Algebraic aspects of the Hirzebruch signature operator and applications to transitive Lie algebroids *

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Abstract

The index of the classical Hirzebruch signature operator on a manifold M is equal to the signature of the manifold. The examples of G.Lusztig [L, 1972] and M.Gromov [Gro, 1985] present the Hirzebruch signature operator for the cohomology of a manifold with coefficients in a flat symmetric or symplectic vector bundle. In [K2] we have a signature operator for the cohomology of transitive Lie algebroids.

In this paper first we present a general approach to the signature operator, and the above four examples are special cases of a one general theorem.

Secondly, due to of the spectral sequence point of view on the signature of the cohomology algebra of some filtered DG-algebras it appears that the Lusztig and Gromov examples are important to the study of the signature of a Lie algebroid. Namely, under some natural simple regularity assumptions on a DG-algebra with a decreasing filtration for which the second term lives in a finite rectangular we obtain that the signature of the second term of the spectral sequence is equal to the signature of the DG algebra. Considering the Hirzebruch-Serre spectral sequence for a transitive Lie algebroid A over a compact oriented manifold for which

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the top group of the real cohomology of A is nontrivial we have that the second term is just identical with the Lusztig or Gromov example (depending on the dimension). Thus we have a second signature operator for Lie algebroids.

1 Preliminary of Lie algebroids, signature of transitive Lie algebroids

Lie algebroids appeared as infinitesimal objects associated to Lie groupoids, principal fibre bundles, vector bundles (Pradines, 1967), TC-foliations and nonclosed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalences are known as Lie pseudo-algebras (Herz 1953) called also Lie-Rinehart algebras (Huebschmann, 1990).

A *Lie algebroid* on a manifold M is a triple

$$A = (A, [\![\cdot,\cdot]\!], \#_A)$$

where A is a vector bundle on M, $(\operatorname{Sec} A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

$$\#_A : A \to TM$$

is a linear homomorphism (called the anchor) of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta, \quad f\in C^\infty(M),\ \xi,\eta\in\operatorname{Sec} A.$$

The anchor is bracket-preserving, $\#_A\circ[\![\xi,\eta]\!] = [\#_A\circ\xi, \#_A\circ\eta]$. A Lie algebroid is called **transitive** if the anchor $\#_A$ is an epimorphism. For a transitive Lie algebroid A we have:

- the Atiyah sequence
$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0, \tag{1}$$
$\boldsymbol{g} := \ker\#_A$,
- the fiber $\boldsymbol{g}_x$ of the bundle $\boldsymbol{g}$ in the point $x\in M$ is the Lie algebra (called the isotropy Lie algebra of A at $x\in M$) with the commutator operation being
$$[v,w] = [\![\xi,\eta]\!](x), \quad \xi,\eta\in\operatorname{Sec} A, \quad \xi(x)=v, \eta(x)=w, \quad v,w\in\boldsymbol{g}_x,$$
- the vector bundle $\boldsymbol{g}$ is a Lie Algebra Bundle (LAB for short), called the *adjoint* of A, the fibres are isomorphic Lie algebras.

Tangent bundles to manifolds and finitely dimensional Lie algebras are simple examples of transitive Lie algebroids.

To an arbitrary (transitive or not) Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \mathrm{Sec} \bigwedge A^*, \text{ - the space of cross-sections of } \bigwedge A^*$$

$$d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

$$\begin{aligned}(d_A\omega)(\xi_0, ..., \xi_k) &= \sum_{j=0}^{k} (-1)^j \left(\#_A \circ \xi_j\right)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k\right),\end{aligned} \tag{2}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \mathrm{Sec}\, A$. The operators d_A^k satisfy

$$d_A(\omega \wedge \eta) = d_A\omega \wedge \eta + (-1)^k \omega \wedge d_A\eta,$$

so they are of first order and the symbol of d_A^k is equal to

$$\begin{aligned} S\left(d_A^k\right)_{(x,v)} &: \bigwedge^k A_x^* \to \bigwedge^{k+1} A_x^* \\ S\left(d_A^k\right)_{(x,v)}(u) &= (v \circ (\#_A)_x) \wedge u, \quad 0 \neq v \in T_x^*M. \end{aligned}$$

In consequence

Proposition 1.1 *The sequence of symbols*

$$\bigwedge^k A_x^* \overset{S\left(d_A^k\right)_{(x,v)}}{\rightarrow} \bigwedge^{k+1} A_x^* \overset{S\left(d_A^{k+1}\right)_{(x,v)}}{\rightarrow} \bigwedge^{k+2} A_x^*$$

is exact if and only if A is transitive. Therefore the complex $\left\{d_A^k\right\}$ is an ***elliptic complex*** *provided that A is transitive.*

Proof. The composition is zero. If $0 \neq v \in T_x^*M$ and A is transitive, then $\tilde{v} = (v \circ (\#_A)_x) \neq 0$ and $S\left(d_A^{k+1}\right)_{(x.v)} = \rho_{\tilde{v}}$, $\rho_{\tilde{v}}(u) = \tilde{v} \wedge u$. From the properties of exterior algebra the sequence of symbols is exact. If A is not transitive, then there exists a covector $0 \neq v \in T_x^*M$ such that $\tilde{v} = (v \circ (\#_A)_x) = 0$. Therefore $\sigma\left(d_A^k\right)_{(x,v)} = \rho_{\tilde{v}} = 0$ for each k and the sequence of symbols is not exact. ■

For the trivial Lie algebroid TM - the tangent bundle of the manifold M - the differential d_{TM} is the usual de-Rham differential d_M of differential forms on M whereas, for $L = \mathfrak{g}$ - a Lie algebra $\mathfrak{g}$ - the differential $d_{\mathfrak{g}}$ is the usual Chevalley-Eilenberg differential, $d_{\mathfrak{g}} = \delta_{\mathfrak{g}}$.

Theorem 1.2 (Kubarski-Mishchenko, 2004) *[K-M-2] For each transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ with the Atiyah sequence (1) over a connected compact oriented manifold M the following conditions are equivalent ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, i.e. $\operatorname{rank} A = m + n$)*

(1) $\mathbf{H}^{m+n}(A) \neq 0$,

(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$,

(3) *A is the so-called invariantly oriented, i.e. there exists a global non-singular cross-section ε of the vector bundle $\bigwedge^n \boldsymbol{g}$, $0 \neq \varepsilon_x \in \bigwedge^n \boldsymbol{g}_{|x}$, invariant with respect to the adjoint representation of A in the vector bundle $\bigwedge^n \boldsymbol{g}$ (which is extending of the adjoint representation ad_A of A in $\boldsymbol{g}$ given by $(ad_A)(\xi) : \mathrm{Sec}\, \boldsymbol{g} \to \mathrm{Sec}\, \boldsymbol{g}, \quad \nu \longmapsto [\![\xi, \nu]\!]$).*

The condition (3) yields that the structure Lie algebras $\boldsymbol{g}_{|x}$ are unimodular. Lie algebroids fulfilling (3) appeared in 1996 [K1] under the name TUIO-Lie algebroids (transitive unimodular invariantly oriented). The connectedness of M implies that any invariant cross-section ε is uniquely determined up to a constant factor. The fibre integral operator

$$\begin{aligned} \int_A &: \quad \Omega^k(A) \to \Omega^{k-n}_{dR}(M), \qquad k \geq n, \\ \left(\int_A \omega\right)_x (w_1, ..., w_{k-n}) &= (-1)^{kn}\, \omega_x(\varepsilon_x, \tilde{w}_1, ..., \tilde{w}_{k-n}), \quad \#_A(\tilde{w}_i) = w_i \end{aligned}$$

commutes with the differentials d_A and d_M if and only if ε is invariant. In this case the fibre integral gives a homomorphism in cohomology

$$\int_A^{\#} : \mathbf{H}^{\bullet}(A) \to \mathbf{H}^{\bullet - n}_{dR}(M)$$

and we have the isomorphism

$$\int_A^{\#} : \mathbf{H}^{m+n}(A) \xrightarrow{\cong} \mathbf{H}^{m}_{dR}(M) = \mathbb{R}.$$

The scalar Poincaré product

$$\mathcal{P}^k_A : \mathbf{H}^k(A) \times \mathbf{H}^{m+n-k}(A) \to \mathbb{R},$$

$$([\omega], [\eta]) \longmapsto \int_A \omega \wedge \eta = \int_M \left(\int_A \omega \wedge \eta\right)$$

is nondegenerated and if $m + n = 4p$ then

$$\mathcal{P}^{2p}_A : \mathbf{H}^{2p}(A) \times \mathbf{H}^{2p}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and it is called the *signature* of A, and is denoted by

$$\mathrm{Sig}(A).$$

Problem 1.3 *Calculate the signature* $\mathrm{Sig}(A)$ *and give some conditions to the equality* $\mathrm{Sig}(A) = 0$.

There are examples for which $\mathrm{Sig}\,(A) \neq 0$. They are based on the example of the flat bundle over surfaces with non-zero signature [Gro, $8\frac{2}{7}$].

In the paper [K2], a Hirzebruch signature operator for the cohomology $\mathbf{H}\,(A)$ is constructed. Below, we look at this operator from more general point of view, as well as we present a general mechanism for the calculation of the signature via spectral sequences, see [K-M-2], which we use to two kinds of spectral sequences associated with Lie algebroids:

a) the spectral sequence of the Čech-de Rham complex,

b) the Hochschild-Serre spectral sequence.

2 General approach to signature via spectral sequences

The idea of using spectral sequences to the signature comes from S.S.Chern, F.Hirzebruch, J-P. Serre [Ch-H-S]. Via spectral sequences the authors proved

Theorem 2.1 *Let $E \to M$ be a fiber bundle, with the typical fiber F, such that the following two conditions are satisfied:*

(1) E, M, F are compact connected oriented manifolds;

(2) the fundamental group $\pi_1\,(M)$ acts trivially on the cohomology ring $\mathbf{H}^*\,(F)$ of F.

Then, if E, M, F are oriented coherently, so that the orientation of E is induced by those of F and M, the index of E is the product of the indices of F and M, that is

$$\mathrm{Sig}\,(E) = \mathrm{Sig}\,(F) \cdot \mathrm{Sig}\,(M)\,.$$

The authors consider the cohomology Leray spectral sequence $E_s^{p,q}$ of the bundle $E \to B$ with the real coefficients. The term E_2 by hypothesis (2) is the bigraded algebra

$$E_2^{p,q} \cong \mathbf{H}^p\,(M;\mathbf{H}^q\,(F)) \cong \mathbf{H}^p\,(M) \otimes \mathbf{H}^q\,(F)\,.$$

Therefore

$$E_2^{p,q} = 0 \ \text{ for } \quad p > m \quad \text{or} \quad q > n.$$

Clearly, E_2 is a **Poincaré algebra** by hypothesis (1). Using the spectral sequences argument the authors noticed that

$$(E_s, d_s, \cdot)\,,\ s \geq 2,$$

are Poincaré algebras with **Poincaré differentiations**. The infinite term $(E_\infty, \cdot)$ is also a Poincaré algebra, and the equality of signatures

$$\mathrm{Sig}\,E_2 = \mathrm{Sig}\,E_3 = ... = \mathrm{Sig}\,E_\infty$$

holds. The last step

$$\operatorname{Sig} E_\infty = \operatorname{Sig} \mathbf{H}(E)$$

is also proved. We add that it is not so trivial since, in general, the algebras E_∞ and $\mathbf{H}(E)$ are not isomorphic (although we have $E_\infty \cong \mathbf{H}(E)$ as bigraded spaces).

We recall that a finitely graded algebra $\left(A^* = \bigoplus_{0\le r\le N} A^N, \cup\right)$ is called a *Poincaré algebra*, if

(1) $\dim A^N = 1$,

(2) $x \cup y = (-1)^{ij}\, y \cup x$ if $x \in A^i$, $y \in A^j$, i.e. $(A, \cup)$ is an anticommutative algebra,

(3) let $0 \neq \xi \in A^N$ be a base element of A^N. The bilinear form

$$\langle \cdot,\cdot \rangle : A^r \times A^{N-r} \to \mathbb{R}$$

relative to ξ (i.e. $\langle x, y\rangle \xi = x \cup y$) is nondegenerate. Therefore $A^r \cong \left(A^{N-r}\right)^*$ and $\dim A^r = \dim A^{N-r}$.

