Lie Groupoid's point of view of linear direct connections and characteristic classes *Poisson Geometry in Mathematics and Physics* Lausanne, Switzerland, July 7-11, 2008

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Linear direct connections in vector bundles and Teleman's theorem

Nicola Teleman in the papers

N.Teleman, *Distance Function, Linear quasi-Connections and Chern Character,* June 2004, IHES/M/04/27

N.Teleman, *Direct Connections and Chern Character,* Proceedings of the International Conference in Honor of Jean-Paul Brasselet, Luminy, May 2005,

shows how the Chern character of the tangent bundle of a smooth manifold may be extracted from the geodesic distance function by means of cyclic homology.

The processing has the following steps:

1. Let M be a smooth Riemannian manifold and let

$$r : M \times M \to [0, \infty)$$

be the induced geodesic distance function.

The function r^2 is smooth on a neighbourhood of the diagonal.

2. Let χ be a cut-off smooth monotone decreasing real valued function, identically 1 on a neighbourhood of 0, having support on a sufficiently small interval, so that $\chi \circ r^2$ be well defined and smooth. For $x, y \in M$ a linear mapping

$$A(y,x) : T_x M \to T_y M$$

is given by the formula

$$A(y,x)\left(\sum_i \xi^i \frac{\partial}{\partial x^i}\right) = \sum_{i,j,k} \xi^i \frac{\partial^2 (\chi \circ r^2)(x,y)}{\partial x^i \partial y^j} g^{jk}(y) \frac{\partial}{\partial y^k}$$

($A(y,x)$ is independent of the local coordinates).

For sufficiently close points x, y,

— $A(y,x)$ is an isomorphism and

— $A(x,x)$ is the identity.

Therefore A is a linear direct connection (=linear quasi-connection), with respect to the definition below.

3. With the object A there is associated the function $\Phi_k : U_{k+1} \to \mathbb{R}$, where U_{k+1} is a neighbourhood of the diagonal in M^{k+1}

$$\Phi_k(x_0, x_1, ..., x_k) := Trace\ A(x_0, x_1) \circ A(x_1, x_2) \circ ... \circ A(x_{k-1}, x_k) \circ A(x_k, x_0).$$

4. Next, N.Teleman studies the function Φ_k in the context of cyclic homology:

— firstly, he notices that Φ_k, k =even, is a cyclic cycle over the algebra $\mathcal{A} = C^\infty(M)$,

— secondly, he uses the Connes' isomorphism which associates with Φ_k a closed differential form

$$\Omega(\Phi_k)(x) = \frac{1}{k!} \sum_{i_1, i_2, .., i_k} \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} ... \frac{\partial}{\partial x_k^{i_k}} \Phi_k(x_0, x_1, ..., x_k)_{x_0 = x_1 = ... = x_k = x}\, dx^{i_1} \wedge ... \wedge dx^{i_k},$$

(we use the same local coordinate system on each factor).

— thirdly, he proves

Theorem 1 *The top degree component of the cyclic homology class of Φ_k is equal to*

$$[\Omega(\Phi_{2k})] = c \cdot Ch_k(M)$$

where c is a constant and $Ch_k(M)$ is the k-component of the Chern character of the tangent bundle of M.

The object A is a particular case of the linear direct connection introduced by N.Teleman.

Definition 2 *Let E be a real or complex smooth vector bundle over the manifold M. A* ***linear direct connection*** *τ in E consists of assigning to any two points $x, y \in M$, sufficiently close one to each other, an isomorphism*

$$\tau(y,x) : E_{|x} \rightarrow E_{|y},$$

such that

$$\tau(x,x) = id,$$

and $\tau(y,x)$ depends smoothly on the pair x, y.

The parallel transport defined by a usual linear connection in E along the small geodesics of an affine connection in M induces a linear direct connection in E (see for example A.Connes and H.Moscovici, "*Cyclic cohomology, the Novikov conjecture and hyperbolic groups*", Topology 29, n 3 345-388, 1990).

-i) As for A with τ there is associated the function Φ_k by the formula

$$\Phi_k(x_0, x_1, ..., x_k) := Trace\ \tau(x_0, x_1) \circ \tau(x_1, x_2) \circ ... \circ \tau(x_{k-1}, x_k) \circ \tau(x_k, x_0).$$

The function

$$\Phi_2(x_0, x_1, x_2) = Trace\ \tau(x_0, x_1) \circ \tau(x_1, x_2) \circ \tau(x_2, x_0)$$

plays a role of the *curvature* of τ and the differential form $\Omega(\Phi_2)$ - the *curvature form* of τ.

