The signature of Lipschitz manifolds from more general point of view and applications to Lie algebroids

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International Conference C*-algebras and elliptic theory. III
Będlewo, January 26 - 31, 2009

Before the presentation of the plan of my talk I give some introduction. Since I would like to describe Hirzebruch signature operators for Lie algebroids, firstly I recall the definition of a Lie algebroid, its signature and some facts concerning to it.

A *Lie algebroid* on a manifold M is a triple

$$A = (A, [\![\cdot, \cdot]\!], \#_A)$$

where A is a vector bundle on M, $(\operatorname{Sec} A, [\![\cdot, \cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

$$\#_A : A \to TM$$

is a linear homomorphism (called the anchor) of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + \#_A(\xi)(f) \cdot \eta, \quad f \in C^\infty(M),\ \xi, \eta \in \operatorname{Sec} A.$$

The anchor is bracket-preserving, $\#_A \circ [\![\xi, \eta]\!] = [\#_A \circ \xi, \#_A \circ \eta]$. A Lie algebroid is called **transitive** if the anchor $\#_A$ is an epimorphism.

For a transitive Lie algebroid A we have:

- the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0, \tag{1}$$

  $\boldsymbol{g} := \ker \#_A$,

- the fiber $\boldsymbol{g}_x$ of the bundle $\boldsymbol{g}$ at the point $x \in M$ is a Lie algebra (called the isotropy Lie algebra of A at $x \in M$) with the commutator operation being

$$[v, w] = [\![\xi, \eta]\!](x), \quad \xi, \eta \in \operatorname{Sec} A, \quad \xi(x) = v, \eta(x) = w, \quad v, w \in \boldsymbol{g}_x,$$

- the vector bundle $\boldsymbol{g}$ is a Lie Algebra Bundle (LAB in short), called the *adjoint* of A, its fibres are isomorphic Lie algebras.

The word "transitive" comes from the theory of differential groupoids. Each differential groupoid

$$\Phi \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$$

on a manifold M with the source $\alpha : \Phi \to M$ and the target $\beta : \Phi \to M$ and the inclusion of M onto the units

$$u : M \to \Phi, \quad x \longmapsto u_x,$$

posseses a Lie algebroid (nontransitive in general) defined as follows: from the submersivity of α it follows that the α-vertical vectors

$$T^{\alpha}\Phi = \ker \alpha_*$$

form a vector bundle. Next we restrict it to the submanifold of units

$$A(\Phi) := u^* (T^{\alpha}\Phi) = (T^{\alpha}\Phi)_{|M}.$$

We take the linear homomorphism called the anchor:

$$\#_A : A \to TM, \quad v \longmapsto \beta_* (v).$$

Any right invariant vector field on Φ determines a crosss-section of $A(\Phi)$ and opposite, the bracket of right invariant vector fields is right invariant therefore the space of cross-sections of $A(\Phi)$ is a Lie algebra. In this way we obtain a Lie algebroid.

What is the image of the anchor $\#_{A(\Phi)} : A(\Phi) \to TM$? Let $R \subset M \times M$ be the equivalence relation defined as follows

$$R = \{(x, y) ; \exists_{h \in \Phi} (\alpha h = x, \ \beta h = y)\} .$$

The equivalence classes are immersed submanifolds and they form a foliation with singularities in the sense of P.Stefan.

The "tangent bundle" to this foliation is just equal to the image of the anchor $\#_{A(\Phi)}$. The groupoid Φ is called transitive if $R = M \times M$.

- Therefore, the Lie algebroid $A(\Phi)$ of a differential groupoid Φ is transitive if and only if the groupoid Φ is transitive.

To an arbitrary (transitive or not) Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \operatorname{Sec} \bigwedge A^*, \quad \text{- the space of cross-sections of } \bigwedge A^*$$

$$d_A^k : \Omega^k(A) \to \Omega^{k+1}(A)$$

$$\begin{aligned} \left(d_A^k \omega\right)(\xi_0, ..., \xi_k) = & \sum_{j=0}^{k} (-1)^j \left(\#_A \circ \xi_j\right)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ & + \sum_{i<j} (-1)^{i+j} \omega\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k\right), \end{aligned} \tag{2}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \operatorname{Sec} A$.

Lemma 1 *For a transitive Lie algebroid A the complex $\left\{d_A^k\right\}$ is an* ***elliptic complex.***

To consider the notion of the signature of a transitive Lie algebroid we need to restrict our considerations to some class of Lie algebroids for which the top cohomology group $\mathbf{H}^{m+n}(A) \neq 0$ ($m = \dim M$, $n = \dim \boldsymbol{g}_x = \operatorname{rank} \boldsymbol{g}$, clearly $\operatorname{rank} A = m+n$, see the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0$).

Theorem 2 (Kubarski-Mishchenko, 2004) *For each transitive Lie algebroid* $(A, [\![\cdot,\cdot]\!], \#_A)$ *with the Atiyah sequence over a connected* ***compact oriented manifold*** M *the following conditions are equivalent* ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, *i.e.* $\operatorname{rank} A = m+n$)

(1) $\mathbf{H}^{m+n}(A) \neq 0$,

(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$,

(3) A *is the so-called invariantly oriented, i.e. there exists a global non-singular cross-section*

$$\boldsymbol{\varepsilon} \in \operatorname{Sec} \bigwedge\nolimits^n \boldsymbol{g},$$

$0 \neq \boldsymbol{\varepsilon}_x \in \bigwedge^n \boldsymbol{g}_{|x}$, *invariant with respect to the adjoint representation of* A *in the vector bundle* $\bigwedge^n \boldsymbol{g}$, *which is extending of the adjoint representation* ad_A *of* A *in* $\boldsymbol{g}$ *given by* $(ad_A)(\xi) : \operatorname{Sec} \boldsymbol{g} \to \operatorname{Sec} \boldsymbol{g}, \quad \nu \longmapsto [\![\xi, \nu]\!]$.

The condition (3) yields that structure Lie algebras $\boldsymbol{g}_{|x}$ are unimodular. These Lie algebroids are called TUIO-Lie algebroids (transitive unimodular invariantly oriented).

The implication (A is *invariantly oriented*)$\Longrightarrow$ ($\mathbf{H}^{m+n}(A) = \mathbb{R}$) comes from Kubarski 1996.

The isomorphism $\mathbf{H}^{m+n}(A) = \mathbb{R}$ is constructed via integral:

$$\begin{array}{rcl} \mathbf{H}^{m+n}(A) & \to & \mathbb{R} \\ [\omega] & \longrightarrow & \displaystyle\int_A \omega = \int_M \fint_A \omega \end{array}$$

where $\fint_A$ is the so-called fibre integral

$$\fint_A : \Omega^k(A) \to \Omega^{k-m}(M), \quad k \geq m, \ \text{ and } \fint_A \omega = 0 \text{ for } |\omega| < m,$$

is defined in such a way that $(\#_A)^* \left(\fint_A \omega\right) = (-1)^{n|\omega|} i_\varepsilon \omega.$

Theorem 3 (Kubarski 2002) *The scalar Poincaré product*

$$\mathcal{P}_A^k : \mathbf{H}^k(A) \times \mathbf{H}^{m+n-k}(A) \to \mathbb{R},$$

$$([\omega], [\eta]) \longmapsto \int_A \omega \wedge \eta = \int_M \left(\fint_A \omega \wedge \eta \right)$$

is nondegenerated. And if $m+n=4p$, then

$$\mathcal{P}_A^{2p} : \mathbf{H}^{2p}(A) \times \mathbf{H}^{2p}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and is called the signature of A, and is denoted by

$$\operatorname{Sig}(A).$$

To investigate the signature of A we can use the techniques of spectral sequences.

