# The signature operator for graded Hilbert subdifferential Hodge spaces

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# 1 Hodge spaces

The purpose of my talk is to present the algebraic aspect of the Hodge theory and of the Hirzebruch signature operator. Of course in suitable places the real analysis is needed, but I would like to avoid analysis as far as possible.

Let $W$ be a real vector space of an arbitrary dimension (finite or infinite).

**Definition 1** *By a* ***Hodge space*** *we mean the system*

$$(W, \langle , \rangle , ( , ) , *_W)$$

*where* $\langle , \rangle$ , $( , ) : W \times W \to \mathbb{R}$ *are* 2*-linear homomorphisms such that*

*(1)* $( , )$ *is symmetric and positive definite (i.e. it is an inner product),*

*(2)* $*_W : W \to W$ *(called* $*$*-Hodge operator) is a linear mapping such that,*

- $*_W$ *is an isometry with respect to* $( , )$ ,
- *for all* $v \in W$, $\langle v, w \rangle = (v, *_W (w))$ .

Clearly, the $*$-Hodge operator $*_W$ is uniquely determined (if exists).

- If $(V, \langle\cdot,\cdot\rangle_V, (\cdot,\cdot)_V, *_V)$ and $(W, \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_W, *_W)$ are Hodge spaces then their tensor product

$$(V \otimes W, \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_V \otimes (\cdot,\cdot)_W, *_V \otimes *_W)$$

  is a Hodge space (i.e. $*_{V\otimes W} = *_V \otimes *_W$ ).

- Let $(W, \langle\cdot,\cdot\rangle)$ be a finite dimensional real vector space equipped with a 2-linear tensor $\langle\cdot,\cdot\rangle : W \times W \to \mathbb{R}$. Then there exists an inner product $(\cdot,\cdot)$ and operator $*_W$ such that the system $(W, \langle,\rangle, (,), *_W)$ is a Hodge space if and only if there exists a basis of $W$ in which the matrix of $\langle,\rangle$ is orthogonal.

— The inner product and the $*$-Hodge operator play an auxiliary role in the study of properties of the pairing $\langle,\rangle$ .

Now we give a some examples of finite dimensional Hodge spaces.

**Example 2 (Classical)** *Let* $(V, G)$ *be a real* $N$*-dimensional oriented vector space with an inner product* $G : V \times V \to \mathbb{R}$ *[i.e. an Euclidean space]. We identify* $\bigwedge^N V = \mathbb{R}$ *via arbitrary positive ON-base* $\{e_i\}_{i=1}^N$ *of* $V$*. We have the classical Hodge space*

$$\left( \bigwedge V = \bigoplus_{K=0}^{N} \bigwedge^k V, \langle , \rangle , ( , ) , * \right)$$

*where for* $v^r \in \bigwedge^r V$

$$\langle v^r, v^s \rangle = \begin{cases} v^r \wedge v^s \in \bigwedge^N V = \mathbb{R}, & \text{if } \ r + s = N, \\ 0 & \text{if } \ r + s \neq N, \end{cases} ,$$

*ffor* $v_i, w_i \in V$

$$(v_1 \wedge .... \wedge v_k, w_1 \wedge ... \wedge w_k) = \det \left[ G \left( v_i, w_k \right) \right],$$

*the subspaces* $\bigwedge^k V$ *are orthogonal,* $* \left( e_{i_1} \wedge ... \wedge e_{i_k} \right) = \varepsilon_{(j_1, ..., j_{n-k})} \cdot e_{j_1} \wedge ... \wedge e_{j_{N-k}}$*, the sequence* $j_1 < ... < j_{N-k}$ *is complementary to* $i_1 < ... < i_k$ *and* $\varepsilon_{(j_1, ..., j_{n-k})} = \operatorname{sgn} \left( j_1, ..., j_{n-k}, i_1, ..., i_k \right).$

**Example 3 (Lusztig example, 1972)** *[L] Let* $\langle , \rangle : E \times E \to \mathbb{R}$ *be a* *symmetric (indefinite in general) nondegenerated scalar product on a finite dimensional vector space* $E$. *Let* $G$ *be an arbitrary positive inner product in* $E$. *Then there exists exactly one direct sum decomposition* $E = E_+ \oplus E_-$ *which is ON with respect to the both scalar products* $\langle , \rangle$ *and* $G$ *and such that* $\langle , \rangle$ *on* $E_+$ *is positive definite and on* $E_-$ *is negative definite. We denote by* $*_E$ *the involution* $*_E : E \to E$ *such that*

$$*_E|E_+ = id, \qquad *_E|E_- = -id.$$

*Then, the quadratic form*

$$\begin{aligned} (,) \; &: \; E \times E \to \mathbb{R} \\ (v, w) \; &: \; = \langle v, *_E w \rangle \end{aligned}$$

*is symmetric and positive definite. The involution* $*_E$ *is an isometry, therefore*

$$(E, \langle , \rangle , (,) , *_E)$$

*is a Hodge-space.*

**Example 4 (Gromov example, 1995)** *[Gro] Let* $\langle , \rangle : E \times E \to \mathbb{R}$, *be a symplectic form on a finite dimensional vector space* $E$ *[i.e. skew-symmetric and nondegenerated]. There exists an anti-involution* $\tau$ *in* $E$, $\tau^2 = -id$ *(i.e. a complex structure) such that*

$$\langle \tau v, \tau w \rangle = \langle v, w \rangle , \quad v, w \in E,$$

$$\langle v, \tau v \rangle > 0 \quad \textit{for all} \quad v \neq 0.$$

*Then the tensor*

$$\begin{aligned} (,) &: E \times E \to \mathbb{R} \\ (v, w) &: = \langle v, \tau w \rangle \end{aligned}$$

*is symmetric and positive defined and* $(\tau v, \tau w) = (v, w)$. *The system*

$$(E, \langle , \rangle , (,) , -\tau)$$

*is a Hodge-space [since* $-\tau$ *is an isometry and* $\langle v, w \rangle_0 = (v, -\tau w)$*].*

Infinite dimensional Hodge spaces can be given on the space of cross-sections of vector bundles (and using differential forms)

**Definition 5** *By the* <u>*Hodge vector bundle*</u> *(or a vector bundle of finitely dimensional Hodge spaces) we mean a system*

$$(\xi, \langle , \rangle, ( , ), *_E)$$

*consisting of a vector bundle* $\xi$ *and two smooth tensor fields*

$$\langle , \rangle, ( , ) : \xi \times \xi \to M \times \mathbb{R}$$

*and linear homomorphism of vector bundles*

$$*_E : \xi \to \xi,$$

*such that for each* $x \in M$ *the system*

$$(\xi_x, \langle , \rangle_x, ( , )_x, *_{E_x})$$

*is a finitely dimensional Hodge space.*

**Example 6 (of Hodge vector bundles)**

- $\xi = \bigwedge T^*M$ *for a Riemannian manifold* $M$*; we use classical Hodge space* $\bigwedge T_x^*M$ *for cotangent space* $T_x^*M$.

