomology space $\mathbf{H}^r(W)$ (in all dimensions) can be obtained after assuming that the Poincaré duality holds. For standard cohomology algebra of differential forms on compact smooth oriented manifold M the Poincaré duality is easily to obtain by a simple argument in algebraic topology: via the so-called Mayer-Vietoris sequences. For the cohomology algebra of elliptic complexes on M the Poincaré duality is a result in the so-called abstract Hodge theory of elliptic complexes [N] (which use L_2-analysis).

Proposition 1.2.4 *[K-M-4] Let* $\left(W = \bigoplus_{r=0}^N W^r, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *, d\right)$ *be a graded anticommutative differential Hodge space of even degree. Then*

1) $**(w^r) = \varepsilon_r \cdot w^r$,

2) the linear operator $\delta : W^r \to W^{r-1}$ *[called coderivative] defined by*

$$\delta^r(w^r) = \varepsilon_r (-1)^r * d * (w^r), \quad w^r \in W^r,$$

is the adjoint operator, i.e.

$$(\delta(w_1)|w_2) = (w_1|d(w_2)),$$

3) the Laplacian $\Delta := (d+\delta)^2 = d\delta + \delta d$ *is homogeneous of degree* 0, *self-adjoint* $(\Delta v|w) = (v|\Delta w)$, *and nonnegative* $(\Delta v|v) \geq 0$.

Definition 1.2.5 *A vector* $v \in W$ *is called* harmonic *if*

$$dv = 0 \quad \textit{and} \quad \delta v = 0.$$

Denote by $\mathcal{H}(W)$ the space of harmonic vectors and $\mathcal{H}^r(W) = \mathcal{H}(W) \cap W^r$ the space of harmonic vectors of degree r.

The following are easy to verify:

- the harmonic vectors form a graded vector space $\mathcal{H}(W) = \bigoplus_{r=0}^{N} \mathcal{H}^r(W)$,
- $\mathcal{H}^r(W) = \ker \Delta^r$ and

$$\mathcal{H}(W) = \ker \Delta = (\operatorname{Im} \Delta)^{\perp},$$

- the spaces $\ker \Delta^r$ and $\operatorname{Im} d^{r-1}$ are orthogonal, therefore the inclusion

$$\mathcal{H}^r(W) = \ker \Delta^r \hookrightarrow \ker d^r$$

induces a monomorphism (called the *Hodge homomorphism*)

$$x^r : \mathcal{H}^r(W) = \ker \Delta^r \rightarrowtail \mathbf{H}^r(W) := \ker d^r / \operatorname{Im} d^{r-1}.$$

The following is the fundamental problem.

Problem 1.2.6 *When the Hodge homomorphism* x^r *is an isomorphism? I.e. when in each cohomology class there is (exactly one) a harmonic vector?*

Theorem 1.2.7 *[K-M-4] If*

$$W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp} \tag{1.2.1}$$

[it means if $W = \operatorname{Im} \Delta \bigoplus \ker \Delta = \operatorname{Im} \Delta \bigoplus \mathcal{H}(W)$*] then*

- $W^k = \ker \Delta^r \bigoplus \operatorname{Im} d^{r-1} \bigoplus \operatorname{Im} \delta^{r+1}$ *(strong Hodge decomposition),*
- $\ker d^r = \ker \Delta^r \bigoplus \operatorname{Im} d^{r-1}$, *in particular, the Hodge homomorphism is an isomorphism*

$$x^r : \mathcal{H}^r(W) = \ker \Delta^r \xrightarrow{\cong} \ker d^r / \operatorname{Im} d^{r-1} = \mathbf{H}^r(W),$$

(particularly, in each cohomology class there is exactly one harmonic vector).

- *(Poincaré Duality Theorem) The cohomology pairing* $\langle \cdot | \cdot \rangle_{\mathbf{H}}^r : \mathbf{H}^r(W) \times \mathbf{H}^{N-r}(W) \to \mathbb{R}$ *for* $k = 0, 1, ..., N$, *is a duality,* $\mathbf{H}(W) \cong \mathbf{H}(W)^*$, *(whence* $\dim \mathbf{H}(W) < \infty$ *and* $\mathbf{H}^r(W) \simeq \mathbf{H}^{N-r}(W)$*).*

Remark 1.2.8 In important examples on manifolds (standard, Lie algebroid, Lusztig's and Gromov's examples, see Subsection 1.4.1 below) the condition $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$ holds thanks to the fact that Laplacian Δ is an elliptic operator.

1.3 Signature and the Hirzebruch operator

Consider a graded anticommutative differential Hodge space of even degree

$$\left(W = \bigoplus_{r=0}^{N=2n} W^k, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *, d\right).$$

We restrict the positive definite product $(\cdot|\cdot)^r : W^r \times W^r \to \mathbb{R}$ and the tensor $\langle\cdot|\cdot\rangle : W^r \times W^{N-r} \to \mathbb{R}$ to the space of harmonic vectors

$$(\cdot|\cdot)^r_{\mathcal{H}} : \mathcal{H}^r(W) \times \mathcal{H}^r(W) \longrightarrow \mathbb{R},$$

$$\mathcal{B}^r = \langle\cdot|\cdot\rangle^r_{\mathcal{H}} : \mathcal{H}^r(W) \times \mathcal{H}^{N-r}(W) \longrightarrow \mathbb{R}.$$

To define the Hirzebruch operator we can use a new operator τ being a small $\pm$ sign-modification of the $*$-Hodge operator.

Theorem 1.3.1 *There exists exactly one operator $\tau : W \to W$ such that*

i) $\tau(u^r) = \tilde{\varepsilon}_r \cdot *_W(u^r),\ u^r \in W^r,$ *for some* $\tilde{\varepsilon}_r \in \{-1, +1\}$,

ii) $\tau \circ \tau = Id$,

iii) $\delta = -\tau \circ d \circ \tau$,

iv) $\tau^n = *|W^n$, *i.e.* $\tilde{\varepsilon}_n = 1$.

If τ fulfils i)-iv) then

$$\tilde{\varepsilon}_r \in (-1)^{\frac{r(r+1)}{2}} (-1)^{\frac{n(n+1)}{2}}.$$

In particular,

- *if* $N = 4p$ *then* $\varepsilon_r = (-1)^r$ *and* $\tilde{\varepsilon}_r = (-1)^{\frac{r(r+1)}{2}} (-1)^p$,
- *if* $N = 4p+2$ *then* $\varepsilon_r = -(-1)^r$ *and* $\tilde{\varepsilon}_r = -(-1)^{\frac{r(r+1)}{2}} (-1)^p$.

Remark 1.3.2 The operator τ can be considered also for odd N, but then we should allow complex numbers $\tilde{\varepsilon}_r \in \{\pm 1, \pm i\}$ and consider the complexification $W \otimes \mathbb{C}$. More precisely (see [K-M-4]) for a quite arbitrary sequence of the coefficients of anticommutativity $\langle v^r | v^{N-r}\rangle = \varepsilon_r \langle v^{N-r} | v^r\rangle$, $\varepsilon_r = \pm 1$, the complex valued operator τ fulfilling i) for $(\tilde{\varepsilon}_r)^2 \in \{+1, -1\}$, ii) and iii) exists [assuming nontriviality condition $d^r \neq 0$ for all r] if and only if the sequence ε_r is equal to $\varepsilon_r = \varepsilon_0 (-1)^{r(N-r)}$ where $\varepsilon_0 = \pm 1$. For each of these sequences ε_r there is exactly two solution $\tilde{\varepsilon}_r \in \{\pm 1, \pm i\}$, namely given by the formula $\tilde{\varepsilon}_r = (-1)^{r \cdot \frac{2N-r-1}{2}} \cdot \tilde{\varepsilon}_0$ where $(\tilde{\varepsilon}_0)^2 = \varepsilon_0 (-1)^{\frac{N(N+1)}{2}}$. The case of even N, $N = 2n$ and conditions $\varepsilon_n = 1 = \tilde{\varepsilon}_n = 1$ yields the unique solution $\varepsilon_r = (-1)^n (-1)^r$ and unique (real) solution $\tilde{\varepsilon}_r \in (-1)^{\frac{r(r+1)}{2}} (-1)^{\frac{n(n+1)}{2}}$ described above.

Consider the symmetric and nondegenerated quadratic forms in the middle degree $\langle\cdot|\cdot\rangle^n$, $\langle\cdot|\cdot\rangle^n_{\mathcal{H}}$ and $\langle\cdot|\cdot\rangle^n_{\mathbf{H}}$.

Remark 1.3.3 Under the assumption (1.2.1), $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$, we have $\mathbf{H}^n(W) \overset{x^n}{\cong} \mathbf{H}^n(W)$, therefore $B^n = \langle\cdot|\cdot\rangle^n_{\mathcal{H}} = \langle\cdot|\cdot\rangle^n_{\mathbf{H}}$. Then we have

$$\operatorname{Sig}(W) = \operatorname{Sig}\langle\cdot|\cdot\rangle^n_{\mathbf{H}} = \operatorname{Sig}\mathcal{B}^n.$$

We put

$$W_{\pm} = \{w \in W;\ \tau w = \pm w\},$$

the eigenspaces corresponding to the eigenvalues $+1$ and -1 of τ, respectively. Denote the operator

$$D = d + \delta$$

and notice that $D[W_+] \subset W_-$.

Definition 1.3.4 *The operator*

$$D_+ = D|W_+ : W_+ \longrightarrow W_-$$

is called the Hirzebruch operator *(or the* signature operator*).*

Take the adjoint to D_+,

$$D_- = D_+^* = D|W_- : W_- \longrightarrow W_+.$$

Clearly, the spaces $\ker(D_+)$ and $\ker(D_-)$ are contained in $\ker D = \mathcal{H}(W)$.

Remark 1.3.5 For arbitrary graded anticommutative differential Hodge space of even degree if $\dim \mathbf{H}(W) < \infty$ then the index

$$\operatorname{Ind} D_+ := \dim_{\mathbb{R}} \ker(D_+) - \dim_{\mathbb{R}} \ker(D_-)$$

is correctly defined (the dimensions are finite).

Simple algebraic arguments give the fundamental theorem on the index of Hirzebruch operator [K-M-4].

Theorem 1.3.6 (Hirzebruch Signature Theorem) *If* $\dim \mathcal{H}(W) < \infty$*, then*

$$\operatorname{Ind} D_+ = \operatorname{Sig}\mathcal{B}^n.$$

If, additionally the condition (1.2.1) holds, then

$$\operatorname{Ind} D_+ = \operatorname{Sig}\mathcal{B}^n = \operatorname{Sig}\langle\cdot|\cdot\rangle^n_{\mathbf{H}} = \operatorname{Sig} W.$$

(Sketch of the proof) Put $\mathcal{H}^n_{\pm}(W) = \{\alpha \in \mathcal{H}^n(W)\,;\ *\alpha = \pm\alpha\}$. Then $\mathcal{H}^n(W) = \mathcal{H}^n_{+}(W) \bigoplus \mathcal{H}^n_{-}(W)$. The subspaces $\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)$ are τ-stable for $s = 0, 1, ..., n-1$ and $\varphi_{\pm} : \mathcal{H}^s(W) \to \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_{\pm}$, $X \longmapsto \frac{1}{2}(X \pm \tau X)$, is an isomorphism of real spaces. The subspaces $W^s + W^{2n-s}$ are also τ-invariant, therefore $W_{\pm} = \bigoplus_{s=0}^{n-1}\left(W^s + W^{2n-s}\right)_{\pm} \bigoplus W^n_{\pm}$ which yields

$$\ker D_{\pm} = W_{\pm} \cap \mathcal{H}(W) = \bigoplus_{s=0}^{n-1}\left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(M)\right)_{\pm} \bigoplus \mathcal{H}^n_{\pm}(W)$$

and in consequence

$$\begin{aligned}
&\dim\ker D_{+} - \dim\ker D_{+}^{*} \\
&= \sum_{s=0}^{n-1} \dim_{\mathbb{R}}\left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_{+} + \dim_{\mathbb{R}}\mathcal{H}^n_{+}(W) \\
&- \sum_{s=0}^{n-1} \dim_{\mathbb{R}}\left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_{-} - \dim_{\mathbb{R}}\mathcal{H}^n_{-}(W) \\
&= \dim_{\mathbb{R}}\mathcal{H}^n_{+}(W) - \dim_{\mathbb{R}}\mathcal{H}^n_{-}(W) \\
&= \operatorname{Sig}(\mathcal{B}^n).
\end{aligned}$$

1.4 Four fundamental examples and their general setting [K-M-4]

1.4.1 Four examples

The above general algebraic approach to the Hirzebruch signature operator can be used to the four fundamental examples.