Theorem 4 (Kubarski-Mishchenko, 2003) . *Let*

$$(B, B^r, \cup, D, B_j)$$

be any DG-algebra with a decreasing filtration B_j *and* $(E_s^{p,q}, d_s)$ *its spectral sequence. Assume*

- *the regularity axiom* $B_0 = B$ *of the filtration,*
- *and that there exist natural numbers* m, n *such that* $m + n = 4p$ *and* $E_2^{j,i} = 0$, *for* $j > m$, *and* $i > n$,
- *the second term* E_2 *is a Poincaré algebra with respect to the total gradation and the top group* $E_2^{(m+n)} = E_2^{m,n}$, *(so* $\dim E_2^{m,n} = 1$*),*

then

- *each term* $\left(E_s^{(*)}, \cup, d_s\right)$ $2 \leq s \leq \infty$, *is a Poincaré algebra with Poincaré differentiation,*

- *the cohomology algebra* $\mathbf{H}(B) = \bigoplus_{r=0}^{m+n} \mathbf{H}^r(B)$ *is a Poincaré algebra,* $\dim \mathbf{H}^{m+n}(B) = 1$ *and*

$$\operatorname{Sig} E_2 = \operatorname{Sig} E_3 = \ldots = \operatorname{Sig} E_\infty . = \operatorname{Sig} \mathbf{H}(B).$$

  - *If* m *and* n *are odd then* $\operatorname{Sig} E_2 = 0$,
  - *if* m *and* n *are even then*

$$\begin{aligned} \operatorname{Sig} E_2 &= \operatorname{Sig}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R} \right) \\ &= \operatorname{Sig}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R} \right). \end{aligned}$$

Using this theorem to the Čech-de Rham complex of a Lie algebroid we prove

Theorem 5 (Kubarski-Mishchenko 2003) $\operatorname{Sig}(A) = 0$ *if the Leray type presheaf of cohomology*

$$\mathcal{H} = \left(U \longmapsto \mathbf{H}\left(A_{|U}\right)\right)$$

(which is locally constant on a good covering) is constant (equivalently, if the monodromy representation $\pi_1(M) \to \operatorname{Aut}(\mathbf{H}(\mathfrak{g}))$ *[*$\mathfrak{g}$ *- the isotropy Lie algebra] is trivial). For example* $\operatorname{Sig}(A) = 0$ *if*

- M *is simply connected,*
- $\operatorname{Aut} G = \operatorname{Int} G$ *where* G *is a simply connected Lie group with Lie algebra* $\mathfrak{g}$ *(for example if* G *is of type* B_l, C_l, E_7, E_8, F_4, G_2 *).*

Remark 6 *There are examples with* $\mathfrak{g}$ *is abelian for which* $\operatorname{Sig}(A) \neq 0$.

In my talk I construct four Hirzbruch signature operators for Lie algebroids.

- Two in the category of smooth differential forms (one of them will be constructed using Hochschild-Serre spectral sequence),
- and analogously two others in the category of L_2-differential forms.

To the case of L_2-technique I present an algebraic point of view on distributional exterior derivative on Lipschitz manifolds and the signature operator.

This permits us to extend our considerations to some other cases, important for Lie algebroids.

PLAN:

1) Let us recall [Kubarski-Mishchenko, 2009, in print] the algebraic point of view on the smooth case and uniform approach to the following four examples. In these examples we have

$$\langle\langle\alpha,\beta\rangle\rangle = \int \alpha\wedge\beta, \quad \text{for } \deg\alpha + \deg\beta = \text{maximal}$$

$$((\alpha,\beta)) = \int (\alpha,\beta), \quad (\alpha,\beta) \text{ is the inner product,} \quad \deg\alpha = \deg\beta$$

$$* \text{ is the Hodge operator such that} \quad \langle\langle\alpha,\beta\rangle\rangle = ((\alpha,*\beta))$$

- classical case of compact oriented manifold M^{4p}, $(\Omega(M), \langle\langle,\rangle\rangle, ((,)), *, d_{dR})$,
- TUIO Lie algebroid A on compact oriented manifold M, $m+n = 4p$, $m = \dim M$, $n = \dim \boldsymbol{g}_x$, $\boldsymbol{g}_x$ is the isotropy Lie algebra of A at x, $(\Omega(A), \langle\langle,\rangle\rangle, ((,)), *, d_A)$
- Lusztig example (1972) of a vector bundle with flat covariant derivative and equipped with nondegenerated indefinite symmetric parallel quadratic form on a compact oriented manifold M^{4p},

$$(\Omega(M,E), \langle\langle,\rangle\rangle, ((,)), *, d_\nabla),$$

• Gromov example (1995) of a f vector bundle with flat covariant derivative and equipped with a parallel symplectic form on a compact oriented manifold M^{4p+2},

$$\left(\Omega\left(M,E\right),\left\langle\left\langle,\right\rangle\right\rangle,\left(\left(,\right)\right),*,d_{\nabla}\right).$$

Lusztig and Gromov examples are very important for calculation of the signature of Lie algebroids, because for the Hochschild-Serre spectral sequence of a TUIO Lie algebroid A over a manifold M the second term E_2 is equal to

$$E_2^{p,q}\cong\mathbf{H}^p_{\nabla^q}\left(M;\mathbf{H}^q\left(\boldsymbol{g}\right)\right)$$

where $\mathbf{H}^q\left(\boldsymbol{g}\right)$ is the vector bundle of the q-cohomology groups of the isotropy Lie algebras of A $\mathbf{H}^q\left(\boldsymbol{g}\right)_x=\mathbf{H}^q\left(\boldsymbol{g}_x\right)$ and ∇^q is a canonical flat covariant derivative. Via suitable theorem on spectral sequences

$$\operatorname{Sig}A=\operatorname{Sig}\mathbf{H}\left(A\right)=\operatorname{Sig}E_2.$$

If n is odd then $\operatorname{Sig}E_2=0$, if $\frac{n}{2}$ is even then we obtain in this way a Lusztig example, while if $\frac{n}{2}$ is odd – a Gromov example.

2) The remaining of Teleman's theory of the distributional exterior derivative (called by me in the sequel shortly a "subderivative") of L_2-differential forms and the signature operator on Lipschitz manifolds (the term "subderivative" is motivated by the fact that it is an operator defined only on some subspace of L_2-forms.) The great value of these theory is that the space of all L_2-forms is Hilbert.

3) Algebraical point of view on Teleman's theory.

4) Some applications of the above algebraical approach to four above examples after passing to the Hilbert completion of the spaces of smooth forms

1 Algebraic aspects of the Hirzebruch signature operator for smooth cases

1.1 Hodge space

Let W be a real vector space of an arbitrary dimension (finite or infinite).

Definition 7 *By a **Hodge space** we mean the system*

$$(W, \langle , \rangle, (,), *_W)$$

where $\langle , \rangle, (,) : W \times W \to \mathbb{R}$ *are* 2*-linear tensors such that*

(1) $(,)$ *is symmetric and positive definite (i.e. it is an inner product),*

(2) $*_W : W \to W$ *(called* $*$*-Hodge operator) is a linear mapping such that,*

- $*_W$ *is an isometry with respect to* $(,)$,
- *for all* $v \in W$, $\langle v, w \rangle = (v, *_W (w))$.

Clearly, the $*$-Hodge operator $*_W$ is uniquely determined (if exists).

Lemma 8 *If* $(V, \langle\cdot,\cdot\rangle_V, (\cdot,\cdot)_V, *_V)$ *and* $(W, \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_W, *_W)$ *are Hodge spaces then their tensor product*

$$(V \otimes W, \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_V \otimes (\cdot,\cdot)_W, *_V \otimes *_W)$$

is a Hodge space (i.e. $*_{V\otimes W} = *_V \otimes *_W$ *).*

1.2 Finitely dimensional Hodge spaces, examples.

Lemma 9 *Let* $(W, \langle\cdot,\cdot\rangle)$ *be a finite dimensional real vector space equipped with a* 2*-linear tensor* $\langle\cdot,\cdot\rangle : W \times W \to \mathbb{R}$*. Then there exists an inner product* $(\cdot,\cdot)$ *and operator* $*_W$ *such that the system* $(W, \langle,\rangle, (,), *_W)$ *is a Hodge space if and only if there exists a basis of* W *in which the matrix of* $\langle,\rangle$ *is orthogonal.*

— The inner product and the $*$-Hodge operator play an auxiliary role in the study of properties of the pairing $\langle,\rangle$.

Now we give a some examples of finite dimensional Hodge spaces.

Example 10 (Classical) *Let (V,G) be a real N-dimensional oriented Euclidean space with an inner product $G : V \times V \to \mathbb{R}$ and the volume tensor $\varepsilon = e_1 \wedge ... \wedge e_N \in \bigwedge^N V$, (where $\{e_i\}_{i=1}^N$ is a positive ON-base of V). Via ε we identify $\bigwedge^N V = \mathbb{R}$. We have the classical Hodge space*

$$\left(\bigwedge V = \bigoplus^{k} \bigwedge^{k} V, \langle , \rangle , (,) , * \right)$$

where

- $\langle \cdot,\cdot \rangle^k : \bigwedge^k V \times \bigwedge^{N-k} V \longrightarrow \bigwedge^N V = \mathbb{R}$ *and* $\left\langle v^k, v^r \right\rangle = 0$ *if* $k + r \neq N$, $v^s \in \bigwedge^s V$,
- $(\cdot,\cdot)^k : \bigwedge^k V \times \bigwedge^k V \to \mathbb{R}, \quad (v_1 \wedge .... \wedge v_k, w_1 \wedge ... \wedge w_k)^k = \det \left[G\left(v_i, w_k\right) \right],$
- *the subspaces* $\bigwedge^k V$, $k = 0, 1, ..., N$ *are orthogonal (by definition),*
- $*\left(e_{i_1} \wedge ... \wedge e_{i_k}\right) = \varepsilon_{(j_1,...,j_{n-k})} \cdot e_{j_1} \wedge ... \wedge e_{j_{n-k}}$ *where* (e_i) *is an ON-base of* V *and* $\varepsilon_{(j_1,...,j_{n-k})} = \operatorname{sgn}\left(j_1, ..., j_{n-k}, i_1, ..., i_k\right)$. *[We notice some slight difference (the sign) with the classical case].*