- *Lusztig example of a vector bundle* $\xi$ *with nondegenerated indefinite symmetric* 2*-linear tensor field* $\langle , \rangle$ *and a flat covariant derivative* $\nabla$ *such that the tensor* $\langle , \rangle$ *is parallel* $\partial_X \langle \sigma, \eta \rangle = \langle \nabla_X \sigma, \eta \rangle + \langle \sigma, \nabla_X \eta \rangle$ ,

- *Gromov example of a vector bundle* $\xi$ *with a symplectic form* $\langle , \rangle$ *and with flat covariant derivative such that* $\langle , \rangle$ *is parallel,*

- *Tensor product* $\bigwedge T^*M\otimes\xi$ *of a Riemann manifold* $M$ *with arbitrary Hodge vector bundle* $\xi$*. This example produces an important and an infinitely dimensional Hodge space on the module of differential forms using derivative of differential* $\xi$*-valued differential forms*

$$\left(\Omega\left(M;\xi\right),\left\langle\left\langle\alpha,\beta\right\rangle\right\rangle,\left(\left(\alpha,\beta\right)\right),*\right)$$

$$\begin{aligned}\left\langle\left\langle\alpha,\beta\right\rangle\right\rangle &= \int_M\left\langle\alpha_x,\beta_x\right\rangle dM=\int_M\alpha\wedge_\varphi\beta,\quad \left(\left(\alpha,\beta\right)\right)=\int_M\left(\alpha_x,\beta_x\right)dM,\\ \left(*\beta\right)_x &= *_{E_x}\left(\beta_x\right),\end{aligned}$$

  *where* $\alpha\wedge_\varphi\beta$ *is the wedge product of differential forms with respect to the multiplication* $\left\langle,\right\rangle_x$ *of the values. The 2-form* $\left(\left(\cdot,\cdot\right)\right)$ *is symmetric and positive definite.*

- *The Lusztig and Gromov examples of Hodge vector spaces are very important for the calculation of the signature of transitive Lie algebroids (thanks to some spectral sequence argument) about which I would like to mention in the end of my talk.*

## 1.1 Graded differential Hodge space

**Definition 7** *By a graded differential Hodge space we mean a system*

$$\left(W=\bigoplus_{k=0}^{N}W^{k},\left\langle \cdot,\cdot\right\rangle ,\left(\cdot,\cdot\right),\ast,d\right)$$

*where* $(W,\left\langle \cdot,\cdot\right\rangle ,\left(\cdot,\cdot\right),\ast)$ *is a Hodge space (finitely or infinitely dimensional) and*

*(1)* $\left\langle \cdot,\cdot\right\rangle |W^{k}\times W^{r}=0$ *if* $k+r\neq N$. *(notation:* $\left\langle ,\right\rangle ^{k}:=\left\langle ,\right\rangle |W^{k}\times W^{N-k}$ *),*

*(2)* $W^{k}$ *are orthogonal with respect to* $\left(\cdot,\cdot\right)$, $\left(W,\left\Vert \cdot\right\Vert =\sqrt{\left(\cdot,\cdot\right)}\right)$ *is not a Hilbert in general.*

*(3) the operator* $d$*, called derivative, is homogeneous of degree* $+1$*, i.e.* $d:W^{k}\rightarrow W^{k+1}$*, and* $d^{2}=0$*,*

*(4)* $\left\langle dw^{k},u^{N-k-1}\right\rangle =\left(-1\right)^{k+1}\left\langle w^{k},du^{N-k-1}\right\rangle$ *for* $w^{k}\in W^{k}$, $u^{N-k-1}\in W^{N-k-1}$.

Clearly,

a) the induced cohomology pairing

$$\langle , \rangle_{\mathbf{H}}^{k} : \mathbf{H}^{k}(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}, \qquad ([u], [v]) \longmapsto \langle [u], [w] \rangle_{\mathbf{H}}^{k} := \langle u, w \rangle^{k},$$

is correctly defined,

b) $* \left[ W^{k} \right] \subset W^{N-k}$, and $\quad * : W^{k} \to W^{N-k} \quad$ is an isomorphism,

c) the above condition (4) in examples coming from differential geometry on an $N$-dimensional compact manifold $M$ without boundary follows from the Stokes theorem $\int_M d\omega^{n-1} = 0$, indeed

$$d\left(\omega^{k} \wedge \eta^{N-k-1}\right) = d\omega^{k} \wedge \eta^{N-k-1} + (-1)^{k} \omega^{k} \wedge d\eta^{N-k-1},$$

so

$$\left\langle d\omega^{k}, \eta^{N-k-1} \right\rangle := \int_M d\omega^{k} \wedge \eta^{N-k-1} = (-1)^{k+1} \int_M \omega^{k} \wedge d\eta^{N-k-1} =: (-1)^{k+1} \left\langle \omega^{k}, d\eta^{N-k-1} \right\rangle.$$

To considering the cohomology pairing in the middle degree (to give a notion of a signature) we assume additionally

(5) $N = 2n$ is even, and the tensor $\langle \cdot, \cdot \rangle$ is anticommutative in the sense

$$\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle, \quad \text{where } \varepsilon_k = (-1)^n (-1)^{k(N-k-1)} = (-1)^n (-1)^k .$$

Particularly, $\varepsilon_n = +1$ so the tensor $\langle , \rangle^n$ on $W^n$ is symmetric.

— Then we say that the space is an **evenly graded anticommutative differential Hodge space.**

— The comomology pairing in the middle degree

$$\langle , \rangle_{\mathbf{H}}^n : \mathbf{H}^n (W) \times \mathbf{H}^n (W) \to \mathbb{R}$$

is symmetric, and

— if $\dim \mathbf{H}^n (W) < \infty$ then we can define the signature of $\langle , \rangle_{\mathbf{H}}^n$.

**Definition 8** *If* $\dim \mathbf{H}^n (W) < \infty$ *we define the signature of* $W$ *as the signature of* $\langle \cdot, \cdot \rangle_{\mathbf{H}}^n$

$$\operatorname{Sig} (W) := \operatorname{Sig} \langle \cdot, \cdot \rangle_{\mathbf{H}}^n .$$

**Remark 9** *The finitness of the dimension of the cohomology space* $\mathbf{H}(W)$ *(in all dimensions) follows from the* ***Poincaré duality,*** *therefore in a suitable place in the second part of my talk we can assume the* ***Poincaré duality.*** *For standard cohomology algebra of differential forms on compact smooth manifold the* ***Poincaré duality*** *is easily to obtain by simple argument in algebraic topology: via the so-called Mayer-Vietoris sequences.*

**Proposition 10** *Let* $\left(W = \bigoplus_{k=0}^{N} W^k, \langle\cdot,\cdot\rangle, (\cdot,\cdot), *, d\right)$ *be an* ***evenly graded anticommutative differential Hodge space.*** *Then*

*1)* $**\left(w^k\right) = \varepsilon_k \cdot w^k$,

*2) the linear operator* $\delta : W^k \to W^{k-1}$ *[called* ***coderivative****] defined by*

$$\delta^k\left(w^k\right) = \varepsilon_k (-1)^k * d * \left(w^k\right), \quad w^k \in W^k,$$

*is the adjoint operator*

$$(\delta(w_1), w_2) = (w_1, d(w_2)),$$

*3) the* ***Laplacian*** $\Delta := (d+\delta)^2 = d\delta + \delta d$ *is homogeneous of degree* $0$, *self-adjoint* $(\Delta v, w) = (v, \Delta w)$, *and nonnegative* $(\Delta v, v) \geq 0$.