$$W^r = \begin{cases} [\textbf{manifold, classical example}], & \\ \left(\Omega^r(M),\ d_{dR}\right);\quad \dim M = N = 4p, & \\ \text{here } \varepsilon_r = (-1)^{r(N-r)} = (-1)^r & \\ & \\ [\textbf{Lie algebroid example}] & \\ \left(\Omega^r(A),\ d_A\right); \qquad A \text{ - TUIO-Lie algebroid} & \\ \qquad\qquad\qquad\qquad \operatorname{rank} A = N = m+n = 4p, & \\ \text{here } \varepsilon_r = (-1)^r & \\ & \\ [\textbf{Lusztig example}] & \\ \qquad\qquad\qquad\qquad (E, \langle\cdot|\cdot\rangle) \text{ - flat vector bundle,} & \\ \left(\Omega^r(M;E),\ d_\nabla\right); \quad \langle\cdot|\cdot\rangle \text{ - symmetric nondegenerated parallel tensor,} & \\ \qquad\qquad\qquad\qquad \dim M = N = 4p, & \\ \text{here } \varepsilon_r = (-1)^r & \\ & \\ [\textbf{Gromov example}] & \\ \qquad\qquad\qquad\qquad (E, \langle\cdot|\cdot\rangle) \text{ - flat vector bundle,} & \\ \left(\Omega^r(M;E),\ d_\nabla\right); \quad \langle\cdot|\cdot\rangle \text{ - symplectic parallel tensor,} & \\ \qquad\qquad\qquad\qquad \dim M = N = 4p+2, & \\ \text{here } \varepsilon_r = -(-1)^r, & \end{cases}$$

The Lusztig anf Gromov examples are important for the Lie algebroid case thanks the following spectral sequence argument:

Theorem 1.4.1 *[K-M-2] Let $(B, B^r, \cup, D, B_j)$ be any DG-algebra with a decreasing filtration B_j and $\left(E_s^{j,i}, d_s\right)$ its spectral sequence. Assume the following regularity $B_0 = 0$ of the filtration B_j and that there exist natural numbers m and n with the following conditions:*

- *$E_2^{j,i} = 0$ for $j > m$ and $i > n$,*
- *E_2 is a Poincaré algebra with respect to the total gradation and the top group $E_2^{(m+n)} = E_2^{m,n}$.*

Then $\mathbf{H}(B) = \bigoplus_{r=0}^{m+n} \mathbf{H}^r(B)$ is a Poincaré algebra, $\dim \mathbf{H}^{m+n}(B) = 1$, and $\operatorname{Sig} E_2 = \operatorname{Sig} \mathbf{H}(B)$.

If m and n are odd, then $\operatorname{Sig} E_2 = 0$. If m and n are even, then

$$\operatorname{Sig} E_2 = \operatorname{Sig}(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \longrightarrow E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}).$$

Using Hochschild-Serre spectral sequence of the Lie algebroid A we have $E_2^{j,i} = \mathbf{H}^j_{\nabla^q}(M; \mathbf{H}^i((\boldsymbol{g}))$ [K-M-3], where ∇^i is an suitable flat structure in the vector bundle of i-group of cohomology of isotropy Lie algebras $\mathbf{H}^i(\boldsymbol{g})$. The multiplication of values $\mathbf{H}^i(\boldsymbol{g}) \times \mathbf{H}^{i'}(\boldsymbol{g}) \to \mathbf{H}^{i+i'}(\boldsymbol{g})$ is taken with respect to multiplication of cohomology classes of isotropy Lie algebras, in particular for the middle degree $\frac{n}{2}$ we have

$$\langle\cdot|\cdot\rangle : \mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\times\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\longrightarrow\mathbf{H}^{n}\left(\boldsymbol{g}\right)=M\times\mathbb{R},$$

and we need to consider two different cases:

- if $\frac{m}{2}$ and $\frac{n}{2}$ even, then the above form is symmetric and we can use Lusztig type example,
- if $\frac{m}{2}$ and $\frac{n}{2}$ are odd, then the above is symplectic and we can use Gromov type example.

In both cases we obtain the Hirzebruch signature operator

$$D_{+}=d_{\nabla^{\frac{n}{2}}}+\delta_{\nabla^{\frac{n}{2}}}:\Omega_{+}(M;\mathbf{H}^{\frac{n}{2}}\left(\mathbf{g}\right))\longrightarrow\Omega_{-}(M;\mathbf{H}^{\frac{n}{2}}\left(\mathbf{g}\right)).$$

1.4.2 General setting, graded anticommutative differential Hodge vector bundles

The above four examples can be considered simultaneosly from the general point of view of graded vector bundle of Hodge spaces over a connected compact oriented Riemannian manifold M [K-M-4].

Definition 1.4.2 *[K-M-4] Let M be a connected compact oriented Riemannian manifold. By a* graded anticommutative differential Hodge vector bundle of even degree $N=2n$ *over the manifold M we mean a system*

$$\left(\xi=\bigoplus\nolimits_{r=0}^{N=2n}\xi^{r},\left\langle\cdot|\cdot\right\rangle,\left(\cdot|\cdot\right),*,d\right)$$

such that

1) $(\xi,\left\langle\cdot|\cdot\right\rangle,\left(\cdot|\cdot\right),*)$ is a Hodge vector bundle and the induced infinitely dimensional Hodge space $\left(\operatorname{Sec}\xi=\bigoplus\nolimits_{r=0}^{N}\operatorname{Sec}\xi^{r},\langle\!\langle\cdot|\cdot\rangle\!\rangle,(\!(\cdot|\cdot)\!),*,d\right)$ (see Lemma 1.1.9) is a graded anticommutative differential Hodge space of even degree,

2) the complex $d=\{d^{r}\}$, $d^{r}:\operatorname{Sec}\xi^{r}\to\operatorname{Sec}\xi^{r+1}$, is a complex of differential operators of first order.

The operator $\delta:\operatorname{Sec}\xi\longrightarrow\operatorname{Sec}\xi$ defined by $\delta\left(\alpha^{r}\right)=\varepsilon_{r}\left(-1\right)^{r}*d*\left(\alpha^{r}\right)$ is the adjoint operator to d, $(\!(\alpha|\delta\beta)\!)=(\!(d\alpha|\beta)\!)$.

If $\left\{d^{k}\right\}$ is an elliptic complex, then the Laplacian $\Delta=\left(d+\delta\right)^{2}$ is a selfadjoint, nonnegative and elliptic operator. In consequence the condition (1.2.1) holds, i.e. $\operatorname{Sec}\xi=\operatorname{Im}\Delta\oplus\left(\operatorname{Im}\Delta\right)^{\perp}$, whence (Th. 1.2.7) we have $\mathcal{H}\left(\operatorname{Sec}\xi\right)\cong\mathbf{H}\left(\operatorname{Sec}\xi,d\right)$ and $\dim\mathcal{H}\left(\operatorname{Sec}\xi\right)<\infty$. Therefore, we get the Hirzubruch operator $D_{+}=\left(d+\delta\right)_{+}:\operatorname{Sec}\xi_{+}\to\operatorname{Sec}\xi_{-}$ and the equality (Th. 1.3.6)

$$\operatorname{Sig}\left(\operatorname{Sec}\xi\right)=\operatorname{Sig}\langle\!\langle\cdot|\cdot\rangle\!\rangle_{\mathbf{H}}^{n}=\operatorname{Ind}D_{+}.$$

The ellipticity of Δ follows from [W, Remark 6.34] since the symbol $\sigma(\delta)_{(x,v)}$ of the adjoint operator of a first order operator d equals $-\sigma(d)^*_{(x,v)}$.

The above four examples (Subsection 1.4.1) can be obtained from the above general setting by putting $\xi = \bigwedge T^*M$, $\bigwedge A^*$ or $\bigwedge T^*M \bigotimes E$, respectively ($E$ is Lusztig or Gromov case). In the above four examples the derivative d is a differential operator of first order with constant coefficients (with respect to some suitable local trivializations).

2 Graded Hilbert subdifferential Hodge spaces

Teleman [T2] noticed that the consideration of the Lipschitz structure on a compact manifold and L_2-differential forms leads to the Hodge theory in an easy algebraic manner (we need only some convolution argument and L_2-Poincaré duality). The cause is that all suitable unitary spaces are then Hilbert. There is only one important difference. On Lipschitz manifolds the derivative of differential L_2-forms is not defined on all space of L_2-differential forms. This is the source of the new algebraic notion of the graded Hilbert subdifferential Hodge spaces (Subsection 2.2).

Firstly, we briefly describe the suitable definitions and facts concerning the Lipschitz manifolds given by Teleman [T2].

2.1 Lipschitz manifolds and distributional exterior derivative (subderivative)

Definition 2.1.1 (Teleman 1983) *[T2] A* Lipschitz structure *on a topological manifold M of dimension n is a maximal atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$, where $\phi_\alpha : M \supset U_\alpha \to V_\alpha \subset \mathbb{R}^n$ (U_α, V_α - open subsets) are homeomorphisms such that the changes coordinates (i.e. transition functions)*

$$\Lambda_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}, \quad \alpha, \beta \in \Lambda$$

are Lipschitz mappings.

Of course, a C^∞-manifold possesses a canonical Lipschitz structure.

Theorem 2.1.2 *(D.Sullivan, 1977 [S]) Any topological manifold of dimension $\neq 4$ admits a Lipschitz structure, and that structure is essentially unique.*

Sullivan's theorem makes then possible to construct signature operators not only on a compact Lipschitz manifold, but on an arbitrary compact, topological manifold of even dimension $\neq 4$ [T3].

The crucial role is played by Rademacher's theorem:

Theorem 2.1.3 *(Rademacher's theorem [Hei]) If $U \to \mathbb{R}$ is a Lipschitz function on an open subset $U \subset \mathbb{R}^n$, then*

- *the partial derivatives* $\frac{\partial f}{\partial x^i}$ *exist almost everywhere,*
- $\frac{\partial f}{\partial x^i}$ *are measurable and bounded.*

Definition 2.1.4 *We say that a Lipschitz manifold with the atlas* $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ *is* orientable *if there exists a subatlas* $\Lambda' \subset \Lambda$ *for which the homeomorphisms* $\Lambda_{\alpha\beta}$ *have positive jacobian (in all points of differentiability). If such an atlas is given we call* M *oriented.*

Since the partial derivatives of a Lipschitz function exists in general not everywhere (only almost everywhere), therefore we can not define traditionally a tangent space at a point to a Lipschitz manifold M and we must use other algebraic ways to define a differential form on M.

Let $L_2^k(U)$ denote the space of k-differential forms of class L_2 on an open subset $U \subset \mathbb{R}^n$ (i.e. the space of differential forms on U with measurable coefficients of square integrable). For the standard Riemannian metric on U the space $L_2^k(U)$ with integral norm (i.e. L_2-norm $\|\omega\| = \sqrt{((\omega|\omega))}$, where $((\omega|\omega)) = \int_U (\omega|\omega))$, is Hilbert.

Definition 2.1.5 *Let* M *be a compact Lipschitz manifold with the atlas* $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$. *By* L_2-differential form *on* M *we mean a system*

$$\omega = \{\omega_\alpha\}_{\alpha\in\Lambda}$$

where ω_α *is a [real]* L_2*-differential form on the open subset* $V_\alpha = \phi_\alpha[U_\alpha] \subset \mathbb{R}^n$, $\alpha \in \Lambda$, *such that the following condition of compatibility holds*

$$\Lambda_{\alpha\beta}^* \omega_\beta = \omega_\alpha.$$

Denote by $L_2(M)$ the vector space of L_2-differential forms on M (modulo equality almost everywhere). A 0-differential form determines a measurable function on M.

For an oriented Lipschitz manifold, using a Lipschitz partition of unity, we define the integral $\int_M \omega$ for $\omega \in L_2^n(M)$ ($n = \dim M$) in a standard way.

Definition 2.1.6 *A* Lipschitz Riemannian metric *on* M *is a collection*

$$\Gamma = \{\Gamma_\alpha\}_{\alpha\in\Lambda}$$

where Γ_α *is a Riemannian metric on* $V_\alpha = \phi_\alpha[U_\alpha] \subset \mathbb{R}^n$ *with measurable components, which satisfies*

- *compatibility condition*
$$(\Lambda_{\alpha\beta})^* \Gamma_\beta = \Gamma_\alpha,$$
- L_2*-norms on* V_α *determined by* Γ_α *and by the standard metric are equivalent.*

Theorem 2.1.7 (Teleman, 1983) *[T2] Any compact Lipschitz manifold M has a Lipschitz Riemannian metric.*

Clearly, any Lipschitz Riemannian metric determines a measure on M.