Example 11 (Lusztig example, 1972) *[L] Let* $(\,,\,)_0 : E \times E \to \mathbb{R}$ *be a symmetric (indefinite in general) nondegenerated tensor on a finite dimensional vector space* E. *Let* G *be an arbitrary positive scalar product in* E. *Then there exists exactly one direct sum decomposition* $E = E_+ \oplus E_-$ *which is ON with respect to the both scalar product* $(\,,\,)_0$ *and* G *and such that* $(\,,\,)_0$ *on* E_+ *is positive definite and on* E_- *is negative definite. We denote by* $*_E$ *the involution* $*_E : E \to E$ *such that*

$$*_E|E_+ = id, \qquad *_E|E_- = -id.$$

Then, the quadratic form

$$\begin{aligned} (\,,\,) \ &: \ E \times E \to \mathbb{R} \\ (v,w) \ &: \ = (v, *_E w)_0 \end{aligned}$$

is symmetric and positive definite. The involution $*_E$ *is an isometry, therefore*

$$(E, (\,,\,)_0\,, (\,,\,)\,, *_E)$$

is a Hodge-space.

Example 12 (Gromov example, 1995) *[Gro] Let* $\langle , \rangle_0 : E \times E \to \mathbb{R}$, *be a symplectic form on a finite dimensional vector space* E. *There exists an anti-involution* τ *in* E, $\tau^2 = -id$ *(i.e. a complex structure) such that*

$$\langle \tau v, \tau w \rangle_0 = \langle v, w \rangle_0 , \quad v, w \in E,$$

$$\langle v, \tau v \rangle_0 > 0 \quad \textit{for all} \quad v \neq 0.$$

Then the tensor

$$\begin{aligned} (,) &: E \times E \to \mathbb{R} \\ (v, w) &: = \langle v, \tau w \rangle_0 \end{aligned}$$

is symmetric and positive defined and $(\tau v, \tau w) = (v, w)$. *The system*

$$(E, \langle , \rangle_0 , (,) , -\tau)$$

is a Hodge-space [since $-\tau$ *is an isometry and* $\langle v, w \rangle_0 = (v, -\tau w)$*].*

Definition 13 *By the* <u>*Hodge vector bundle*</u> *we mean a system*

$$(\xi, \langle , \rangle , (,) , *_E)$$

consisting of a vector bundle ξ *and two smooth tensor fields*

$$\langle , \rangle , (,) : \xi \times \xi \to M \times \mathbb{R}$$

and linear homomorphism

$$*_E : \xi \to \xi,$$

such that for each $x \in M$ *the system* $(\xi_x, \langle , \rangle_x , (,)_x , *_{E_x})$ *is a finitely dimensional Hodge space.*

Example 14 (of Hodge vector bundles)

- $\xi = \bigwedge T^*M$ *for a Riemannian manifold* M,
- *Lusztig example of a vector bundle* ξ *with flat covariant derivative, equipped with nondegenerated indefinite symmetric parallel quadratic form,*
- *Gromov example of a vector bundle* ξ *with flat covariant derivative and equipped with a parallel symplectic form.*

Example 15 *Consider an arbitrary Riemannian oriented manifold M of dimension N and a Hodge vector bundle $(\xi, \langle , \rangle, (,), *_E)$ [for example of Lusztig or Gromov vector bundle]. Then for any point $x \in M$ we take the tensor product of Hodge spaces*

$$\bigwedge T_x^* M \otimes \xi_x.$$

Assuming compactness of M we can define by integration along M two 2-$\mathbb{R}$-linear tensors

$$((\alpha, \beta)), \ \langle\langle \alpha, \beta \rangle\rangle : \Omega(M; \xi) \times \Omega(M; \xi) \to \mathbb{R},$$

$$((\alpha, \beta)) = \int_M (\alpha_x, \beta_x)\, dM, \qquad \langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha_x, \beta_x \rangle\, dM = \int_M \alpha \wedge_\varphi \beta$$

where $\varphi_x = \langle \cdot, \cdot \rangle_x^k : \bigwedge^k T_x^* M \otimes \xi_x \times \bigwedge^{N-k} T_x^* M \otimes \xi_x \to \bigwedge^N T_x^* M = \mathbb{R}$ is the wedge product with respect to the multiplication $\langle , \rangle_x$ of the values. The 2-form $((\cdot, \cdot))$ is symmetric and positive definite and the triple

$$(\Omega(M; \xi), \langle\langle \alpha, \beta \rangle\rangle, ((\alpha, \beta)), *)$$

is a Hodge space with the $*$-Hodge operator $(*\beta)_x = *_{E,x}(\beta_x)$.

Example 16 *Let A be a TUIO Lie algebroid. For any $\alpha, \beta \in \Omega(A) = \Gamma\left(\bigwedge A^*\right)$ we put*

$$\langle\langle \alpha, \beta \rangle\rangle = \int_A \alpha \wedge \beta, \qquad |\alpha| + |\beta| = N = m + n,$$

Theorem 17 (Kubarski 2008) *There exists an inner product $(,)_x$ in A_x, $x \in M$, such that $(\bigwedge A_x^*, \langle,\rangle_x, (,)_x, *_x)$ is finite dimensional Hodge space where*

$$\langle,\rangle_x^k : \bigwedge^k A_x^* \times \bigwedge^{N-k} A_x^* \to \bigwedge^N A_x^* = \mathbb{R}$$

After integration along M

$$((\alpha, \beta)) := \int (\alpha, \beta)$$

gives an inner product in $\Omega^k(A)$ for which the $*$-Hodge operator is an isometry and the condition

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$$

holds. It follows that

$$(\Omega(A), \langle\langle,\rangle\rangle, ((,)), *)$$

is a Hodge space.

1.3 Graded differential Hodge space

Definition 18 *By a graded differential Hodge space we mean a system*

$$\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *, d\right)$$

where $(W, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *)$ *is a Hodge space (finitely or infinitely dimensional) and*

(1) $\langle \cdot, \cdot \rangle | W^k \times W^r = 0$ *if* $k + r \neq N$. *(notation:* $\langle , \rangle^k := \langle , \rangle | W^k \times W^{N-k}$ *)*

(2) W^k *are orthogonal with respect to* $(\cdot, \cdot)$,

(3) d *is homogeneous of degree* $+1$, *i.e.* $d : W^k \to W^{k+1}$, *and* $d^2 = 0$,

(4) $\langle dw, u \rangle = (-1)^{k+1} \langle w, du \rangle$ *for* $w \in W^k$, $u \in W^{N-k-1}$.

Clearly, a) the induced cohomology pairing

$$\langle , \rangle^k_{\mathbf{H}} : \mathbf{H}^k (W) \times \mathbf{H}^{N-k} (W) \to \mathbb{R}, \quad ([u], [v]) \longmapsto \langle [u], [w] \rangle^k_{\mathbf{H}} := \langle u, w \rangle^k,$$

is correctly defined,

b) $* \left[W^k\right] \subset W^{N-k}$, and $* : W^k \to W^{N-k}$ is an isomorphism.

Proposition 19 *Let* $\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *, d\right)$ *be a graded differential Hodge space. Let* $\varepsilon_k \in \{-1, +1\}$ *be given such that* $\varepsilon_k = \varepsilon_{N-k}$, $k = 0, ..., N$. *Assume* ε*-anticommutativity of* $\langle , \rangle^k$, *i.e.*

$$\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle, \qquad \textit{for} \qquad v^k \in W^k,\ v^{N-k} \in W^{N-k},$$

then

1) $**\left(w^k\right) = \varepsilon_k \cdot w^k$,

2) *the linear operator* $\delta : W^k \to W^{k-1}$ *defined by*

$$\delta^k \left(w^k\right) = \varepsilon_k \left(-1\right)^k * d * \left(w^k\right), \quad w^k \in W^k,$$

is the adjoint operator

$$\left(\delta \left(w_1\right), w_2\right) = \left(w_1, d\left(w_2\right)\right),$$

3) *the Laplacian* $\Delta := (d + \delta)^2 = d\delta + \delta d$ *is homogeneous of degree* 0, *self-adjoint* $(\Delta v, w) = (v, \Delta w)$, *and nonnegative* $(\Delta v, v) \geq 0$.