**Definition 11** *A vector $v \in W$ is called **harmonic** if*

$$dv = 0 \quad \textit{and} \quad \delta v = 0.$$

*Denote*

$$\begin{aligned} \mathcal{H}(W) &= \{v \in W;\ dv = 0,\ \delta v = 0\}, \\ \mathcal{H}^k(W) &= \left\{v \in W^k;\ d^k v = 0,\ \delta^k v = 0\right\}. \end{aligned}$$

- The harmonic vectors form a graded vector space $\mathcal{H}(W) = \bigoplus_{k=0}^{N} \mathcal{H}^k(W)$.
- $\mathcal{H}^k(W) = \ker \Delta^k$ and $\mathcal{H}(W) = \ker \Delta = (\operatorname{Im} \Delta)^{\perp}$.
- The spaces $\ker \Delta^k$ and $\operatorname{Im} d^{k-1}$ are orthogonal, therefore the inclusion

  $$\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$$

  induces a monomorphism (called the Hodge homomorphism)

  $$\mathcal{H}^k(W) = \ker \Delta^k \rightarrowtail \mathbf{H}^k(W) := \ker d^k / \operatorname{Im} d^{k-1}.$$

**Problem 12** *When the Hodge homomorphism is an isomorphism? i.e. when in each cohomology class there is (exactly one) a harmonic vector?*

**Theorem 13** *If*

$$\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp} \quad \textit{it means if } W = \operatorname{Im}\Delta \bigoplus \ker\Delta = \operatorname{Im}\Delta \bigoplus \mathcal{H}(W)},$$

*then*

- $W^k = \ker\Delta^k \bigoplus \operatorname{Im} d^{k-1} \bigoplus \operatorname{Im}\delta^{k+1}$ *(strong Hodge decomposition),*
- $\ker d^k = \ker\Delta^k \bigoplus \operatorname{Im} d^{k-1}$, *in particular, the Hodge homomorphism is an isomorphism*

  $$\mathcal{H}^k(W) = \ker\Delta^k \xrightarrow{\cong} \ker d^k / \operatorname{Im} d^{k-1} = \mathbf{H}^k(W),$$

  *It means that in each cohomology class there is exactly one harmonic vector.*

- ***(Poincaré Duality Theorem)***

  $\mathbf{H}^k(W) \simeq \mathbf{H}^{N-k}(W)$, *and* $\mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$ *is nondegenerated and* $\underline{\dim\mathbf{H}(W) < \infty}$.

In important examples belonging to differential geometry (standard, Lie algebroid, Lusztig's and Gromov's examples) the condition $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$ holds thanks to the fact that the Laplacian $\Delta$ is an elliptic operator.

**Theorem 14** *Let $\xi$ be a Riemannian vector bundle over a compact oriented Riemannian manifold $M$. If $\Delta : \operatorname{Sec}\xi \to \operatorname{Sec}\xi$ is a* ***self-adjoint nonnegative*** ***elliptic*** ***operator*** *then* $\ker\Delta$ *is a finite dimensional space and*

$$\operatorname{Sec}\xi = \operatorname{Im}\Delta \bigoplus \ker\Delta = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}.$$

*In particular, this holds if $\Delta$ comes from an elliptic complex*

$$\ldots \xrightarrow{d^{k-1}} \Gamma\left(\xi^{k}\right) \xrightarrow{d^{k}} \Gamma\left(\xi^{k+1}\right) \xrightarrow{d^{k+1}} \ldots.$$

$$\xi = \bigoplus \xi^{k}$$

*(as for example in our four cases below).*

## 1.2 Signature and the Hirzebruch operator

Consider an **evenly graded anticommutative differential Hodge space**,

$$\left(W=\bigoplus_{k=0}^{N=2n}W^k,\langle\cdot,\cdot\rangle,(\cdot,\cdot),*,d\right).$$

We restrict the positive definite product $(\cdot,\cdot):W^k\times W^k\to\mathbb{R}$ to the space of harmonic vectors

$$(\cdot,\cdot)_{\mathcal{H}}:\mathcal{H}^k(W)\times\mathcal{H}^k(W)\to\mathbb{R},$$

and we restrict the tensor $\langle\cdot,\cdot\rangle:W^k\times W^{N-k}\to\mathbb{R}$ also to harmonic vectors

$$\mathcal{B}^k=\langle\cdot,\cdot\rangle_{\mathcal{H}}:\mathcal{H}^k(M)\times\mathcal{H}^{N-k}(M)\to\mathbb{R}.$$

**Theorem 15** *There exists exactly an operator $\tau:W\to W$ such that*

*i)* $\tau\left(u^k\right)=\tilde{\varepsilon}_k\cdot *_W\left(u^k\right)$, *where* $\tilde{\varepsilon}_k\in\{-1,+1\}$,

*ii)* $\tau\circ\tau=Id$, *iii)* $\delta=-\tau\circ d\circ\tau$,

*iv)* $\tau^n=*_W$, *i.e.* $\tilde{\varepsilon}_n=1$.

*The operator $\tau$ fulfilling i)-iv) is defined by*

$$\tau\left(u^{k}\right)=\tilde{\varepsilon}_{k}\cdot *_{W}\left(u^{k}\right), \quad \textit{for} \quad \tilde{\varepsilon}_{k}\in(-1)^{\frac{k(k+1)}{2}}(-1)^{\frac{n(n+1)}{2}}.$$

*In particularly*

- *If $N=4p$ then $\varepsilon_{k}=(-1)^{k}$ and $\tilde{\varepsilon}_{k}=(-1)^{\frac{k(k+1)}{2}}(-1)^{p}$.*
- *If $N=4p+2$ then $\varepsilon_{k}=-(-1)^{k}$ and $\tilde{\varepsilon}_{k}=-(-1)^{\frac{k(k+1)}{2}}(-1)^{p}$.*

Concider the symmetric and nondegenerated quadratic forms in the middle degree.

- $\langle\cdot,\cdot\rangle^{n}: W^{n}\times W^{n}\rightarrow\mathbb{R}$ ,
- $\langle\cdot,\cdot\rangle_{\mathcal{H}}^{n}=\mathcal{B}^{n}:\mathcal{H}^{n}(W)\times\mathcal{H}^{n}(W)\rightarrow\mathbb{R}$ ,
- $\langle\cdot,\cdot\rangle_{\mathbf{H}}^{n}:\mathbf{H}^{n}(W)\times\mathbf{H}^{n}(W)\rightarrow\mathbb{R}$.

**Remark 16** *Under the assumption*

$$\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}}$$

*we have* $\mathcal{H}^n(W) \cong \mathbf{H}^n(W)$, *therefore* $\mathcal{B}^n = \langle\cdot,\cdot\rangle_{\mathcal{H}}^n = \langle\cdot,\cdot\rangle_{\mathbf{H}}^n$. *Then we have*

$$\operatorname{Sig}(W) = \operatorname{Sig}\langle\cdot,\cdot\rangle_{\mathbf{H}}^n = \operatorname{Sig}\mathcal{B}^n.$$

We put

$$W_{\pm} = \{w \in W;\ \tau w = \pm w\},$$

the eigenspaces corresponding to the eigenvalues $+1$ and $-1$ of $\tau$. Denote the operator $D = d + \delta$. We notice that

$$D[W_+] \subset W_-.$$

**Definition 17** *The operator*

$$D_+ = d + \delta : W_+ \to W_-$$

*is called the* ***Hirzebruch operator*** *(or the signature operator).*

Take the adjoint one to $D_+$,

$$\begin{aligned} D_+^* &= D_- : W_- \to W_+, \\ D_- &= d + \delta : W_- \to W_+. \end{aligned}$$

Clearly, $\ker(D_+) \subset \ker(d+\delta) = \mathcal{H}(W)$, and $\ker(D_-) \subset \ker(d+\delta) = \mathcal{H}(W)$.