Let $*_{\alpha,x}$ be a Hodge star isomorphism in $\bigwedge(\mathbb{R}^n)^*$ defined by the metric Γ_α at $x \in \mathbb{R}^n$

$$*_{\alpha,x}\left(e_{i_1} \wedge ... \wedge e_{i_r}\right) = \varepsilon_{(j_1,...,j_{n-r})} \cdot e_{j_1} \wedge ... \wedge e_{j_{n-r}},$$

$\{e_i\}_{i=1}^n$ is $\Gamma_\alpha(x)$-ON frame and we take $\varepsilon_{(j_1,...,j_{n-r})}$ identically as in Example 1.1.3.

Definition 2.1.8 *For a Lipschitz Riemannian metric $\Gamma = \{\Gamma_\alpha\}$ on a compact Lipschitz manifold M and $\omega \in L_2^r(M)$, $\omega = \{\omega_\alpha\}$, we define*

- *L_2-differential form $*_\Gamma\omega = \{*_\alpha\omega_\alpha\}_\alpha$,*
- *for ω, η of the same degree we define the inner product $(\omega|\eta)_\Gamma := \{(\omega_\alpha|\eta_\alpha)_\alpha\}$ (it is a 0-form, i.e. a measurable function on M).*
- *the unitary structure $((\omega|\eta))_\Gamma := \int_M (\omega|\eta)_\Gamma$*

Theorem 2.1.9 (Teleman 1983) *[T2] The space $L_2^k(M)$ with unitary structure $((\cdot|\cdot))_\Gamma$ is Hilbert, two Lipschitz Riemannian metrics define equivalent norms in $L_2^k(M)$.*

Introducing the pairing of differential forms in complementary degrees by integration of the wedge product

$$\langle\langle\omega|\eta\rangle\rangle = \int_M \omega \wedge \eta$$

we have $\langle\langle\omega|\eta\rangle\rangle = ((\omega| *_\Gamma \eta))_\Gamma$ which means that $\left(L_2(M), \langle\langle\cdot|\cdot\rangle\rangle, ((\cdot|\cdot)), *\right)$ is a Hilbert Hodge space (Def. 1.1.1).

Definition 2.1.10 *Let $\sigma \in L_2^r(U)$ be any L_2-differential form on $U \subset \mathbb{R}^n$ of degree $r < n$. We say that σ has* distributional (or weak) exterior derivative *in the class L_2 if there exists an L_2-differential form of degree $r+1$*

$$\bar{d}\sigma \in L_2^{r+1}(U)$$

such that for any C^∞-differential form φ of degree $n-1-r$ with compact support in U

$$\int_U \bar{d}\sigma \wedge \varphi = (-1)^{r+1} \int_U \sigma \wedge d\varphi.$$

If $r = n$, we put $\bar{d}\sigma = 0$ for each $\sigma \in L_2^n(U)$.

Distributional exterior derivative $\bar{d}\sigma$ is uniquely determined and $\bar{d}\sigma$ has zero distributional exterior derivative, $\bar{d}\left(\bar{d}\sigma\right) = 0$. If σ is of the form $\sigma =$

$\Sigma_{i_1<\ldots.<i_r}\sigma^{i_1\ldots i_r}dx_{i_1}\wedge\ldots\wedge dx_{i_r}\in L_2^r(U)$, the distributional exterior derivative $\bar{d}\sigma$ (if exists) looks exactly as in the smooth case: $\bar{d}\sigma=\Sigma_j\Sigma_{i_1<\ldots.<i_r}\frac{\partial\sigma^{i_1\ldots i_r}}{\partial x_j}dx_j\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_r}$, where $\frac{\partial\sigma^{i_1\ldots i_r}}{\partial x_j}$ is the usual distributional derivative of the function $\sigma^{i_1\ldots i_r}$, and although the components of $\bar{d}\sigma$ in the base $dx_{i_1}\wedge\ldots\wedge dx_{i_r}\wedge dx_{i_{r+1}}$ are required to belong to L_2, it is not true, in general, that each distributional derivative $\frac{\partial\sigma^{i_1\ldots i_r}}{\partial x_j}$ belongs to L_2 [T2].

Via the convolution argument and a mollifying sequence (see Remark 3.2.1 below adopted for the exterior derivative of differential forms and vector bundles $\xi=\operatorname{Sec}\bigwedge^r T^*U$, $\eta=\operatorname{Sec}\bigwedge^{r+1}T^*U$) we have (compare [T4]):

Theorem 2.1.11 *A form $\sigma=\Sigma\sigma^{i_1\ldots i_r}dx_{i_1}\wedge\ldots\wedge dx_{i_r}\in L_2^r(U)$ of degree $r<n$ has distributional exterior derivative $\bar{d}\sigma\in L_2^{r+1}(U)$ if and only if there exists a sequence $\sigma_n\in L_2^r(U)\cap\Omega^r(U)$ of smooth L_2-differential forms on U such that $\sigma_n\xrightarrow{L_2}\sigma$ and $d\sigma_n$ is convergent in L_2 (then $\lim_{n\to\infty}d\sigma_n$ is independent of the chooice of the sequence σ_n and $\lim_{n\to\infty}d\sigma_n=\bar{d}\sigma$). For σ_n we can take the differential form $\sigma_n=\Sigma\sigma_n^{i_1\ldots i_r}dx_{i_1}\wedge\ldots\wedge dx_{i_r}$ such that*

$$\sigma_n^{i_1\ldots i_r}=\lambda_{\frac{1}{n}}*\sigma^{i_1\ldots i_r}\tag{2.1.1}$$

where $\lambda_{\frac{1}{n}}*\sigma^{i_1\ldots i_r}$ is the convolution operation (see Subsection 3.2.1) of $\lambda_{\frac{1}{n}}$ and $\sigma^{i_1\ldots i_r}$ when for $t>0$ $\lambda_t(x)=\frac{1}{t}\varphi\left(\frac{x}{t}\right)$ and $\varphi\in C_0^\infty(\mathbb{R}^n)$, $\varphi\geq 0$, $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$, $\operatorname{supp}\varphi=B(0,1)$.

Proposition 2.1.12 *[T2] If $\omega=\{\omega_\alpha\}_{\alpha\in\Lambda}$ is an L_2-differential form on M of degree r and $\bar{d}\omega_\alpha\in L_2(V_\alpha)$ is the distributional exterior derivative of ω_α, then*

$$\bar{d}\omega:=\left\{\bar{d}\omega_\alpha\right\}_{\alpha\in\Lambda}$$

is an L_2-differential form on M of degre $r+1$.

Denote by $\Omega_d^r(M)\subset L_2^r(M)$ the subspace of L_2-differential forms of degree r possessing the distributional exterior derivative

$$\Omega_d^r(M)=\left\{\omega\in L_2^r(M);\ \bar{d}\omega\in L_2^{r+1}(M)\right\}.$$

It is easy to see that $\left(\bar{d}\right)^2=0$ na $\Omega_d(M)$. We obtain a cohomology complex

$$0\longrightarrow\Omega_d^0(M)\longrightarrow\ldots\longrightarrow\Omega_d^r(M)\xrightarrow{\bar{d}}\Omega_d^{r+1}(M)\longrightarrow\ldots\longrightarrow\Omega_d^n(M)=L_2^n(M)\longrightarrow 0.$$

Theorem 2.1.13 (Teleman 1983, L_2-Poincaré duality) *[T2] For a compact oriented Lipschitz manifold M of dimension n the pairing*

$$\langle\cdot,\cdot\rangle_{\mathbf{H},d}^r:\mathbf{H}^r\left(\Omega_d^\bullet(M)\right)\times\mathbf{H}^{n-r}\left(\Omega_d^\bullet(M)\right)\longrightarrow\mathbb{R},\quad([\omega],[\eta])\longmapsto\int_M\omega\wedge\eta$$

is strongly nondegenerated, i.e. $\mathbf{H}^r\left(\Omega_d^\bullet(M)\right)=\left(\mathbf{H}^{n-r}\left(\Omega_d^\bullet(M)\right)\right)^*$. Therefore $\mathbf{H}\left(\Omega_d^\bullet(M)\right)=\left(\mathbf{H}\left(\Omega_d^\bullet(M)\right)\right)^*$, whence $\dim\mathbf{H}\left(\Omega_d^\bullet(M)\right)<\infty$.

The first step of Teleman's proof demonstrates that the inclusion $L^{*}(M) \hookrightarrow \Omega_{d}^{\bullet}(M)$ of the Whitney subcomplex $L^{\bullet}(M)$ (consisting of the so-called flat differential forms, i.e. differential forms $\omega = \{\omega_{\alpha}\}$ such that ω_{α} and $\bar{d}\omega_{\alpha}$ have bounded measurable components [Wh]) into $\Omega_{d}^{\bullet}(M)$ induces an isomorphism in cohomology. It follows from the fact that associated differential sheaves of germs $\underline{L^{\bullet}(M)}$ and $\underline{\Omega_{d}^{\bullet}(M)}$ are fine and are resolutions of the $\mathbb{R}$-constant sheaf (thanks the suitable versions of the Poincaré lemmas). The second step demonstrates the Poincaré duality for $L^{\bullet}(M)$ which is easy to see by the Whitney's theory [Wh].

Theorem 2.1.14 (Teleman 1983) *[T2] (1) Suppose that $\omega \in \Omega_{d}^{r}(M)$ and $\eta \in \Omega_{d}^{n-r-1}(M)$, $r < n$, then we can switch the distributional derivatives*

$$\left\langle\!\left\langle \omega | \bar{d}\eta \right\rangle\!\right\rangle = (-1)^{r+1} \left\langle\!\left\langle \bar{d}\omega | \eta \right\rangle\!\right\rangle .$$

(2) Let $\omega \in L_{2}^{r}(M)$, $r < n$, and if there exist $\omega' \in L_{2}^{r+1}(M)$ such that

$$\left\langle\!\left\langle \omega | \bar{d}\eta \right\rangle\!\right\rangle = (-1)^{r+1} \langle\!\langle \omega' | \eta \rangle\!\rangle$$

for all $\eta \in \Omega_{d}^{n-r-1}(M)$, then $\omega \in \Omega_{d}^{r}(M)$ and ω' is the distributional exterior derivative of ω, $\bar{d}\omega = \omega'$.

In particular, if $\left\langle\!\left\langle \omega | \bar{d}\eta \right\rangle\!\right\rangle = 0$ for all $\eta \in \Omega_{d}^{n-r-1}(M)$ then $\bar{d}\omega = 0$.

The first part of the above theorem is reduced to a local problem where ω is supported in an open set in $\mathbb{R}^{n}$ (using a Lipschitz partition of unity). The local problem is proved by the convolution argument (see Theorem 2.1.11). We notice that this argument easily implies that $\Omega_{d}(M)$ is an algebra and the usual formula for the derivative of the wedge products holds. The operator $\bar{d}$ remains local.

Notice that Teleman's considerations [T2] concerning Lipschitz manifolds to obtain the signature operator has algebraic nature which enables us to give some generalizations.

In the end we notice that if M is a smooth compact manifold and $M_{\mathfrak{L}}$ is corresponding Lipschitz manifold then $L_{2}(M) \cong L_{2}(M_{\mathfrak{F}})$ as Hilbert spaces.

2.2 Hilbert anticommutative graded subdifferential Hodge spaces

We start with the following definition.

Definition 2.2.1 *By a* Hilbert anticommutative graded subdifferential Hodge space *of degree N we mean a system*

$$\left(W = \bigoplus_{r=0}^{N} W^{r}, \langle \cdot | \cdot \rangle, (\cdot | \cdot), *_{W}, \bar{d} : W_{d} \longrightarrow W_{d} \right) \tag{2.2.1}$$

consisting of a Hodge space $(W, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *_W)$ *and a subderivative* $\bar{d}$ *defined on some subspace with gradation*

$$W_d = \bigoplus_{r=0}^{N} W_d^r \subset W, \qquad W_d^r = W^r \cap W_d,$$

satisfying the conditions:

(1_H) the unitary space $\left(W, \|\cdot\| = \sqrt{(\cdot|\cdot)}\right)$ *is complete, i.e.* W *is Hilbert, the subspaces* W^k *are orthogonal with respect to* $(\cdot|\cdot)$,

(2_H) $\langle\cdot|\cdot\rangle |W^k \times W^r = 0$ *if* $k + r \neq N$ *and* $\left\langle v^r, v^{N-r}\right\rangle = \varepsilon_r \left\langle v^{N-r}, v^r\right\rangle$, *where* $\varepsilon_r := \varepsilon_0 (-1)^{r(N-r)}$ *for* $\varepsilon_0 \in \{+1, -1\}$.

(3_H) W_d *is dense in* W,

(4_H) $\bar{d}$ *is degree* $+1$, $\bar{d}^r = d|W_d^r : W_d^r \longrightarrow W_d^{r+1}$, *and* $\bar{d} \circ \bar{d} = 0$,

(5_H) $\left\langle \bar{d}w|u\right\rangle = (-1)^{r+1} \left\langle w|\bar{d}u\right\rangle$ *for* $w \in W_d^r$, $u \in W_d^{N-r-1}$,

(6_H) If for $w \in W^r$ *there exists* $w' \in W^{r+1}$, *such that* $\langle w'|u\rangle = (-1)^{r+1} \left\langle w|\bar{d}u\right\rangle$ *for each* $u \in W_d^{N-r-1}$, *then* $w \in W_d^r$ *and* $\bar{d}w = w'$.