The key to the further investigation is the notion of a *Poincaré differentiation*, i.e. a linear homomorphism $d : A \to A$ satisfying the conditions:

(1) $d^2 = 0$,

(2) $d\left[A^r\right] \subset A^{r+1}$,

(3) d is antiderivation,

(4) $d\left[A^{N-1}\right] = 0$ (in particular, if $x \in A^r$, $y \in A^{N-r-1}$, then $dx \cup y = -(-1)^r\, x \cup dy$).

In analogy with the signature of an oriented manifold we have the *signature* of a finitely dimensional Poincaré algebra $(A = \bigoplus A^r, \cup)$ relative to $0 \neq \xi \in A^N$. It is to be zero if $N \neq 0 \pmod 4$ and if $N = 4k$, $\operatorname{Sig} A$ is the signature of the symmetric nondegenerated function $\langle\cdot,\cdot\rangle^{2k,2k} : A^{2k} \times A^{2k} \to \mathbb{R}$ defined relatively to ξ

$$\operatorname{Sig} A = \operatorname{Sig}\langle\cdot,\cdot\rangle^{2k,2k}.$$

The following lemma will be very useful below.

Lemma 2.2 *[Ch-H-S] If $(A, \cup, d)$ is a finitely dimensional Poincaré algebra with Poincaré differentiation, then the cohomology graded algebra $(\mathbf{H}^*(A), \cup)$ is a Poincaré algebra and relative to the same element $0 \neq \xi \in A^N = \mathbf{H}^N(A, d)$ the equality holds*

$$\operatorname{Sig} A = \operatorname{Sig} \mathbf{H}(A).$$

Example 2.3 (1) Let E be any finitely dimensional vector space. Then the exterior algebra $\bigwedge E$ is a Poincaré algebra. Its signature is zero.

(2) Let $\mathfrak{g}$ be any real Lie algebra. Then the system

$$\left(\bigwedge \mathfrak{g}^*, \wedge, \delta_{\mathfrak{g}}\right)$$

with the Chevalley-Eilenberg differentiation $\delta_{\mathfrak{g}}$ is a Poincaré algebra with Poincaré differentiation if and only if $\mathfrak{g}$ is unimodular. The above lemma yields: if $\mathfrak{g}$ is unimodular, then the cohomology algebra $\mathbf{H}(\mathfrak{g})$ is a Poincaré algebra and

$$\operatorname{Sig}\mathbf{H}(\mathfrak{g}) = \operatorname{Sig}\bigwedge \mathfrak{g}^* = 0.$$

It appears that the Chern-Hirzebruch-Serre arguments used to prove the above theorems on the signature of the total space of the bundle $E \to M$ are purely algebraic and lead to the following general theorems [K-M-2].

Theorem 2.4 *Let $((A, \langle , \rangle), A^r, \cup, D, A_j)$ be any DG-algebra with a gradation A^r and a decreasing filtration A_j and let $(E_s^{p,q}, d_s)$ be its spectral sequence. We assume that there exist natural numbers m and n such that:*

- *$E_2^{p,q} = 0$ for $p > m$ and $q > n$, $m + n = 4k$,*
- *E_2 is a Poincaré algebra with respect to the total gradation and the top group $E_2^{(m+n)} = E_2^{m,n}$.*

Then each term $\left(E_s^{(*)}, \cup, d_s\right)$ $2 \leq s < \infty$, is a Poincaré algebra with Poincaré differentiation, the infinite term $\left(E_\infty^{(*)}, \cup\right)$ is also a Poincaré algebra and

$$\operatorname{Sig} E_2 = \operatorname{Sig} E_3 = \ldots = \operatorname{Sig} E_\infty.$$

If m and n are odd, then $\operatorname{Sig} E_2 = 0$. If m and n are even, then

$$\begin{aligned} \operatorname{Sig} E_2 &= \operatorname{Sig}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \\ &= \operatorname{Sig}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right). \end{aligned}$$

It remains to prove the equality $\operatorname{Sig} E_\infty = \operatorname{Sig}\mathbf{H}(A)$. The same arguments as in the original work [Ch-H-S] give the following general theorem:

Theorem 2.5 *Let $(A, A^r, \cup, D, A_j)$ be any DG-algebra with a gradation A^r and with a decreasing filtration A_j compatible with the DG structure, i.e.*

$$A_i A_j \subset A_{i+j}, \;\; D(A_j) \subset A_j, \;\; A_j = \bigoplus_r A^r \cap A_j,$$

and satisfying the regularity condition $A_0 = A$

$$A = A_0 \supset \cdots \supset A_j \supset A_{j+1} \supset \cdots$$

Let $(E_s^{p,q}, d_s)$ be the spectral sequence associated to this graded differential filtered algebra A. We assume that

— the infinite term $E_\infty^{p,q}$ lives in the rectangular $0 \leq p \leq m$, $0 \leq q \leq n$,

— $\dim E_\infty^{m,n} = 1$,

— E_∞ is a Poincaré algebra with respect to the total gradation, in particular, $\dim E_\infty$ is finite.

Under the above assumptions on the graded differential filtered algebra A, the cohomology algebra $\mathbf{H}(A)$ satisfies the conditions:

(1) $\mathbf{H}^{m+n}(A) \cong E_{\infty}^{m,n}$*,i.e. in particular,* $\dim \mathbf{H}^{m+n}(A) = 1$,

(2) the algebra $\mathbf{H}(A) = \bigoplus_{r=0}^{m+n} \mathbf{H}^r(A)$ *is a Poincaré algebra,*

(3) the signature of the cohomology of $\mathbf{H}(A)$ *is equal to the signature of the term* E_∞,

$$\operatorname{Sig} E_\infty = \operatorname{Sig} \mathbf{H}(A)$$

under suitable choice of generators of the top groups.

Therefore, under some natural simple regularity assumptions on a DG-algebra A^r we have: if $E_2^{p,q}$ is a Poincaré algebra and live in a finite rectangular, then

$$\operatorname{Sig}(E_2) = \operatorname{Sig}(\mathbf{H}(A)).$$

We use this mechanism to

(a) the spectral sequence for the Čech-de Rham complex of the Lie algebroid A [K-M-2],

(b) the Hochshild-Serre spectral sequences [K-M-3].

(a): For the details see [K-M-2]. Let $\mathcal{H}^*(A) = \left(U \longmapsto \mathbf{H}^*\left(A_{|U}\right)\right)$ be the Leray type presheaf of cohomology, locally constant on a good covering $\mathfrak{U}$, with values in the cohomology algebra $\mathbf{H}^*(\mathfrak{g})$ of the structural Lie algebra $\mathfrak{g}$. Then $E_1^{p,q} = C^p(\mathfrak{U}, \mathcal{H}^q(A))$, $d_1 = \delta^\# : E_1^{p,q} \to E_1^{p+1,q}$, where δ is the coboundary homomorphism, $E_2^{p,q} = \mathbf{H}^p_{\delta^\#}(\mathfrak{U}, \mathcal{H}^q(A))$. If the monodromy representation $\rho : \pi_1(M) = \pi_1(N(\mathfrak{U})) \to \operatorname{Aut}(\mathbf{H}(\mathfrak{g}))$ of the presheaf $\mathcal{H}(A)$ is trivial, then

$$E_2^{p,q} \cong \mathbf{H}^p_{dR}(M) \otimes \mathbf{H}^q(\mathfrak{g})$$

(the isomorphisms are canonical isomorphisms of bigraded algebras). Therefore $\operatorname{Sig} E_2 = \operatorname{Sig}(\mathbf{H}(M) \otimes \mathbf{H}(\mathfrak{g})) = \operatorname{Sig} \mathbf{H}(M) \cdot \operatorname{Sig} \mathbf{H}(\mathfrak{g})$. Hence as the isotropy Lie algebra $\mathfrak{g}$ is unimodular, i.e. $\dim \mathbf{H}^n(\mathfrak{g}) = 1$, we have $\operatorname{Sig} \mathbf{H}(\mathfrak{g}) = \operatorname{Sig} \bigwedge \mathfrak{g}^* = 0$ and therefore

$$\operatorname{Sig}(A) = \operatorname{Sig} H(A) = \operatorname{Sig} E_2 = \operatorname{Sig}(M) \cdot \operatorname{Sig}(\mathfrak{g}) = 0.$$

Example 2.6 The condition of the triviality of the monodromy (in consequence $\operatorname{Sig}(A) = 0$) holds if:

- M is simply connected,
- $\operatorname{Aut} G = \operatorname{Int} G$, where G is a simply connected Lie group with the Lie algebra $\mathfrak{g}$, for example if $\mathfrak{g}$ is a simple Lie algebra of type $B_l, C_l, E_7, E_8, F_4, G_2$.
- the adjoint Lie algebra bundle $\boldsymbol{g}$ is trivial in the category of flat bundles (the bundle $\mathbf{H}(\boldsymbol{g})$ of cohomology of isotropy Lie algebras with the typical fibre $H(\mathfrak{g})$ possess canonical flat covariant derivative - which will be important for studying of the Hochshild-Serre spectral sequence). For example, $\boldsymbol{g}$ is trivial in the category of flat bundles, for the Lie algebroid $A(G; H)$ of the the TC-foliation of left cosets of a nonclosed Lie subgroup H in any Lie group G.

(b): Following G.Hochschild and J.-P.Serre [H-S], for a pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g},\mathfrak{k})$ one can consider a graded cochain group of $\mathbb{R}$-linear alternating functions $A_{\mathbb{R}}(P)=\bigoplus_{k\geq 0}A^{k}(P)$, $A^{k}(P)=C^{k}(\mathfrak{g},P)$ with values in a $\mathfrak{g}$-module P, $\mathfrak{g}\times P\rightarrow P$, with the standard $\mathbb{R}$-differential operator d of degree 1 and the Hochschild-Serre filtration $A_{j}\subset A_{\mathbb{R}}(P)$ as follows:

— $A_{j}=A_{\mathbb{R}}(P)$ for $j\leq 0$,

— if $j>0$, $A_{j}=\bigoplus_{k\geq j}A_{j}^{k}$, $A_{j}^{k}=A_{j}\cap A^{k}$, where A_{j}^{k} consists of all those k-cochains f for which $f(\gamma_{1},...,\gamma_{k})=0$ whenever $k-j+1$ of the arguments γ_{i} belongs to $\mathfrak{k}$.