**Remark 18** *In general case if* $\dim \mathcal{H}(W) < \infty$ *then the index*

$$\operatorname{Ind} D_+ := \dim_{\mathbb{R}} \ker(D_+) - \dim_{\mathbb{R}} \ker(D_-)$$

*is correctly defined (the dimensions are finite).*

Simple algebraic arguments give the fundamental

**Theorem 19 (Hirzebruch Signature Theorem)** *If* $\dim \mathcal{H}(W) < \infty$*, then*

$$\operatorname{Ind} D_+ = \operatorname{Sig}\left(\mathcal{B}^n : \mathcal{H}^n(W) \times \mathcal{H}^n(W) \to \mathbb{R}\right).$$

*If, additionally,* $\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}}$*, then*

$$\operatorname{Ind} D_+ = \operatorname{Sig} \mathcal{B}^n = \operatorname{Sig} \langle \cdot, \cdot \rangle_{\mathbf{H}}^n = \operatorname{Sig} W.$$

## 1.3 Four fundamental examples

The above general algebraic approach to the Hirzebruch signature operator can be used to the four above mentioned fundamental examples.

$$
W^k = \left\{ \begin{array}{|l|}
\hline
\text{[}\textbf{manifold, classical example}\text{],} \\
\left(\Omega^k\left(M\right),\ d_{dR}\right); \quad \dim M = N = 4p, \\
\quad \text{here } \varepsilon_k = (-1)^{k(N-k)} = (-1)^k \\
\\
\hline
\text{[}\textbf{Lie algebroid example}\text{]} \\
\left(\Omega^k\left(A\right),\ d_A\right); \quad \operatorname{rank} A = N = m + n = 4p, \quad A \text{ - a TUIO-Lie algebroid} \\
\quad \text{here } \varepsilon_k = (-1)^k \\
\\
\hline
\text{[}\textbf{Lusztig's example}\text{]} \\
\left(\Omega^k\left(M; E\right),\ d_{\nabla}\right); \quad E, \nabla, \langle , \rangle \text{ flat vector bundle,} \\
\qquad \langle , \rangle \text{ - symmetric nondegenerated parallel tensor}, \quad \dim M = N = 4p, \\
\quad \text{here } \varepsilon_k = (-1)^k \\
\\
\hline
\text{[}\textbf{Gromov's example}\text{]} \\
\left(\Omega^k\left(M; E\right),\ d_{\nabla}\right); \quad (E, \langle , \rangle) \text{ flat vector bundle,} \\
\qquad \langle , \rangle \text{ -symplectic parallel tensor}, \quad \dim M = N = 4p + 2, \\
\quad \text{here } \varepsilon_k = -(-1)^k, \\
\hline
\end{array} \right.
$$

The Lusztig anf Gromov examples are important for the Lie algebroid case, since $\operatorname{Sig}\mathbf{H}(A)=\operatorname{Sig}E_2$ - the signature of the second term of a suitable spectral sequence. and using Hochschild-Serre spectral sequence of the Lie algebroid we have $E_2^{j,i}=\mathbf{H}_{\nabla^q}^{j}\left(M;\mathbf{H}^i\left(\boldsymbol{g}\right)\right)$ with respect to some natural flat structure $\nabla^i$ in the vector bundle of $i$-group of cohomology of isotropy Lie algebras $\mathbf{H}^i\left(\boldsymbol{g}\right)$. The multiplication of values is taken with respect to multiplication of cohomology classes

$$\langle,\rangle:\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\times\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\rightarrow\mathbf{H}^n\left(\boldsymbol{g}\right)=M\times\mathbb{R}.$$

We need to consider two different cases:

- $\frac{m}{2}$ and $\frac{n}{2}$ even, then the above form is symmetric and we can use Lusztig type Example to obtain the Hirzebruch signature operator

$$D_+=d_{\nabla^{\frac{n}{2}}}+\delta_{\nabla^{\frac{n}{2}}}:\Omega_+\left(M;\mathbf{H}^{\frac{n}{2}}\left(\mathbf{g}\right)\right)\rightarrow\Omega_-\left(M;\mathbf{H}^{\frac{n}{2}}\left(\mathbf{g}\right)\right),$$

- $\frac{m}{2}$ and $\frac{n}{2}$ are odd, then the above is symplectic and we can use Gromov type example to obtain the Hirzebruch signature operator

$$D_+=d_{\nabla^{\frac{n}{2}}}+\delta_{\nabla^{\frac{n}{2}}}:\Omega_+\left(M;\mathbf{H}^{\frac{n}{2}}\left(\mathbf{g}\right)\right)\rightarrow\Omega_-\left(M;\mathbf{H}^{\frac{n}{2}}\left(\mathbf{g}\right)\right).$$

# 2 Lipschitz manifolds and distributional exterior derivative (subderivative)

## 2.1 Lipschitz manifolds

On Lipschitz manifolds the derivative of differential $L_2$-forms is not defined on all space of differential forms.

We briefly recall the notion of a Lipschitz manifold and differential forms of the class $L_2$ on them.

**Definition 20 (Teleman 1983)** *A Lipschitz structure on a topological manifold $M$ of dimension $n$ is a maximal atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$, where $\phi_\alpha : M \supset U_\alpha \to V_\alpha \subset \mathbb{R}^n$ ($U_\alpha$, $V_\alpha$ - open subsets) are homeomorphisms such that the changes coordinates (i.e. transition functions)*

$$\Lambda_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}, \quad \alpha, \beta \in \Lambda$$

*are Lipschitz mappings.*

Of course, a $C^\infty$-manifold possesses a canonical Lipschitz structure.

**Theorem 21** *(D.Sullivan) Any topological manifold of dimension* $\neq 4$ *admits a Lipschitz structure, and that structure is essentially unique.*

Sullivan's theorem makes then possible to construct signature operators not only on Lipschitz manifold, but on an arbitrary compact, topological manifold of even dimension $\neq 4$.

The crucial role is played by the Rademacher theorem:

**Theorem 22 (Rademacher)** *If* $U \to \mathbb{R}$ *is a Lipschitz function on an open subset* $U \subset \mathbb{R}^n$*, then*

- *the partial derivatives* $\frac{\partial f}{\partial x^i}$ *exist almost everywhere,* ***[not everywhere, therefore we can not define a tangent space at a point to a Lipschitz manifold*** $M$ ***and we must use other algebraic ways to define a differential form on*** $M$***],***
- $\frac{\partial f}{\partial x^i}$ *are measurable and bounded.*

**Definition 23** *We say that a Lipschitz manifold with the atlas* $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$ *is orientable if there exists a subatlas* $\Lambda' \subset \Lambda$ *for which the homeomorphisms* $\Lambda_{\alpha\beta}$ *have positive jacobian (in all points of differentiability). If such an atlas is given we call* $M$ *oriented.*

## 2.2 Differential forms

Let $L_2^k(U)$ denote the space of differential forms of the class $L_2$ on an open subset $U \subset \mathbb{R}^n$.

**Definition 24** *Let $M$ be a Lipschitz manifold with the atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$, . By $L_2$-differential form on $M$ we mean a system*

$$\omega = \{\omega_\alpha\}_{\alpha\in\Lambda}$$

*where $\omega_\alpha$ is a [real] $L_2$-differential form on the open subset $V_\alpha = \phi_\alpha[U_\alpha] \subset \mathbb{R}^n$, $\alpha \in \Lambda$,* ***such that the following condition of compatibility holds***

$$\Lambda^*_{\alpha\beta}\omega_\beta = \omega_\alpha.$$

$L_2(M)$ - the vector space of $L_2$-differential forms on $M$.

The 0-differential form determines a measurable function on $M$.

For oriented Lipschitz manifold, using the Lipschitz partition of unity, we define the integral $\int_M \omega$ for $\omega \in L_2^n(M)$ $(n = \dim M)$ in a standard way.