A Lipschitz manifold M leads to the Hilbert anticommutative graded subdifferential Hodge space with $\varepsilon_0 = +1$ for $W = L_2^r(M)$ and $W_d = \Omega_d(M)$.

Notation 2.2.2 *Let* W *be a Hilbert anticommutative graded subdifferential Hodge space of degree* N. *We put*

$$W_\delta^{N-r} := *_W [W_d^r].$$

This space is dense in W^{N-r} *and* $*_W : W_d^r \to W_\delta^{N-r}$ *is an isometry. By a cosubderivative of degree* r *we mean the oparator*

$$\bar{\delta}^r : W_\delta^r \longrightarrow W_\delta^{r-1}$$

defined by

$$\bar{\delta}^r (w^r) = \varepsilon_r (-1)^r * \bar{d} * (w^r), \quad w^r \in W_\delta^r.$$

Standardly, we can prove the following lemma.

Lemma 2.2.3 *The operators* $\bar{d}$ *and* $\bar{\delta}$ *are adjoint*

$$(\bar{\delta}^{N-r} v|w) = (v|\bar{d}^{N-r-1} w), \quad v \in W_\delta^{N-r},\ w \in W_d^{N-r-1}.$$

Proof.

$$\begin{aligned} & (\bar{\delta}^{N-r} v|w) = (\varepsilon_{N-r} (-1)^{N-r} * \bar{d} * (w^r) |w) = ((-1)^{N-r} * \bar{d}^r \left(*^{-1} v\right) |w) \\ = \quad & \langle w| (-1)^{N-r} \bar{d}^r \left(*^{-1} v\right)\rangle = (-1)^{N-r} \left\langle w|\bar{d}^r \left(*^{-1} v\right)\right\rangle \overset{(5_H)}{=} \left\langle \bar{d}^{N-r-1} w| *^{-1} v\right\rangle \\ = \quad & (\bar{d}^{N-r-1} w| * *^{-1} v) = (\bar{d}^{N-r-1} w|v) = (v|\bar{d}^{N-r-1} w). \end{aligned}$$

■

Notation 2.2.4 $W_1^r = W_d^r \cap W_\delta^r = \left\{ w \in W^r : \ w \in W_d^r, \ *w \in W_d^{N-r} \right\}$ - *the Sobolev space of order 1 on the Hilbert anticommutative graded subdifferential Hodge space W.*

Lemma 2.2.5 *(a) W_1^r is a Hilbert space with respect to the diagonal norm $\|w\|_1$:*

$$\|w\|_1^2 = \|w\|^2 + \left\|\bar{d}w\right\|^2 + \left\|\bar{\delta}w\right\|^2 .$$

(b) The operator

$$\bar{D}^r = \bar{d} + \bar{\delta} : W_1^r \longrightarrow W^r$$

is a bounded operator (i.e. it is continuous).

Proof. (a) Let $w_i \in W_1^r$ be a Cauchy sequence with respect to the diagonal norm $\|w\|_1$. For arbitrary $\varepsilon > 0$ there exists n_1 such that $\|w_i - w_j\|_1 < \varepsilon$ for $i, j > n_1$. Then $||w_i - w_j||^2 < \varepsilon^2$, $\left|\left|\bar{d}\left(w_i - w_j\right)\right|\right|^2 < \varepsilon^2$, $\left|\left|\bar{\delta}\left(w_i - w_j\right)\right|\right|^2 < \varepsilon^2$. Whence w_i is Cauchy in W. From assumption (1_H) there exists a limit $w = \lim w_i \in W$. The sequences $\bar{d}w_i$ and $\bar{\delta}w_i$ are Cauchy in W, too. Let $w_d = \lim \bar{d}w_i$ and $w_\delta = \lim \bar{\delta}w_i$. It is enough to check that $w_d = \bar{d}w$ and $w_\delta = \bar{\delta}w$.

a1) $w_d = \bar{d}w$. Take arbitrary $u \in W_d^{N-r-1}$, then

$$\langle w_d, u\rangle = \lim \left\langle \bar{d}w_i, u\right\rangle \overset{(5_H)}{=} \lim \left(-1\right)^{r+1} \left\langle w_i, \bar{d}u\right\rangle = \left(-1\right)^{r+1} \left\langle w, \bar{d}u\right\rangle .$$

By axiom (6_H) we have $w \in W_d$ and $w_d = \bar{d}w$.

a2) $w_\delta = \bar{\delta}w$. To show that $w \in W_\delta$ it is enough to check that $*w \in W_d$. Firstly, we notice that the sequence $\bar{d}\left(*w_i\right)$ is convergent because $\bar{d}\left(*w_i\right) = \pm * \bar{\delta}w_i \to \pm * w_\delta$. Secondly,

$$\left\langle \lim d\left(*w_i\right), u\right\rangle = \lim \left\langle d\left(*w_i\right), u\right\rangle = \lim \left(-1\right)^{N-r+1} \left\langle *w_i, du\right\rangle = \left(-1\right)^{N-k+1} \left\langle *w, du\right\rangle .$$

By (6_H) we have $*w \in W_d$ and $\bar{d}\left(*w\right) = \lim d\left(*w_i\right)$. Therefore $*\bar{d} * w = \lim *\bar{d}\left(*w_i\right)$, whence $\bar{\delta}w = \lim \bar{\delta}w_i = w_\delta$.

(b) evident. ■

Let $\mathbf{H}_d\left(W\right) = \bigoplus_{r=0}^N \mathbf{H}_d^r\left(W\right)$ be the graded cohomology space of the complex $\left(W_d, \bar{d}\right)$ and let

$$\langle\cdot|\cdot\rangle_{\mathbf{H},d}^r : \mathbf{H}^r\left(W_d^\bullet\right) \times \mathbf{H}^{N-r}\left(W_d^\bullet\right) \longrightarrow \mathbb{R}, \qquad \left(\left[w\right], \left[v\right]\right) \longmapsto \langle w|v\rangle ,$$

be the pairing induced by the tensor $\langle\cdot|\cdot\rangle$, $r = 0, 1, ..., N$ (thanks (5_H)). We define now the spaces of harmonic vectors

$$\mathcal{H}_d^r = \left\{ w \in W_1^r ; \ \bar{d}\omega = 0 = \bar{\delta}\omega \right\} .$$

Clearly

$$* : \mathcal{H}_d^r \to \mathcal{H}_d^{N-r}$$

is an isomorphism. Any harmonic vector is a cocycle, therefore there exists a Hodge homomorphism

$$x_d^r : \mathcal{H}_d^r \longrightarrow \mathbf{H}^r\left(W_d^\bullet\right).$$

As in the paper by Teleman (1983) we can prove in an algebraic way the following theorem

Theorem 2.2.6 *If (2.2.1) is a Hilbert anticommutative graded subdifferential Hodge space of degree N then*

(a) the subspaces $\mathcal{H}_d^r(W)$ and $\operatorname{Im}\bar{d}^{r-1}$ are perpendicular,

(b) x_d^r is a monomorphism,

(c) the subspace $\operatorname{Ker}\bar{d}^r$ is closed in W^r, therefore $\operatorname{Ker}\bar{d}^r$ is a Hilbert space,

(d) $\mathcal{H}_d^r(W) = \{h \in W^r;\quad h \in \operatorname{Ker} d^r,\ h \perp \operatorname{Im} d^{r-1}\}$, i.e. $\mathcal{H}_d^r = \left(\operatorname{Im} d^{r-1}\right)^{\perp}$ in $\operatorname{Ker} d^r$,

(e) if the cohomology pairing $\langle\cdot|\cdot\rangle^r_{\mathbf{H},d}$ is a weak nondegenerated then the subspace $\operatorname{Im}\bar{d}^{r-1}$ is closed in W^r, therefore $\operatorname{Im}\bar{d}^{r-1}$ is a Hilbert space,

(f) (Hodge Theorem) if the cohomology pairing $\langle\cdot|\cdot\rangle^r_{\mathbf{H},d}$ is a weak nondegenerated then $\operatorname{Im}\bar{d}^{r-1}(\subset \operatorname{Ker}\bar{d}^r)$ is a closed subspace of the Hilbert space $\operatorname{Ker}\bar{d}^r$ and

$$\operatorname{Ker}\bar{d}^r = \operatorname{Im}\bar{d}^{r-1} \oplus \left(\operatorname{Im}\bar{d}^{r-1}\right)^{\perp} = \operatorname{Im}\bar{d}^{r-1} \oplus \mathcal{H}_d^r$$

which means that

$$\mathcal{H}_d^r(W) \cong \operatorname{Ker}\bar{d}^r / \operatorname{Im}\bar{d}^{r-1} = \mathbf{H}^r\left(W_d^{\bullet}\right),$$

i.e. the Hodge homomorphism x_d^r is an isomorphism,

(g) if the cohomology pairing $\langle\cdot|\cdot\rangle^r_{\mathbf{H},d}$ is a weakly nondegenerated and the subspace of harmonic tensors $\mathcal{H}_d^r(W)$ is finitely dimensional then

(g1) the cohomology space $\mathbf{H}_d(W) = \bigoplus_{r=0}^{N} \mathbf{H}_d^r(W)$ of the complex $\left(W_d, \bar{d}\right)$ is finite dimensional and fulfills the Poincaré duality, i.e. the pairing

$$\langle\cdot|\cdot\rangle^r_{\mathbf{H},d} : \mathbf{H}^r\left(W_d^{\bullet}\right) \times \mathbf{H}^{N-r}\left(W_d^{\bullet}\right) \longrightarrow \mathbb{R}, \qquad ([w],[v]) \longmapsto \langle w|v\rangle,$$

is a duality, $\mathbf{H}^r\left(W_d^{\bullet}\right) \cong \left(\mathbf{H}^{N-r}\left(W_d^{\bullet}\right)\right)^*$, for $r = 0, 1, ..., N$,

(g2) there is a strong Hodge decomposition of closed perpendicular subspaces

$$W^r = \mathcal{H}_d^r(W) \oplus \bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right], \tag{2.2.2}$$

and

$$W^r = \mathcal{H}_d^r(W) \oplus \bar{d}\left[W_1^{r-1}\right] \oplus \bar{\delta}\left[W_1^{r+1}\right]. \tag{2.2.3}$$

Proof. (a) If $w \in \mathcal{H}_d^r$ and $v \in W_d^{r-1}$ then by Lemma 2.2.3 $\left(w|\bar{d}v\right) = \left(\bar{\delta}w|v\right) = 0$.

(b) Follows by (a). Independently: if $w \in \mathcal{H}_d^r$ and $x_d^r(w) = [w] = 0$, i.e. there exists $v \in W_d$ such that $w = \bar{d}^{r-1}(v)$ then by Lemma 2.2.3 $(w|w) = \left(w|\bar{d}v\right) = \left(\bar{\delta}w|v\right) = 0$, so $w = 0$.

(c) Assume $w_i \in W_d^r$, $\bar{d}w^i = 0$, and let $w_i \to w \in W^r$. For arbitrary vector $u \in W_d^{N-r-1}$ we have

$$\left\langle w|\bar{d}u\right\rangle = \lim \left\langle w_i|\bar{d}u\right\rangle \overset{(5_H)}{=} \lim \left(-1\right)^{r+1} \left\langle \bar{d}w_i|u\right\rangle = 0 = \left(-1\right)^{r+1} \left\langle 0|u\right\rangle .$$

Axiom (6_H) implies that $w \in W_d^r$ and $\bar{d}w = 0$.

(d) "$\subset$" follows from (a). "$\supset$" Let $h \in \operatorname{Ker} d^r$ and $h \perp \operatorname{Im} d^{r-1}$. Then for arbitrary $w \in W_d^{r-1}$ we have $(w|h) = \varepsilon_r \left\langle w| * h\right\rangle$, therefore

$$0 = \left(\bar{d}w|h\right) = \varepsilon_r \left\langle \bar{d}w| * h\right\rangle .$$

From Axiom (6_H) we have $*h \in W_d$ and $\bar{d}\left(*h\right) = 0$, therefore $\bar{\delta}h = \varepsilon_r \left(-1\right)^r * \bar{d} * h = 0$.