In this way we have obtained a graded filtered differential $\mathbb{R}$-vector space

$$\big(A_{\mathbb{R}}=\bigoplus_{k\geq 0}A^{k},d,A_{j}\big) \tag{3}$$

and we can use its spectral sequence

$$\big(E_{s}^{p,q},d_{s}\big). \tag{4}$$

For a transitive Lie algebroid $A=(A,[\![\cdot,\cdot]\!],\#_{A})$ with the Atiyah sequence $0\rightarrow\boldsymbol{g}\hookrightarrow A\overset{\#_{A}}{\rightarrow}TM\rightarrow 0$ we will consider the pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g},\mathfrak{k})$ where

$$\mathfrak{g}=\operatorname{Sec}(A)\,,\quad \mathfrak{k}=\operatorname{Sec}(\boldsymbol{g})\,.$$

Following K.C.M.Mackenzie (1987) [M], V.Itskov, M.Karashev, and Y.Vorobjev (1998), [I-K-V]), J.Kubarski, A.S.Mishchenko (2004) [K-M-3] we will consider the $C^{\infty}(M)$-submodule of $C^{\infty}(M)$-linear alternating cochains

$$\Omega^{k}(A)\subset C^{k}(\mathfrak{g},C^{\infty}(M))$$

with values in the trivial $\mathfrak{g}$-module $C^{\infty}(M)$ (i.e. with respect to the trivial representation $\partial_{\xi}(X)=\#_{A}(\xi)(X)$) and the induced filtration

$$\Omega_{j}=\Omega_{j}(A)=A_{j}\cap\Omega(A)$$

of $C^{\infty}(M)$-modules. In this way we obtain a graded filtered differential space

$$\big(\Omega(A)=\bigoplus_{k}\Omega^{k}(A)\,,d_{A},\Omega_{j}\big) \tag{5}$$

and its spectral sequence

$$\big(E_{A,s}^{p,q},d_{A,s}\big). \tag{6}$$

Assume as above

$$m=\dim M,\ n=\dim\boldsymbol{g}_{|x},\quad \text{i.e. }\ \operatorname{rank}A=m+n.$$

The multiplication $\wedge$ and differentiation d_{A} of differential form, defined by (2), preserves gradations and filtrations $\wedge:\Omega_{j}^{k}\times\Omega_{i}^{r}\rightarrow\Omega_{j+i}^{k+r}$, $d_{A}:\Omega_{j}^{k}\rightarrow\Omega_{j}^{k+1}$. We have

$$\begin{aligned}
E_{A,0}^{p,q} &= \Omega_{p}^{p+q}(A)/\Omega_{p+1}^{p+q}\\
d_{A,0}^{p,q} &: E_{A,0}^{p,q}\rightarrow E_{A,0}^{p,q+1},\quad [\omega]\longmapsto[d_{A}\omega]\,.
\end{aligned}$$

Taking an arbitrary connection $\lambda : TM \to A$ in the Lie algebroid A we obtain an isomorphism of $C^{\infty}(M)$-modules [K-M-3, Con. 5.2]

$$a_A^{p,q} : E_{A,0}^{p,q} \xrightarrow{\cong} \Omega^p \left(M; \bigwedge^q \boldsymbol{g}^* \right),$$

$$a_A^{p,q}([\omega])_x (v_1, ..., v_p)(\sigma_1, ..., \sigma_q) = \omega_x (\sigma_1, ..., \sigma_q, \lambda v_1, .., \lambda v_p), \quad v_i \in T_x M, \ \sigma_i \in \boldsymbol{g}_{|x}.$$

Since $\omega \in \Omega_j^{p+q}$, then $a_A^{p,q}$ does not depend on λ.

Through the isomorphism $a_A^{p,q}$ the differential $d_{A,0} : E_{A,0}^{p,q} \to E_{A,0}^{p,q+1}$ can be identified with a differentiation

$$\tilde{d}_{A,0}^{p,*} : \Omega^p \left(M; \bigwedge^q \boldsymbol{g}^* \right) \to \Omega^p \left(M; \bigwedge^{q+1} \boldsymbol{g}^* \right)$$

of differential forms with values in $\bigwedge^q \boldsymbol{g}^*$ with respect to the Chevalley-Eilenberg differential at any point for the isotropy Lie algebra $\boldsymbol{g}_{|x}$, $\mathbf{H}\left(\Omega^p \left(M; \bigwedge^{\bullet} \boldsymbol{g}^* \right), \tilde{d}_{A,0} \right) = \Omega^p \left(M; \mathbf{H}^{\bullet}(\boldsymbol{g}) \right)$. Therefore

$$E_{A,1}^{p,\bullet} \cong \mathbf{H}\left(E_{A,0}^{p,\bullet}, d_{A,0} \right) \overset{b^{p,\bullet}}{\cong} \Omega^p \left(M; \mathbf{H}^{\bullet}(\boldsymbol{g}) \right).$$

where the isomorphism $b^{p,q}$ is given by $[\omega] \longmapsto [\bar{\omega}_p]$ for $\omega \in \Omega_p^{p+q}; \ d_A \omega \in \Omega_{p+1}^{p+q+1}$ and $\bar{\omega}_p \in \Omega^p \left(M; \bigwedge^q \boldsymbol{g}^* \right)$ is equal to $\bar{\omega}_p (v_1, ..., v_p)(\sigma_1, ..., \sigma_q) = \omega(\sigma_1, ..., \sigma_q, \lambda v_1, .., \lambda v_p)$. We carry over the differentials $d_{A,1}^{p,q} : E_{A,1}^{p,q} \to E_{A,1}^{p+1,q}$ to $\Omega^p \left(M; \mathbf{H}^q(\boldsymbol{g}) \right)$ via isomorphisms $b^{p,q}$. In the vector bundle $\mathbf{H}^q(\boldsymbol{g})$ there is a flat covariant derivative ∇^q such that $d_{A,1}^{*,q} = (-1)^q d_{\nabla^q}$ [K-M-3, Prop. 5.9]. The flat covariant derivative ∇^q is defined by $\nabla_X^q [f] = [\mathcal{L}_X f]$ for $f \in \Omega^p \left(M; Z\left[\bigwedge^q \boldsymbol{g}^* \right] \right)$, $[f] \in \Omega^p \left(M; \mathbf{H}^q(\boldsymbol{g}) \right)$ where

$$(\mathcal{L}_X f)(\sigma_1, ..., \sigma_q) = \partial_X \left(f(\sigma_1, ..., \sigma_q) \right) - \sum_{i=1}^{q} f(\sigma_1, ..., [\![\lambda X, \sigma_i]\!], ..., \sigma_q)$$

(we recall that $\lambda : TM \to A$ is an arbitrary auxiliary connection in A). Therefore

$$E_{A,2}^{p,q} \cong \mathbf{H}^p \left(E_{A,1}^{\bullet,q}, d_{A,1}^{\bullet,q} \right) \cong \mathbf{H}_{\nabla^q}^p \left(M; \mathbf{H}^q(\boldsymbol{g}) \right)$$

and the second isomorphism is given by

$$[\omega] \longmapsto [\ [\bar{\omega}]]\,..$$

Summing up, we have obtained

Theorem 2.7 *If A is a TUIO-Lie algebroid such that $m + n = 4p$ ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$), then*

a) if m and n are odd, then $\operatorname{Sig} A = 0$, *b) if m and n are even, then*

$$\begin{aligned}\operatorname{Sig} A &= \operatorname{Sig} E_2 = \operatorname{Sig}\left(E_2^{(2p)} \times E_2^{(2p)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \\ &= \operatorname{Sig}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right)\end{aligned}$$

where $E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)$ *and*

$$\mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbf{H}^{m}_{\nabla^{n}}\left(M; \mathbf{H}^{n}(\boldsymbol{g})\right) = \mathbb{R}$$

is defined via the usual multiplication of differential forms with respect to the multiplication of cohomology class for Lie algebras.

$$\phi : \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \times \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \to \mathbf{H}^{n}(\boldsymbol{g}) = M \times \mathbb{R}.$$

We notice that if $\frac{n}{2}$ is even, then ϕ is symmetric nondegenerated (in this way we obtain a Lusztig example), while if $\frac{n}{2}$ is odd, then ϕ is symplectic (in this way we obtain a Gromov example). However, $\mathbf{H}^{\frac{m}{2}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{\frac{m}{2}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$ *is always symmetric nondegenerated.*

3 Algebraic aspects of the Hirzebruch signature operator

Below, we give a common algebraic approach to the calculation the signature $\operatorname{Sig}(W)$ via the Hirzebruch signature operator.

3.1 Hodge space

In this subsection we present algebraic point of view on the $*$-Hodge operator, Hodge theorem and Hirzebruch signature operator.

Definition 3.1 *By a* ***Hodge space*** *we mean the triple* $(W, \langle , \rangle, (,))$ *where W is a real vector space ($\dim W$ is finite or infinite),* $\langle , \rangle, (,) : W \times W \to \mathbb{R}$ *are 2-linear tensors such that*

(1) $(,)$ is symmetric and positive definite (i.e. is an inner product),

(2) there exists a linear homomorphism

$$*_W : W \to W$$

called $*$-Hodge operator fulfilling properties:

(i) for all $v \in V$,

$$\langle v, w \rangle = (v, *_W(w)),$$

(ii) $*_W$ is an isometry with respect to $(,)$, i.e.

$$(v, w) = (*_W v, *_W w),$$

Clearly, the $*$-Hodge operator is uniquely determined (if exists).

Two 2-tensors $f : V \times V \to \mathbb{R}$ and $g : W \times W \to \mathbb{R}$ determine the tensor product

$$f \otimes g : (V \otimes W) \times (V \otimes W) \to \mathbb{R}$$

which is 2-linear.

Lemma 3.2 ([Gre]) *The tensor $f \otimes g$ is symmetric and positive definite if both f and g are symmetric and positive definite (the dimensions of V and W can be infinite).*

From the above we have:

Lemma 3.3 *If $(V, \langle \cdot, \cdot \rangle_V, (\cdot, \cdot)_V)$ and $(W, \langle \cdot, \cdot \rangle_W, (\cdot, \cdot)_W)$ are Hodge spaces, then their tensor product*

$$(V \otimes W, \langle \cdot, \cdot \rangle_V \otimes \langle \cdot, \cdot \rangle_W, (\cdot, \cdot)_V \otimes (\cdot, \cdot)_W)$$

is a Hodge space and

$$*_{V \otimes W} = *_V \otimes *_W.$$

3.2 Finitely dimensional Hodge spaces, examples.

Lemma 3.4 *Let $(W, \langle \cdot, \cdot \rangle)$ be a finite dimensional real vector space equipped with a 2-tensor $\langle \cdot, \cdot \rangle$. Then there exists an inner product $(\cdot, \cdot)$ such that the system $(W, \langle , \rangle, (,))$ is a Hodge space if and only if there exists a basis of W in which the matrix of $\langle , \rangle$ is orthogonal.*

Proof. Standard calculations. ∎

It is an important observation that the calculation of the signature (in standard cases) via the idea of Hirzebruch operator is restricted to such 2-tensors $\langle \cdot, \cdot \rangle$ (in fibres of some vector bundles) for which there exists an auxiliary scalar product $(\cdot, \cdot)$ with respect to the system $(W, \langle , \rangle, (,))$ is a Hodge space.

Now we give examples of a number of finite dimensional Hodge spaces.

Example 3.5 (Classical) Let (V, G) be a real N-dimensional oriented Euclidean space with an inner product $G : V \times V \to \mathbb{R}$ and the volume tensor $\varepsilon = e_1 \wedge ... \wedge e_N \in \bigwedge^N V$, (where $\{e_i\}_{i=1}^N$ is a positive ON base of V). We identify $\bigwedge^N V = \mathbb{R}$ via the isomorphism $\rho : \bigwedge^N V \xrightarrow{\cong} \mathbb{R}, \quad s \cdot \varepsilon \longmapsto s$. We have the classical Hodge space

$$\left(\bigwedge V, \langle , \rangle, (,) \right)$$

where

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad &: \quad \bigwedge V \times \bigwedge V \to \mathbb{R}, \\ \langle \cdot, \cdot \rangle^k \quad &: \quad \bigwedge^k V \times \bigwedge^{N-k} V \to \bigwedge^N V = \mathbb{R}, \\ \left\langle v^k, v^{N-k} \right\rangle \quad &= \quad \rho\left(v^k \wedge v^{N-k}\right), \end{aligned}$$

$\langle,\rangle = 0$ outside the pairs of degree $(k, N-k)$, and

$$(\cdot,\cdot)^k : \bigwedge^k V \times \bigwedge^k V \to \mathbb{R}, \qquad (v_1 \wedge .... \wedge v_k, w_1 \wedge ... \wedge w_k)^k = \det\left[G\left(v_i, w_k\right)\right],$$

the subspaces $\bigwedge^k V$, $k = 0, 1, ..., N$ are orthogonal (by definition).

The $*$-Hodge operator exists and it is determined via an ON base $\{e_i\}_{i=1}^N$ by the formula

$$*\left(e_{i_1} \wedge ... \wedge e_{i_k}\right) = \varepsilon_{(j_1,...,j_{n-k})} \cdot e_{j_1} \wedge ... \wedge e_{j_{n-k}}$$

where $i_1 < ... < i_k$ and $j_1 < ... < j_{n-k}$ and the sequence $(j_1, ..., j_{n-k})$ is complementary to $(i_1, ..., i_k)$ and $\varepsilon_{(j_1,...,j_{n-k})} = \operatorname{sgn}\left(j_1, ..., j_{n-k}, i_1, ..., i_k\right).$

The above classical example is used:

- for $V = T_x M$ or $V = T_x^* M$ where M is a Riemannian manifold;
- for $V = A_x$ where A is a TUIO-Lie algebroid over a Riemann manifold (see below).