## 2.3 Lipschitz Riemannian metric

**Definition 25** *A Lipschitz Riemannian metric on $M$ is a collection*

$$\Gamma = \{\Gamma_\alpha\}_{\alpha\in\Lambda}$$

*where $\Gamma_\alpha$ is a Riemannian metric on $V_\alpha = \phi_\alpha [U_\alpha] \subset \mathbb{R}^n$ with measurable components, which satisfy*

- *compatibility condition*
$$(\Lambda_{\alpha\beta})^* \Gamma_\beta = \Gamma_\alpha,$$
- *$L_2$-norms on $V_\alpha$ determined by $\Gamma_\alpha$ and by standard metric are equivalent.*

**Theorem 26 (Teleman, 1983)** *Any compact Lipschitz manifold $M$ has a Lipschitz Riemannian metric.*

Clearly, any Lipschitz Riemannian metric detrmines a measure on $M$.

Let $*_{\alpha,x}$ be a Hodge star isomorphism in $\bigwedge (\mathbb{R}^n)^*$ defined of the metric $\Gamma_\alpha$ at $x \in \mathbb{R}^n$ defined as in the previous sections

$$*_{\alpha,x} (e_{i_1} \wedge ... \wedge e_{i_k}) = \varepsilon_{(j_1,\dots,j_{n-k})} \cdot e_{j_1} \wedge ... \wedge e_{j_{n-k}}, \quad (e_i) \text{ is } \Gamma_\alpha (x)\text{-ON.}$$

**Definition 27** *For a Lipschitz Riemannian metric* $\Gamma = \{\Gamma_\alpha\}$ *and* $\omega \in L_2^r(M)$, $\omega = \{\omega_\alpha\}$, *we define*

- $L_2$*-differential form* $*_\Gamma \omega = \{*_\alpha \omega_\alpha\}_\alpha$, *where* $*_{\alpha,x}$ *a Hodge star isomorphism in* $\bigwedge (\mathbb{R}^n)^*$ *defined of the metric* $\Gamma_\alpha$ *at* $x \in \mathbb{R}^n$
- *for* $\omega, \eta$ *of the same degree we define the inner product* $(\omega, \eta)_\Gamma := \{(\omega_\alpha, \eta_\alpha)_\alpha\}$ *(it is a* $0$*-form, i.e. a function on* $M$ *).*
- $((\omega, \eta))_\Gamma := \int_M (\omega, \eta)_\Gamma$.

**Theorem 28 (Teleman 1983)** *The space* $L_2^k(M)$ *with unitary structure* $((,))_\Gamma$ *is Hilbert, two Lipschitz Riemannian metrics define equivalent norms in* $L_2^k(M)$.

Introducing the pairing (via the wedge product) of differential forms in complementary degrees by

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \omega \wedge \eta$$

we have

$$\langle\langle \omega, \eta \rangle\rangle = ((\omega, *_\Gamma \eta))_\Gamma$$

which means that

$$\left(L_2\left(M\right), \langle\langle,\rangle\rangle, ((,)), *\right)$$

is a Hodge space.

## 2.4 Distributional exterior derivative

**Definition 29** *Let $\sigma \in L_2^r(U)$ be any $L_2$-differential form on $U \subset \mathbb{R}^n$ of degree $r < n$. We say that $\sigma$ has distributional exterior derivative in the class $L_2$ if there exists an $L_2$-differential form of degree $r+1$*

$$\bar{d}\sigma \in L_2^{r+1}(U)$$

*such that for any $C^\infty$-differential form $\varphi$ of degree $n-1-r$ with compact support in $U$*

$$\int_U \bar{d}\sigma \wedge \varphi = (-1)^{r+1} \int_U \sigma \wedge d\varphi.$$

If $r = n$, we put $\bar{d}\sigma = 0$ for each $\sigma \in L_2^n(U)$.

Distributional exterior derivative $\bar{d}\sigma$ is uniquely determined and clearly $\bar{d}\left(\bar{d}\sigma\right)$ exists and $\bar{d}\left(\bar{d}\sigma\right) = 0$.

**Proposition 30** *If $\omega = \{\omega_\alpha\}_{\alpha\in\Lambda}$ is an $L_2$-differential form on $M$ of degree $r$ and $d\omega_\alpha \in L_2(V_\alpha)$ is the distributional exterior derivative of $\omega_\alpha$, then*

$$\bar{d}\omega := \left\{\bar{d}\omega_\alpha\right\}_{\alpha\in\Lambda}$$

*is an $L_2$-differential form on $M$ of degre $r+1$* ***(i.e. the suitable compatibility condition holds).***

Denote by $\Omega_d^r(M) \subset L_2^r(M)$ the subspace of $L_2$-differential forms of degree $r$ possessing the distributional exterior derivative

$$\Omega_d^r(M) = \left\{\omega \in L_2^r(M);\ \bar{d}\omega \in L_2^{r+1}(M)\right\}.$$

$\bar{d}^2 = 0$ na $\Omega_d^r(M)$. We obtain a cohomology complex

$$0 \to \Omega_d^0(M) \to \Omega_d^1(M) \to ... \to \Omega_d^r(M) \to \Omega_d^{r+1}(M) \to ... \to \Omega_d^n(M) = L_2^n(M) \to 0.$$

**Theorem 31 (Teleman (1983))** *For a compact oriented Lipschitz manifold $M$*

- *the pairing*

$$\mathbf{H}_r\left(\Omega_d^\bullet(M)\right) \times \mathbf{H}_{dimM-r}\left(\Omega_d^\bullet(M)\right) \to \mathbb{R}, \quad ([\omega],[\eta]) \to \int_M \omega \wedge \eta$$

  *is nondegenerated and* $\mathbf{H}_r\left(\Omega_d^\bullet(M)\right) = \left(\mathbf{H}_{dimM-r}\left(\Omega_d^\bullet(M)\right)\right)^*$. *Therefore* $\dim \mathbf{H}\left(\Omega_d^\bullet(M)\right) < \infty$ *($L_2$-Poincaré duality),*

- *for a $C^\infty$ manifold $M$ and induced Lipschitz structure, the inclusion*

$$j : \Omega^\bullet(M) \hookrightarrow \Omega_d^\bullet(M)$$

  *induces isomorphism in cohomology* $j_\# : \mathbf{H}(M) \xrightarrow{\cong} \mathbf{H}\left(\Omega_d^\bullet(M)\right)$ *(mainly thanks the $L_2$-Poincaré Lemma [L.Hörmander] ).*

Using the so-called convolution argument **(I explain it further)** Teleman proved the theorem

**Theorem 32 (Teleman (1983))** *Suppose that* $\omega$ *and* $\eta$ *possess distributional exterior derivatives* $\bar{d}\omega$ *and* $\bar{d}\eta$ *and* $|\omega| + |\eta| = n - 1$, ***then we can switch the distributional derivative***

$$\left\langle\left\langle\omega, \bar{d}\eta\right\rangle\right\rangle = (-1)^{|\omega|+1}\left\langle\left\langle\bar{d}\omega, \eta\right\rangle\right\rangle.$$

*Let* $\omega$ *be a given* $L_2$*-form and let there exist* $\omega'$ *such that*

$$\left\langle\left\langle\omega, \bar{d}\eta\right\rangle\right\rangle = (-1)^{|\omega|+1}\left\langle\left\langle\omega', \eta\right\rangle\right\rangle$$

*for all* $\eta$ *with distributional exterior derivative* $\bar{d}\eta$, *then* $\omega'$ *is the distributional exterior derivative of* $\omega$, $\bar{d}\omega = \omega'$ *.*

*In particular, if* $\left\langle\left\langle\omega, , \bar{d}\eta\right\rangle\right\rangle = 0$ *for all* $\eta$ *with distributional exterior derivative* $\bar{d}\eta$ *then* $\bar{d}\omega = 0$.