-ii) Any two smooth linear direct connections in a smooth vector bundle are smoothly homotopic. The results above implay

Theorem 3 ***(N.Teleman)*** *For any smooth linear direct connection τ in the smooth vector bundle E over the manifold M,*

- *-i) Φ_k, $k =$even, is a cyclic cycle over the algebra $C^{\infty}(M)$,*
- *-ii) the cohomology class of $\Omega(\Phi_{2k})$ is (up to a multiplicative constant) is the k-component of the Chern character of E.*

1 Underlying linear connection ∇^τ and a direct proof of this theorem

In the paper

J.Kubarski, N.Teleman, *Linear direct connections,* Banach Center Publications, 2007, in print,

we study the geometry of direct connections τ:

- we construct the "infinitesimal part" ∇^τ and show that ∇^τ is a usual linear connection. We next determine the curvature tensor R of ∇^τ and show that the **equality of differential forms** holds

$$\Omega\left(\Phi_{2k}\right) = c \cdot Tr\ R^k.$$

We intend to extract from a direct connection its infinitesimal part along the diagonal.

Definition 4 *Let X be a smooth tangent field over M and ϕ a smooth section in E. Let x_0 be an arbitrary point in M and let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ be an integral path of the field X with the initial condition $\gamma(0) = x_0$. We define*

$$\nabla^\tau_{X(x_0)}(\phi) = \frac{d}{dt}\left\{\tau(\gamma(0), \gamma(t))\left(\phi\left(\gamma\left(t\right)\right)\right)\right\}_{|t=0} \in E_{|x_0}.$$

Theorem 5 *The right hand side of the above formula depends only on the value of X at x_0. The operator $\nabla^\tau_{X(x_0)}(\phi)$ is a usual linear connection in E.*

The linear connection ∇^τ will be called *associated, or underlying, linear connection to the direct connection τ.*

Proposition 6 *Let $R = (\nabla^\tau)^2$ be the curvature tensor of the connection ∇^τ. R is given by*

$$R = (\frac{\partial^2}{\partial x^\alpha \partial y^\beta}\tau^j_i(x|y)_{y=x} - \frac{\partial^2}{\partial x^\beta \partial y^\alpha}\tau^j_i(x|y)_{y=x} +$$
$$+ \frac{\partial}{\partial y^\alpha}\tau^j_k(x|y)_{y=x} \cdot \frac{\partial}{\partial y^\beta}\tau^k_i(x|y)_{y=x} - \frac{\partial}{\partial y^\beta}\tau^j_k(x|y)_{y=x} \cdot \frac{\partial}{\partial y^\alpha}\tau^k_i(x|y)_{y=x})dx^\alpha \wedge dx^\beta.$$

Although, $\tau(x, y) = (\tau(y, x))^{-1}$ is not true in general, it is true, however, that it holds infinitesimally. In fact, we have the

Proposition 7 *For any direct connection τ, its matrix components satisfy the identities*

-i)

$$\frac{\partial}{\partial x^\alpha}\tau_i^j(x|y)_{y=x} + \frac{\partial}{\partial y^\alpha}\tau_i^j(x|y)_{y=x} = 0.$$

-ii)

$$\frac{\partial}{\partial x^\alpha}\{\tau(x|y)\circ\tau(y|x)\}_{y=x} = 0 = \frac{\partial}{\partial y^\alpha}\{\tau(x|y)\circ\tau(y|x)\}_{y=x}.$$

As $\tau(x|x) = Id.$, we get that the directional derivative $(\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial y^\alpha})$ of τ along the diagonal **vanishes**. This proves -i). The second identity is a consequence of the first.

The above properties of any direct connection are fundamental for comparing the curvature tensor R to the differential form $\Omega\left(\Phi_{2k}^\tau\right)$.