Example 3.6 (Lusztig example, 1972) [L] Let $(,)_0 : E \times E \to \mathbb{R}$ be a symmetric (indefinite in general) nondegenerated tensor on a finite dimensional vector space E. Let G be an arbitrary positive scalar product in E. Then there exists exactly one direct product $E = E_+ \oplus E_-$ which is ON with respect to the both scalar product $(,)_0$ and G and such that $(,)_0$ on E_+ is positive and on E_- is negative. We denote by $*_E$ the involution $*_E : E \to E$ such that

$$*_E | E_+ = id, \quad *_E | E_- = -id.$$

Then, the quadratic form

$$\begin{aligned} (,) \quad &: \quad E \times E \to \mathbb{R} \\ (v, w) \quad &: \quad = (v, *_E w)_0 \end{aligned}$$

is symmetric and positive definite. The involution $*_E$ is an isometry

$$(*_E v, *_E w) = \left(*_E v, *_E^2 w\right)_0 = (*_E v, w)_0 = (w, *_E v)_0 = (w, v) = (v, w).$$

Therefore $(E, (,)_0, (,))$ is a Hodge-space.

Example 3.7 (Gromov example, 1995) [Gro] Let $\langle,\rangle_0 : E \times E \to \mathbb{R}$, be a skew-symmetric nondegerated tensor on a finite dimensional vector space E. There exists an anti-involution τ in E, $\tau^2 = -id$ (i.e. a complex structure) such that

$$\langle \tau v, \tau w \rangle_0 = \langle v, w \rangle_0, \quad v, w \in E,$$

$$\langle v, \tau v \rangle_0 > 0 \quad \text{for all} \quad v \neq 0.$$

Namely, there exists a base of E for which the matrix of $\langle,\rangle_0$ is orthogonal and is of the form

$$\begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}$$

and τ is given by the formula

$$\begin{aligned} \tau(v_i) &= v_{n+i}, \\ \tau(v_{n+i}) &= -v_i. \end{aligned}$$

Then the tensor

$$\begin{aligned} (,) &: E \times E \to \mathbb{R} \\ (v,w) &: = \langle v, \tau w\rangle_0 \end{aligned}$$

is symmetric and positive definite and

$$(\tau v, \tau w) = (v, w),$$

i.e. τ preserves both forms $\langle,\rangle_0$ and $(,)$. The system $(E, \langle,\rangle_0, (,))$ is a Hodge-space since the operator $-\tau$ is the $*$-Hodge operator

$$\langle v, w\rangle_0 = \left\langle v, -\tau^2 w\right\rangle_0 = \langle v, \tau(-\tau w)\rangle_0 = (v, -\tau w),$$

and $-\tau$ is an isometry $(-\tau v, -\tau w) = (\tau v, \tau w) = (v, w)$.

Definition 3.8 *By the Hodge vector bundle we mean a system $(\xi, \langle,\rangle, (,))$ consisting of a vector bundle ξ and two smooth tensor fields $\langle,\rangle$, $(,) : \xi \times \xi \to \mathbb{R}$ (sections of $(\xi \otimes \xi)^*$) such that for each $x \in M$ the system $(\xi_x, \langle,\rangle_x, (,)_x)$ is a finitely dimensional Hodge space and the family of Hodge operators $*_x : \xi_x \to \xi_x$, $x \in M$, gives a smooth linear homomorphism of vector bundles, $* : \xi \to \xi$.*

Example 3.9 (Important example) Consider an arbitrary Riemannian oriented manifold M of dimension N and a Hodge vector bundle $(\xi, \langle,\rangle, (,))$. Then for any point $x \in M$ we take the tensor product of Hodge spaces $\bigwedge T_x^* M \otimes \xi_x$. Assuming the compactness of M we can define two 2-$\mathbb{R}$-linear tensors

$$((\alpha, \beta)),\ \langle\langle \alpha, \beta\rangle\rangle : \Omega(M;\xi) \times \Omega(M;\xi) \to \mathbb{R},$$

by integrating along the Riemannian manifold

$$((\alpha, \beta)) = \int_M (\alpha_x, \beta_x)\, dM, \qquad \langle\langle \alpha, \beta\rangle\rangle = \int_M \langle \alpha_x, \beta_x\rangle\, dM = \int_M \alpha \wedge_\varphi \beta$$

where

$$\varphi_x = \langle\cdot,\cdot\rangle_x^k : \bigwedge^k T_x^* M \otimes \xi_x \times \bigwedge^{N-k} T_x^* M \otimes \xi_x \to \bigwedge^N T_x^* M = \mathbb{R}$$

is the wedge product with respect to the multiplication $\langle,\rangle_x$ of the values. The 2-form $((\cdot,\cdot))$ is symmetric and positive definite and the triple

$$(\Omega(M;W), \langle\langle \alpha, \beta\rangle\rangle, ((\alpha, \beta)))$$

is a Hodge space with the $*$-Hodge operator $\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$ defined point by point

$$(*\beta)_x = *_x (\beta_x) .$$

Indeed,

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha_x, \beta_x \rangle \, dM = \int_M (\alpha_x, *_x \beta_x) \, dM = ((\alpha, *\beta)) .$$

3.3 Graded differential Hodge space

Definition 3.10 *By a* ***graded differential Hodge space*** *we mean a system*

$$\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), d \right)$$

where $(W, \langle \cdot, \cdot \rangle, (\cdot, \cdot))$ is a Hodge space (finitely or infinitely dimensional) and

(1) $\langle \cdot, \cdot \rangle^k : W^k \times W^{N-k} \to \mathbb{R}$ and $\langle , \rangle = 0$ outside the pairs $(k, N-k)$,
(2) W^k are orthogonal with respect to $(\cdot, \cdot)$,
(3) d is homogeneous of degree $+1$, i.e. $d : W^k \to W^{k+1}$, and $d^2 = 0$,
(4) $\langle dw, u \rangle = (-1)^{k+1} \langle w, du \rangle$ for $w \in W^k$.

Clearly,

a) the induced cohomology pairing

$$\langle , \rangle_{\mathbf{H}}^k : \mathbf{H}^k (W) \times \mathbf{H}^{N-k} (W) \to \mathbb{R},$$

$$\langle [u], [w] \rangle_{\mathbf{H}}^k = \langle u, w \rangle^k ,$$

is correctly defined,

b) $* \left[W^k \right] \subset W^{N-k}$, and $\quad * : W^k \to W^{N-k} \quad$ is an isomorphism.

Assume that W is a **graded differential Hodge spacer,** and let $d^* : W \to W$ be the adjoint operator with respect to $(,)$, i.e. the one such that

$$(d^* (w_1), w_2) = (w_1, d (w_2)) .$$

We assume there exists d^*. It is easy to see that d^* is of degree -1, $d^* : W^{k+1} \to W^k$. Using standard calculations we can show that the operator (called the Laplacian)

$$\Delta := (d + d^*)^2 = dd^* + d^* d$$

is homogeneous of degree 0, i.e. $\Delta [W^r] \subset W^r$, is self-adjoint $(\Delta v, w) = (v, \Delta w)$, nonnegative $(\Delta v, v) \geq 0$, and we have

$$\{v \in W;\ (\Delta v, v) = 0\} = \{v \in W;\ dv = 0 = d^* v\} .$$

Definition 3.11 *A vector $v \in W$ is called* ***harmonic*** *if $dv = 0$ and $d^* v = 0$, or equivalently if $v \perp (\Delta v)$. Denote*

$$\mathcal{H}(W) = \{v \in W;\ dv = 0,\ d^* v = 0\},$$
$$\mathcal{H}^k(W) = \left\{v \in W^k;\ dv = 0,\ d^* v = 0\right\}.$$

The harmonic vectors forms a graded vector space

$$\mathcal{H}(W) = \bigoplus_{k=0}^{N} \mathcal{H}^k(W).$$

Lemma 3.12 $\mathcal{H}^k(W) = \ker\left\{d + d^* : W^k \to W\right\} = \ker\left\{\Delta^k : W^k \to W^k\right\}$, *i.e.*

$$\mathcal{H}(W) = \ker \Delta = (\operatorname{Im} \Delta)^{\perp}.$$

Proof. Standard calculations. ■

$\ker \Delta$ is the eigenspace of the operator Δ corresponding to the zero value of the eigenvalue.

If W is a Hilbert space and $Y \subset W$ is a closed subset, then W is the direct sum $Y \bigoplus Y^{\perp}$. For a Riemannian vector bundle ξ over a Riemannian manifold, the space $W = \operatorname{Sec}(\xi)$ is not a Hilbert one (because it is not complete). But we have the following well known important theorem, see for example [L-M].

Theorem 3.13 *Let ξ be a Riemannian vector bundle over a compact oriented Riemannian manifold M. If $\Delta : \operatorname{Sec} \xi \to \operatorname{Sec} \xi$ is a* ***self-adjoint nonnegative elliptic operator*** *then $\ker \Delta$ is a finite dimensional space and*

$$\operatorname{Sec} \xi = \operatorname{Im} \Delta \bigoplus \ker \Delta = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}.$$

In the sequel $\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), d\right)$ denotes an arbitrary graded differential Hodge space. The spaces $\ker \Delta^k$ and $\operatorname{Im} d^{k-1}$ are orthogonal, in particular $\ker \Delta^k \cap \operatorname{Im} d^{k-1} = 0$. Therefore the inclusion

$$\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$$

induces a monomorphism

$$\ker \Delta^k \rightarrowtail \mathbf{H}^k(W).$$

Below we notice that the algebraic assumption

$$W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp} \tag{7}$$

implies that the above monomorphism is an isomorphism, i.e. that in each cohomology class there is (exactly one) a harmonic vector.

Theorem 3.14 *If* $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$, *then*

(1)

$$W^k = \ker \Delta^k \bigoplus \operatorname{Im}\Delta^k,$$

(2)

$$W^k = \ker \Delta^k \bigoplus \operatorname{Im} d^{k-1} \bigoplus \operatorname{Im}(d^*)^{k+1},$$

(3)

$$\ker d^k = \ker \Delta^k \bigoplus \operatorname{Im} d^{k-1}.$$

In particular, if $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$, *the inclusion*

$$\ker \Delta^k \hookrightarrow \ker d^k$$

induces an isomorphism

$$\ker \Delta^k \hookrightarrow \ker d^k \to \ker d^k / \operatorname{Im} d^{k-1} = \mathbf{H}^k(W).$$

It means that in each cohomology class there is exactly one harmonic vector.

(4) The equation $\Delta w = u$, *for a given* u, *has a solution if and only if* $u \in (\ker\Delta)^{\perp}$, *equivalently*

$$\operatorname{Im}\Delta = (\ker\Delta)^{\perp}.$$

Proof. (1) evident.

(2) Since $\operatorname{Im}\Delta^k \subset \operatorname{Im} d^{k-1} + \operatorname{Im} d^{*(k+1)}$ $(\Delta u = d(d^*u) + d^*(du) \in \operatorname{Im} d^{k-1} + \operatorname{Im} d^{*(k+1)}$) then

$$W^k = \ker\Delta^k \bigoplus \operatorname{Im}\Delta^k = \ker\Delta^k + \operatorname{Im} d^{k-1} + \operatorname{Im}(d^*)^{k+1}.$$

So we need only to check (which is very easy) that these three subspaces are ON.

(3) Since $\ker\Delta^k$ and $\operatorname{Im} d^{k-1}$ are ON and $\ker\Delta^k + \operatorname{Im} d^{k-1} \subset \ker d^k$, we only need to show that $\ker\Delta^k + \operatorname{Im} d^{k-1} \subset \ker d^k$. Let $u^k \in \ker d^k$ and expand u^k thanks to (2) $u^k = w_1 + dw_2 + d^*w_3, \quad \Delta w_1 = 0$. In particular, $dw_1 = 0$ and $0 = dd^*w_3$. From the equality $(d^*w_3, d^*w_3) = (w_3, dd^*w_3) = 0$ we have $d^*w_3 = 0$ and $u^k = w_1 + dw_2 \in \ker\Delta^k + \operatorname{Im} d^{k-1}$.