Definition 20 *A vector $v \in W$ is called **harmonic** if*

$$dv = 0 \quad \text{and} \quad \delta v = 0,$$

or equivalently if $v \perp (\Delta v)$. Denote

$$\mathcal{H}(W) = \{v \in W;\ dv = 0,\ \delta v = 0\},$$
$$\mathcal{H}^k(W) = \left\{v \in W^k;\ d^k v = 0,\ \delta^k v = 0\right\}.$$

- The harmonic vectors forms a graded vector space $\mathcal{H}(W) = \bigoplus_{k=0}^{N} \mathcal{H}^k(W)$.
- $\mathcal{H}^k(W) = \ker \Delta^k$ and $\mathcal{H}(W) = \ker \Delta = (\operatorname{Im} \Delta)^{\perp}$.
- The spaces $\ker \Delta^k$ and $\operatorname{Im} d^{k-1}$ are orthogonal, therefore the inclusion

  $$\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$$

  induces a monomorphism (called the Hodge homomorphism)

  $$\mathcal{H}^k(W) = \ker \Delta^k \rightarrowtail \mathbf{H}^k(W) := \ker d^k / \operatorname{Im} d^{k-1}.$$

Problem 21 *When the Hodge homomorphism is an isomorphism? i.e. when in each cohomology class there is (exactly one) a harmonic vector?*

Theorem 22 *If*

$$\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}},$$

i.e. $W = \operatorname{Im}\Delta \bigoplus \ker\Delta,$ *then*

- $W^k = \ker\Delta^k \bigoplus \operatorname{Im} d^{k-1} \bigoplus \operatorname{Im}\delta^{k+1}$ *(strong Hodge decomposition),*
- $\ker d^k = \ker\Delta^k \bigoplus \operatorname{Im} d^{k-1}$, *in particular, the Hodge homomorphism is an isomorphism*

  $$\mathcal{H}^k(W) = \ker\Delta^k \xrightarrow{\cong} \ker d^k / \operatorname{Im} d^{k-1} = \mathbf{H}^k(W),$$

  It means that in each cohomology class there is exactly one harmonic vector.
- ***(Poincaré Duality Theorem)*** $\mathbf{H}^k(W) \simeq \mathbf{H}^{N-k}(W),$

  $\mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$ *is nondegenerated.*

In all four examples given above (standard, Lie algebroid, Lusztig's and Gromov's) we have $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$ according to the well-known theorem.

Theorem 23 *Let ξ be a Riemannian vector bundle over a compact oriented Riemannian manifold M. If $\Delta : \operatorname{Sec}\xi \to \operatorname{Sec}\xi$ is a* ***self-adjoint nonnegative elliptic operator*** *then $\ker\Delta$ is a finite dimensional space and*

$$\operatorname{Sec}\xi = \operatorname{Im}\Delta \bigoplus \ker\Delta = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}.$$

In particular, this holds if Δ comes from an elliptic complex $d^k : \Gamma\left(\xi^k\right) \to \Gamma\left(\xi^{k+1}\right)$, $\xi = \bigoplus \xi^k$ (as for example in our four cases).

1.4 Signature and the Hirzebruch operator

Consider a **graded differential Hodge space** assuming an ε-anticommutativity of $\langle , \rangle^k$,

$$\left(W = \bigoplus\nolimits_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *, d\right).$$

We restrict the positive definite product $(\cdot, \cdot) : W^k \times W^k \to \mathbb{R}$ to the space of harmonic vectors

$$(\cdot, \cdot)_{\mathcal{H}} : \mathcal{H}^k(W) \times \mathcal{H}^k(W) \to \mathbb{R},$$

and we restrict the tensor $\langle \cdot, \cdot \rangle : W^k \times W^{N-k} \to \mathbb{R}$ also to harmonic vectors

$$\mathcal{B}^k = \langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H}^k(M) \times \mathcal{H}^{N-k}(M) \to \mathbb{R}.$$

We want to find an operator $\tau : W \to W$ defined by the formula

$$\tau\left(u^k\right) = \tilde{\varepsilon}_k \cdot *\left(u^k\right),$$

for some complex numver $\tilde{\varepsilon}_k \in \mathbb{C}$ such that $|\tilde{\varepsilon}_k| = 1$.fulfilling the condition:
i) $\tau^2 = \mathrm{Id}$, ii) $\delta = -\tau d \tau$.

Theorem 24 • *Operator τ fulfilling i) and ii) exists if and only if the coefficient ε_k of ε- antycommutativity is equal to*

$$\varepsilon_k = (-1)^{k(N-k)}, \quad k \leq N \quad \text{or} \quad \varepsilon_k = -(-1)^{k(N-k)}, \quad k \leq N.$$

(Remark: in the proof of the part "$\Longrightarrow$" we have to add the "natural" assumption that $d^k \neq 0$ for all $k < N$)

- *For a given ε there are two possibilities of τ :*
  - *if $\varepsilon_k = (-1)^{k(N-k)}$ then*

$$\tilde{\varepsilon}_k = \begin{cases} \pm 1 & gdy \quad N = 4p,\ 4p+1, \\ \pm i & gdy \quad N = 4p+2,\ 4p+3. \end{cases}$$

  - *if $\varepsilon_k = -(-1)^{k(N-k)}$ then opposite*

$$\tilde{\varepsilon}_k = \begin{cases} \pm 1 & gdy \quad N = 4p+2,\ 4p+3. \\ \pm i & gdy \quad N = 4p,\ 4p+1. \end{cases}$$

The above justify the use of (but only sometimes !) the complex valued differential forms.

From the point of view of the signature we need to consider only even N and the additional conditions $\varepsilon_n = +1$

$$N = 2n \quad \text{and} \quad \varepsilon_n = +1 \qquad \text{(the pairing } \langle \cdot, \cdot \rangle^n \text{ is then symmetric).}$$

We additionally assume

iii) $\tau^n = *$, i.e. $\tilde{\varepsilon}_n = 1$.

Theorem 25 *If $N = 2n$ and $\varepsilon_n = +1$, then the operator τ fulfilling i), ii), and iii) exists if and only if*

$$\varepsilon_k = (-1)^k (-1)^{\frac{N(N-1)}{2}} = (-1)^k (-1)^n ,$$

and then τ is unique, and

$$\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{n(n+1)}{2}} \quad \text{/real number}$$

Particularly

- *If $N = 4p$ then $\varepsilon_k = (-1)^k$ and $\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} (-1)^p$.*
- *If $N = 4p + 2$ then $\varepsilon_k = -(-1)^k$ and $\tilde{\varepsilon}_k = -(-1)^{\frac{k(k+1)}{2}}$.*

Assume the natural (for the signature theory) case

$$N = 2n \quad \text{and} \quad \varepsilon_n = +1.$$

Then

- $\langle\cdot,\cdot\rangle^n : W^n \times W^n \to \mathbb{R}$,
- $\langle\cdot,\cdot\rangle^n_{\mathcal{H}} = \mathcal{B}^n : \mathcal{H}^n(W) \times \mathcal{H}^n(W) \to \mathbb{R}$,
- $\langle\cdot,\cdot\rangle^n_{\mathbf{H}} : \mathbf{H}^n(W) \times \mathbf{H}^n(W) \to \mathbb{R}$

are symmetric and nondegenerated quadratic forms.

Definition 26 *If*

$$\dim \mathbf{H}^n(W) < \infty$$

we define the signature of W *as the signature of* $\langle\cdot,\cdot\rangle^n_{\mathbf{H}}$

$$\operatorname{Sig}(W) := \operatorname{Sig}\langle\cdot,\cdot\rangle^n_{\mathbf{H}}.$$

Remark 27 *Under the assumption*

$$\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}}$$

we have $\mathcal{H}^n(W) \cong \mathbf{H}^n(W)$, *therefore* $\mathcal{B}^n = \langle\cdot,\cdot\rangle_{\mathcal{H}}^n = \langle\cdot,\cdot\rangle_{\mathbf{H}}^n$.

Then if $\dim \mathbf{H}^n(W) < \infty$ *we have*

$$\operatorname{Sig}(W) = \operatorname{Sig}\langle\cdot,\cdot\rangle_{\mathbf{H}}^n = \operatorname{Sig}\mathcal{B}^n.$$

Assume in the sequel that

$$N = 2n, \qquad \varepsilon_k = (-1)^n (-1)^k,$$

and take the suitable operator τ

$$\tau^k\left(w^k\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{n(n+1)}{2}} \cdot * \left(w^k\right).$$

uniquely determined by the conditions

i) $\tau \circ \tau = Id$,

ii) $d^* = -\tau \circ d \circ \tau$,

iii) $\tau^n = *$, i.e. $\tilde{\varepsilon}_n = 1$.

We put

$$W_{\pm} = \{w \in W;\ \tau w = \pm w\},$$

the eigenspaces corresponding to the eigenvalues $+1$ and -1 of τ. We notice that

$$(d + \delta)[W_+] \subset W_-.$$

Definition 28 *The operator*

$$D_+ = d + \delta : W_+ \to W_-$$

is called the ***Hirzebruch operator*** *(or the signature operator).*

Take the adjoint one to D_+,

$$\begin{aligned} D_+^* &= D_- : W_- \to W_+, \\ D_- &= d + \delta : W_- \to W_+. \end{aligned}$$

Remark 29 *If* $\dim \mathcal{H} < \infty$ *then the index*

$$\operatorname{Ind} D_+ := \dim_{\mathbb{R}} \ker(D_+) - \dim_{\mathbb{R}} \ker(D_-)$$

is correctly defined (the dimensions are finite).