We obtain an important explicit link between $\Omega\left(\Phi_{2k}\right)$ and the classical Chern-Weil forms, at the level of **differential forms** rather than **cohomology classes.**

Theorem 8 *Let τ be a direct connection and let ∇^τ be its underlying linear connection. Then*

$$\Omega(\Phi_2^\tau) = \frac{1}{4}\cdot Tr\ R,$$

and more generally,

$$\Omega(\Phi_{2k}^\tau) = \frac{1}{(2k)!}\cdot\frac{1}{2^k}\cdot Tr\ R^k.$$

In consequence, the mentioned above Teleman's theorem follows from this directly.

2 Groupoids point of view and groupoids generalizations

N.Teleman in yours papers said:

"*The arguments discussed here may be extended to the language of groupoids*".

My further talk is the first step in this direction.

2.1 Direct connections and the Lie groupoid $GL(E)$

Let E be a real or complex smooth vector bundle over the manifold M. Consider the **transitive Lie groupoid**

$$\Phi = GL(E)$$

of all linear fibre isomorphisms $h : E_{|x} \to E_{|y}$ of the vector bundle E, with the source α, $\alpha(h) = x$, and the target β, $\beta(h) = y$, and the unit $u_y = \mathrm{id}_{E_{|y}}$.

Remark 9 *A linear direct connection in a vector bundle E is equivalently a smooth mapping*

$$\tau : U \to GL(E)$$

where $U \subset M \times M$ is an open neighborhood of the diagonal $\Delta = \{(x,x)\,;\ x \in M\}$, such that

$$\tau(x,y) : E_{|y} \to E_{|x}$$

i.e.

$$\alpha \circ \tau(x,y) = y, \quad \beta \circ \tau(y,x) = x,$$

and

$$\tau(x,x) = \mathrm{id} : E_{|x} \to E_{|x}.$$

2.2 Lie Groupoids and point of view of linear direct connections and the using of the Lie algebroids

According to the Pradines definition, the **Lie algebroid** of an arbitrary transitive Lie groupoid Φ is equal to the vector bundle

$$A(\Phi) = u^*(T^\alpha \Phi)$$

where $u : M \to \Phi$, $y \to u_y$, and $T^\alpha \Phi = \ker \alpha_*$, equipped with the suitable structures: the bracket of cross-sections and the anchor.

Let $\Phi = GL(E)$ be the Lie groupoid of all linear fibre isomorphisms of fibres of E.

For $y \in M$ the submanifold $\Phi_y = GL(E)_y \subset GL(E)$ of all elements $u \in GL(E)$ for which $\alpha(u) = y$,

$$GL(E)_y = \alpha^{-1}(y),$$

is a $GL(E_y)$-principal fibre bundle.

- Lie algebroid of the Lie groupoid is the infinitesimal object and play analogous role to that of Lie algebras for Lie groups.
- The space [Lie algebra] of global cross-sections $Sec\left(A\left(\Phi\right)\right), \Phi = GL\left(E\right)$ where E is a vector bundle, is naturally isomorphic to the Lie algebra of all Covariant Derivative Operators, i.e. to the space of differential operators of the rank ≤ 1

  $$\mathfrak{L} : SecE \rightarrow SecE$$

  such that $\mathfrak{L}\left(f \cdot \xi\right) = f \cdot \mathfrak{L}\left(\xi\right) + X\left(f\right) \cdot \xi$, for a vector field X called the anchor of $\mathfrak{L}$, $f \in C^{\infty}\left(M\right), \xi \in SecE$.

Let $\tau : \left(M \times M\right)_{|U} \rightarrow GL\left(E\right)$ (where $U \subset M \times M$ is an open neighbourhood of the diagonal $\Delta = \left\{\left(x, x\right);\ x \in M\right\}$) be a linear quasi-connection,

$$\tau_{(x,y)} : E_{|y} \rightarrow E_{|x},$$

so $\alpha\left(\tau_{(x,y)}\right) = y$ and $\beta\left(\tau_{(x,y)}\right) = x$ and let ∇^{τ} be the **underlying linear connection of τ in E.**

Now, we fix y and take

$$\tau\left(\cdot, y\right) : M \rightarrow GL\left(E\right)_{y}, \quad x \longmapsto \tau\left(x, y\right).$$