(4) Assume that the equation $\Delta w = u$ has a solution w for a given u. Then for each $v \in \ker\Delta$

$$(u, v) = (\Delta w, v) = (w, \Delta v) = (w, 0) = 0.$$

Therefore $u \in (\ker\Delta)^{\perp}$. For the converse, take $u \in (\ker\Delta)^{\perp}$ and assume that $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$. Representing u in this direct sum as $u = u_1 + u_2$, $u_1 \in \operatorname{Im}\Delta$, $u_2 \in (\operatorname{Im}\Delta)^{\perp} = \ker\Delta$ we have: $u \in (\ker\Delta)^{\perp}$ and $u_2 \in \ker\Delta$. Therefore if $u_1 = \Delta h$, then $0 = (u, u_2) = (\Delta h, u_2) + (u_2, u_2) = (h, \Delta u_2) + (u_2, u_2) = (u_2, u_2)$ from which $u_2 = 0$, i.e. $u = u_1 \in \operatorname{Im}\Delta$, $u = \Delta h$. ∎

Remark 3.15 *The above fact (4) means that the condition* $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$ *is sufficient to show the equality* $\operatorname{Im}\Delta = (\ker\Delta)^{\perp}$ *. We can ask: Is this condition necessary?*

Now we will try to formulate a condition assuring the existence of the adjoint operator d^* in a graded differential Hodge space.

Theorem 3.16 *Let* $\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), d\right)$ *be a graded differential Hodge space. Let* $\varepsilon : \{0, 1, ..., N\} \to \{-1, 1\}$ *be an arbitrary function such that* $\varepsilon_k = \varepsilon_{N-k}$*, for each* k*. Assume* ε*-anticommutativity of* $\langle , \rangle$*, i.e.*

$$\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle$$

for $v^k \in W^k$*,* $v^{N-k} \in W^{N-k}$*, [remark: if* $\langle , \rangle^{k,N-k}$ *is nontrivial, then* $\varepsilon_k \varepsilon_{N-k} = +1$*, i.e. the condition* $\varepsilon_k = \varepsilon_{N-k}$ *holds] then*

(a)

$$**\left(w^k\right) = \varepsilon_k \cdot w^k;$$

in particular,

$$*^{-1}\left(u^k\right) = \varepsilon_k \cdot *\left(u^k\right)$$
$$\left(*^{N-k}\right)^{-1}\left(u^k\right) = \varepsilon_k \cdot *^k\left(u^k\right).$$

(b) the adjoint operator d^* *exists and is given by the formula*

$$d^*\left(w^k\right) = \varepsilon_k (-1)^k * d * \left(w^k\right), \quad w^k \in W^k,$$

where $*$ *is the* $*$*-Hodge operator in W.*

(c)

$$\begin{aligned} (*d^*)\left(w^k\right) &= \varepsilon_k (-1)^k \varepsilon_{N-k+1} (d*)\left(w^k\right), \\ (d^**)\left(w^k\right) &= (-1)^{N-k} \varepsilon_k \varepsilon_{N-k} (*d)\, w^k = (-1)^{N-k} (*d)\, w^k. \end{aligned}$$

(d) if $\varepsilon_{k-1} = \varepsilon_{k+1}$ *then* $*\Delta = \pm\Delta*$*, to be precise*

$$*\Delta w^k = \varepsilon_{k-1}\varepsilon_k (-1)^{N+1} \Delta * w^k,$$

and we then conclude that

$$*\left[\mathcal{H}^k(W)\right] \subset \mathcal{H}^{N-k}(W),$$

and

$$* : \mathcal{H}^k(W) \to \mathcal{H}^{N-k}(W)$$

is an isomorphism.

Proof. (a) Simply calculations.

(b) $(,)$ is symmetric and positive definite (i.e. it is an inner product). Since the tensor $(,)$ is an inner product, it is sufficient to prove that the operator $\tilde{d}\left(w^{k}\right) = \varepsilon_k \left(-1\right)^k * d * \left(w^k\right)$, $w^k \in W^k$, is adjoint to d. Take auxiliarily $w^{k-1} \in W^{k-1}$. Therefore, since $*d*\left(w^k\right) \in W^{k-1}$, then, by (a)

$$\begin{aligned}
\left(w^{k-1}, \tilde{d}\left(w^k\right)\right) &= \left(w^{k-1}, \varepsilon_k \left(-1\right)^k * d * \left(w^k\right)\right) \\
&= \left(w^{k-1}, \varepsilon_k \left(-1\right)^k * d \varepsilon_k *^{-1} \left(w^k\right)\right) \\
&= \left(-1\right)^k \left(w^{k-1}, *d *^{-1} \left(w^k\right)\right) \\
&= \left(-1\right)^k \left\langle w^{k-1}, d *^{-1} \left(w^k\right)\right\rangle \qquad / \ \left\langle dw^k, u\right\rangle = \left(-1\right)^{k+1} \left\langle w^k, du\right\rangle \\
&= \left\langle dw^{k-1}, *^{-1} \left(w^k\right)\right\rangle \\
&= \left(dw^{k-1}, * *^{-1} \left(w^k\right)\right) \\
&= \left(dw^{k-1}, w^k\right)
\end{aligned}$$

(c) Easy calculations.

(d) We calculate

$$\begin{aligned}
\Delta * \left(w^k\right) &= \left(-1\right)^{N-k} d * dw^k + \varepsilon_{k-1} \left(-1\right)^{N-k+1} * d * d * \left(w^k\right) \\
*\Delta \left(w^k\right) &= \varepsilon_{k+1}\varepsilon_k \left(-1\right)^{N+1} \left(\left(-1\right)^{N-k} d * dw^k + \varepsilon_{k+1} \left(-1\right)^{N-k+1} * d * d * \left(w^k\right)\right)
\end{aligned}$$

which imply the equality $*\Delta w^k = \varepsilon_{k-1}\varepsilon_k \left(-1\right)^{N+1} \Delta * w^k$ for $\varepsilon_{k-1} = \varepsilon_{k+1}$. ■

As a corollary from (d) above we obtain the following theorem.

Theorem 3.17 (Duality Theorem) *If* $W = \operatorname{Im}\Delta \bigoplus \left(\operatorname{Im}\Delta\right)^{\perp}$, *then*

$$\mathbf{H}^k\left(W\right) \simeq \mathbf{H}^{N-k}\left(W\right).$$

The composition

$$\mathbf{H}^k\left(W\right) \underset{\cong}{\longleftarrow} \mathcal{H}^k\left(W\right) \underset{\cong}{\overset{*}{\longrightarrow}} \mathcal{H}^{N-k}\left(W\right) \underset{\cong}{\longrightarrow} \mathbf{H}^{N-k}\left(W\right)$$

$$[v] \longmapsto [*v]$$

is an isomorphism given by the above formula for harmonic vectors only !

We restrict the scalar positive product $(\cdot,\cdot) : W^k \times W^k \to \mathbb{R}$ to the space of harmonic vectors

$$(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H}^k\left(W\right) \times \mathcal{H}^k\left(W\right) \to \mathbb{R},$$

and we restrict the tensor $\langle\cdot,\cdot\rangle : W^k \times W^{N-k} \to \mathbb{R}$ also to harmonic vectors

$$\mathcal{B}^k = \langle\cdot,\cdot\rangle_{\mathcal{H}} : \mathcal{H}^k\left(M\right) \times \mathcal{H}^{N-k}\left(M\right) \to \mathbb{R}.$$

From the properties of the $*$-Hodge operator we have the commuting diagram:

$$\begin{array}{rccc}
\langle\cdot,\cdot\rangle_{\mathcal{H}} : & \mathcal{H}^k\left(M\right) \times \mathcal{H}^{N-k}\left(M\right) & \longrightarrow & \mathbb{R} \\
& \downarrow \mathrm{id} \times * & & \\
(\cdot,\cdot)_{\mathcal{H}} : & \mathcal{H}^k\left(W\right) \times \mathcal{H}^k\left(W\right) & \longrightarrow & \mathbb{R}.
\end{array}$$

3.4 Signature and the Hirzebruch operator

Let $\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), d\right)$ be a graded differential Hodge space. Let $\varepsilon : \{0, 1, ..., N\} \to \{-1, 1\}$ be an arbitrary function such that $\varepsilon_k = \varepsilon_{N-k}$, for each k. Assume ε-antycommutativity of $\langle , \rangle$. From the point of view of the signature we need to consider even N,

$$N = 2n \quad \text{and} \quad \varepsilon_n = +1.$$

Then

$$\langle \cdot, \cdot \rangle^n : W^n \times W^n \to \mathbb{R}$$

and

$$\langle \cdot, \cdot \rangle^n_{\mathcal{H}} = \mathcal{B}^n : \mathcal{H}^n (W) \times \mathcal{H}^n (W) \to \mathbb{R}$$

are symmetric and nondegenerated. Therefore in cohomology, the tensor

$$\langle \cdot, \cdot \rangle^n_{\mathbf{H}} : \mathbf{H}^n (W) \times \mathbf{H}^n (W) \to \mathbb{R}$$

is also symmetric and is an extension of $\langle \cdot, \cdot \rangle^n_{\mathcal{H}}$.

Definition 3.18 *If*

$$\dim \mathbf{H}^n (W) < \infty$$

we define the signature of W *as the signature of* $\langle \cdot, \cdot \rangle^n_{\mathbf{H}}$

$$\operatorname{Sig} (W) := \operatorname{Sig} \langle \cdot, \cdot \rangle^n_{\mathbf{H}}.$$

Remark 3.19 *Under the assumption* $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$ *we have* $\mathcal{H}^n (W) \cong \mathbf{H}^n (W)$ *and*

$$\mathcal{B}^n = \langle \cdot, \cdot \rangle^n_{\mathbf{H}}.$$

Therefore if $\dim \mathbf{H}^n (W) < \infty$, *then*

$$\operatorname{Sig} (W) = \operatorname{Sig} \langle \cdot, \cdot \rangle^n_{\mathbf{H}} = \operatorname{Sig} \mathcal{B}^n$$

because under the identification $\mathbf{H}^k (W) = \mathcal{H}^k (W)$ *we have* $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle^n_{\mathbf{H}}$.

Remark 3.20 *The condition* $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$ *does not imply the relation* $\dim \ker \Delta < \infty$. *Indeed, if* $d = 0$, *then* $d^* = 0$, $\Delta = 0$, $\ker \Delta = W$. *Therefore* $W = 0 \bigoplus 0^{\perp}$ *and* $\dim \mathcal{H} = \dim \mathbf{H} = \dim W$ *can be arbitrary.*

In the construction of the Hirzebruch signature operator the fundamental role is played by an operator (small modification of the $*$-Hodge operator)

$$\begin{aligned} \tau &: W \to W, \\ \tau^k &: W^k \to W^{N-k}, \\ \tau^k (w) &= \tilde{\varepsilon}_k \cdot *w, \quad \tilde{\varepsilon}_k \in \{-1, 1\}. \end{aligned} \tag{8}$$

such that

i) $\tau \circ \tau = Id$,

ii) $d^{*} = -\tau \circ d \circ \tau$,

iii) $\tau^{n} = *$, i.e. $\tilde{\varepsilon}_{n} = 1$, $\quad (n = \frac{1}{2}N)$.

We check the existence of τ and prove the uniqueness (assuming $d^{k} \neq 0$). (If consider odd N, it is necessary to admit complex $\tilde{\varepsilon}_{k} \in \mathbb{C}$ which have the absolute value 1).

Theorem 3.21 *If $N = 2n$ and*

$$\varepsilon_{k} = (-1)^{n} (-1)^{k(N-k)} = (-1)^{n} (-1)^{k}, \tag{9}$$

then there exists the operator τ fulfilling i), ii) iii) and it is given by

$$\tau^{k}\left(w^{k}\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{n(n+1)}{2}} \cdot * \left(w^{k}\right).$$

Conversely, if $d^{k} \neq 0$ for all $k = 0, 1, ..., N-1$ and τ exists, then ε_{k} is given by (9). The function ε satisfies $\varepsilon_{k-1} = \varepsilon_{k+1}$.