Theorem 30 (Hirzebruch Signature Theorem) *If* $\dim \mathcal{H} < \infty$, *then*

$$\operatorname{Ind} D_+ = \operatorname{Sig}\left(\mathcal{B}^n : \mathcal{H}^n(W) \times \mathcal{H}^n(W) \to \mathbb{R}\right).$$

If, additionally, $\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}}$, *then*

$$\operatorname{Ind} D_+ = \operatorname{Sig} W.$$

Proof. (a) $\mathcal{H}^n(W) = \mathcal{H}^n_+(W) \bigoplus \mathcal{H}^n_-(W)$ for

$$\mathcal{H}^n_\pm(W) = \left\{\alpha \in \mathcal{H}^n(W);\ *\alpha = \pm\alpha\right\}.$$

(b) The subspaces $\mathcal{H}^s(W)+\mathcal{H}^{2n-s}(W)$ are τ-stable and for $s = 0, 1, ..., n-1$

$$\begin{aligned} \varphi_\pm &: \mathcal{H}^s(W) \to \left(\mathcal{H}^s(W) + \mathcal{H}^{2n-s}(W)\right)_\pm \\ X &\longmapsto \frac{1}{2}(X \pm \tau X) \end{aligned}$$

is an isomorphism of real spaces.

(c) The subspaces $W^s + W^{2n-s}$ are τ-invariant. Therefore

$$W_\pm = \bigoplus_{s=0}^{n-1} \left(W^s + W^{2n-s}\right)_\pm \bigoplus W^n_\pm$$

which yields

$$\ker D_{\pm} = \bigoplus_{s=0}^{n-1} \left(\mathcal{H}^{s}\left(W\right) + \mathcal{H}^{4k-s}\left(M\right)\right)_{\pm} \bigoplus \mathcal{H}^{n}\left(W\right)_{\pm}$$

and in consequence (since $\tau^{n} = *^{n}$ then $W_{\pm}^{n} \cap \mathcal{H}^{n} = \mathcal{H}_{\pm}^{n}$)

$$\dim \ker D_{+} - \dim \ker D_{-} \overset{\text{(b)}}{=} \dim_{\mathbb{R}} \mathcal{H}_{+}^{n}\left(W\right) - \dim_{\mathbb{R}} \mathcal{H}_{-}^{n}\left(W\right) = \operatorname{Sig}\left(\mathcal{B}^{n}\right).$$

■

1.5 Four fundamental examples

The above general algebraic approach to the Hirzebruch signature operator can be used to the four above mentioned fundamental examples.

$$W^k = \left\{ \begin{array}{|l|} \hline \quad [\textbf{manifold, classical example}], \\ \left(\Omega^k\left(M\right),\ d_{dR}\right); \quad \dim M = N = 4p, \\ \qquad \text{here } \varepsilon_k = (-1)^{k(N-k)} = (-1)^k \\ \\ \hline [\textbf{Lie algebroid example}] \\ \left(\Omega^k\left(A\right),\ d_A\right); \quad \operatorname{rank} A = N = m+n = 4p, \quad A \text{ - a TUIO-Lie algebroid} \\ \qquad \text{here } \varepsilon_k = (-1)^k \\ \\ \hline [\textbf{Lusztig's example}] \\ \left(\Omega^k\left(M;E\right),\ d_\nabla\right); \quad \left(E, \nabla, (,)_0\right) \text{ flat vector bundle,} \\ \qquad\qquad (,)_0 \text{ - symmetric nondegenerated parallel,} \quad \dim M = N = 4p, \\ \quad \text{here } \varepsilon_k = (-1)^k \\ \\ \hline [\textbf{Gromov's example}] \\ \left(\Omega^k\left(M;E\right),\ d_\nabla\right); \quad \left(E, \langle,\rangle_0\right) \text{ flat vector bundle,} \\ \qquad\qquad \langle,\rangle_0 \text{ -symplectic parallel,} \quad \dim M = N = 4p+2, \\ \quad \text{here } \varepsilon_k = -\left(-1\right)^k, \\ \hline \end{array} \right.$$

The Lusztig anf Gromov examples are important for Lie algebroids.

Example 31 *For a TUIO-Lie algebroid A over a compact oriented manifold M for which $m = \dim M, \quad n = \operatorname{rank} \mathbf{g} = \dim \mathbf{g}_x$, and under the assumption $\mathbf{H}^{m+n}(A) \neq 0$ and $m + n = 4p$ we have two signature Hirzebruch operators:*

- *The first one.*
$$D_+ = d_A + \delta_A : \Omega(A)_+ \to \Omega(A)_-$$
where δ_A is adjoint to d_A with respect to the scalar product $((\alpha, \beta)) = \int_M (\alpha, \beta)$ with respect to the suitable inner product $(,)$.

- *The second one. We can use the mentioned above theorem on spectral sequences:*
$$\operatorname{Sig} \mathbf{H}(A) = \operatorname{Sig} E_2$$
for the second term E_2, of the Hochschild-Serre spectral sequence of the Lie algebroid and
$$E_2^{j,i} = \mathbf{H}^j_{\nabla^q}\left(M; \mathbf{H}^i(\boldsymbol{g})\right).$$
$\mathbf{H}^i(\boldsymbol{g})$ is the flat vector bundle of q-group of cohomology of isotropy Lie algebras $\mathbf{H}^i(\boldsymbol{g})_x = \mathbf{H}^i(\boldsymbol{g}_x)$ with respect to some natural flat structure ∇^i depending on the structure of the Lie algebroid A.

Let $m+n=4p$, $m=\dim M$, $\ n=\operatorname{rank}\boldsymbol{g}=\dim\boldsymbol{g}_x$. The signature $\operatorname{Sign}E_2$ is equal to the signature of the quadratic form

$$\langle\langle,\rangle\rangle^{\frac{m}{2}}:\mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)\times\mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)\to\mathbf{H}^{m}_{\nabla^{n}}\left(M;\mathbf{H}^{n}(\boldsymbol{g})\right)=\mathbb{R},$$

The bundle $\mathbf{H}^n(\boldsymbol{g})$ is trivial, $\mathbf{H}^n(\boldsymbol{g})\cong M\times\mathbb{R}$, the connection ∇^n is equal to ∂, and the multiplication of values is taken with respect to multiplication of cohomology classes

$$\langle,\rangle:\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\times\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\to\mathbf{H}^{n}(\mathbf{g})=M\times\mathbb{R}.$$

We need to consider two different cases:

- $\frac{m}{2}$ and $\frac{n}{2}$ even, then the above form is symmetric and we can use Lusztig type Example to obtain the Hirzebruch signature operator

$$D_+=d_{\nabla^{\frac{n}{2}}}+\delta_{\nabla^{\frac{n}{2}}}:\Omega_+\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right)\to\Omega_-\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right),$$

- $\frac{m}{2}$ and $\frac{n}{2}$ are odd, then the above is symplectic and we can use Gromov type Example to obtain the Hirzebruch signature operator

$$D_+=d_{\nabla^{\frac{n}{2}}}+\delta_{\nabla^{\frac{n}{2}}}:\Omega_+\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right)\to\Omega_-\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right).$$

2 Lipschitz manifolds and distributional exterior derivative (subderivative)

2.1 Lipschitz manifolds

We briefly recall the notion of a Lipschitz manifold and differential forms of the class L_2 on them.

Definition 32 (Teleman 1983) *A Lipschitz structure on a topological manifold M of dimension n is a maximal atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$, where $\phi_\alpha : M \supset U_\alpha \to V_\alpha \subset \mathbb{R}^n$ (U_α, V_α - open subsets) are homeomorphisms, such that changes coordinates*

$$\Lambda_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}, \quad \alpha, \beta \in \Lambda$$

are Lipschitz mappings.

Of course, a C^∞-manifold possesses a canonical Lipschitz structure.

The crucial role is played by the Rademacher theorem:

Theorem 33 (Rademacher) *If $U \to \mathbb{R}$ is a Lipschitz function on an open subset $U \subset \mathbb{R}^n$, then*

- *the partial derivatives $\frac{\partial f}{\partial x^i}$ exist almost everywhere,*
- *$\frac{\partial f}{\partial x^i}$ are measurable and bounded.*

Definition 34 *We say that a Lipshitz manifold with the atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$ is orientable if there exists a subatlas $\Lambda' \subset \Lambda$ for which the homeomorphisms $\Lambda_{\alpha\beta}$ have positive jakobian (in all points of differentiability). If such an atlas is given we call M oriented. .*

2.2 Differential forms

Let $L_2^k(U)$ denote the space of differential forms of the class L_2 on an open subset $U \subset \mathbb{R}^n$. If $\Lambda : U \to U'$ is a Lipschitz homeomorphism and $\omega \in L_2(U')$ then the pullback $\Lambda^*(\omega) \in L_2(U)$ (defined point by point in all points of the differentiability of Λ).