It is a smooth mapping such that $\beta \circ \tau\left(\cdot, y\right) = \mathrm{id}$. Therefore the composition of the differential

$$\tau\left(\cdot, y\right)_{*x} : T_{x}M \rightarrow T_{\tau(x,y)}\left(GL\left(E\right)_{y}\right)$$

with the differential of $\beta | GL\left(E\right)_{y} \rightarrow M$ is identity

$$\mathrm{id} : T_{x}M \overset{\tau(\cdot,y)_{*x}}{\longrightarrow} T_{\tau(x,y)}\left(GL\left(E\right)_{y}\right) \overset{\beta_{*}}{\longrightarrow} T_{x}M.$$

Taking $x = y$ and using the fact $\tau\left(y, y\right) = u_{y} = \mathrm{id}_{E_{y}}$ we see that

$$\tau\left(\cdot, y\right)_{*y} : T_{y}M \rightarrow T_{u_{y}}\left(GL\left(E\right)_{y}\right).$$

Therefore τ determines a usual connection

$$\begin{aligned} \bar{\nabla}^{\tau} &: TM \rightarrow u^{*}\left(T^{\alpha}\Phi\right) \\ \bar{\nabla}^{\tau}\left(v_{y}\right) &= \tau\left(\cdot, y\right)_{*y}\left(v_{y}\right). \end{aligned}$$

in the Lie algebroid $u^{*}(T^{\alpha}\Phi)$ $(\Phi = GL(E))$, i.e. a splitting of the Atiyah sequence

$$0 \to \boldsymbol{g} \to A(\Phi) \underset{\overleftarrow{\nabla^{\tau}}}{\overset{\beta_{*}}{\longrightarrow}} TM \to 0.$$

$\bar{\nabla}^{\tau}$ is the "usual covariant derivative" since the anchor of the Covariant Derivative Operator $\bar{\nabla}^{\tau}(X) : SecE \to SecE$ is just equal to X, therefore noticing $\bar{\nabla}^{\tau}(X)(\xi)$ in the form

$$\bar{\nabla}^{\tau}_{X}(\xi)$$

the usual axioms for covariant derivative are fulfilled.

Theorem 10 $\bar{\nabla}^{\tau} = \nabla^{\tau}$, *i.e. the connection* $\bar{\nabla}^{\tau}$ *is equal to the* ***underlying linear connection of*** τ ***in*** E.

Proof. (a sketch) Since we need to prove it at any point $y \in M$ so we can prove it locally for $E = M \times \mathbb{R}^{n}$ and $M = \mathbb{R}^{m}$. Then $GL(E) = M \times GL(\mathbb{R}^{n}) \times M$, $\alpha^{-1}(y) = GL(E)_{y} = M \times GL(\mathbb{R}^{n}) \times \{y\}$. Let $\{e_i\}_{i=1}^{n}$ be a trivial local basis of E, then the induced linear connection ∇^{τ} is determined by

$$\nabla^{\tau}_{\frac{\partial}{\partial x^{k}}|y} e_i = \frac{\partial \tau_i^j}{\partial x^{m+k}}(y,y) \cdot e_j .$$

We can obtain the same results for $\bar{\nabla}^{\tau}$. ■

2.3 Groupoids generalization

The above consideration has "groupoids sense" so we can it generalize to any transitive Lie groupoids.

Let Φ be an arbitrary transitive Lie groupoid with the anchor α and the target β. We denote by u_y the unit of Φ at y.

Definition 11 *By a linear direct connection in* Φ *we mean a mapping*

$$\tau : (M \times M)_{|U} \to \Phi,$$

such that $\alpha \circ \tau(x,y) = y,\quad \beta \circ \tau(x,y) = x,$ *and* $\tau(x,x) = u_x$.

For y the submanifold $\Phi_y \subset \Phi$ of all elements $h \in \Phi$ for which $\alpha(h) = y$. Now, we fix y and take

$$\tau(\cdot, y) : M \to \Phi_y, \quad x \longmapsto \tau(x, y).$$

It is a smooth mapping such that $\beta \circ \tau(\cdot, y) = \mathrm{id}$. Taking the differential

$$\tau(\cdot, y)_{*x} : T_x M \to T_{\tau(x,y)}(\Phi_y).$$

τ determines a splitting of the Atiyah sequence of Φ, i.e. a usual connection in the Lie algebroid $A(\Phi) = u^*(T^\alpha \Phi)$,

$$\nabla^\tau : TM \to u^*(T^\alpha \Phi) = A(\Phi)$$

$$\nabla^\tau(v_y) = \tau(\cdot, y)_{*y}(v_y)$$

The connection ∇^τ will be called the **underlying linear connection of the linear direct connection** τ. .