(e) Let $\bar{d}^{r-1}w_i \to w \in W^r$, $w_i \in W_d^{r-1}$. Then $\bar{d}^{r-1}w_i$ are cocycles. The part (c) yields that w is a cocycle too. For an arbitrary cocycle $h \in W_d^{N-r}$, $\bar{d}h = 0$, we see that

$$\left\langle w|h\right\rangle = \lim \left\langle \bar{d}^{r-1}w_i|h\right\rangle \overset{(5_H)}{=} \left(-1\right)^r \lim \left\langle w_i|\bar{d}h\right\rangle = 0.$$

The weak nondegenerance of the cohomology pairing implies that $[w] = 0$, i.e. $w \in \operatorname{Im} \bar{d}^{r-1}$.

(f) follows from (d) and (e).

(g1) it follows by (f) and the fact that the weakly nondegenerated pairing for finitely dimensional vector spaces is a duality,

(g2) (2.2.2) $\operatorname{Im} \bar{d}^{r-1}$ is closed from (d), $\operatorname{Im} \bar{\delta}^{r+1} = * \left[\operatorname{Im} \bar{d}^{N-r+1}\right]$ is closed by (e) and that $*$ is an homeomorphism. All subspaces are then closed and pair-wise perpendicular. It remains to check that $\mathcal{H}_d^r\left(W\right) = \left(\bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right]\right)^{\perp}$.
The inclusion "$\subset$" is evident by Lemma 2.2.3. To prove "$\supset$" take $h \in \left(\bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right]\right)^{\perp}$.
Clearly $h \perp \operatorname{Im} \bar{d}^{r-1}$ and $h \perp \operatorname{Im} \bar{\delta}^{r+1}$. According to (d) we need only to notice that $h \in \operatorname{Ker} \bar{d}^r$. But $\left(\bar{d}h|\bar{d}h\right) = \left(h|\bar{\delta}\bar{d}h\right) = 0$ since $\bar{\delta}\bar{d}h \in \operatorname{Im} \bar{\delta}$ and $h \perp \operatorname{Im} \bar{\delta}$. Therefore $\bar{d}h = 0$.

To prove that the strong Hodge decomposition (2.2.2) can be presented in another way (2.2.3) we notice (analogously to Cor. 4.4 from [T1]) that

$$\text{(i)} \quad \bar{d}\left[W_d^{r-1}\right] = \bar{d}\left[W_1^{r-1}\right] \qquad \text{and} \qquad \text{(ii)} \quad \bar{\delta}\left[W_\delta^{r+1}\right] = \bar{\delta}\left[W_1^{r+1}\right] .$$

Since $W_d \cap \bar{\delta}\left[W_\delta\right] \subset W_d \cap W_\delta = W_1 \subset W_d$ therefore to check (i) it is sufficient to prove the equality

$$\bar{d}\left[W_d\right] = \bar{d}\left[W_d \cap \bar{\delta}\left[W_\delta\right]\right] .$$

It is clear that $\bar{d}\left[W_d \cap \bar{\delta}\left[W_\delta\right]\right] \subset \bar{d}\left[W_d\right]$. Conversely, suppose $w \in \bar{d}\left[W_d\right]$, i.e. $w = \bar{d}u$, $u \in W_d$. From the first decomposition $W^r = \mathcal{H}_d^r\left(W\right) \oplus \bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right]$ we deduce that there exist $h \in \mathcal{H}_d\left(W\right)$, $\alpha \in W_d$ and $\beta \in W_\delta$ such that $u = h + \bar{d}\alpha + \bar{\delta}\beta$. But u, h, $\bar{d}\alpha \in W_d$, it follows that $\bar{\delta}\beta \in W_d$ too, whence $\bar{\delta}\beta \in W_d \cap \bar{\delta}\left[W_\delta\right]$. We get

$$w = \bar{d}u = \bar{d}\left(h + \bar{d}\alpha + \bar{\delta}\beta\right) = \bar{d}\left(\bar{\delta}\beta\right) \in \bar{d}\left[W_d \cap \bar{\delta}\left[W_\delta\right]\right]$$

which prove (i). The equality (ii) can be proved similarly. ■

Analogously as in Teleman's paper we show

Theorem 2.2.7 *The operator* $\bar{D}^r = \bar{d} + \bar{\delta} : W_1^r \to W^r$ *is a continuous Fredholm operator such that*

(1) $\operatorname{Ker} \bar{D}^r = \mathcal{H}_d^r(W)$,

(2) $\operatorname{Im} \bar{D}^r = \bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right]$ *(so* $\operatorname{Coker} \bar{D}^r \cong \mathcal{H}_d^r$ *) and* $\operatorname{index} \bar{D} = 0$.

Proof. (1). The inclusion "$\supset$" is evident. To see the opposite inclusion take $h \in \operatorname{Ker} \bar{D}^r$. Then

$$0 = \left(\bar{D}h|\bar{D}h\right) = \left(\left(\bar{d} + \bar{\delta}\right) h|\left(\bar{d} + \bar{\delta}\right) h\right) = \left(\bar{d}h|\bar{d}h\right) + \left(\bar{\delta}h|\bar{\delta}h\right)$$

therefore $\bar{d}h = 0 = \bar{\delta}h$. In conclusion $\dim \operatorname{Ker} \bar{D} < \infty$.

(2). The inclusion "$\subset$" holds by definition. To prove the opposite inclusion it is sufficient to check that $\bar{d}\left[W_d^{r-1}\right] \subset \operatorname{Im} \bar{D}^r$ and $\bar{\delta}\left[W_\delta^{r+1}\right] \subset \operatorname{Im} \bar{D}^r$.

a) $\bar{d}\left[W_d^{r-1}\right] \subset \operatorname{Im} \bar{D}^r$. Let $a \in W_d \subset W = \mathcal{H}_d(W) \oplus \bar{d}\left[W_d\right] \oplus \bar{\delta}\left[W_\delta\right]$, and put $a = h + \bar{d}\alpha + \bar{\delta}\beta$. Since a, h and $d\alpha$ have derivative then $\bar{\delta}\beta$ has also the derivative $\bar{\delta}\beta \in W_d$. Therefore

$$\bar{d}a = \bar{d}h + \bar{d}^2\alpha + \bar{d}\bar{\delta}\beta = \bar{d}\bar{\delta}\beta = \left(\bar{d} + \bar{\delta}\right)\left(\bar{\delta}\beta\right) = \bar{D}\left(\bar{\delta}\beta\right) \in \operatorname{Im} \bar{D}.$$

b) $\bar{\delta}\left[W_\delta^{r+1}\right] \subset \operatorname{Im} \bar{D}^r$ - analogously. ■

Definition 2.2.8 *A Hilbert anticommutative graded subdifferential Hodge space of degree* N *will be called* regular *if it satisfies the following additional properties*

($7'_H$) the cohomology pairing $\langle\cdot|\cdot\rangle_{\mathbf{H},d}^r$ *is weakly nondegenerated,*

($7''_H$) the subspaces of harmonic tensors $\mathcal{H}_d^r(W)$ *are finitely dimensional.*

Properties ($7'_H$)-($7''_H$) are equivalent to the property

(7_H) the cohomology space $\mathbf{H}_d(W) = \bigoplus_{r=0}^N \mathbf{H}_d^r(W)$ of the complex $\left(W_d, \bar{d}\right)$ fulfils the Poincaré duality (i.e. the pairings

$$\langle\cdot|\cdot\rangle_{\mathbf{H},d}^r : \mathbf{H}^r\left(W_d^\bullet\right) \times \mathbf{H}^{N-r}\left(W_d^\bullet\right) \longrightarrow \mathbb{R}$$

are dualities).

2.3 The signature operator for regular Hilbert anticommutative graded subdifferential Hodge spaces of even degree

Let W be a regular Hilbert anticommutative graded subdifferential Hodge spaces of even degree $N = 2n$.

Since $\dim \mathbf{H}(\Omega_d^\bullet)$ is finite (from the Poincaré duality) we can define the signature of W, and

$$\mathrm{Sig}\,(W) := \mathrm{Sig}\,\langle\cdot|\cdot\rangle_{\mathbf{H}}^n = \mathrm{Sig}\,\mathcal{B}_d^n.$$

Puting (for the operator τ defined identically as in Subsection 1.3)

$$W_\pm = \{w \in W;\ \tau w = \pm w\}, \quad W_{1\pm} = W_\pm \cap W_1,$$

we notice that

$$\bar{D}[W_{1+}] \subset W_-, \quad \bar{D}[W_{1-}] \subset W_+.$$

Definition 2.3.1 *The operator*

$$\bar{D}_+ = \bar{D}|W_{1+} : W_{1+} \to W_-$$

is called the signature operator *for W. Also we define the operator*

$$\bar{D}_- = \bar{D}|W_{1-} : W_{1-} \to W_+$$

called adjoint to $\bar{D}_+$.

The condition of duality holds

$$\left(\bar{D}_+\alpha|\beta\right) = \left(\alpha|\bar{D}_-\beta\right), \qquad \text{for} \qquad \alpha \in W_{1+}, \beta \in W_{1-}.$$

By the index of the operator $\bar{D}_+$ we mean $\mathrm{index}\,\bar{D}_+ := \dim_{\mathbb{R}} \ker\left(\bar{D}_+\right) - \dim_{\mathbb{R}} \ker\left(\bar{D}_-\right)$. Analogously as in Subsection 1.3 (i.e. see [K-M-4]) we prove the signature theorem

Theorem 2.3.2 *For the regular Hilbert anticommutative graded subdifferential Hodge space W of even degree we have*

$$\mathrm{Sig}\,(W) = \mathrm{index}\,\bar{D}_+.$$

3 Hilbert completion of the graded anticommutative differential Hodge space of even degree

3.1 Algebraic setting

For a graded anticommutative differential Hodge space of even degree we define its completion to the Hilbert space. Then we can obtain two signature operators and we want to compare them. It is the case of Hodge vector bundles on compact smooth manifolds and associated L_2-theory, and in particular of Lie algebroids.

Consider a graded anticommutative differential Hodge space of even degree

$$\left(W = \bigoplus\nolimits_{r=0}^{N=2n} W^r, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *, d\right),$$

(i.e. $(W, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *)$ is a Hodge space and axioms (1)-(5) from Definition 1.2.1 are satisfied).

Now, we complete the unitary space $(W, (\cdot|\cdot))$ to the Hilbert one $\bar{W}$. The inner product and the norm in $\bar{W}$ will be denoted by the same symbol. We extend the $*$-Hodge isometry to the isometry $* : \bar{W} \to \bar{W}$ and the pairing $\langle\cdot|\cdot\rangle$ to a new one denoting by the same symbol $\langle\cdot|\cdot\rangle : \bar{W} \times \bar{W} \to \mathbb{R}$. Of course this pairing remains continuous. We obtain a new Hilbert graded anticommutative Hodge space of even degree

$$\left(\bar{W} = \bigoplus_{r=0}^{N=2n} \bar{W}^r, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *\right).$$

Now we extend the derivative $d^r : W^r \to W^{r+1}$ to some bigger subspace $\bar{W}_d^r \subset \bar{W}^r$ in a "distributional manner".

Definition 3.1.1 *We say that a vector $w \in \bar{W}^r$ has a* distributional derivative *if there exists a vector belonging to $\bar{W}^{r+1}$ denoted by $\bar{d}w$ such that for each vector $v \in W^{N-r-1}$ the following condition*

$$\left\langle w|d^{N-r-1}v\right\rangle = (-1)^{r+1} \left\langle \bar{d}^r w|v\right\rangle$$

holds.

The derivative $\bar{d}w$ is unique (if it exists). The vector space of vectors v possessing distributional derivative will be denoted by $\bar{W}_d$. Clearly, if $w \in W^k$ then $\bar{d}w$ exists and $\bar{d}w = dw$. If $w \in \bar{W}_d^r$ then $\bar{d}^r w \in \bar{W}_d^{r+1}$ and $\bar{d}^{r+1}\left(\bar{d}^r(w)\right) = 0$.

It is evident that

- axioms (1_H)-(4_H) and (6_H) are fulfilled.
- $\bar{W}$ is a Hilbert anticommutative graded subdifferential Hodge space if condition (5_H) holds,
- $\bar{W}$ is a regular if and only if the axioms (5_H), (7_H) are satisfied.

Before considering axioms (5_H) and (7_H) we introduce the fundamental diagram joining Hodge homomorphisms for W and its extension $\bar{W}$. First, we take the subspace $\bar{W}_\delta^r := *\left[\bar{W}_d^{N-r}\right]$ and the coderivative $\bar{\delta}^r : \bar{W}_\delta^r \longrightarrow \bar{W}_\delta^{r-1}$, $\bar{\delta}^r := \varepsilon_r (-1)^r * \bar{d}^{N-r} *$. The operator $\bar{\delta}$ is an extension of δ.