Definition 35 *Let M be a Lipshitz manifold with the atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$, . By L_2-differential form on M we mean a system*

$$\omega = \{\omega_\alpha\}_{\alpha\in\Lambda}$$

where ω_α is a [real] L_2-differential form on the open subset $V_\alpha = \phi_\alpha[U_\alpha] \subset \mathbb{R}^n$, $\alpha \in \Lambda$, such that

$$\Lambda_{\alpha\beta}^*\omega_\beta = \omega_\alpha.$$

$L_2(M)$ - the vector space of L_2-differential forms on M.

The 0-differential form determines a measurable function on M.

For oriented Lipschitz manifold, using the Lipschitz partition of unity, we define the integral $\int_M \omega$ dla $\omega \in L_2^n(M)$ $(n = \dim M)$ in a standard way.

2.3 Lipschitz Riemannian metric

Definition 36 *A Lipschitz Riemannian metric on M is a collection*

$$\Gamma = \{\Gamma_\alpha\}_{\alpha \in \Lambda}$$

where Γ_α is a Riemannian metric on $V_\alpha = \phi_\alpha [U_\alpha] \subset \mathbb{R}^n$ with measurable components, which satisfy

- *compatibility condition*
$$(\Lambda_{\alpha\beta})^* \Gamma_\beta = \Gamma_\alpha,$$
- *L_2-norms on V_α determined by Γ_α and by standard metric are equivalent.*

Theorem 37 (Teleman, 1983) . *Any compact Lipschitz manifold M has Lipschitz Riemannian metric.*

Clearly, any Lipschitz Riemannian metric detrmines a measure on M.

Let $*_{\alpha,x}$ be a Hodge star isomorphism in $\bigwedge(\mathbb{R}^n)^*$ defined of the metric Γ_α at $x \in \mathbb{R}^n$ defined as in the previous sections

$$*_{\alpha,x}(e_{i_1} \wedge ... \wedge e_{i_k}) = \varepsilon_{(j_1,...,j_{n-k})} \cdot e_{j_1} \wedge ... \wedge e_{j_{n-k}}, \qquad (e_i) \text{ is } \Gamma_\alpha(x)\text{-ON}.$$

The family $\{*_{\alpha,x}\}_x$ determines the $*$-Hodge operator for differential forms from $L_2(V)$, V is open in $\mathbb{R}^n$.

Definition 38 *For a Lipschitz Riemannian metric* $\Gamma = \{\Gamma_\alpha\}$ *and* $\omega \in L_2^r(M)$, $\omega = \{\omega_\alpha\}$, *we define*

- L_2*-differential form* $*_\Gamma\omega = \{*_\alpha\omega_\alpha\}_\alpha$.
- *for* ω, η *of the same degree we define the inner product* $(\omega,\eta)_\Gamma := \{(\omega_\alpha,\eta_\alpha)_\alpha\}$ *(it is a* 0*-form, i.e. a function on* M *).*
- $((\omega,\eta))_\Gamma := \int_M (\omega,\eta)_\Gamma$.

Clearly $\quad ((\omega,\eta))_\Gamma = \varepsilon_k \int_M \omega \wedge *_\Gamma\eta, \qquad$ where $\varepsilon_k = (-1)^{k(N-k)}$ and

$$*_\Gamma *_\Gamma = \varepsilon_k \cdot Id.$$

Theorem 39 (Teleman 1983) *The space $L_2^k(M)$ with unitary structure $((,))_\Gamma$ is Hilbert, two Lipschitz Riemannian metrics define equivalent norms in $L_2^k(M)$.*

Introducing the pairing of differential forms in complementary degrees by

$$\langle\langle\omega,\eta\rangle\rangle = \int_M \omega\wedge\eta$$

we have

$$\langle\langle\omega,\eta\rangle\rangle = ((\omega,*_\Gamma\eta))_\Gamma$$

which means that

$$(L_2(M), \langle\langle,\rangle\rangle, ((,)), *)$$

is a Hodge space. The operations $\langle\langle\cdot,\cdot\rangle\rangle$ and $*_\Gamma$ are continuous.

2.4 Distributional exterior derivative

Definition 40 *Let $\sigma \in L_2^r(U)$ be any L_2-differential form on $U \subset \mathbb{R}^n$ of degree $r < n$. We say that σ has distributional exterior derivative in the class L_2 if there exists an L_2-differential form of degree $r+1$*

$$\bar{d}\sigma \in L_2^{r+1}(U)$$

such that for any C^∞-differential form φ of degree $n-1-r$ with compact support in U

$$\int_U \bar{d}\sigma \wedge \varphi = (-1)^{r+1} \int_U \sigma \wedge d\varphi.$$

If $r = n$, we put $\bar{d}\sigma = 0$ for each $\sigma \in L_2^n(U)$.

Distributional exterior derivative $\bar{d}\sigma$ is uniquely determined and clearly $\bar{d}\left(\bar{d}\sigma\right)$ exists and $\bar{d}\left(\bar{d}\sigma\right) = 0$.

Proposition 41 *If $\omega = \{\omega_\alpha\}_{\alpha\in\Lambda}$ is an L_2-differential form on M of degree r and $d\omega_\alpha \in L_2(V_\alpha)$ is the distributional exterior derivative of ω_α, then*

$$\bar{d}\omega := \left\{\bar{d}\omega_\alpha\right\}_{\alpha\in\Lambda}$$

is an L_2-differential form on M of degre $r+1$.

Denote by $\Omega_d^r(M) \subset L_2^r(M)$ the subspace of L_2-differential forms of degree r possessing the distributional exterior derivative

$$\Omega_d^r(M) = \left\{\omega \in L_2^r(M);\ \bar{d}\omega \in L_2^{r+1}(M)\right\}.$$

$\bar{d}^2 = 0$ na $\Omega_d^r(M)$. We obtain a cohomology complex

$$0 \to \Omega_d^0(M) \to \Omega_d^1(M) \to ... \to \Omega_d^r(M) \to \Omega_d^{r+1}(M) \to ... \to \Omega_d^n(M) = L_2^n(M) \to 0.$$

Theorem 42 (Teleman (1983)) *For a compact oriented Lipschitz manifold M*

- *the pairing*

$$\mathbf{H}_r\left(\Omega_d^\bullet(M)\right) \times \mathbf{H}_{dimM-r}\left(\Omega_d^\bullet(M)\right) \to \mathbb{R}, \quad ([\omega], [\eta]) \to \int_M \omega \wedge \eta$$

  is nondegenerated and $\mathbf{H}_r\left(\Omega_d^\bullet(M)\right) = \left(\mathbf{H}_{dimM-r}\left(\Omega_d^\bullet(M)\right)\right)^*$. *Therefore* $\dim \mathbf{H}\left(\Omega_d^\bullet(M)\right) < \infty$ *(L_2-Poincaré duality),*

- *for a C^∞ manifold M and induced Lipschitz structure, the inclusion*

$$j : \Omega^\bullet(M) \hookrightarrow \Omega_d^\bullet(M)$$

  induces isomorphism in cohomology $j_\# : \mathbf{H}(M) \xrightarrow{\cong} \mathbf{H}\left(\Omega_d^\bullet(M)\right)$.

Theorem 43 (Teleman (1983)) *Suppose that ω and η possesse distributional exterior derivatives $\bar{d}\omega$ and $\bar{d}\eta$ and $|\omega| + |\eta| = n - 1$, then*

$$\left\langle\left\langle \omega, \bar{d}\eta \right\rangle\right\rangle = (-1)^{|\omega|+1} \left\langle\left\langle \bar{d}\omega, \eta \right\rangle\right\rangle .$$

Let ω be a given L_2-form and let there exist ω' such that

$$\left\langle\left\langle \omega, \bar{d}\eta \right\rangle\right\rangle = (-1)^{|\omega|+1} \langle\langle \omega', \eta \rangle\rangle$$

for all η with distributional exterior derivative $\bar{d}\eta$, then ω' is the distributional exterior derivative of ω, $\bar{d}\omega = \omega'$.

In particular, if $\left\langle\left\langle \omega, , \bar{d}\eta \right\rangle\right\rangle = 0$ for all η with distributional exterior derivative $\bar{d}\eta$ then $\bar{d}\omega = 0$.

The remaining elements needed to construct the signature operator are of algebraic nature only.