# 3 Algebraic aspect of the signature operator on Lipschitz manifolds

## 3.1 Graded Hilbert subdifferential Hodge space

**Definition 33** *By a* ***Hilbert anticommutative graded subdifferential Hodge space****, we mean a system*

$$\left(W = \bigoplus_{k=0}^{N=2n} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *_W, \bar{d} : W_d \to W_d\right)$$

*consisting of an* ***anticommutative graded*** *Hodge space* $(W, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *_W)$ *and a subderivative* $\bar{d}$ *defined on some subspace with gradation*

$$W_d = \bigoplus_{k=0}^{N} W_d^k \subset W, \qquad W_d^k = W^k \cap W_d,$$

*such that*

*(1) the space* $\left(W, \|\cdot\| = \sqrt{(\cdot, \cdot)}\right)$ *is complete, i.e. it is Hilbert,*

*(2)* $\langle \cdot, \cdot \rangle | W^k \times W^r = 0$ *if* $k + r \neq N$, *the subspaces* $W^r$ *are orthogonal with respect to* $(\cdot, \cdot)$,

*(3)* $W_d$ *is dense in* $W$,

*(4)* $\bar{d}$ *is degree* $+1$, $\bar{d}^r = d|W_d^r : W_d^r \to W_d^{r+1}$,

*(5)* $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$ *for* $w \in W_d^r$, $u \in W_d^{N-r-1}$,

*(6)* *If for* $w \in W^r$ *there exists* $w' \in W^{r+1}$, *such that* $\langle w', u \rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$ *for each* $u \in W_d^{N-r-1}$, *then* $w \in W_d^r$ *and* $\bar{d}w = w'$,

*(7)* *the cohomology space* $\mathbf{H}_d(W) = \bigoplus_{k=0}^{N} \mathbf{H}_d^k(W)$ *of the complex* $\left(W_d, \bar{d}\right)$ *fulfills the Poincaré duality, i.e. the pairing*

$$\mathbf{H}^r\left(W_d^\bullet\right) \times \mathbf{H}^{N-r}\left(W_d^\bullet\right) \to \mathbb{R}, \qquad ([w], [v]) \longmapsto \langle w, v \rangle$$

*is nondegenerated, and* $\dim \mathbf{H}(W_d) < \infty$.

**Notation 34** *Let $W$ be a Hilbert* ***anticommutative*** *graded subdifferential Hodge space. We put*

$$W_\delta^{N-r} := *_W \left[W_d^r\right].$$

*This space is dense in $W^{N-r}$ and $*_W : W_d^r \to W_\delta^{N-r}$ is an isometry. By a* ***cosubderivative*** *of the degree $r$ we mean the oparator*

$$\bar{\delta}^r : W_\delta^r \to W_\delta^{r-1}$$

*defined by*

$$\bar{\delta}^r \left(w^k\right) = \varepsilon_r \left(-1\right)^r * \bar{d} * \left(w^r\right), \quad w^r \in W_\delta^r.$$

It is easy to prove that $\bar{d}^r$ and $\bar{\delta}^{N-r}$ are adjoint

$$\left(\bar{\delta}^{N-r} v, w\right) = \left(v, \bar{d}^{N-r-1} w\right).$$

**Definition 35** *We define now the spaces of harmonic vectors*

$$\mathcal{H}_d^r = \left\{ w \in W_1^r;\ \bar{d}\omega = 0 = \bar{\delta}\omega \right\}.$$

Clearly

$$* : \mathcal{H}_d^r \to \mathcal{H}_d^{N-r}$$

is an isomorphism. Any harmonic vector is a cocycle, therefore there exists a Hodge homorphism

$$x_d^r : \mathcal{H}_d^r \to \mathbf{H}^r\left(W_d^\bullet\right).$$

As in the paper by Teleman (1983) we can prove in an algebraic way that

- The subspaces $\mathcal{H}^r$ and $\operatorname{Im}\bar{d}^{r-1}$ are perpendicular, which gives that $x_d^r$ is a monomorphism.
- The subspace $\operatorname{Ker}\bar{d}^r$ is closed in $W^r$, therefore it is a Hilbert space.
- The subsapce $\operatorname{Im}\bar{d}^{r-1}$ is closed in $W^r$, therefore it is a Hilbert space.
- **(Hodge Theorem)** $\operatorname{Im}\bar{d}^{r-1}(\subset \operatorname{Ker}\bar{d}^r)$ is a closed subspace of the Hilbert space $\operatorname{Ker}\bar{d}^r$, therefore

$$\operatorname{Ker}\bar{d}^r = \operatorname{Im}\bar{d}^{r-1} \oplus \left(\operatorname{Im}\bar{d}^{r-1}\right)^\perp = \operatorname{Im}\bar{d}^{r-1} \oplus \mathcal{H}_d^r$$

which means that

$$\mathcal{H}_d^r = \operatorname{Ker} \bar{d}^r / \operatorname{Im} \bar{d}^{r-1} = \mathbf{H}^r \left( W_d \left( M \right) \right),$$

i.e. the Hodge homomorphism is an <u>isomorphism.</u>

**Theorem 36** *There is a strong Hodge decomposition.*

$$W^r = \mathcal{H}_d^r \oplus \bar{d} \left[ W_d^{r-1} \right] \oplus \bar{\delta} \left[ W_\delta^{r+1} \right].$$

## 3.2 The signature operator for graded Hilbert subdifferential Hodge space

For the uniformity of notation we put

- $W_0^r = W^r$ with the norm $||w|| = \sqrt{(w,w)}$.

  — **We notice that in analysis on vector bundles the norm $||w||$ means the $L_2$ norm.**

- $W_1^r := W_d^r \cap W_\delta^r$ with the norm $||w_1||$, such that $\|w\|_1^2 = \|w\|^2 + \left\|\bar{d}w\right\|^2 + \left\|\bar{\delta}w\right\|^2$. The both are Hilbert.

Analogously as in Teleman's paper we show

**Theorem 37** *The operator*

$$\bar{D}^r = \bar{d} + \bar{\delta} : W_1^r \to W_0^r$$

*is a continuous Fredholm operator,*

$$\mathrm{Ker}\, \bar{D}^r = \mathcal{H}_d^r$$

*and*

$$\mathrm{Im}\, \bar{D}^r = \bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right]$$

*(so* Coker $\bar{D}^r \cong \mathcal{H}_d^r$ *).*

Since $\dim \mathbf{H}\left(\Omega_d^\bullet\right)$ is finite [from the Poincaré duality] we define as above the signature of $W$, and

$$\mathrm{Sig}\,(W) := \mathrm{Sig}\,\langle , \rangle_{\mathbf{H}}^n = \mathrm{Sig}\,\mathcal{B}_d^n.$$

Puting

$$\begin{aligned} W_\pm &= \{w \in W;\ \tau w = \pm w\}, \\ W_{1,\pm} &= W_\pm \cap W_1 \\ W_{0,\pm} &= W_\pm \end{aligned}$$

we notice that

$$\begin{aligned}\left(\bar{d}+\bar{\delta}\right)[W_{1,+}] &\subset W_{0,-},\\ \left(\bar{d}+\bar{\delta}\right)[W_{1,-}] &\subset W_{0,+},\end{aligned}$$

**Definition 38** *The operator*

$$\bar{D}_+ = \bar{d}+\bar{\delta} : W_{1,+} \to W_{0,-}$$

*is called the signature operator. Also we consider the adjoint one*

$$\bar{D}_- = \bar{d}+\bar{\delta} : W_{1,-} \to W_{0,+},$$

*for which the condition of duality holds*

$$\left(\bar{D}_+\alpha, \beta\right) = \left(\alpha, \bar{D}_-\beta\right), \qquad \textit{for} \quad \alpha \in W_{1,+}, \beta \in W_{1,-}.$$

Analogously as in the previous part we prove the signature theorem

**Theorem 39**

$$\mathrm{Sig}\,(W) := \mathrm{Sig}\,\langle,\rangle_{\mathbf{H}}^n = \mathrm{Sig}\,(\mathcal{B}_d^n) = \dim_{\mathbb{R}}\ker\left(\bar{D}_+\right) - \dim_{\mathbb{R}}\ker\left(\bar{D}_-\right) := \mathrm{index}\,\bar{D}$$

We see that in the Hilbert case there are very simple considerations to obtain a Hodge theorem (no analysis !, only algebraic topology, the exception being the condition $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$ which uses the convolution argument).