Now we can ask on a very important question:

- How can we reconstruct the curvature tensor of ∇ from the linear direct connection in the Lie groupoid Φ? And next how can we reconstruct the Chern-Weil homomorphism of Lie groupoids Φ (i.e. equivalently of the principal bundle Φ_y) from arbitrary taken linear direct connection τ?

2.4 Curvature tensor of the linear direct connection in transitive Lie groupoids

Take any transitive Lie groupoid Φ and its Lie algebroid $A(\Phi)$ with the Atiyah sequence. For a linear direct connection τ in Φ denote by $\nabla^\tau : TM \to A(\Phi)$ the underlying linear connection in the Lie algebroid $A(\Phi)$ induced by τ. Consider the curvature tensor $\Omega^\tau \in \Omega^2(M; \boldsymbol{g})$ of ∇^τ

$$\Omega^\tau(X, Y) = [\![\nabla^\tau_X, \nabla^\tau_Y]\!] - \nabla^\tau_{[X,Y]}.$$

The linear direct connection τ determines the mapping

$$\Psi^\tau_k : \left(\underbrace{M \times ... \times M}_{k+1} \right)_{|U} \to \Phi,$$

$$\Psi^\tau_k(x_0, x_1, ..., x_k) = \tau(x_0, x_1) \cdot \tau(x_1, x_2) \cdot ... \cdot \tau(x_{k-1}, x_k) \cdot \tau(x_k, x_0)$$

having the values in the associated **Lie group bundle,**

$$\Psi_k^\tau(x_0, x_1, ..., x_k) \in \Phi_{x_0}^{x_0}.$$

For example, for $k = 2$, the function

$$\Psi_2^\tau : (M \times M \times M)_{|U} \to \Phi,$$
$$\Psi_2^\tau(x_0, x_1, x_2) = \tau(x_0, x_1) \cdot \tau(x_1, x_2) \cdot \tau(x_2, x_0)$$

is called the *curvature* of τ. Analogously to the previous cases we can associate some **differential form** to the function Ψ_k. Namely, fixing a point x_0 we define

$$\Psi_k^\tau(x_0) : \left(\underbrace{M \times ... \times M}_{k}\right)_{|U} \to \Phi_{x_0}^{x_0},$$
$$(x_1, ..., x_k) \longmapsto \Psi_k^\tau(x_0, x_1, ..., x_k) \in \Phi_{x_0}^{x_0}.$$

Next, we take a coordinate system $(x^1, ..., x^m)$ $(dimM = m)$ on an open neighborhood $\mathcal{V}$ of the point x_0. Using the same local coordinate system on each factors of the direct product $M \times \cdots \times M$ we take for $(x_1, ..., x_k) \in \mathcal{V} \times ... \times \mathcal{V}$

$$\frac{\partial}{\partial x_k^j} \Psi_k^\tau(x_0) \in T_{\Psi_k(x_0, x_1, ..., x_k)} \Phi_{x_0}^{x_0}.$$

This vector we can translate via **right** translation to the unit, therefore we can write

$$\frac{\partial}{\partial x_k^j} \Psi_k^\tau(x_0) \in T_{u_{x_0}}\left(\Phi_{x_0}^{x_0}\right) = \boldsymbol{g}_{|x_0}.$$

The function obtained

$$(x_1, ..., x_{k-1}, x_k) \longmapsto \frac{\partial}{\partial x_k^j} \Psi_k^\tau(x_0) \in \boldsymbol{g}_{|x_0}$$

can be differentiated usually as a vector valued function.

$$(x_1, ..., x_{k-1}, x_k) \longmapsto \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} ... \frac{\partial}{\partial x_{k-1}^{i_{k-1}}} \frac{\partial}{\partial x_k^{i_k}} \Psi_k^\tau(x_0, x_1, ..., x_k) \in \boldsymbol{g}_{|x_0}.$$

We put

$$\Omega(\Psi_k^\tau)(x) = \frac{1}{k!} \sum_{i_1, i_2, .., i_k} \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} \frac{\partial}{\partial x_k^{i_k}} \Psi_k^\tau(x_0, x_1, ..., x_k)_{x_0 = x_1 = ... = x_k = x} dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

It is a k-form on M with values in the vector bundle $\boldsymbol{g}$

$$\Omega\left(\Psi_k^\tau\right) \in \Omega^k\left(M; \boldsymbol{g}\right)$$

Considering $k = 2$ we obtain a 2-form with values in $\boldsymbol{g}$,

$$\Omega\left(\Psi_2^\tau\right) \in \Omega^2\left(M; \boldsymbol{g}\right),$$

called the **curvature form** of τ.