3 Algebraic aspect of the signature operator on Lipschitz manifolds

3.1 Graded Hilbert subdifferential Hodge space

Definition 44 *By a Hilbert graded subdifferential Hodge space we mean a system*

$$\left(W=\bigoplus_{k=0}^{N}W^{k},\left\langle \cdot,\cdot\right\rangle ,\left(\cdot,\cdot\right),*_{W},\bar{d}:W_{d}\rightarrow W_{d}\right)$$

consisting of a Hodge space $(W,\left\langle \cdot,\cdot\right\rangle ,\left(\cdot,\cdot\right),*_{W})$ *with gradation* $W=\bigoplus_{k=0}^{N}W^{k}$ *and a subdifferential* $\bar{d}$ *defined on some subspace with gradation* $W_{d}=\bigoplus_{k=0}^{N}W_{d}^{k}\subset W$, $W_{d}^{k}=W^{k}\cap W_{d}$, *such that*

(1) the unitary space $(W,(,))$ *is complete (i.e. it is Hilbert),*

(2) $\left\langle \cdot,\cdot\right\rangle |W^{k}\times W^{r}=0$ *if* $k+r\neq N$, *the subspaces* W^{r} *are orthogonal with respect to* $\left(\cdot,\cdot\right)$,

(3) W_{d} *is dense in* W,

(4) $\bar{d}$ *is degree* $+1$, $\bar{d}^{r}=d|W_{d}^{r}:W_{d}^{r}\rightarrow W_{d}^{r+1}$,

(5) $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$ *for* $w \in W_d^r$, $u \in W_d^{N-r-1}$,

(6) If for $w \in W^r$ *there exists* $w' \in W^{r+1}$, *such that* $\langle w', u \rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$ *for each* $u \in W_d^{N-r-1}$, *then* $w \in W_d^r$ *and* $\bar{d}w = w'$,

(7) the cohomology space $\mathbf{H}_d(W) = \bigoplus_{k=0}^{N} \mathbf{H}_d^k(W)$ *of the complex* $\left(W_d, \bar{d}\right)$ *fulfills the Poincaré duality, i.e. the pairing*

$$\mathbf{H}^r\left(W_d^\bullet(M)\right) \times \mathbf{H}^{N-r}\left(W_d^\bullet(M)\right) \to \mathbb{R}, \qquad ([w], [v]) \longmapsto \langle w, v \rangle$$

is nondegenerated, i.e. $\mathbf{H}^r\left(W_d^\bullet(M)\right) = L\left(\mathbf{H}^{N-r}\left(W_d^\bullet(M)\right), \mathbb{R}\right)$, *what follows* $\mathbf{H}\left(W_d^\bullet(M)\right) = L\left(\mathbf{H}\left(W_d^\bullet(M)\right), \mathbb{R}\right)$ *and* $\dim \mathbf{H}(W_d) < \infty$.

– Clearly, the operation $\langle \cdot, \cdot \rangle$ and $*$ are continuous in the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$.

Notation 45 *Let W be a Hilbert graded subdifferential Hodge space. We put*

$$W_\delta^{N-r} := *_W \left[W_d^r \right].$$

This space is dense in W^{N-r} and $*_W : W_d^r \to W_\delta^{N-r}$ is an isometry.

Let $w_n \to w$ and let $w_n \in W_d^r$. **Question:** when the limit w possesses a subdifferential ?

Theorem 46 *Let w_n possess a subdifferential $\bar{d}w_n$ and assume that the sequence (w_n) is Cauchy and $w = \lim w_n$. Then w has a subdifferential if and only if the sequence $\left(\bar{d}w_n\right)$ is partially weak convergent to some wektor w' with respect to the space W_δ^{r+1} in the following sense:*

— for each $h \in W_\delta^{r+1}$ we have

$$\left(\bar{d}w_n, h\right) \to \left(\bar{d}w, h\right).$$

$\boxed{\text{We assume}}$ the ε-antycommutativity of $\langle , \rangle$ $\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle$ for some $\varepsilon_k \in \{-1, 1\}$. Then we recall that $**\left(u^k\right) = \varepsilon_k u^k$.

Definition 47 *By a cosubdifferential of the degree $N-r$ in Hilbert Hodge space with gradation and subdiffefrential and with ε-antycommutativity we mean the oparator*

$$\bar{\delta}^{N-r} : W_{\delta}^{N-r} \to W_{\delta}^{N-r-1}$$

defined by

$$\bar{\delta}^{N-r} = (-1)^{N-r} *_W^r \bar{d}^r (*_W)^{-1} .$$

It is easy to prove that the condition of jointness holds

$$\left(\bar{\delta}^{N-r} v, w\right) = \left(v, \bar{d}^{N-r-1} w\right),$$

for v having a cosubdifferential and w having a subdifferential.

Notation 48 $W_1^r = W_d^r \cap W_\delta^r = \left\{ w \in W^r;\ w \in W_d^r,\ *w \in W_d^{N-r} \right\}.$

Theorem 49 *The space W_1^r with the norm $||\omega||_1$ defined as*

$$||\omega||_1^2 = ||\omega||^2 + ||d\omega||^2 + ||\delta\omega||^2$$

is Hilbert.

Definition 50 *We define now the spaces of harmonic vectors*

$$\mathcal{H}_d^r = \left\{ w \in W_1^r;\ \bar{d}\omega = 0 = \bar{\delta}\omega \right\}.$$

Clearly

$$* : \mathcal{H}_d^r \to \mathcal{H}_d^{N-r}$$

is an isomorphism. Any harmonic vector is a cocycle, therefore there exists a Hodge homorphism

$$x_d^r : \mathcal{H}_d^r \to \mathbf{H}^r \left(W_d^\bullet \right).$$

As in the paper by Teleman (1983) we can prove

Lemma 51 *The subspaces* $\mathcal{H}^r$ *and* $\operatorname{Im} \bar{d}^{r-1}$ *are perpendicular, therefore* $\mathcal{H}^r \cap \operatorname{Im} d^{r-1} = \{0\}$ *, which gives that* x_d^r *is a monomorphism.*

From Axiom (6) we see that

Lemma 52 *The subspace* $\operatorname{Ker} \bar{d}^r$ *is closed in* W^r*, therefore it is a Hilbert space.*

From Axiom (7) [Poincaré duality for $\mathbf{H}\,(W_d)$] we see that

Lemma 53 *The subsapce* $\operatorname{Im} \bar{d}^{r-1}$ *is closed in* W^r*, therefore it is a Hilbert space.*

From Axiom (5)

Lemma 54 $\mathcal{H}_d^r = \left\{w \in W^r;\quad h \in \operatorname{Ker} \bar{d}^r,\ h \perp \operatorname{Im} \bar{d}^{r-1}\right\}$ *, .i.e.* $\mathcal{H}_d^r = \left(\operatorname{Im} \bar{d}^{r-1}\right)^{\perp}$ *in* $\operatorname{Ker} \bar{d}^r$.

Conclusion 55 *(Hodge Theorem)* $\operatorname{Im}\bar{d}^{r-1}$*(*$\subset \operatorname{Ker}\bar{d}^{r}$*) is a closed subspace of the Hilbert space* $\operatorname{Ker}\bar{d}^{r}$*, therefore*

$$\operatorname{Ker}\bar{d}^{r} = \operatorname{Im}\bar{d}^{r-1} \oplus \left(\operatorname{Im}\bar{d}^{r-1}\right)^{\perp} = \operatorname{Im}\bar{d}^{r-1} \oplus \mathcal{H}_{d}^{r}$$

which means that

$$\mathcal{H}_{d}^{r} = \operatorname{Ker}\bar{d}^{r} / \operatorname{Im}\bar{d}^{r-1} = \mathbf{H}^{r}\left(W_{d}\left(M\right)\right),$$

i.e. the Hodge homomorphism is an isomorphism.

Theorem 56 *There is a strong Hodge decomposition.*

$$W^{r} = \mathcal{H}_{d}^{r} \oplus \bar{d}\left[W_{d}^{r-1}\right] \oplus \bar{\delta}\left[W_{\delta}^{r+1}\right].$$

3.2 The signature operator for graded Hilbert subdifferential Hodge space

Consider a graded Hilbert subdifferential Hodge space and ε-antycommutativity

$$\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), *_W, \bar{d} : W_d \to W_d,\ W_d \subset W,\ W_d = \bigoplus_{k=0}^{N} W_d^k\right),$$

$$W_\delta^{N-r} := *_W \left[W_d^r\right],\ \bar{\delta}^{N-r} : W_\delta^{N-r} \to W_\delta^{N-r-1} \text{ – the cosubdifferential.}$$

- As an example can serve a space of L_2-differential forms on a Lipschitz Riemannian compact oriented manifold.

For the uniformity of notation we put

- $W_0^r = W^r(M)$ with the norm $||w|| = \sqrt{(w, w)}$.
- $W_1^r = W_d^r(M) \cap W_\delta^r(M)$ with the norm $||w_1||$, such that $\|w\|_1^2 = \|w\|^2 + \left\|\bar{d}w\right\|^2 + \left\|\bar{\delta}w\right\|^2$. The both are Hilbert.

