Hirzebruch signature operator for transitive Lie algebroids

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10 October 2007

Abstract

The aim of the paper is to construct Hirzebruch signature operator for transitive invariantly oriented Lie algebroids

1 Signature of Lie algebroids

1.1 Definition of Lie algebroids, Atiyah sequence

Lie algebroids appeared as infinitesimal objects of Lie groupoids, principal fibre bundles, vector bundles (Pradines, 1967), TC-foliations and nonclosed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalents are known as Lie pseudo-algebras (Herz 1953) called also further as Lie-Rinehart algebras (Huebschmann, 1990).

A *Lie algebroid* on a manifold M is a triple

$$A = (A, [\![\cdot,\cdot]\!], \#_A)$$

where A is a vector bundle on M, $(\operatorname{Sec} A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

$$\#_A : A \to TM$$

is a linear homomorphism (called the *anchor*) of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta, \quad f\in C^\infty(M),\ \xi,\eta\in \operatorname{Sec} A.$$

The anchor is bracket-preserving [B-K-W], [H]

$$\#_A\circ[\![\xi,\eta]\!] = [\#_A\circ\xi, \#_A\circ\eta].$$

A Lie algebroid is called *transitive* if $\#_A$ is an epimorphism.

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0,$$

$\boldsymbol{g} := \ker \#_A$. The fiber $\boldsymbol{g}_x$ of the bundle $\boldsymbol{g}$ in the point $x \in M$ is the Lie algebra with the commutator operation being

$$[v, w] = [\![\xi, \eta]\!](x), \quad \xi, \eta \in \operatorname{Sec} A, \quad \xi(x) = v, \eta(x) = w, \quad v, w \in \boldsymbol{g}_x.$$

The Lie algebra $\boldsymbol{g}_x$ is called the isotropy Lie algebra of A at $x \in M$. The vector bundle $\boldsymbol{g}$ is a Lie Algebra Bundle (LAB in short), called the *adjoint* of A, the fibres are isomorphic Lie algebras.

TM is a Lie algebroid with $id : TM \to TM$ as the anchor,

$\mathfrak{g}$ -finitely dimensional Lie algebra - is a Lie algebroid over a point $M = \{*\}$.

1.2 Cohomology algebra, ellipticity of the complex of exterior derivatives $\left\{d_A^k\right\}$

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \operatorname{Sec} \bigwedge A^*, \ \text{ - the space of cross-sections of } \bigwedge A^*$$

$$d_A : \Omega^{\bullet}(A) \to \Omega^{\bullet+1}(A)$$

$$\begin{aligned}(d_A \omega)(\xi_0, ..., \xi_k) &= \sum_{j=0}^{k} (-1)^j \left(\#_A \circ \xi_j\right)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \, \omega\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k\right),\end{aligned}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \operatorname{Sec} A$. The operators d_A^k satisfy

$$d_A(\omega \wedge \eta) = d_A \omega \wedge \eta + (-1)^k \omega \wedge d_A \eta,$$

so they are of first order and the symbol of d_A^k is equal to

$$\begin{aligned} S\left(d_A^k\right)_{(x,v)} &: \bigwedge^k A_x^* \to \bigwedge^{k+1} A_x^* \\ S\left(d_A^k\right)_{(x,v)}(u) &= \left(v \circ (\#_A)_x\right) \wedge u, \quad 0 \neq v \in T_x^* M.\end{aligned}$$

In consequence the sequence of symbols

$$\bigwedge^k A_x^* \overset{S\left(d_A^k\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+1} A_x^* \overset{S\left(d_A^{k+1}\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+2} A_x^*$$

is exact if and only if A is transitive and then the complex $\left\{d_A^k\right\}$ is an elliptic complex.

The exterior derivative d_A introduces the cohomology algebra

$$\mathbf{H}(A) = \mathbf{H}(\Omega(A), d_A).$$

For the trivial Lie algebroid TM - the tangent bundle of the manifold M - the differential d_{TM} is the usual de-Rham differential d_M of differential forms on M whereas, for $L = \mathfrak{g}$ - a Lie algebra $\mathfrak{g}$ - the differential $d_{\mathfrak{g}}$ is the usual Chevalley-Eilenberg differential, $d_{\mathfrak{g}} = \delta_{\mathfrak{g}}$.

1.3 Invariantly oriented Lie algebrois and signature

The following theorem describes the class of transitive Lie algebroids (over compact oriented manifold) for which $\mathbf{H}^{\mathrm{top}}(A) \neq 0$.