Consider

$$\begin{aligned} \bar{W}_1 &= \bar{W}_d \cap \bar{W}_\delta, \\ \bar{D} &= \bar{d} + \bar{\delta} : \bar{W}_1 \longrightarrow \bar{W}, \\ \mathcal{H}_d(W) &= \ker \bar{D} = \left\{w \in \bar{W}_1;\ \bar{d}w = 0 = \bar{\delta}w\right\}. \end{aligned} \tag{3.1.1}$$

The inclusion $J : W \hookrightarrow \bar{W}_d$ commutes with derivations therefore induces a homomorphism in cohomology

$$J_\# : \mathbf{H}(W) \longrightarrow \mathbf{H}\left(\bar{W}_d\right).$$

We have the commutative diagram joining the Hodge homomorphisms x^r and x_d^r

$$\begin{array}{ccc} \mathcal{H}^r(W) & \overset{x^r}{\rightarrowtail} & \mathbf{H}^r(W) \\ \cap & & \downarrow J_\#^r \\ \mathcal{H}_d^r(W) & \overset{x_d^r}{\longrightarrow} & \mathbf{H}^r\left(\bar{W}_d\right). \end{array} \tag{3.1.2}$$

We recall that Axiom (5_H) says that we can

- switch the distributional derivatives $\bar{d}$, $\left\langle \bar{d}w|u\right\rangle = (-1)^{r+1}\left\langle w|\bar{d}u\right\rangle$ for $w \in W_d^r$, $u \in W_d^{N-r-1}$.

Theorem 2.1.11 suggests that we should add to the set of axioms (1)-(5) the following

(6) if $w \in \bar{W}_d^r$ $(r = 0, 1, ..., N)$ then there exists a Cauchy sequence $w_i \in W^r$ convergent to w, $w_i \to w$, such that dw_i is convergent in $\bar{W}^{r+1}$.

Lemma 3.1.2 *If W fulffills (6) then (5_H) is satisfied (i.e. $\bar{W}$ is a Hilbert anticommutative graded subdifferential Hodge space) and $x_d^r : \mathcal{H}_d^r(W) \to \mathbf{H}^r\left(\bar{W}_d^\bullet\right)$ is a monomorphism. Then $J_\#^r$ is a monomorphism too.*

Proof. Let Axiom (6) be fulfilled, and let $w \in \bar{W}_d^r$. If $w_i \in W^r$ is a sequence such that $w_i \to w$ and dw_i is convergent in $\bar{W}^{r+1}$, then $\lim dw_i = \bar{d}w$. Indeed, for arbitrary $v \in W^{N-r-1}$

$$\langle w|dv\rangle = \lim\langle w_i|dv\rangle \overset{(4)}{=} \lim(-1)^{r+1}\langle dw_i|v\rangle = (-1)^{r+1}\langle \lim dw_i, v\rangle,$$

therefore, see Def. 3.1.1, $\lim dw_i = \bar{d}w$. To prove that (5_H) is satisfied take $w \in W_d^r$, $u \in W_d^{N-r-1}$ and let $w_i \in W^r$ be a sequence such that $w_i \to w$ and dw_i is convergent. Then $\lim dw_i = \bar{d}w$ and

$$\left\langle \bar{d}w|u\right\rangle = \lim\langle dw_i|u\rangle = \lim(-1)^{r+1}\left\langle w_i|\bar{d}u\right\rangle = (-1)^{r+1}\left\langle w|\bar{d}u\right\rangle.$$

For the monomorphicity of the Hodge homomorphism x_d^r see Theorem 2.2.6. ■

Conclusion 3.1.3 *According to Lemma 2.2.3 (and the pre-sentence) the condition (6) implies that the operators $\bar{d}$ and $\bar{\delta}$ are adjoint. i.e. $(\bar{\delta}v|w) = (v|\bar{d}w)$, $v \in \bar{W}_\delta$, $w \in \bar{W}_d$, whence $\bar{D}$ is self-adjoint in the sense that for $u, w \in \bar{W}_1$ the equality holds*

$$\left(\bar{D}u, w\right) = \left(u, \bar{D}w\right).$$

Condition (6) holds on the spaces of crooss-sections of vector bundles on manifolds and for linear differential operators of finite rank, see the next subsection, see also Theorem 2.1.11.

In the sequel we assume that W fulfils properties (6).

Now we pass to the problem of the Poincaré duality for $\mathbf{H}\left(\bar{W}_d\right)$. In many problems we have from the beginning that Poincaré duality property holds for the cohomology $\mathbf{H}(W)$ (for example we have this in all four examples considerded in Subsection 1.4.1 proved simply by algebraic topology methods [i.e. without using analytic methods of elliptic operators - although it follows also by the latter ones].

Definition 3.1.4 *The graded anticommutative differential Hodge space of even degree W is called* regular *if:*

(7) *the Poincaré duality property for $\mathbf{H}(W)$ holds: i.e. $\langle\cdot|\cdot\rangle^r_{\mathbf{H}} : \mathbf{H}^r(W) \times \mathbf{H}^{N-r}(W) \to \mathbb{R}$, $r = 0, ..., N$, are dualities, $\mathbf{H}^r(W) \cong \left(\mathbf{H}^{N-r}(W)\right)^*$ (particularly, the spaces $\mathbf{H}(W)$ and $\mathcal{H}(W)$ is finitely dimensional).*

Clearly, Lemma 3.1.2 and axiom (7) yield: if

(8) $J_{\#} : \mathbf{H}(W) \longrightarrow \mathbf{H}\left(\bar{W}_d\right)$ is an epimorphism,

then $J_{\#}$ is an isomorphisms and the Poincaré duality for $\mathbf{H}\left(\bar{W}_d\right)$ holds (as a consequence of that in $\mathbf{H}(W)$). Condition (8) says that in each cohomology class in $\mathbf{H}\left(\bar{W}_d\right)$ there exists a d-cocycle $w \in W$.

If (8) holds then we have by Theorem 2.2.6 the Hodge isomorphism $\mathcal{H}^r_d(W) \xrightarrow[\cong]{x^r_d} \mathbf{H}^r\left(\bar{W}_d\right)$ and the Hirzebruch operator $\bar{D}_+$ for the signature of $\bar{W}$ is defined (according to the previous section), and

$$\operatorname{index} D_+ = \operatorname{Sig}\mathcal{B}^n \leq \operatorname{Sig}(W) = \operatorname{Sig}\left(\bar{W}_d\right) = \operatorname{index}\bar{D}_+.$$

From the point of view of the theory of elliptic operators on manifolds the following axiom (9), see below, seems to be natural (for elliptic operators on manifolds (9) follows from the so-called Weyl Lemma (on regularity) [N] for elliptic operators defined on $\mathbb{R}^n$ - or on open subsets of $\mathbb{R}^n$). We recall this lemma

Theorem 3.1.5 (Weyl Lemma)) *[N, Th. 10.3.6; Cor. 10.3.11] Let $L : C^\infty(\mathbb{R}^n;\mathbb{R}^s) \to C^\infty(\mathbb{R}^n;\mathbb{R}^s)$ be an elliptic operator of order k,*

$$Lu = \sum_{|\beta|\leq k} A_\alpha(x)\,\partial^\alpha u(x)$$

with smooth coefficients $A_\alpha(x)$. Denote by $(\cdot|\cdot)$ the natural metric on Euclidean spaces. Let $p \in (1, \infty)$.

(1) *If $u \in L_{p,loc}(\mathbb{R}^n;\mathbb{R}^s)$, $v \in L_{p,loc}(\mathbb{R}^n;\mathbb{R}^s)$, and u is a L_p-weak solution of the equation*

$$Lu = v, \tag{3.1.3}$$

i.e.

$$\int_{\mathbb{R}^n} (u(x)\,|L^*\phi(x))\,dx = \int_{\mathbb{R}^n} (v(x)\,|\phi(x))\,dx, \quad \textit{for all } \phi \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^s),$$

where L^* is a formal adjoint to L, then u is an L_p-strong solution of (3.1.3), i.e. $u \in L_{k,p,loc}(\mathbb{R}^n;\mathbb{R}^s)$, and (3.1.3) hold almost everywhere, where the partial derivatives of u are understood in generalized [distributional] sense.

(2) *If $u \in L_{p,loc}(\mathbb{R}^n;\mathbb{R}^s)$ is an L_p-weak solution of (3.1.3) and v is smooth, then u must be smooth (particularly, u must be smooth if $Lu = 0$).*

Finally we can formulate the mentioned above axiom.

(9) $\mathcal{H}(W) = \mathcal{H}_d(W)$ (equivalently $\ker \bar{D} \subset W$).

Proposition 3.1.6 *Assume (1)-(7) and (9). Then the following conditions are equivalent.*

(i) all homomorphisms in the fundamental diagram (3.1.2) are isomorphisms,

(ii) $J_\#$ is an epimorphism (i.e. (8) holds),

(iii) $\mathbf{H}(\bar{W}_d)$ fulfils Poincaré duality (i.e. (7_H) holds),

(iv) the cohomology pairings $\langle\cdot|\cdot\rangle^r_{\mathbf{H},d}$ are weakly nondegenerated for each r (i.e. ($7'_H$) holds),

(v) there is a strong Hodge decomposition $\bar{W}^r = \mathcal{H}^r_d(W) \oplus \bar{d}\left[W_1^{r-1}\right] \oplus \bar{\delta}\left[W_1^{r+1}\right]$ for each r,

If some of these conditions hold then

$$\operatorname{Sig}(W) = \operatorname{Sig}(\bar{W}_d) = \operatorname{index} \bar{D}_+ = \operatorname{index} D_+$$

and the strong Hodge decomposition for W is satisfied, i.e.

$$W = \mathcal{H}(W) \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta = \mathcal{H}(W) \oplus \operatorname{Im} \Delta.$$

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are evident. For (iv) $\Rightarrow$ (v) see Theorem 2.2.6 (g2). We prove now that (iv) $\Rightarrow$ (i). From (iv) together with the condition $\dim \mathcal{H}_d(W) = \dim \mathcal{H}(W) < \infty$ we obtain that $\bar{W}$ is regular Hilbert anticommutative graded subdifferential Hodge space of even degree. By Theorem 2.2.6 the Hodge homomorphism x^r_d is an isomorphism, therefore the remaining homomorphisms in the diagram (3.1.2) are isomorphisms, too.

To complete the proof it is sufficient to check that (v) $\Rightarrow$ (ii). Let $P^r : \bar{W} \to \mathcal{H}_d$ denote the orthogonal projection with respect to the strong Hodge decomposition. To prove that $J^r_\# : \mathbf{H}^r(W) \to \mathbf{H}^r(\bar{W}_d)$ is an epimorphism, take arbitraly $u \in \bar{Z}^r_d := \ker\left(\bar{d}^r : \bar{W}^r_d \to \bar{W}\right)$. Then $u - P^r(u) \in \bar{B}^r = \operatorname{Im}\left(\bar{d}^{r-1}\right)$. Indeed, using the above decomposition there exists $v \in \bar{W}_1^{r-1}$ and $w \in \bar{W}_1^{r+1}$ such that

$$u = P^r(u) + \bar{d}^{r-1}(v) + \bar{\delta}^{r+1}(w).$$

Clearly, $\bar{\delta}^{r+1}(w) = u - P^r(u) - \bar{d}^{r-1}(v) \in \bar{W}_d$. Applying $\bar{d}^r$ on both sides of this equality we get

$$\begin{aligned} 0 &= \bar{d}^r(u) = \bar{d}^r(P^r(u)) + \bar{d}^r\left(\bar{d}^{r-1}(v)\right) + \bar{d}^r\left(\bar{\delta}^{r+1}(w)\right) \\ &= \bar{d}^r\left(\bar{\delta}^{r+1}(w)\right). \end{aligned}$$

The above implies (see Lemma 2.2.3)

$$(\bar{\delta}^{r+1}w|\bar{\delta}^{r+1}w) = (\bar{d}^r\bar{\delta}^{r+1}w|w) = 0,$$

then $\bar{\delta}^{r+1}w = 0$ and $u = P^r(u) + \bar{d}^{r-1}v$. Therefore $[u] = [P^r(u)]$ in the cohomology space $\mathbf{H}^r\left(\bar{W}_d\right)$ which means that $J_{\#} : \mathbf{H}(W) \to \mathbf{H}\left(\bar{W}_d\right)$ is an epimorphism, i.e. (8) holds. ■

Problem 3.1.7 *Assume that the equivalent conditions (i)-(v) hold. Does it then imply the axiom "sequence compactness" i.e.: for a sequence $\alpha_n \in W$ such that that $\|\alpha_n\| < C$ and $\|\Delta\alpha_n\| < C$ for some $C > 0$ there exists a Cauchy subsequence α_{n_k}?*

Thanks to the above Proposition (3.1.6) to prove that index $D_+ =$ index $\bar{D}_+$ we can go (for a given W) in three equivalent directions:

(I) to prove that $J_{\#}$ is an epimorphism, or

(II) to prove the Poincaré duality for $\mathbf{H}\left(\bar{W}_d\right)$ (or less that the cohomology pairings $\langle\cdot|\cdot\rangle^r_{\mathbf{H},d}$ are weakly nondegenerated), or

(III) to prove the strong Hodge decomposition for $\bar{W}$.