Proof. Easy calculations show that for an arbitrary natural number N, even or odd, the operator τ defined by (8)

(a) satisfies condition i) if and only if

$$\tilde{\varepsilon}_{k} \tilde{\varepsilon}_{N-k} = \varepsilon_{k}, \quad k \in \{0, 1, ..., N\}. \tag{10}$$

(b) satisfies condition ii) if and only if

$$\tilde{\varepsilon}_{k} \tilde{\varepsilon}_{N-k+1} = (-1)^{k+1} \varepsilon_{k}, \quad k \geq 1 \tag{11}$$

Now we prove that for a sequence $\varepsilon_{k} \in \{-1, 1\}$ there exists a complex sequence $\tilde{\varepsilon}_{k} \in \mathbb{C}$ satisfying (10) and (11) if and only if ε_{k} is given by the formula

$$\varepsilon_{k} = (-1)^{k(N-k)} (-1)^{\frac{N(N-1)}{2}} \left(\tilde{\varepsilon}_{0}\right)^{2} \tag{12}$$

for some $\tilde{\varepsilon}_{0} \in \{1, -1, i, -i\}$. Each value $\tilde{\varepsilon}_{0} \in \{1, -1, i, -i\}$ determines a sequence $\tilde{\varepsilon}_{k}$ uniquely by

$$\tilde{\varepsilon}_{k} = (-1)^{\frac{2N-k-1}{2} k} \tilde{\varepsilon}_{0}.$$

First, assume that for ε_{k} there exists $\tilde{\varepsilon}_{k} \in \mathbb{C}$ fulfilling (10) and (11). Substituting (10) into (11) we obtain (for $k = 1, 2, ..., N$)

$$\begin{aligned} \tilde{\varepsilon}_{k} \tilde{\varepsilon}_{N-k+1} &= (-1)^{k+1} \tilde{\varepsilon}_{k} \tilde{\varepsilon}_{N-k} \\ \tilde{\varepsilon}_{N-k+1} &= (-1)^{k+1} \tilde{\varepsilon}_{N-k} \quad /k \rightsquigarrow N-k+1 \\ \tilde{\varepsilon}_{k} &= (-1)^{N-k} \tilde{\varepsilon}_{k-1}. \end{aligned}$$

It follows that $\tilde{\varepsilon}_k = (-1)^{\frac{N-k+N-1}{2}k} \tilde{\varepsilon}_0$. Next $\varepsilon_0 \overset{(10)}{=} \tilde{\varepsilon}_0 \tilde{\varepsilon}_N = (-1)^{\frac{N(N-1)}{2}} (\tilde{\varepsilon}_0)^2$ and

$$\varepsilon_k = \tilde{\varepsilon}_k \tilde{\varepsilon}_{N-k} = (-1)^{k(N-k)} (-1)^{\frac{N(N-1)}{2}} (\tilde{\varepsilon}_0)^2 .$$

Since $\varepsilon_k \in \{-1, 1\}$, then $\tilde{\varepsilon}_0 \in \{1, -1, i, -i\}$.

Conversely, let ε_k fulfil (12) and $\tilde{\varepsilon}_k = (-1)^{\frac{N-k+N-1}{2}k} \tilde{\varepsilon}_0$. We easily see that the condition $\tilde{\varepsilon}_k \tilde{\varepsilon}_{N-k} = \varepsilon_k$ is fulfilled and $\tilde{\varepsilon}_k \tilde{\varepsilon}_{N-k+1} = (-1)^{k+1} \varepsilon_k$. Adding k and noticing that $(-1)^{\frac{N(N-1)}{2}} (\tilde{\varepsilon}_0)^2 \in \{-1, 1\}$ we see that (assuming $d^k \neq 0$) there are only two possibilities on ε_k for which there exists a suitable τ :

$$\varepsilon_k = (-1)^{k(N-k)} \qquad \text{or} \qquad \varepsilon_k = -(-1)^{k(N-k)} .$$

For the case $N = 2n$ and $\varepsilon_n = +1$ we obtain

$$\begin{aligned} \varepsilon_k &= (-1)^{k(N-k)} (-1)^{\frac{N(N-1)}{2}} (\tilde{\varepsilon}_0)^2 = (-1)^k (-1)^{\frac{N(N-1)}{2}} (\tilde{\varepsilon}_0)^2 \\ 1 &= \varepsilon_n = (\tilde{\varepsilon}_0)^2 . \end{aligned}$$

Therefore $\tilde{\varepsilon}_0 \in \{-1, 1\}$ and $\varepsilon_k = (-1)^k (-1)^{\frac{N(N-1)}{2}} = (-1)^k (-1)^n$ which yields two possible τ

$$\tilde{\varepsilon}_k = (-1)^{\frac{2N-k-1}{2}k} \tilde{\varepsilon}_0 = (-1)^{\frac{k(k+1)}{2}} \tilde{\varepsilon}_0, \quad \tilde{\varepsilon}_0 \in \{-1, 1\} .$$

Finally, we need take $\tilde{\varepsilon}_0$ in such a way that $\tilde{\varepsilon}_n = +1$, so $\tilde{\varepsilon}_0 = (-1)^{\frac{n(n+1)}{2}}$. Therefore

$$\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{n(n+1)}{2}} .$$

■

Example 3.22 (Classical) For differential forms the ε-anticommutativity is defined by $\varepsilon_k = (-1)^{k(N-k)}$. Then, the operator τ such that $\tau \circ \tau = \mathrm{Id}$, and $d^* = -\tau \circ d \circ \tau$, exists and we need to take $(-1)^{\frac{N(N-1)}{2}} (\tilde{\varepsilon}_0)^2 = 1$, i.e. $(\tilde{\varepsilon}_0)^2 = (-1)^{\frac{N(N-1)}{2}}$. The operator τ is real if and only if $(-1)^{\frac{N(N-1)}{2}} = +1$ which is equivalent to $N = 4k$ or $N = 4k + 1$. We observe that $\tilde{\varepsilon}_0$ is given then by

$$\tilde{\varepsilon}_0 = \begin{cases} \pm 1 & \text{for } N = 4k \text{ or } \ N = 4k+1 \\ \pm i & \text{for } N = 4k+2 \ \text{ or } \ N = 4k+3 \end{cases}$$

Assume in the sequel $N = 2n$, $\varepsilon_k = (-1)^n (-1)^k$, and take the suitable operator τ. We take

$$W_\pm = \{w \in W; \ \tau w = \pm w\} ,$$

the eigenspaces corresponding to the eigenvalues $+1$ and -1 of τ. We notice that

$$(d + d^*) [W_+] \subset W_- .$$

Definition 3.23 *The operator*

$$D_+ = d + d^* : W_+ \to W_-$$

is called the ***Hirzebruch signature operator****.*

Remark 3.24 *If* $\dim \mathcal{H} < \infty$ *then the index*

$$\operatorname{Ind} D_+ := \dim_{\mathbb{R}} \ker (D_+) - \dim_{\mathbb{R}} \ker \left(D_+^*\right)$$

is correctly defined (the dimensions are finite)

$$\ker (D_+) = W_+ \cap \mathcal{H}(W),$$

and analogously for the adjoint operator $(D_+)^* = D_- : W_- \to W_+$

$$\ker (D_-) = W_- \cap \mathcal{H}(W).$$

Theorem 3.25 (Hirzebruch Theorem on signature) *If* $\dim \mathcal{H} < \infty$, *then*

$$\operatorname{Ind} D_+ = \operatorname{Sig} (\mathcal{B}^n : \mathcal{H}^n (W) \times \mathcal{H}^n (W) \to \mathbb{R}).$$

If, additionally, $W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}$, *then* $\operatorname{Ind} D_+ = \operatorname{Sig} W$.

Proof. For the subspace $V \subset W$ stable under τ, $\tau [V] \subset V$, we put $V_+ = \{v \in V;\ \tau v = v\}$ and analogously $V_- = \{v \in V;\ \tau v = -v\}$. The mapping $\langle \cdot, \cdot \rangle_{\mathcal{H}}^n = \mathcal{B}^n : \mathcal{H}^n (W) \times \mathcal{H}^n (W) \to \mathbb{R}$ is nondegenerated. It is easily to see that

(a) $\mathcal{H}^n (W) = V_1 \bigoplus V_2$ for

$$\begin{aligned} \mathcal{H}_+^n (W) &= V_1 = \{\alpha \in \mathcal{H}^n (W);\ *\alpha = \alpha\}, \\ \mathcal{H}_-^n (W) &= V_2 = \{\alpha \in \mathcal{H}^n (W);\ *\alpha = -\alpha\}. \end{aligned}$$

(b) The subspaces $\mathcal{H}^s (W) + \mathcal{H}^{2n-s} (W)$ are τ-stable and for $s = 0, 1, ..., n-1$ and

$$\begin{aligned} \varphi_{\pm} &: \mathcal{H}^s (W) \to \left(\mathcal{H}^s (W) + \mathcal{H}^{2n-s} (W)\right)_{\pm} \\ X &\longmapsto \frac{1}{2} (X \pm \tau X) \end{aligned}$$

is an isomorphism of real spaces.

(c) The subspaces $W^s + W^{2n-s}$ are τ-invariant. Therefore

$$W_{\pm} = \bigoplus\nolimits_{s=0}^{n-1} \left(W^s + W^{2n-s}\right)_{\pm} \bigoplus W_{\pm}^n$$

which yields

$$\begin{aligned} &\ker D_+ \\ &= W_+ \cap \ker (d + d^* : W \to W) \\ &= W_+ \cap \mathcal{H}(W) \\ &= \bigoplus\nolimits_{s=0}^{n-1} \left(W^s + W^{2n-s}\right)_+ \bigoplus W_+^n \\ &\cap \bigoplus\nolimits_{s=0}^{n-1} \left(\mathcal{H}^s (W) + \mathcal{H}^{2n-s} (W)\right) \oplus \mathcal{H}^n (W) \\ &= \bigoplus\nolimits_{s=0}^{n-1} \left(\mathcal{H}^s (W) + \mathcal{H}^{4k-s} (M)\right)_+ \bigoplus \mathcal{H}^n (W)_+ \end{aligned}$$

and in consequence (since $\tau^n = *^n$ then $W^n_{\pm} \cap \mathcal{H}^n = \mathcal{H}^n_{\pm}$)

$$\begin{aligned}
&\dim\ker D_+ - \dim\ker D_+^* \\
&= \sum_{s=0}^{n-1} \dim_{\mathbb{R}} \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_+ + \dim_{\mathbb{R}} \mathcal{H}^n_+(W) \\
&- \sum_{s=0}^{n-1} \dim_{\mathbb{R}} \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_- - \dim_{\mathbb{R}} \mathcal{H}^n_-(W) \\
&= \dim_{\mathbb{R}} \mathcal{H}^n_+(W) - \dim_{\mathbb{R}} \mathcal{H}^n_-(W) \\
&= \mathrm{Sig}\left(\mathcal{B}^n\right).
\end{aligned}$$

■

4 Four fundamental examples and their general settings

4.1 Four fundamental examples

In the previous section we have described a general algebraic approach to the Hirzebruch signature operator. Thanks to it the following four fundamental examples can be understood as special cases of the general setting. Here are the four examples of the spaces with gradation and differential operator $\left(W = \oplus_{k=0}^{N} W^k, d\right)$ in which M is a connected compact oriented Riemannian manifold

$$W^k = \begin{cases}
\left(\Omega^k(M),\ d_{dR}\right); \quad N = 4p, \quad [\textbf{classical example}] \\
\left(\Omega^k(A),\ d_A\right); \quad N = m + n = 4p, \quad \begin{array}{l} A \text{ - a TUIO-Lie algebroid} \\ [\textbf{Lie algebroid example}] \end{array} \\
\left(\Omega^k(M;E),\ d_{\nabla}\right); \quad \left(E, (,)_0\right) \text{ flat vector bundle,} \\
\qquad (,)_0 \text{ - symmetric nondegenerated parallel, } N = 4p \\
\qquad\qquad [\textbf{Lusztig example}] \\
\left(\Omega^k(M;E),\ d_{\nabla}\right); \quad \left(E, \langle,\rangle_0\right) \text{ flat vector bundle,} \\
\qquad \langle,\rangle_0 \text{ -symplectic parallel, } N = 4p + 2 \\
\qquad\qquad [\textbf{Gromov example}]
\end{cases}$$

In the above, all cases the sequences of differentials $\left\{d^k_{dR}\right\}, \left\{d^k_A\right\}, \left\{d^k_{\nabla}\right\}$ are elliptic complexes, $\dim \mathbf{H}^k(W) < \infty$ and the pairing

$$\mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$$

is defined, which in the middle degree $\frac{N}{2}$ is symmetric. Its signature, $\mathrm{Sig}(W)$, is defined to be the signature of W.