# 4 Completion of the graded Hodge differential space

**For an evenly graded anticommutative differential Hodge space we define its completion to the Hilbert case (which is easier since all spaces are Hilbert and the complementation to closed subspace give a direct sum). Then we have two signature operators and we want to compare them. Especially we have in mind a smooth and suitable $L_2$ theory on differential compact manifolds.**

Consider an **evenly graded anticommutative differential Hodge space with the Poincaré duality**

$$\left(W = \bigoplus_{k=0}^{N=2n} W^k, \langle\cdot,\cdot\rangle, (\cdot,\cdot), *, d\right),$$

i.e $(W, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *)$ is a Hodge space (finitely or infinitely dimensional) it means $\langle \cdot, \cdot \rangle$ is given 2-linear tensor whose via $d$ determines quadratic form on the cohomology in the middle degree [we want to calculate the sugnature of its], $(\cdot, \cdot)$ is an auxiliary inner product and $*$ is a Hodge isometry such that $\langle \cdot, w \rangle = (\cdot, *w)$, and

(1) $\langle \cdot, \cdot \rangle | W^k \times W^r = 0$ if $k + r \neq N$. (notation: $\langle , \rangle^k := \langle , \rangle | W^k \times W^{N-k}$ ),

**Definition 40** •

*(2)* $W^k$ *are orthogonal with respect to* $(\cdot, \cdot)$,

*(3) the operator* $d$*, called derivative, is homogeneous of degree* $+1$*, i.e.* $d : W^k \to W^{k+1}$*, and* $d^2 = 0$,

*(4)* $\left\langle dw^k, u^{N-k-1} \right\rangle = (-1)^{k+1} \left\langle w^k, du^{N-k-1} \right\rangle$ *for* $w^k \in W^k$, $u^{N-k-1} \in W^{N-k-1}$,

(5) $N = 2n$ is even, and the tensor $\langle \cdot, \cdot \rangle$ is anticommutative in the sense

$$\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle, \qquad \text{where } \varepsilon_k = (-1)^n (-1)^k,$$

(6) Poincaré duality axiom holds: i.e. $\mathbf{H}^k(W)$ are finitely dimensional and $\mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$ are nondegenerated.

We complete the unitary space $(W, (,))$ to Hilbert one $\bar{W}$. The inner product and the norm in $\bar{W}$ will be denoted by the same symbol. We extend the $*$-Hodge isometry to the isometry $* : \bar{W} \to \bar{W}$ and the pairing $\langle , \rangle$ to a new one $\langle \cdot, \cdot \rangle : \bar{W} \times \bar{W} \to \mathbb{R}$ denoting by the same symbol. Of course this pairing remains continuous.

We obtain a new Hilbert **evenly graded anticommutative Hodge space** $\left(\bar{W} = \bigoplus_{k=0}^{N=2n} \bar{W}^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *\right)$.

Now we extend the derivative $d^k : W^k \to W^{k+1}$ to some bigger subspace $\bar{W}_d^k \subset \bar{W}^k$ in a "distributional manner".

**Definition 41** *We say that a vector $w \in \bar{W}^k$ has a distributional derivative if there exists a vector belonging to $\bar{W}^{k+1}$ denoted by $\bar{d}w$ such that for each vector $v \in W^{N-k-1}$ the following condition*

$$\left\langle w, d^{N-k-1} v \right\rangle = (-1)^{k+1} \left\langle \bar{d}w, v \right\rangle$$

*holds.*

The derivative $\bar{d}w$ is unique (if it exists). The vector space of vectors $v$ possessing distributional derivative will be denoted by $\bar{W}_d$. Clearly, if $w \in W^k$ then $\bar{d}w$ exists and $\bar{d}w = dw$, as well as $\bar{d}^{k+1}\left(\bar{d}^k(w)\right) = 0$ for $w \in \bar{W}_d$.

It is easy to see that $\bar{d}\bar{d} = 0$ and that the inclusion $J : W \to \bar{W}_d$ induces a homomorphism in cohomology

$$J_{\#} : \mathbf{H}(W) \to \mathbf{H}\left(\bar{W}_d\right).$$

To see that we obtain a **Hilbert anticommutative graded subdifferential Hodge space** $\bar{W}$ we must check two conditions only:

- the **switch** of the distributional derivative $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1}\left\langle w, \bar{d}u \right\rangle$ for $w \in W_d^r$, $u \in W_d^{N-r-1}$, and
- the Poincaré duality for the complex $\left(W_d, \bar{d}\right)$.

**Problem 42** ***(1)*** *Does* $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1}\left\langle w, \bar{d}u \right\rangle$ *for* $w \in \bar{W}_d^r$, $u \in \bar{W}_d^{N-r-1}$ *for a given Hodge graded differential space?*

To solve this problem we need to add the next condition

**(7)** Additional condition: if $w \in \bar{W}_d^r$ then there exists a Cauchy sequence $w_n \in W^r$ convergent to $w$ such that $dw_n \to \bar{d}w$.

Clearly, from this condition the above equation holds:

$$\left\langle \bar{d}w, u \right\rangle = \lim \left\langle dw_n, u \right\rangle = \lim \left(-1\right)^{k+1} \left\langle w_n, \bar{d}u \right\rangle = \left(-1\right)^{k+1} \left\langle w, \bar{d}u \right\rangle .$$

**Remark 43** *In local analysis on $\mathbb{R}^n$ and global analysis on manifolds condition (7) holds for differential operators thanks the convolution argument using the so-called mollifying sequence. To be more precise, we take any differential operator of rank $k$ (we use it for $k = 1$ ) between modules od cross-sections of vector bundles $d : \operatorname{Sec}\xi \to \operatorname{Sec}\eta$ .*

*First we look at this locally. In arbitrary local base of the vector bundles and identifying coordinate neighbourhood $U$ with open set in $\mathbb{R}^n$ an differential operator $d : \operatorname{Sec}\xi \to \operatorname{Sec}\eta$ can be written in the form ($p = \operatorname{rank}\xi$, $q = \operatorname{rank}\eta$ )*

$$d_U = \sum_{|\alpha| \le k} A_\alpha \left(x\right) \partial^\alpha \quad \textit{for} \quad A_\alpha \in C^\infty \left(U, \operatorname{Hom}\left(\mathbb{R}^p, \mathbb{R}^q\right)\right).$$

*We exploit the convolution operation*

$$f * g\left(x\right) = \int_{\mathbb{R}^n} f\left(x - y\right) g\left(y\right) dy,$$

*and a mollifying sequence*

$$\lambda_t(x) = \frac{1}{t}\varphi\left(\frac{x}{t}\right), \quad t > 0,$$

*where* $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1, \operatorname{supp}\varphi = B(0,1)$. *We have fundamental theorem*