The fundamental role is playing by the following

Theorem 12 *For an arbitrary linear direct connection $\tau : (M \times M)_{|U} \to \Phi$ in the Lie groupoid Φ the curvature form of τ and the curvature form of the underlying connection in $A(\Phi)$ are differs on a constant*

$$\Omega\left(\Psi_2^\tau\right) = \frac{1}{4} \cdot \Omega^\tau.$$

2.5 Characteristic classes

The last theorem gives that we can extract the Chern-Weil homomorphism of Φ via any local direct connection τ on the level of differential forms. The Chern-Weil homomorphism of Φ is really the Chern-Weil homomorphism of the Lie algebroid $A(\Phi)$ of Φ.

We recall the construction of the Chern-Weil homomorphism for Lie algebroids

- Jan Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids*, UNIVERSITE CLAUDE BERNARD – LYON 1, Publications du Départment de Mathématiques, nouvelle série, 1991, 1-70.

Consider a transitive Lie algebroid A with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A \to TM \to 0$$

with the adjoint bundle of Lie algebras $\boldsymbol{g}$.

The Chern-Weil homomorphism for transitive Lie algebroid A is defined as follows:

$$\begin{aligned} h_A : \bigoplus^{k \geq 0} \left(\operatorname{Sec} \bigvee^{k} \boldsymbol{g}^*\right)_{I^0} &\longrightarrow \mathbf{H}_{dR}(M) \\ \Gamma &\longmapsto \left[\frac{1}{k!}\langle \Gamma, \Omega \vee .... \vee \Omega \rangle\right] \end{aligned}$$

where $\Omega \in \Omega^2_E(M;\boldsymbol{g})$ is the curvature tensor of any connection in A, whereas $\left(\operatorname{Sec}\bigvee^k \boldsymbol{g}^*\right)_{I^0}$ is the space of invariant cross-sections of $\bigvee^k \boldsymbol{g}^*$ with respect to the adjoint representation of A.

We explain also that $\Omega \vee ... \vee \Omega$ is the usual skew multiplication of differential forms with values multiplying symmetrically.

Take a local direct connection τ in Φ and consider once again the curvature form $\Omega(\Psi^\tau_2) \in \Omega^2(M;\boldsymbol{g})$. We known that τ induces a usual connection ∇^τ in $A(\Phi)$ and that the curvature of it is a constant time the form $\Omega(\Psi^\tau_2)$,

$$\Omega^\tau = 4 \cdot \Omega(\Psi^\tau_2).$$

In conclusion, the Chern-Weil homomorphism of Φ (i.e. of the $A(\Phi)$) can be extracted via τ on the **level of differential forms** by

$$\langle \Gamma, \Omega \vee ... \vee \Omega \rangle = 4^k \langle \Gamma, \Omega(\Psi^\tau_2) \vee ... \vee \Omega(\Psi^\tau_2) \rangle.$$

Problem 13 *How can we express the form*

$$\Omega(\Psi^\tau_2) \vee ... \vee \Omega(\Psi^\tau_2) \in \Omega^{2k}\left(M; \bigvee \boldsymbol{g}\right)$$

with the help of $\Omega(\Psi^\tau_{2k})$*?*

Example 14 *Let* $\Phi = GL(E)$ *be a Lie groupoid of all linear fibre isomorphisms.*

The equality holds

$$\Omega(\Psi^\tau_{2k}) = c \cdot \Omega(\Psi^\tau_2) \circ ... \circ \Omega(\Psi^\tau_2) \quad (\ k \ \textit{times})$$

or equivalently

$$\Omega(\Psi^\tau_{2k}) = c_1 \cdot \Omega^\tau \circ .... \circ \Omega^\tau \quad (\ k \ \textit{times}).$$