3.2 L_2-Hirzebruch operator for graded anticommutative differential Hodge vector bundles of even degree; applications to Lie algebroids

Let M be a connected compact oriented Riemannian manifold M. Take into consideration a graded anticommutative differential Hodge vector bundle of even degree $N = 2n$ (Def. 1.4.2)

$$\left(\xi = \bigoplus_{r=0}^{N=2n} \xi^r, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *, d\right). \tag{3.2.1}$$

Put $W = \operatorname{Sec}\xi$ and $W^r = \operatorname{Sec}\xi^r$. Let $\left(W = \bigoplus_{r=0}^{N} W^r, \langle\!\langle\cdot|\cdot\rangle\!\rangle, (\!(\cdot|\cdot)\!), *, d\right)$ be the induced a graded anticommutative differential Hodge space.

We can use L_2-theory [N]: $\bar{W} = L_2(\xi)$. Condition (6) from the previous subsection holds thanks to the so-called convolution argument, see below. Whence $\bar{W}$ together with $\bar{d}$ is a Hilbert anticommutative graded subdifferential Hodge space of even degree. The operator $\bar{d}$ remains local.

3.2.1 Convolution argument

In local analysis on $\mathbb{R}^n$ and global analysis on manifolds condition (6) holds for differential operators thanks the convolution argument using the so-called mollifying sequence. We briefly describe it here. Take any linear differential operator $d : \operatorname{Sec} \xi \to \operatorname{Sec} \eta$ of rank k (in our considerations we use the case of $k = 1$, only) between modules od cross-sections of vector bundles ξ and η.

First, we look at this locally. In arbitrary local base of the vector bundles and identifying coordinate neighbourhood U with open set in $\mathbb{R}^n$ any differential operator $d : \operatorname{Sec} \xi \to \operatorname{Sec} \eta$ of rank k can be written in the form ($p = \operatorname{rank} \xi$, $q = \operatorname{rank} \eta$)

$$d_U = \sum_{|\alpha| \leq k} A_\alpha (x) \partial^\alpha \quad \text{for} \quad A_\alpha \in C^\infty (U, \operatorname{Hom} (\mathbb{R}^p, \mathbb{R}^q)).$$

We exploit the convolution operation

$$f * g (x) = \int_{\mathbb{R}^n} f (x - y) g (y) dy,$$

and a mollifying sequence

$$\lambda_t (x) = \frac{1}{t} \varphi \left(\frac{x}{t} \right), \quad t > 0,$$

where $\varphi \in C_0^\infty (\mathbb{R}^n)$, $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi (x) dx = 1$, $\operatorname{supp} \varphi = B (0, 1)$. We have the well known fundamental theorems

Theorem 3.2.1 *The convolution operation has the following properties:*

- *for any $f \in L_{1,loc} (\mathbb{R}^n)$ the convolution $\lambda_t * f$ is a smooth, and if f has the distributional derivative $\frac{\partial f}{\partial x_i} \in L_{1,loc} (\mathbb{R}^n)$ then*

$$\frac{\partial (\lambda_t * f)}{\partial x_i} = \lambda_t * \frac{\partial f}{\partial x_i},$$

- *if $f \in L_p (\mathbb{R}^n)$, $p \geq 1$, then*

$$\lambda_t * f \xrightarrow[t \to 0]{L_p} f,$$

- *if additionally f has the distributional derivative $\frac{\partial f}{\partial x_i} \in L_p (\mathbb{R}^n)$ then*

$$\frac{\partial (\lambda_t * f)}{\partial x_i} \xrightarrow[t \to 0]{L_p} \frac{\partial f}{\partial x_i}.$$

We will use it for $p = 2$. Let d be a differential operator of rank k locally written by $d_U = \sum_{|\alpha| \leq k} A_\alpha (x) \partial^\alpha$ and let $A_\alpha (x)$ belong to L_2 (restricting U if needed). Then, according to the above theorem, for a sequence of functions

$\omega = \left(f^1, ..., f^p\right)$ of the class L_2 possesing distributional derivative of the class L_2 (i.e. $\omega \in L_{1,2}$) the elements of the sequence $\lambda_t * \omega := \left(\lambda_t * f^1, ..., \lambda_t * f^p\right)$ are smooth, and

$$\begin{aligned} d_U\left(\lambda_t * \omega\right) &= \sum_{|\alpha|\leq k} A_\alpha\left(x\right) \partial^\alpha\left(\lambda_t * \omega\right) \\ &= \sum_{|\alpha|\leq k} A_\alpha\left(x\right)\left(\lambda_t * \partial^\alpha \omega\right) \xrightarrow[t\to 0]{L_p} \sum_{|\alpha|\leq k} A_\alpha\left(x\right) \partial^\alpha\left(\omega\right) = d_U\left(\omega\right). \end{aligned}$$

On a compact manifold we take a partition of unity ρ_β subordinated to the suitable finite covering $\{U_\beta\}$ of M. Representing a cross-section ω of the vector bundle ξ in local trivializations $\xi_{U_\beta} \cong U_\beta \times \mathbb{R}^p$ we take $\lambda_t \tilde{*} \omega := \Sigma_\beta \lambda_t * \left(\rho_\beta \cdot \omega\right)$ (it depends on the trivializations) and we have

$$\lambda_t \tilde{*} \omega \xrightarrow[t\to 0]{L_2} \omega$$

and

$$\begin{aligned} d\left(\lambda_t \tilde{*} \omega\right) &= d\left(\Sigma_\beta \lambda_t * \left(\rho_\beta \cdot \omega\right)\right) \\ &= \Sigma_\beta d_{U_\beta}\left(\lambda_t * \left(\rho_\beta \cdot \omega\right)\right) \xrightarrow[t\to 0]{L_2} \Sigma_\beta d_{U_\beta}\left(\rho_\beta \cdot \omega\right) = d\left(\Sigma_\beta \rho_\beta \cdot \omega\right) = d\omega. \end{aligned}$$

Unfortunately, the operations $d(\cdot)$ and $\lambda_t \tilde{*} (\cdot)$ on a compact manifold do not commute even if the operator d has locally constant coefficients.

3.2.2 Condition (9)

We continue with our considerations of the graded anticommutative differential Hodge vector bundle of even degree (3.2.1). The operator $D = d + \delta$ is elliptic and the previously defined operator $\bar{D} = \bar{d} + \bar{\delta} : \bar{W}_1 \to \bar{W}$ (3.1.1) is a continuous Fredholm operator. According to Conclusion 3.1.3 we have $(\!(\bar{D}u|w)\!) = (\!(u|\bar{D}w)\!)$ for $u, w \in \bar{W}_1$, i.e

$$\int_M \left(\bar{D}u|w\right) = \int_M \left(u|\bar{D}w\right). \tag{3.2.2}$$

Let $u \in \bar{W}_1$ and $v := \bar{D}u \in \bar{W} = L_2(\xi)$. Considering integral equality (3.2.2) only for all smooth cross-sections $w \in W = \operatorname{Sec}\xi$ and taking into account that D is self-adjoint we see that u is a week solution of the equation $\bar{D}u = v$. Considering w supported in a coordinated neighbourhood we can use Weyl lemma and check that $\ker \bar{D} \subset \operatorname{Sec}\xi$, i.e. $\mathcal{H}_d(W) = \ker \bar{D}$ consists only smooth cross-sections which yields $\mathcal{H}(W) = \mathcal{H}_d(W) = \ker \bar{D} \subset \operatorname{Sec}\xi$. Whence (9) holds.

3.2.3 Condition (8)

Now, we pass to condition (8).

Method (III) Let $u \in \bar{W}_1$ and $v := \bar{D}u \in \bar{W} = L_2(\xi)$. The integral equality (3.2.2) used for all smooth cross-sections $w \in W = \operatorname{Sec} \xi$ and Theorem [N, Theorem 10.3.6.] (see Theorem 3.1.5) say that $u \in L_{1,2}(\xi)$, i.e. $\bar{W}_1 \subset L_{1,2}(\xi)$. The opposite inclusion follows from the more general fact that the so-called analytical realization of an elliptic operator is defined on $L_{1,2}(\xi)$ [N, Definition 10.4.3.]. There is the strong Hodge orthogonal decomposition [N, Th. 10.4.29]:

$$L_2(\xi^r) = \mathcal{H}^r(W) \bigoplus \operatorname{Im} \bar{d}_1^{r-1} \bigoplus \operatorname{Im} \bar{\delta}_1^{r+1}$$

where we view both $\bar{d}_1$ and $\bar{\delta}_1$ as bounded operators $L_{1,2}(\xi) \to L_2(\xi)$. The Theorem 3.1.6 implies that $J_\# : \mathbf{H}(W) \to \mathbf{H}(\bar{W}_d)$ is an epimorphism, i.e. (8) holds. Finally, in diagram 3.1.2 all arrows are isomorphisms. Hence $\operatorname{Sig}(W) = \operatorname{Sig}(\bar{W}_d) = \operatorname{index} \bar{D}_+ = \operatorname{index} D_+$.

Method (II) or (I) Sometimes we can use simpler algebraic topology methods (do not require results from elliptic operators). Our aim is to consider the four examples (see Subsection 1.4.1) $W = \Omega(M)$ for a manifold M, $W = \Omega(A)$ for a Lie algebroid A or $W = \Omega(M, E)$ for the Lusztig or the Gromov vector bundle E. We pass to the spaces of L_2-differential forms $\bar{W} = L_2(\bigwedge T^*M)$, $L_2(\bigwedge A^*)$, $L_2(\bigwedge T^*M \otimes E)$ and we obtain graded Hilbert subdifferential Hodge bundles. First, we start with a classical example.

Example 3.2.2 (classical) *N.Teleman has proved (using only algebraic topology methods and Whitney's complex of the so-called flat differential forms) that the cohomology pairings $\langle \cdot | \cdot \rangle^r_{\mathbf{H},d}$ are dualities for the space of L_2-differential forms on Lipschitz (especially on smooth) manifolds [T2, Th. 2.1 (iii)] (see Theorem 2.1.13 and subsequent sentences), i.e. (II) holds. We can also notice that $J_\# : \mathbf{H}(W) \to \mathbf{H}(\bar{W}_d)$ is an isomorphism, i.e. (I) holds since the associated differential sheaves of germs of smooth differential forms from $W = \Omega(M)$ and L_2-differential forms from $\bar{W}_d$ are fine and are resolutions of the real constant sheaf thanks the usual smooth Poincaré lemma and the L_2-Poincaré lemma given by L.Hörmander [H] in the version by N.Teleman [T1], compare [T2, Th. 2.1 (i)]. This ends the Hodge theory for smooth manifolds (using analysis only in the convolution argument, Weyl lemma and L_2-Poincaré lemma in the version by N.Teleman).*

We recall the L_2-Poincaré lemma.

Theorem 3.2.3 (L_2-Poincaré lemma) *[H], [T1]. If U is a convex domain in $\mathbb{R}^n$, for any differential form $u \in L^r_{2,loc}(U)$ such that $\bar{d}u = 0$ (in distributional sense), then exists a differential form $v \in H_1^{r-1}(U)$ such that $\bar{d}v = u$ in the strong sense.*

$H_1^{r-1}(U)$ denotes the space of all differential forms of degree $r - 1$ whose components belong to $H_{1,loc}(U)$.

Example 3.2.4 (Lusztig, Gromov) *The idea given in Example 3.2.2 can be used also in Lusztig and Gromov examples since the associated differential sheaves of germs of smooth and L_2-differential forms with values in the flat vector bundle (E, ∇) are fine and are resolutions of the sheaf of local ∇-constant cross-sections of E.*

In a more general case (including the Lie algebroid example) to prove (I) we can use the Mayer-Vietoris technique or spectral sequences and the comparison theorem.

Firstly, we describe an application of the Mayer-Vietoris technique (it is a simple case of the very general technique in algebraic topology of spectral sequences or sheaves). The MV-technique has limited application to the case when the associated presheave of cohomology is locally constant on a good covering. Hovewer, all four important examples considered in the paper (see Subsection 1.4.1) are of this nature. The main defect of this method is necessity to use cohomology on noncompact manifolds.