- *for any* $f \in L_{1,loc}(\mathbb{R}^n)$ *the convolution* $\lambda_t * f$ *is a smooth, and if* $f$ *has the distributional derivative* $\frac{\partial f}{\partial x_i} \in L_{1,loc}(\mathbb{R}^n)$ *then*

$$\frac{\partial(\lambda_t * f)}{\partial x_i} = \lambda_t * \frac{\partial f}{\partial x_i},$$

- *if* $f \in L_p(\mathbb{R}^n)$, $p \geq 1$, *then*

$$\lambda_t * f \xrightarrow[t\to 0]{L_p} f,$$

*if additionally* $f$ *has the distributional derivative* $\frac{\partial f}{\partial x_i} \in L_p(\mathbb{R}^n)$ *then*

$$\frac{\partial(\lambda_t * f)}{\partial x_i} \xrightarrow{L_p} \frac{\partial f}{\partial x_i}.$$

*We take advantage it for $p=2$. Let $d$ be a differential operator of rank $k$ locally written by $d_U=\sum_{|\alpha|\le k}A_\alpha(x)\partial^\alpha$ and let $A_\alpha(x)$ belongs to $L_2$. Then, for the sequence of functions $f=(f^1,...,f^p)$ of the class $L_2$ possesing distributional derivative of the class $L_2$ (i.e. $f\in L_{1,2}$ ) we have $\lambda_t * f$ is smooth, and*

$$d_U(\lambda_t * f)=\sum_{|\alpha|\le k}A_\alpha(x)\partial^\alpha(\lambda_t * f)=\sum_{|\alpha|\le k}A_\alpha(x)(\lambda_t * \partial^\alpha f)\xrightarrow{L_p}\sum_{|\alpha|\le k}A_\alpha(x)\partial^\alpha(f)=d_U(f).$$

*On a compact manifold we take a partition of unity $\rho_\beta$ subordinated to the suitable finite covering $\{U_\beta\}$ of $M$. Representing a cross-section $\omega$ of the vector bundle $\xi$ in local trivializations $\xi_{U_\beta}\cong U_\beta\times\mathbb{R}^p$ we take $\lambda_t\tilde{*}\omega:=\Sigma_\beta\lambda_t*(\rho_\beta\cdot\omega)$ (it depends on the trivializations) and we have*

$$\lambda_t\tilde{*}\omega\xrightarrow[t\to0]{L_2}\omega$$

*and*

$$d(\lambda_t\tilde{*}\omega)=d\left(\Sigma_\beta\lambda_t*(\rho_\beta\cdot\omega)\right)=\Sigma_\beta d_{U_\beta}\left(\lambda_t*(\rho_\beta\cdot\omega)\right)\xrightarrow[t\to0]{L_2}\Sigma_\beta d_{U_\beta}(\rho_\beta\cdot\omega)=d\left(\Sigma_\beta\rho_\beta\cdot\omega\right)=d\omega.$$

*Unfortunately, the operation $d$ and $\lambda_t\tilde{*}(\cdot)$ on a compact manifold do not commutate even if the operator $d$ has locally constant coefficients.*

We introduce

$$\bar{W}_{\delta}^{N-k} = * \left[ \bar{W}_{d}^{k} \right]$$

and coderivative $\bar{\delta}^{N-k} : \bar{W}_{\delta}^{N-k} \to \bar{W}_{\delta}^{N-k-1}$ by the formula

$$\bar{\delta}^{N-k} := (-1)^{N-k} * \bar{d}^{k} *^{-1} .$$

$\bar{\delta}$ is an extension of $\delta$.

**Problem 44** *(2) Does the inclusion $J : W \to \bar{W}_d$ (which of course commutes with derivatives $d$ and $\bar{d}$ ) induces an isomorphism $\mathbf{H}(W) \to \mathbf{H}\left(\bar{W}_d\right)$ in cohomology?*

Particularly, if "YES", then the Poincaré duality holds for the cohomology $\mathbf{H}\left(\bar{W}_d\right)$ and we can give Hirzebruch operator for the signature of $\bar{W}$ and compare this operator and signature for $W$. Clearly, if "YES" then

$$\operatorname{Sig}(W) = \operatorname{Sig}\left(\bar{W}_d\right) = \operatorname{index} \bar{D}, \quad \bar{D} = \bar{d} + \bar{\delta},$$

but $\operatorname{index} D = \operatorname{Sig} \mathcal{B}^n$ of the quadratic form on harmonic tensors and $\operatorname{Sig} \mathcal{B}^n \leq$

Sig $(W)$ . See the commutative diagram joining the Hodge homomorphisms

$$\begin{array}{ccc} \mathcal{H}^k(W) & \rightarrowtail & \mathbf{H}^k(W) \\ \cap & & \downarrow\cong \\ \mathcal{H}^k\left(\bar{W}_d\right) & \xrightarrow{\cong} & \mathbf{H}^k\left(\bar{W}_d\right) \end{array}$$

Now for differential operators we can use the so-called Weyl-Lemma saying that $L_2$-harmonic tensor is smooth (this lemma for local theory and for operators with constant coefficients is not too difficult as for the general case).

In conclusion, we obtain for **evenly graded anticommutative differential Hodge space $W$ with the Poincaré duality** two Hirzebruch operators, the usual $D$ and extended $\bar{D}$ having the same indexes equal to the signature of $W$.

When Problem 2) has a positive answer? Firstly it was given by Teleman for the de Rham cohomology (thanks to $L_2$-Poincaré lemma given by L.Hörmander.) In the general case we use spectral sequences and the comparison theorem

- If there exist spectral sequences $E_p^{j,i}$ for $W$ and $\bar{E}_p^{j,i}$ for $\bar{W}_d$ converging to $\mathbf{H}(W)$ and $\mathbf{H}\left(\bar{W}_d\right)$ , respectively, such that the homomorphism $J$ :

$W \rightarrow \bar{W}_d$ induces an isomorphism on the first terms $J_1^{j,i} : E_1^{j,i} \xrightarrow{\cong} \bar{E}_1^{j,i}$ then $J$ induces isomorphism on cohomology $J_\# : \mathbf{H}(W) \xrightarrow{\cong} \mathbf{H}\left(\bar{W}_d\right)$,

used for the Čech-de Rham double complexes.

For the case of a Lie algebroid, independently, we can define the Hoshchild-Serre type spectral sequence for $L_2$-theory on TUIO-Lie algebroids (for smooth case see [K-M-3]) and use the comparison theorem for the second terms.

**Conclusion 45** *Consider the four examples* $W = \Omega(M)$, *for a manifold* $M$, $\Omega(A)$ *for a Lie algebroid* $A$, $\Omega(M, E)$ *for the Lusztig or the Gromov vector bundle* $E$. *Now we pass to the spaces of* $L_2$*-differential forms* $\bar{W} = \overline{\Omega(M)}$, $\overline{\Omega(A)}$, $\overline{\Omega(M, E)}$. *We obtain graded Hilbert subdifferential Hodge spaces.*

*In consequence, the signature in that four cases can be calculated via two Hirzebruch signature operators, the usual smooth and second of the* $L_2$ *class. So for Lie algebroid we have four Hirzebruch operators, two smooth using the space* $\Omega(A)$ *and* $\Omega(M, E)$ *[Lusztig or Gromov subcases] and two* $L_2$ *Hirzebruch operators, using graded Hilbert subdifferential Hodge spaces* $\overline{\Omega(A)}$ *or* $\overline{\Omega(M, E)}$, *respectively.*

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