Example 15 ***Pontryagin classes.*** *Take the* ***invariant*** *cross section* $C_k \in \Gamma\left(\mathrm{Sec}\bigvee^k End\left(E\right)^*\right)$ *by*

$$C_{k|x} = tr\left(\varphi_1 \Box ... \Box \varphi_k\right)$$

where for $\varphi_i \in End\left(E_{|x}\right)$ *the linear mapping* $\varphi_1 \Box ... \Box \varphi_k : \bigwedge^k E_{|x} \to \bigwedge^k E_{|x}$ *is defined [Greub-Halperin-Vanstone] by*

$$\varphi_1 \Box ... \Box \varphi_k\left(v_1 \wedge ... \wedge v_k\right) = \sum_{\sigma \in \Sigma^k} sgn\sigma \cdot \varphi_1\left(v_{\sigma_1}\right) \wedge ... \wedge \varphi_k\left(v_{\sigma_k}\right).$$

Then the Ponryagin class is equal to

$$p_k\left(E\right) = p_k\left(\Phi\right) = h\left(C_{2k}\right) = \frac{1}{(2k)!}\left[\langle C_{2k}, \underbrace{\Omega \vee .... \vee \Omega}_{2k\ \ times}\rangle\right]_{dR}.$$

The class $p_k\left(E\right)$ *is represented by the differential form*

$$c \cdot tr\left(\Omega \Box ... \Box \Omega\right).$$

According to the notation of Greub-Halperin-Vanstone, the forms $\Omega \vee .... \vee \Omega$ *and* $\Omega \Box ... \Box \Omega$ *are the usual skew multiplication of differential forms for which the values are multiplicated by the suitable mappings*

$$\vee \ : \ End\left(E\right) \times ... \times End\left(E\right) \to \bigvee^{2k} End\left(E\right).$$

$$\Box \ : \ End\left(E\right) \times ... \times End\left(E\right) \to End\left(\bigwedge^{2k} E\right).$$

Trace classes. *Take the* ***invariant*** *cross section* $Tr_k \in \Gamma\left(\mathrm{Sec}\bigvee^k End\left(E\right)^*\right)$ *by*

$$Tr_k\left(\varphi_1, ..., \varphi_k\right) = \sum_{\sigma \in \Sigma^k} tr\left(\varphi_{\sigma_1} \circ ... \circ \varphi_{\sigma_k}\right).$$

Then the trace class is equal to

$$tr_k\left(E\right) = tr_k\left(\Phi\right) = h\left(Tr_{2k}\right) = \frac{1}{(2k)!}\left[\langle Tr_{2k}, \underbrace{\Omega \vee .... \vee \Omega}_{2k\ \ times}\rangle\right]_{dR}.$$

The class $tr_k\left(E\right)$ *is represented by the differential form*

$$c \cdot tr\left(\Omega \circ ... \circ \Omega\right)$$

(the values of the skew multiplication of $\Omega \circ ... \circ \Omega$ are multilied by the composing of the linear mapping.

Pffafian class for oriented $2k$***-dimensional vector bundle*** E. *Take the* ***invariant*** *cross-section* $pf \in \Gamma \left(\mathrm{Sec} \bigvee^{k} Sk\left(E\right)^{*} \right)$

$$pf^{F}\left(\varphi_{1},...,\varphi_{k}\right) = \langle e, \beta^{-1}\left(\varphi_{1}\right) \wedge ... \wedge \beta^{-1}\left(\varphi_{k}\right) \rangle$$

where $\beta : \bigwedge^{2}\left(F\right) \overset{\cong}{\longrightarrow} Sk_{F},\ \beta\left(x \wedge y\right)\left(z\right) = \langle x, z\rangle y - \langle y, z\rangle x$ *and* $e \in \bigwedge^{k} F$ *determine the orientation and* $|\langle e, e\rangle| = 1$. *Then the Atiyah sequence of* $A\left(IsoE\right)$ *is*

$$0 \rightarrow Sk\left(E\right) \rightarrow A\left(IsoE\right) \rightarrow TM \rightarrow 0.$$

and the Pfaffian class is equal to $i^{k} \cdot h\left(pf\right)$ *and it is represented by*

$$c \cdot \langle \Delta, \left(\beta^{-1}\Omega\right) \wedge ... \wedge \left(\beta^{-1}\Omega\right) \rangle.$$