Theorem 1 *[K-M1]For each transitive Lie algebroid* $(A, [\![\cdot,\cdot]\!], \#_A)$ *with the Atiyah sequence*

$$0 \to \boldsymbol{g} \to A \overset{\#_A}{\longrightarrow} TM \to 0,$$

over a compact oriented manifold M *the following conditions are equivalent* ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, *i.e.* $\operatorname{rank} A = m + n$)

(a) $\mathbf{H}^{m+n}(A) \neq 0$,

(b) $\mathbf{H}^{m+n}(A) \cong \mathbb{R}$ *and* $\mathbf{H}(A)$ *is an Poincaré algebra, i.e. the pairing* $\mathbf{H}^j(A) \times \mathbf{H}^{m+n-j}(A) \to \mathbf{H}^{m+n}(A) \cong \mathbb{R}$ *is nondegenerate,* $\mathbf{H}^j(A) \cong \left(\mathbf{H}^{m+n-j}(A)\right)^*$,

(c) there exists a global nonsingular cross-section $\varepsilon \in \operatorname{Sec}(\bigwedge^n \boldsymbol{g})$ *invariant with respect to the adjoint representation* ad_A*, that is,* A *is the so-called a TUIO-Lie algebroid, see [K1], (shortly,* A *is invariantly oriented),*

(d) the vector bundle $\boldsymbol{g}$ *is orientable and the modular class of* A *is trivial,* $\theta_A = 0$.

We recall the definition of the isomorphism $\mathbf{H}^{m+n}(A) \cong \mathbb{R}$ (for invariantly oriented transitive Lie algebroids). In [K1] there is defined (for arbitrary transitive Lie algebroids) the so-called *fibre integral operator*

$$\fint_A : \Omega^\bullet(A) \to \Omega_{dR}^{\bullet - n}(M)$$

by the formula

$$\left(\fint_A \omega^k\right)_x (w_1, ..., w_{k-n}) = (-1)^{kn} \omega_x(\boldsymbol{\varepsilon}_x, \tilde{w}_1, ..., \tilde{w}_{k-n}), \quad \#_A(\tilde{w}_i) = w_i,$$

where $\varepsilon \in \operatorname{Sec}(\bigwedge^n \boldsymbol{g})$ is a nonsingular cross-section. The operator $\fint_A$ commutes with the differentials d_A and d_M if and only if ε is invariant. Then, the fibre integral gives a homomorphism in cohomology

$$\fint_A^{\#} : \mathbf{H}^\bullet(A) \to \mathbf{H}_{dR}^{\bullet - n}(M).$$

Assume in the sequel that a transitive Lie algebroid A over compact oriented manifold M is invariantly oriented and $\varepsilon \in \operatorname{Sec}(\bigwedge^n \boldsymbol{g})$ is an invariant cross-section. The scalar Poincaré product

$$\mathcal{P}_A^k : \mathbf{H}^k(A) \times \mathbf{H}^{m+n-k}(A) \to \mathbb{R},$$

$$([\omega], [\eta]) \longmapsto \int_A \omega \wedge \eta \quad \left(:= \int_M \left(\fint_A \omega \wedge \eta\right)\right)$$

is defined and is nondegenerated [K2]; in consequence

$$\begin{aligned}\mathbf{H}^k(A) &\cong \mathbf{H}^{m+n-k}(A),\\ \mathbf{H}^{m+n}(A) &\cong \left(\mathbf{H}^0(M)\right)^* = \mathbb{R},\\ \dim \mathbf{H}(M) &< \infty,\end{aligned}$$

and we can consider an isomorphism

$$\mathbf{H}^{m+n}(A) \cong \mathbb{R}, \quad [\omega] \longmapsto \int_A \omega.$$

The pairing of A-differential forms

$$\begin{gathered}\langle\langle\cdot,\cdot\rangle\rangle^k : \Omega^k(A) \times \Omega^{m+n-k}(A) \to \mathbb{R} \\ \langle\langle\omega,\eta\rangle\rangle^k = \int_M \int_A \omega \wedge \eta\end{gathered} \tag{1}$$

has the property

$$\langle\langle\omega,\eta\rangle\rangle^k = (-1)^{k(m+n-k)} \langle\langle\eta,\omega\rangle\rangle^{m+n-k}$$

and

$$\langle\langle d_A\omega,\eta\rangle\rangle = (-1)^{k+1} \langle\langle\omega, d_A\eta\rangle\rangle \quad \text{for } \omega \in \Omega^k(A),\ \eta \in \Omega^{m+n-(k+1)}(A).$$