4.2 General setting of the above four examples

We give some applications of the above algebraic theory and theorems to vector bundles over manifolds. Other applications to more general objects than manifolds are probably available, see the last section.

Consider a graded vector bundle $\xi = \bigoplus_{k=0}^{N} \xi^k$ of Hodge spaces over a connected compact oriented Riemannian manifold M,

$$\left(\xi = \bigoplus\nolimits_{k=0}^{N} \xi^k, \langle , \rangle , (,) , d\right)$$

where

1) $\langle , \rangle , (,)$ are fields of smooth 2-tensors in ξ such that

$$\left(\xi_x, \langle , \rangle_x , (,)_x\right)$$

is a Hodge space, $x \in M$, with a $*$-Hodge operator $*_x : \xi_x \to \xi_x$, and assume that $\langle v, w\rangle = 0$ if $v \in \xi^r$, $w \in \xi^s$, $r + s \neq N$, and that subbundles ξ^k are orthogonal with respect to $(,)$,

2) the axiom ε-anticommutativity holds

$$\left\langle v^k, v^{N-k}\right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k\right\rangle$$

where

$$\varepsilon_k \in \{-1, +1\} .$$

By integration along M we define 2-linear tensors

$$\begin{aligned} \langle\langle , \rangle\rangle , ((,)) &: \mathrm{Sec}\,(\xi) \times \mathrm{Sec}\,(\xi) \to \mathbb{R}, \\ \langle\langle \alpha, \beta\rangle\rangle &:= \int_M \langle \alpha_x, \beta_x\rangle \, dM \\ ((\alpha, \beta)) &:= \int_M (\alpha_x, \beta_x)\, dM. \end{aligned}$$

Then $((,))$ is a positive definite scalar product in $\mathrm{Sec}\,(\xi)$, the $*$-Hodge operator is an isometry

$$((\alpha, \beta)) = ((*\alpha, *\beta))$$

and

$$\langle\langle \alpha, \beta\rangle\rangle = ((\alpha, *\beta)) .$$

3) d is a differential in $\mathrm{Sec}\,(\xi)$, $d^2 = 0$, of the degree $+1$,

$$d^k : \mathrm{Sec}\left(\xi^k\right) \to \mathrm{Sec}\left(\xi^{k+1}\right),$$

such that, by definition

3a) d^k are differential operators of first order,

3b) $\langle\langle dw, u\rangle\rangle = (-1)^{k+1} \langle\langle w, du\rangle\rangle \qquad$ for $w \in \mathrm{Sec}\left(\xi^k\right)$, $u \in \mathrm{Sec}(\xi)$.

Therefore

$$\left(\mathrm{Sec}(\xi) = \bigoplus_{k=0}^{N} \mathrm{Sec}(\xi)^k, \langle\langle\cdot,\cdot\rangle\rangle, ((\cdot,\cdot)), d\right)$$

is a graded differential Hodge space. Then the adjoint operator $d^* : \mathrm{Sec}(\xi) \to \mathrm{Sec}(\xi)$

$$((\alpha, d^*\beta)) = ((d\alpha, \beta))$$

exists and $d^*\left(\alpha^k\right) = \varepsilon_k (-1)^k * d * \left(\alpha^k\right)$.

Theorem 4.1 *If $\left\{d^k\right\}$ is an elliptic complex, then the Laplacian Δ is a selfadjoint, nonnegative and elliptic operator. In consequence,*

$$\mathrm{Sec}\,\xi = \mathrm{Im}\,\Delta \oplus (\mathrm{Im}\,\Delta)^{\perp}$$

$$\begin{aligned} \mathcal{H}(\mathrm{Sec}\,\xi) &\cong \mathbf{H}(\mathrm{Sec}\,\xi, d) \\ \dim \mathcal{H}(\mathrm{Sec}\,\xi) &< \infty. \end{aligned}$$

If we assume that $N = 2n$ and $\varepsilon_k = (-1)^n (-1)^k$ we get the Hirzubruch operator $D_+ = d + d^* : \mathrm{Sec}\,\xi_+ \to \mathrm{Sec}\,\xi_-$ and the equality

$$\mathrm{Sig}\,\langle\langle,\rangle\rangle_{\mathbf{H}}^{n} = \mathrm{Ind}\,D_+.$$

The ellipticity of Δ follows from [War, Remark 6.34]. The fact that the symbol $\sigma(D^*)_{(x,v)}$ of the adjoint operator of a first order operator $D : \mathrm{Sec}\,\xi \to \mathrm{Sec}\,\eta$ equals $-\sigma(D)^*_{(x,v)}$ is well know and may be easily checked. Indeed, the symbol $\sigma(D)_{(x,v)} : \xi_x \to \eta_x$ is a linear mapping such that

$$D(fW)_x = \sigma(D)_{\left(x,(df)_x\right)}(W_x) + f(x)\,D(W)_x,$$

$f \in C^\infty(M), W \in \mathrm{Sec}\,\xi$. Let D^* be the adjoint operator for D, i.e. $((D^*(V), W)) = ((V, D(W)))$, $W \in \mathrm{Sec}\,\xi$, $V \in \mathrm{Sec}\,\eta$. To prove that $\sigma(D^*)_{(x,v)} = -\sigma(D)^*_{(x,v)}$ we need only to notice that for f, W, V as above

$$\begin{aligned} &\left(\left(x \longmapsto -\sigma(D)^*_{\left(x,(df)_x\right)}(V_x) + f(x)\,D^*(V)_x, W\right)\right) \\ &= \int_M \left(-\sigma(D)^*_{\left(x,(df)_x\right)}(V_x) + f(x)\,D^*(V)_x, W_x\right) \\ &= \int_M \left(-V_x, \sigma(D)_{\left(x,(df)_x\right)}(W_x)\right) + \int_M (V_x, D(fW)_x) \\ &= \int_M (V_x, f(x)\,D(W)_x) = ((fV, D(W))) = ((D^*(fV), W)). \end{aligned}$$

5 Applications to Lie algebroids

In all four above examples the complexes of differentials, $\left\{d^k\right\}$, $\left\{d^k_A\right\}$, $\left\{d^k_\nabla\right\}$ are elliptic, since the sequences of symbols are exact.

We describe four fundamental examples of graded differential Hodge space. The fundamental idea is as follows: we have a 2-tensor $\langle , \rangle$ and we want to find a positive definite scalar tensor $(,)$ for which the $*$-Hodge operator exists and is an isometry.

Example 5.1 (standard) M is compact oriented Riemannian manifold,

$$\dim M = 4p.$$

(a) $W^k = \Omega^k(M) = \operatorname{Sec}\left(\bigwedge^k T^*M\right)$,

(b) $\langle\langle , \rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}$, $(\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta$.

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge \beta = (-1)^{k(N-k)} \int_M \beta \wedge \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}.$$

In the middle degree $2p$, the tensor $\langle\langle , \rangle\rangle$ is symmetric.

(c) $d : W^k \to W^{k+1}$ is a differentiation of differential forms and

(d) $\langle\langle d\alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$ for $\alpha \in W^k$, $\beta \in W^{N-(k+1)}$ (which follows from $\int_M d(\alpha \wedge \beta) = 0$).

With respect to the standardly defined inner product in $\bigwedge T^*_x M$ we have a finite dimensional Hodge-space

$$\left(\bigwedge T^*_x M, \langle , \rangle_x, (,)_x\right).$$

By integrating along the Riemannian manifold M we obtain 2-linear tensors

$$\langle\langle , \rangle\rangle,\ ((,)) : \Omega(M) \times \Omega(M) \to \mathbb{R},$$

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \, dM = \int_M \alpha \wedge \beta, \quad ((\alpha, \beta)) = \int_M (\alpha, \beta) \, dM$$

and the equality

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$$

holds giving a graded Hodge-space with a differential operator $(\Omega(M), \langle\langle , \rangle\rangle, ((,)), d)$. The signature $\operatorname{Sig} M = \operatorname{Sig} \langle\langle , \rangle\rangle^{2p}_{\mathbf{H}}$ can be calculated as the index of the Hirzebruch operator.

$$D_+ = d_{dR} + d^*_{dR} : \Omega(M)_+ \to \Omega(M)_-$$

(d^*_{dR} is the adjoint operator to d_{dR} with respect to the scalar product $((,))$).

Example 5.2 [K1] Let A be a transitive Lie algebroid over a compact oriented manifold M and let

$$\operatorname{rank} A = N = 4p = m + n, \quad m = \dim M, \quad n = \dim \mathbf{g}_{|x}.$$

We assume that A is invariantly oriented via a volume tensor

$$\boldsymbol{\varepsilon} \in \operatorname{Sec}\left(\bigwedge^{n} \mathbf{g}\right)$$

invariant with respect to the adjoint representation Ad_A.

(a) $W^k = \Omega^k(A) = \operatorname{Sec}\left(\bigwedge^k A^*\right)$,

(b) $\langle\langle,\rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}, \ \ (\alpha, \beta) \longmapsto \int_M \fint_A \alpha \wedge \beta.$

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \fint_A \alpha \wedge \beta = (-1)^{k(N-k)} \int_M \fint_A \beta \wedge \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}.$$

This tensor is symmetric in the middle degree $2p$.

(c) $d_A : W^k \to W^{k+1}$ is the differentiation of A-differential forms, and

(d) $\langle\langle d_A\alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d_A\beta \rangle\rangle$ for $\alpha \in W^k$, $\beta \in W^{N-(k+1)}$.

There exists a scalar product $((,))$ in $W = \Omega(A)$ such that then

$$(\Omega(A), \langle\langle,\rangle\rangle, ((,)))$$

is a graded Hodge space with a differential. Indeed [K2], let G' be an arbitrary Riemannian tensor in $\boldsymbol{g} = \ker \#_A$. Then the volume tensor $\boldsymbol{\varepsilon}_{G'}$ of G' is equal to $f \cdot \boldsymbol{\varepsilon}$ for some smooth function $f > 0$. The tensor $G := f^{-\frac{2}{n}} G'$ is a Riemannian tensor in $\boldsymbol{g}$ for which $\boldsymbol{\varepsilon}$ is the volume tensor. Let G_2 be any Riemannian tensor on M. Taking an arbitrary connection $\lambda : TM \to A$ in A and taking the horizontal space $H = \operatorname{Im} \lambda \subset A$ we have $A = \boldsymbol{g} \bigoplus H$. Define a Riemannian tensor G on $A = \boldsymbol{g} \bigoplus H$ such that $\boldsymbol{g}$ and H are orthogonal. On $\boldsymbol{g}$ we have G_1 but on H we have the pullback $\lambda^* G_2$. The vector bundle A is oriented (since $\boldsymbol{g}$ and M are oriented). At each point $x \in M$ we consider the defined above scalar product G_x on $A_{|x}$ and the multiplication of tensors

$$\langle,\rangle_x^k : \bigwedge^k A_x^* \times \bigwedge^{N-k} A_x^* \to \bigwedge^N A_x^* \xrightarrow{\rho_x} \mathbb{R}$$

where ρ_x is defined via the volume form for G_x.

We can notice that $\rho_x = \rho_{G_2 x} \circ \fint_{A_x}$. The scalar product G_x in A_x can be extended to a scalar product in $\bigwedge A_x^*$ and we can notice that we obtain the classical finite dimensional Hodge-space

$$\left(\bigwedge A_x^*, \langle,\rangle_x, (,)_x\right)$$

and two $C^\infty(M)$-tensors $\langle,\rangle, (,) : \Omega(A) \times \Omega(A) \to C^\infty(M)$ defined as above point by point. Integrating along M we get a graded Hodge-space with differential operator

$$(\Omega(M), \langle\langle,\rangle\rangle, ((,)), d_A).$$

The tensor $\langle\langle,\rangle\rangle$ induces a 2-tensor in cohomology

$$\langle\langle,\rangle\rangle_{\mathbf{H}} : \mathbf{H}^{k}(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$$

which in the middle degree

$$\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p} : \mathbf{H}^{2p}(M) \times \mathbf{H}^{2p}(M) \to \mathbb{R}$$

is symmetric. The dimension $\dim \mathbf{H}(A)$ is finite (Kubarski, Mishchenko, 2003, [K-M-1]). Therefore, the signature of $\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p}$ can be calculated as the index of the Hirzebruch operator

$$D_{+} = d_A + d_A^{*} : \Omega(A)_{+} \to \Omega(A)_{-}$$

where d_A^{*} is the adjoint to d_A with respect to the scalar product $((,))$.