We set a graded anticommutative differential Hodge vector bundle $W = \left(\xi = \bigoplus_{r=0}^{N=2n} \xi^r, \langle\cdot|\cdot\rangle, (\cdot|\cdot), *, d\right)$ (3.2.1) of even degree $N = 2n$ over a connected finite type (compact) oriented Riemannian manifold M. Then $\bar{W} = L_2(\xi)$ together with $\bar{d} : L_2(\xi)_d \to L_2(\xi)_d$ is a Hilbert anticommutative graded subdifferential Hodge space. Since the operators d and $\bar{d}$ are local we can define (using compact support smooth cross-sections) the operators $d_U^r : (\mathrm{Sec}_2\, \xi_U^r)_d \to \left(\mathrm{Sec}_2\, \xi_U^{r+1}\right)_d$ (where $(\mathrm{Sec}_2(\eta))_d = \mathrm{Sec}(\eta) \cap (L_2(\eta))_d$ - the subspace of smooth L_2-integrated cross-sections u whose derivative du is also L_2-integrable) and a graded subspaces

$$(L_2(\xi_U))_d = \bigoplus_{r=0}^{N=2n} (L_2(\xi_U^r))_d$$

where $(L_2(\xi_U^r))_d$ consists of all $u \in L_2(\xi_U^r)$ such that there exists $u' \in L_2\left(\xi_U^{r+1}\right)$ satisfying

$$\int_U \langle u'|\phi\rangle = (-1)^{r+1} \int_U \langle u|d_U\phi\rangle, \ \phi \in \mathrm{Sec}_0\left(\xi_U^{N-r-1}\right)$$

and operators

$$\begin{gathered}\bar{d}_U^r : (L_2(\xi_U^r))_d \longrightarrow \left(L_2\left(\xi_U^{r+1}\right)\right)_d, \\ \bar{d}_U^r(u) := u'.\end{gathered}$$

The manifold M as a compact one is of finite type (i.e. can be covered by finite good covering $\mathfrak{U} = \{U_\beta\}_{\beta\in\{1,..,m\}}$). Now we define a category $\mathcal{O}_M$ of open finite type subsets $U \subset M$. The morphisms of this category are inclusions $i_{U,V} : U \hookrightarrow V$. The Mayer-Vietoris principle [N, Theorem 7.1.41] says that if $\mathcal{F}$ and $\mathcal{G}$ are two MV-funtors from $\mathcal{O}_M$ to the category of graded vector spaces and Φ is a natural transformation of $\mathcal{F}$ and $\mathcal{G}$ such that $\Phi_U^r : \mathcal{F}^r(U) \to \mathcal{G}^r(U)$ is an isomorphisms (for any r) for each $U \cong \mathbb{R}^n$ then Φ is a natural equivalence. Particularly $\Phi_M^r : \mathcal{F}^r(M) \to \mathcal{G}^r(M)$ is an isomorphisms for any r. In application to our problem consider open subsets $U, V, U \cup V \in \mathcal{O}_M$ and take into consideration two standartly defined short Mayer-Vietoris sequences of cochain complexes (for smooth and L_2-category)

$$0 \longrightarrow \left(\mathrm{Sec}_2\left(\xi^r_{U\cup V}\right)\right)_d \longrightarrow \left(\mathrm{Sec}_2\left(\xi^r_{U}\right)\right)_d \oplus \left(\mathrm{Sec}_2\left(\xi^r_{V}\right)\right)_d \longrightarrow \left(\mathrm{Sec}_2\left(\xi^r_{U\cap V}\right)\right)_d \longrightarrow 0$$

$$0 \longrightarrow \left(L_2\left(\xi^r_{U\cup V}\right)\right)_d \longrightarrow \left(L_2\left(\xi^r_{U}\right)\right)_d \oplus \left(L_2\left(\xi^r_{V}\right)\right)_d \longrightarrow \left(L_2\left(\xi^r_{U\cap V}\right)\right)_d \longrightarrow 0$$

They produce suitable connecting morphisms ∂^r and and therefore two MF-functors

$$\begin{aligned} \mathcal{F}(U) &= \bigoplus\nolimits_{r=0}^{N} \mathbf{H}^r\left(\left(\mathrm{Sec}_2\left(\xi^\bullet_{U\cap V}\right)\right)_d, d\right), \\ \mathcal{G}(U) &= \bigoplus\nolimits_{r=0}^{N} \mathbf{H}^r\left(\left(L_2\left(\xi^\bullet_{U\cap V}\right)\right)_{d,c}, \bar{d}\right) \end{aligned}$$

for which morphisms are determined by inlusions and restrictions.

There is a natural transformation of functors $\mathcal{F}$ and $\mathcal{G}$

$$\Phi(U) : \mathbf{H}^r\left(\left(\mathrm{Sec}_2\left(\xi^\bullet_U\right)\right)_d, d\right) \longrightarrow \left(\mathbf{H}^r\left(\left(L_2\left(\xi^\bullet_U\right)\right)_d, \bar{d}\right)\right), \quad U \in \mathcal{O}_M.$$

The Mayer-Vietoris principle yields the following theorem.

Theorem 3.2.5 *The natural transformation* $\Phi(U)$ *is a natural equivalence (particularly* $\Phi(M) : \mathbf{H}^r\left(\left(\mathrm{Sec}\left(\xi^\bullet\right)\right), d\right) \to \left(\mathbf{H}^r\left(\left(L_2\left(\xi^\bullet\right)\right)_d, \bar{d}\right)\right)$ *is an isomorphism) provided that* $\Phi(U)$ *is an isomorphism for each* $U \cong \mathbb{R}^n$.

Applications to Examples 1-3 from the list given in Subsection 1.4.1 are given in Examples (3.2.2) and (3.2.4) [via the language of sheaves]. To consider the last example of a Lie algebroid it is sufficient to notice that Künneth smooth formula for trivial Lie algebroid $A = TU \times \mathfrak{g}$ [K2, Lemat 6.1 and Corollary 6.1] can be written for $L_2(A)_d$-differential forms: $L_2(U)_d \otimes \Omega(\mathfrak{g}) = L_2(A)_d$, $\mathbf{H}\left(L_2(U)_d \otimes \Omega(\mathfrak{g})\right) = \mathbf{H}\left(L_2(A)_d\right)$ therefore using the case for differential forms on manidolds we can obtain a results for Lie algebroids. The identical results can be obtained via the spectral sequence argument and the following comparison theorem used for the Čech-de Rham double complexes.

- If there exist spectral sequences $E^{j,i}_p$ for W and $\bar{E}^{j,i}_p$ for $\bar{W}_d$ converging to $\mathbf{H}(W)$ and $\mathbf{H}\left(\bar{W}_d\right)$, respectively, such that the homomorphism $J : W \to \bar{W}_d$ induces an isomorphism on the first terms $J^{j,i}_1 : E^{j,i}_1 \xrightarrow{\cong} \bar{E}^{j,i}_1$ then J induces isomorphism on cohomology $J_\# : \mathbf{H}(W) \xrightarrow{\cong} \mathbf{H}\left(\bar{W}_d\right)$.

Independently, we can define the Hoshchild-Serre type spectral sequence for L_2-theory on TUIO-Lie algebroids (for smooth case see [K-M-3]) and use the comparison theorem for the second terms. Consider a graded cochain group

$$L_2(A)_d = \bigoplus\nolimits_r L^r_2(A)_d, \quad L^r_2(A)_d = L_2\left(\bigwedge\nolimits^r A^*\right)_d$$

of L_2-differential forms on A possessing derivative of the class L_2 with the distributional derivative operator $\bar{d}_A$ of degree 1

$$\bar{d}^r_A : L_2\left(\bigwedge\nolimits^r A^*\right)_d \longrightarrow L_2\left(\bigwedge\nolimits^{r+1} A^*\right)_d.$$

Each cochain $f \in L_2\left(\bigwedge^r A^*\right)_d$ is a measurable r-differential form $f : \operatorname{Sec} A \times \ldots \times \operatorname{Sec} A \to L_1(M)$ ($f\left(v^1, ..., v^r\right)$ is a measurable function such that

$$\int_M \left|f\left(v^1, ..., v^r\right)\right| < \infty$$

for $v^i \in \operatorname{Sec} A$). In the space $L_2(A)_d$ we have the Hochschild-Serre filtration $A_j := \left(L_2(A)_d\right)_j \subset L_2(A)_d$ as follows: $A_j = \left(L_2(A)_d\right)_j = L_2(A)_d$ for $j \leq 0$. If $j > 0$, $A_j = \left(L_2(A)_d\right)_j = \bigoplus_{r \geq j} A_j^r$, where $A_j^r := \left(L_2^r(A)_d\right)_j$ consists of all those r-cochains f for which $f\left(v^1, ..., v^r\right) = 0$ whenever $r - j + 1$ of the arguments v^i belongs to $\operatorname{Sec} \boldsymbol{g}$. In this way we have obtained a graded filtered differential $\mathbb{R}$-vector space

$$\left(L_2(A)_d = \bigoplus_r L_2^r(A)_d, \bar{d}_A, A_j\right)$$

and we can use its spectral sequence

$$\left(\bar{E}_s^{j,i}, \bar{d}_s\right).$$

Take auxiliarily $\tilde{\alpha}_0^{j,i} : A_j^{j,i} \to L_2^j\left(M; \bigwedge^i \boldsymbol{g}^*\right)$, by definition

$$\tilde{\alpha}_0^{j,i}(f)(X_1, ..., X_j)(\sigma_1, ..., \sigma_i) = f(\sigma_1, ..., \sigma_i, \lambda X_1, ..., \lambda X_j)$$

where $\lambda : TM \to A$ is a fixed (smooth) connection in A and put

$$L_2^j\left(M; \bigwedge^i \boldsymbol{g}^*\right)_d := \operatorname{Im} \tilde{\alpha}_0^{j,i}.$$

The homomorphism

$$\alpha_0^{j,i} : \bar{E}_0^{j,i} = A_j^{j+i}/A_{j+1}^{j+i} \longrightarrow L_2^j\left(M; \bigwedge^i \boldsymbol{g}^*\right)_d, \ [f] \longmapsto \tilde{\alpha}_0^{j,i}(f),$$

is an isomorphism. This generalize the smooth case ([K-M-3]). Through isomorphism $\alpha_0^{j,i}$ the differential $\bar{d}_0^{j,i}$ becomes a differentiation of values with respect to the usual Chevalley–Eilenberg differential $d_{\mathfrak{g}}$. From the above we obtain isomorphisms

$$\alpha_1^{j,i} = \left(\alpha_0^{j,i}\right)_\# : \bar{E}_1^{j,i} = \mathbf{H}^i\left(\bar{E}_0^{j,*}, \bar{d}_0^{j,*}\right) \xrightarrow{\cong} L_2^j\left(M, \mathbf{H}^i(\boldsymbol{g})\right)_d$$

and $L_2^j\left(M, \mathbf{H}^i(\boldsymbol{g})\right)_d$ is equal to those L_2^j-differential forms with values in the bundle $\mathbf{H}^i(\boldsymbol{g})$ for which there exists the distributional derivative $\bar{d}_{\nabla^i}$ where ∇^i is the standard flat structure in the vector bundle $\mathbf{H}^i(\boldsymbol{g})$ (see [K3]). Through isomorphism $\alpha_1^{j,i}$ the differential $\bar{d}_1^{j,i}$ becomes a derivative $(-1)^i \bar{d}_{\nabla_\iota}$. In consequence

$$\begin{aligned}\bar{E}_2^{j,i} &\xrightarrow{\cong} \mathbf{H}^j\left(L_2^*\left(M, \mathbf{H}^i(\boldsymbol{g})\right)_d, \bar{d}_{\nabla^i}\right) \underset{\cong}{\overset{(\text{Ex. } 3.2.4)}{\longrightarrow}} \mathbf{H}^j\left(M, \mathbf{H}^i_{\nabla^i}(\boldsymbol{g})\right) \\ = \ & E_2^{j,i} \text{ (the second term of Hochschild-Serre spectral sequence for smooth case).}\end{aligned}$$

Therefore $J_2^{j,i} : E_2^{j,i} \xrightarrow{\cong} \bar{E}_2^{j,i}$ is an isomorphism.

3.3 Applications to Lie algebroids

For our four examples we have

Conclusion 3.3.1 *Consider the four examples $W = \Omega(M)$ for a manifold M, $W = \Omega(A)$ for a Lie algebroid A or $W = \Omega(M, E)$ for the Lusztig or the Gromov vector bundle E. Now we pass to the spaces of L_2-differential forms $\bar{W} = L_2(\bigwedge T^*M)$, $L_2(\bigwedge A^*)$, $L_2(\bigwedge T^*M \otimes E)$. We obtain graded Hilbert subdifferential Hodge bundles.*

In consequence, the signature in that four cases can be calculated via two Hirzebruch signature operators, the usual smooth and second of the L_2 class. So for a Lie algebroid we have four Hirzebruch operators, two smooth using the space $\Omega(A)$ and $\Omega(M, E)$ [Lusztig or Gromov subcases] and two L_2 Hirzebruch operators, using graded Hilbert subdifferential Hodge spaces $L_2(\bigwedge A^*)$, or $L_2(\bigwedge T^*M \otimes E)$, respectively.

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