Analogously as in Teleman paper we show

Theorem 57 *The operator*

$$D^r = \bar{d} + \bar{\delta} : W_1^r \to W_0^r$$

is a continuous Fredholm operator,

$$\operatorname{Ker} D^r = \mathcal{H}_d^r$$

and

$$\operatorname{Im} D^r = \bar{d}\left[W_d^{r-1}\right] \oplus \bar{\delta}\left[W_\delta^{r+1}\right]$$

(so Coker $D \cong \mathcal{H}_d^r$ *).*

As in the previous part we assume

$$\underline{N = 2n}, \qquad \underline{\varepsilon_n = +1}$$

and use the operator $\tau : W \to W$ defined by

$$\tau\left(u^k\right) = \tilde{\varepsilon}_k * \left(u^k\right), \quad |\tilde{\varepsilon}_k| = 1,\ \tilde{\varepsilon}_k \in \mathbb{C},$$

such that

i) $\tau^2 = Id$,
ii) $\bar{\delta} = -\tau \bar{d} \tau$ on the subspapce Ω_δ, of course.
iii) $\tilde{\varepsilon}_n = 1$.

For this we must assume that $\varepsilon_k = (-1)^k (-1)^{\frac{N(N-1)}{2}} = (-1)^k (-1)^n$, and then τ is uniquely determined by

$$\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{n(n+1)}{2}} \quad \text{/real number}$$

As previously since $\dim \mathbf{H}\left(\Omega_b^\bullet\right)$ is finite [from the Poincaré duality] we define as above the signature of W, and

$$\operatorname{Sig}(W) := \operatorname{Sig} \langle , \rangle_{\mathbf{H}}^n = \operatorname{Sig} \mathcal{B}_d^n.$$

Puting

$$\begin{aligned} W_\pm &= \{w \in W;\ \tau w = \pm w\}, \\ W_{1,\pm} &= W_\pm \cap W_1 \\ W_{0,\pm} &= W_\pm \end{aligned}$$

we notice that

$$\begin{aligned} \left(\bar{d} + \bar{\delta}\right)[W_{1,+}] &\subset W_{0,-}, \\ \left(\bar{d} + \bar{\delta}\right)[W_{1,-}] &\subset W_{0,+}, \end{aligned}$$

Definition 58 *The operator*

$$D_{1,+} = \bar{d} + \bar{\delta} : W_{1,+} \to W_{0,-}$$

is called the signature operator. Also we consider the adjoint one

$$D_{1,-} = \bar{d} + \bar{\delta} : W_{1,-} \to W_{0,+},$$

for which the condition of duality holds

$$(D_{1,+}\alpha, \beta) = (\alpha, D_{1,-}\beta), \qquad \text{for} \quad \alpha \in W_{1,+}, \beta \in W_{1,-}.$$

Analogously as in the previous part we prove the signature theorem

Theorem 59

$$\operatorname{Sig}(W) := \operatorname{Sig}\langle , \rangle_{\mathbf{H}}^{n} = \operatorname{Sig}(\mathcal{B}_d^n) = \dim_{\mathbb{R}} \ker(D_{1,+}) - \dim_{\mathbb{R}} \ker(D_{1,-}).$$

We see that in the Hilbert case there are very simple considerations to obtain a Hodge theorem (no analysis !, only algebraic topology, may be with the exception of the condition like $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$).

4 Completion of the graded Hodge differential space

Consider a graded Hodge differential space

$$\left(W = \bigoplus\nolimits_{k=0}^{N} W^k, \langle\cdot,\cdot\rangle, (\cdot,\cdot), *, d\right),$$

i.e

(0) $\langle\cdot, w\rangle = (\cdot, *w)$, $*$ is Hodge operator $(v, w) = (*v, *w)$,

(1) W^k are mutually orthogonal, and $\langle\cdot,\cdot\rangle \mid : W^k \times W^r = 0$ if $k + r \neq N$,

(2) d is of degree $+1$,

(3) $\langle dw, u\rangle = (-1)^{k+1} \langle w, du\rangle$ for $w \in W^k$ and $u \in W^{N-k-1}$.

We complete the unitary space $(W, (,))$ to Hilbert one $\bar{W}$. The inner product and the norm in $\bar{W}$ will be denoted by the same symbol. We extend the $*$-Hodge isometry to the isometry $* : \bar{W} \to \bar{W}$ and the pairing $\langle,\rangle$ to a new one denoting by the same symbol. Of course this pairing remains continuous.

We obtain a Hilbert graded Hodge space $\left(\bar{W} = \bigoplus_{k=0}^{N} \bar{W}^k, \langle\cdot,\cdot\rangle, (\cdot,\cdot), *\right)$.

Now we extend the differential $d^k : W^k \to W^{k+1}$ to some bigger subspace $\bar{W}_d^k \subset \bar{W}^k$ in a "distributional manner".

Definition 60 *We say that a vector* $w \in \bar{W}^k$ *has a distributional differential if there exists a vector belonging to* W^{k+1} *denoted by* $\bar{d}w$ *such that for each vector* $v \in W^{N-k-1}$ *the following condition*

$$\left\langle w, d^{N-k-1} v \right\rangle = (-1)^{k+1} \left\langle \bar{d}w, v \right\rangle$$

holds, equivalently if

$$(w, \delta h) = \left(\bar{d}w, h\right)$$

for each $h \in W^{k+1}$, *where*

$$\delta := (-1)^{N-k} * d^k *^{-1} .$$

The differential $\bar{d}w$ is unique (if it exists). The vector space of vectors v possessing distributional differential will be denoted by $\bar{W}_d$. Clearly, if $w \in W^k$ then $\bar{d}w$ exists and $\bar{d}w = dw$, as well as $\bar{d}^{k+1}\left(\bar{d}^k(w)\right) = 0$ for $w \in \bar{W}_d$.

Theorem 61 *Let* $w_n \in W$ *and assume that* (w_n) *is Cauchy and* $w_n \to w \in \bar{W}$. *Then* w *possesses distrbutional differential if and only if there exists a vector* $w' \in \bar{W}^{k+1}$ *such that for each* $v \in W^{k+1}$ *the condition*

$$(dw_n, v) \to (w', v)$$

holds (i.e. it is the condition of <u>*partially weak convergence*</u> *of* d *with respect to* W^{k+1} *). The vector* w' *is then* <u>*a distributional differential*</u> *of* w, $w' = \bar{d}w$.

It is easy to see that $\bar{d}\bar{d} = 0$ and that the inclusion $i : W \to \bar{W}_d$ induces a homomorphism in cohomology

$$i_{\#} : \mathbf{H}(W) \to \mathbf{H}\left(\bar{W}_d\right).$$

Problem 62 *(1) Does* $\left\langle \bar{d}w, u \right\rangle = (-1)^{r+1} \left\langle w, \bar{d}u \right\rangle$ *for* $w \in \bar{W}_d^r$, $u \in \bar{W}_d^{N-r-1}$ *for a given Hodge graded differential space?*

We introduce

$$\bar{W}_\delta^{N-k} = * \left[\bar{W}_d^k\right]$$

and codifferential $\bar{\delta}^{N-k} : \bar{W}_\delta^{N-k} \to \bar{W}_\delta^{N-k-1}$ by the formula

$$\bar{\delta}^{N-k} := (-1)^{N-k} * \bar{d}^k *^{-1}.$$

$\bar{\delta}$ is an extension of δ.

Problem 63 *(2) Does the inclusion* $j : H \to \bar{H}$ *(which of course commutes with differentials* d *and* $\bar{d}$ *) induces an isomorphism* $\mathbf{H}(H) \to \mathbf{H}\left(\bar{H}\right)$ *in cohomology? Particularly, then the space* $\mathbf{H}\left(\bar{H}\right)$ *is with Poincare duality.*

Remark 64 *Now consider the four examples* $W = \Omega(M)$, $\Omega(A)$ *for a Lie algebroid* A, $\Omega(M, E)$ *for the Lusztig or the Gromov vector bundle. Now we pass to the spaces of* L_2*-differential forms* $\bar{W} = \overline{\Omega(M)}$, $\overline{\Omega(A)}$, $\overline{\Omega(M, E)}$. *The first one has been considered as a special case in Teleman's theory (each smooth manifold possesses a Lipschitz structure, so an* L_2*-signature operator).*

We need to check only whether the problems (1) and (2) in the remaining three cases have also a positive answer. We can use to solve (1) some local calculations but to (2) the comparison theorem for suitable spectral sequences coming from Čech-de Rham complexes.

Conclusion 65 *Thus we obtain graded Hilbert subdifferential Hodge space. Consequently, each of four examples considered above: manifold, Lie algebroid, Lusztig and Gromov examples produces easily such a space.*

In consequence, the signature of A *can be calculated as the index of the two* L_2*-Hirzebruch signature operators using graded Hilbert subdifferential Hodge spaces* $\overline{\Omega(A)}$ *or* $\overline{\Omega(M, E)}$, *respectively.*

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