Our previous considerations on the signature of a Lie algebroid via **Hochschild-Serre spectral sequence** of A permit us to calculate the signature of a Lie algebroid using a second Hirzebruch operator following the Lusztig and Gromov examples.

Example 5.3 Lusztig (1971) [L], Gromov (1995) [Gro]. Signature for flat bundles. Let M be a compact oriented $N = 4p$-dimensional manifold and $E \to M$ a flat bundle equipped with a flat covariant derivative ∇ and a nondegenerated indefinite symmetric tensor

$$G_0 = (,)_0 : E \times E \to M \times \mathbb{R}, \quad (,)_{0x} : E_x \times E_x \to \mathbb{R},$$

constant for ∇, i.e. satisfying $\partial_X (\sigma,\eta)_0 = (\nabla_X \sigma, \eta)_0 + (\sigma, \nabla_X \eta)_0$. We take

- $W^k = \Omega^k(M; E)$,
- the differential operator $d_\nabla : W^k \to W^{k+1}$ defined standardly via ∇.

From $\nabla G_0 = 0$ we have

$$d(\alpha \wedge_{G_0} \beta) = d_\nabla \alpha \wedge_{G_0} \beta + (-1)^{|\alpha|} (\alpha \wedge_{G_0} d_\nabla \beta),$$

therefore if $|\alpha| + |\beta| = N - 1$, then

$$\int_M (d_\nabla \alpha) \wedge_{G_0} \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge_{G_0} (d_\nabla \beta). \tag{13}$$

Define the duality

$$\begin{aligned} \langle\langle \cdot,\cdot \rangle\rangle^k &: W^k \times W^{N-k} \to \mathbb{R} \\ \langle\langle \alpha,\beta \rangle\rangle^k &= \int_M \alpha \wedge_{G_0} \beta. \end{aligned}$$

and we see that

$$\langle\langle d_\nabla \alpha, \beta\rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$$

is satisfied. Since G_0 is symmetric we have

$$\langle\langle \alpha, \beta\rangle\rangle^k = \int_M \alpha \wedge_{G_0} \beta = \int_M (-1)^{k(N-k)} \beta \wedge_{G_0} \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha\rangle\rangle^{N-k}.$$

This tensor is symmetric in the middle degree. There is a scalar product $((,))$ in W^k for which the $*$-Hodge operator for $(W, \langle\langle,\rangle\rangle, ((,)))$ is an isometry. Indeed [L], [Gro], we fix some positive definite scalar product $(,)'$ on E. Then we take a unique splitting $E = E_+ \oplus E_-$ of the vector bundle E which is both $(,)_0$ and $(,)'$ orthogonal and such that $(,)_0$ on E_+ is positive and $(,)_0$ on E_- is negative definite. We denote by τ the involution $\tau : E \to E$ ($\tau^2 = id$) such that $\tau|E_+ = id$, $\tau|E_- = -id$. Then, the quadratic form

$$(v, w) = (v, \tau w)_0$$

is symmetric positive definite and $(E_x, (,)_{0x}, (,)_x)$ is a Hodge-space.

In each fibre $\bigwedge T_x^* M \bigotimes E_x$ we introduce the tensor product of the classical Hodge-space $\bigwedge T_x^* M$ and the above one in E_x.Point by point we obtain tensors

$$\begin{gathered}\langle,\rangle,\ (,) : \Omega(M; E) \times \Omega(M; E) \to C^\infty(M), \\ * : \Omega(M; E) \to \Omega(M; E)\end{gathered}$$

such that

$$\langle \alpha, \beta\rangle = (\alpha, *\beta)$$

and integrating along M we obtain a Hodge-space $(\Omega(M; E), \langle\langle,\rangle\rangle, ((,)))$.

$$\begin{aligned}\langle\langle \alpha, \beta\rangle\rangle &= \int_M \langle \alpha, \beta\rangle\, dM = \int_M \alpha \wedge_{G_0} \beta, \\ ((\alpha, \beta)) &= \int_M (\alpha, \beta)\, dM\end{aligned}$$

with $* : \Omega(M) \to \Omega(M)$, $*(\alpha)(x) = *_x(\alpha_x)$, and

$$\langle\langle \alpha, \beta\rangle\rangle = ((\alpha, *\beta)).$$

Let d_∇^* be the adjoint operator to d_∇ with respect to $((,))$. The tensor $\langle\langle,\rangle\rangle$ induces a 2-tensor in cohomology $\langle\langle,\rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$ which in the middle degree

$$\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p} : \mathbf{H}^{2p}(M; E) \times \mathbf{H}^{2p}(M; E) \to \mathbb{R}$$

is symmetric and the signature of it is the index of the Hirzebruch operator

$$D_+ = d_\nabla + d_\nabla^* : \Omega(M; E)_+ \to \Omega(M; E)_-$$

Example 5.4 Gromov (1995) [Gro]. Signature for a symplectic bundle. Let M be a compact oriented manifold M of dimension $\dim M = N = 4p + 2$ and let $E \to M$ be a symplectic vector bundle, i.e. the one equipped with a flat covariant derivative ∇ and parallel symplectic structure [i.e. skew symmetric nondegenerated] $S = \langle,\rangle : E \times E \to M \times \mathbb{R}$, $\nabla S = 0$. We take

- $W^k = \Omega^k(M;E)$,
- $d_\nabla : W^k \to W^{k+1}$ - the differential operator defined via ∇.

The condition

$$\int_M (d_\nabla \alpha) \wedge_S \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge_S d_\nabla \beta. \tag{14}$$

holds for $|\alpha| + |\beta| = N - 1$.

$\langle\langle \cdot, \cdot \rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}$ is defined by

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge_S \beta$$

and $\langle\langle d_\nabla \alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$ is fulfilled. Since S is skew symmetric, then

$$\alpha \wedge_S \beta = -(-1)^{k(N-k)} \beta \wedge_S \alpha$$

and

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}.$$

There is a scalar product $((,))$ in W^k for which $(W, \langle\langle, \rangle\rangle, ((,)))$ is a Hodge space. Namely [V, p.56] there exists an anti-involution τ in E, $\tau^2 = -\tau$ (i.e. a complex structure) such that

- $\langle \tau v, \tau w \rangle = \langle v, w \rangle$, $v, w \in E_x$,
- $\langle v, \tau v \rangle > 0$ for all $v \neq 0$.

Then the tensor $(v, w) := \langle v, \tau w \rangle$ is symmetric, positive definite, and $(\tau v, \tau w) = (v, w)$, i.e., τ preserves both forms $\langle, \rangle$ and $(,)$. The operator $-\tau$ is the $*$-Hodge operator in $(E_x, \langle, \rangle_x, (,)_x)$. In consequence, the system $(E_x, \langle, \rangle_x, (,)_x)$ is a Hodge-space.

At each point $x \in M$, we take the tensor product $\bigwedge T_x^* M \bigotimes E_x$ of the classical Hodge space $\bigwedge T_x^* M$ and the above E_x. Analogously to the above example, we obtain a graded Hodge-space $(\Omega(M;E), \langle\langle, \rangle\rangle, ((,)), d)$ with a differential (where the $*$-Hodge operator is defined point by point $* : \Omega(M) \to \Omega(M)$, $*(\alpha)(x) = *_x(\alpha_x)$). Passing to cohomology we obtain $\langle\langle, \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$,

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}$$

which in the middle degree $2p+1$ is symmetric (thanks to the fact that $\langle, \rangle$ is skew symmetric)

$$\langle\langle, \rangle\rangle_{\mathbf{H}}^{2p+1} : \mathbf{H}^{2p+1}(M;E) \times \mathbf{H}^{2p+1}(M;E) \to \mathbb{R}$$

$$\langle\langle\alpha,\beta\rangle\rangle^{2p+1} = -(-1)^{2p+1}\langle\langle\beta,\alpha\rangle\rangle^{2p+1} = \langle\langle\beta,\alpha\rangle\rangle^{2p+1}.$$

We can calculate the signature of $\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p+1}$ as the index of the Hirzebruch operator $D_+ = d_\nabla + d_\nabla^* : \Omega(M;E)_+ \to \Omega(M;E)_-$.

Example 5.5 In consequence, for a TUIO-Lie algebroid A over a compact oriented manifold M for which $m = \dim M$, $\quad n = \operatorname{rank}\mathbf{g} = \dim \mathbf{g}_x$, and under the assumption $\mathbf{H}^{m+n}(A) \neq 0$ and $m+n = 4p$ we have two Hirzebruch signature operators:

(I) The first one. $D_+ = d_A + d_A^* : \Omega(A)_+ \to \Omega(A)_-$ where d_A^* is the adjoint to d_A with respect to the scalar product $((\alpha,\beta)) = \int_M (\alpha,\beta)$ defined in Example 2 above, and $W_\pm = \{\alpha \in \Omega(A);\ \tau\alpha = \pm\alpha\}$, for $\tau(\alpha^k) = (-1)^{\frac{k(k+1)}{2}}(-1)^p \cdot * (\alpha^k)$.

(II) The second one. We use the equality

$$\operatorname{Sig}\mathbf{H}(A) = \operatorname{Sig}E_2$$

for the second term E_2, $E_2^{p,q} = \mathbf{H}^p_{\nabla^q}(M;\mathbf{H}^q(\boldsymbol{g}))$, of the Hochschild-Serre spectral sequence. The flat covariant derivative ∇^q in the cohomology vector bundle $\mathbf{H}^q(\boldsymbol{g})$ depends on the structure of the Lie algebroid A.

Let $m+n = 4p$. The signature $\operatorname{Sign}E_2$ is equal to the signature of the quadratic form

$$E_2^{2p} \times E_2^{2p} \to E_2^{m+n} = \mathbb{R},$$

and

a) if n is odd, then $\operatorname{Sig}E_2 = 0$,

b) if n is even, then

$$\operatorname{Sig}E_2 = \operatorname{Sig}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{m+n} = E_2^{m,n} = \mathbb{R}\right)$$

where

$$E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right).$$

Consider the form $\langle\langle,\rangle\rangle : \mathbf{H}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$,

$$\langle\langle,\rangle\rangle^k : \mathbf{H}^k_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{m-k}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbf{H}^m_{\nabla^n}(M;\mathbf{H}^n(\boldsymbol{g})) = \mathbb{R},$$

which is symmetric in the middle degree $k = \frac{m}{2}$ and its signature is equal to the signature of A. For $k = n$, the bundle $\mathbf{H}^n(\boldsymbol{g})$ is trivial, $\mathbf{H}^n(\boldsymbol{g}) \cong M \times \mathbb{R}$, the connection ∇^n is equal to ∂, and the multiplication of values is taken with respect to $\langle,\rangle : \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$.

We have $\frac{m}{2} + \frac{n}{2} = 2p$. We need to consider two different cases:

(a) $\frac{m}{2}$ and $\frac{n}{2}$ even, then the form

$$\mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$$

is symmetric and we can use Example 5.3 (Lusztig type) to obtain the Hirzebruch signature operator $D_+ = d_{\nabla^{\frac{n}{2}}} + d^*_{\nabla^{\frac{n}{2}}} : \Omega_+\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right) \to \Omega_-\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right)$,

(b) $\frac{m}{2}$ and $\frac{n}{2}$ are odd, then the form $\mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \rightarrow \mathbf{H}^{n}(\mathbf{g}) = M \times \mathbb{R}$ is symplectic and we can use Example 5.4 (Gromov type) to obtain the Hirzebruch signature operator $D_{+} = d_{\nabla^{\frac{n}{2}}} + d^{*}_{\nabla^{\frac{n}{2}}}$.

For each of the cases the index of D_{+} is equal to the signature of A. Therefore, the Atiyah-Singer formula for the index can be used.

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