If

$$m + n = 4p$$

then

$$\mathcal{P}_A^{2p} : \mathbf{H}^{2p}(A) \times \mathbf{H}^{2p}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and is called the signature of A, and is denoted by

$$\operatorname{Sign}(A).$$

The problem is [K2]:

- to calculate the signature $\operatorname{Sign}(A)$ and give some conditions to the equality $\operatorname{Sign}(A) = 0$. There are examples for which $\operatorname{Sign}(A) \neq 0$ (this is announced in [K-M2]).

2 $*$-Hodge operator and exterior coderivative d_A^*

2.1 Associated scalar product and $*$-Hodge operator

Consider

- any Riemannian tensor G_1 in the vector bundle $\boldsymbol{g} = \ker \#_A$ for which $\boldsymbol{\varepsilon}$ is the volume tensor (such a tensor exists).

- any Riemannian tensor G_2 on M.

Next, taking an arbitrary connection $\lambda : TM \to A$ in the Lie algebroid A i.e. a splitting of the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \underset{\lambda}{\overset{\#_A}{\longrightarrow}} TM \longrightarrow 0,$$

and the horizontal space

$$\begin{aligned} H &= \operatorname{Im}\lambda, \\ A &= \boldsymbol{g} \bigoplus H \end{aligned}$$

we define a Riemannian tensor G (called *scalar product associated* to ε) on $A = \boldsymbol{g} \bigoplus H$ such that $\boldsymbol{g}$ and H are orthogonal, on $\boldsymbol{g}$ we have G_1 but on H we have the pullback $\lambda^* G_2$. The vector bundle A is oriented (since $\boldsymbol{g}$ and M are oriented).

At each point $x \in M$ we consider the scalar product G_x on $A_{|x}$ and the pairing of tensors

$$\langle , \rangle_x^k : \bigwedge^k A_x^* \times \bigwedge^{m+n-k} A_x^* \to \bigwedge^{m+n} A_x^* \overset{\rho_x}{\to} \mathbb{R}$$

where ρ_x is defined via the volume form for G_x.

We can notice that ρ_x is equal to the composition

$$\begin{array}{ccccc} \bigwedge^{m+n} A_x^* & & \overset{\rho_x}{\longrightarrow} & & \mathbb{R} \\ \downarrow (-1)^{(m+n)n}\, i_{\boldsymbol{\varepsilon}_x} & \searrow \oint_{A_p} & & \nearrow \rho_{G_2 x} & \\ \bigwedge^m A_x^* & & \overset{\rho_{\lambda x}}{\longrightarrow} & \bigwedge^m T_x^* M & \end{array}$$

$$\begin{aligned} i_{\boldsymbol{\varepsilon}_x}\omega_x\left((v_1,...,v_{k-n})\right) &= \omega_x\left(\boldsymbol{\varepsilon}_x, v_1,...,v_{k-n}\right), \\ \rho_{\lambda x}\left(\omega_x\right)\left(w_1,...,w_{k-n}\right) &= \omega_x\left(\boldsymbol{\varepsilon}_x, \lambda_x\left(w_1\right),...,\lambda_x\left(w_{k-n}\right)\right). \end{aligned}$$

Standartly, we can extend the scalar product G_x in A_x to a scalar product $(\cdot,\cdot)_x$ in $\bigwedge A_x^*$. There exists exactly one the so-called $*$-Hodge operator $*_x : \bigwedge^k A_x^* \to \bigwedge^{m+n-k} A_x^*$ such that

$$\langle \alpha_x, \beta_x \rangle = \left(\alpha_x, *_x \beta_x\right),$$

and it is given by

$$*_x\left(e_{i_1}^* \wedge ... \wedge e_{i_k}^*\right) = \operatorname{sgn}\left(j_1,...,j_{m+n-k}, i_1,..,i_k\right) e_{j_1}^* \wedge ... \wedge e_{j_{m+n-k}}^*$$

where $1 \leq i_1 < ... < i_k \leq m+n$, $1 \leq j_1 < ... < j_{m+n-k} \leq m+n$ and $\{i_1,...,i_k\} \cap \{j_1,...,j_{m+n-k}\} = \varnothing$ (for ON positive frame e_i in A_x). Using all points $x \in M$ we obtain two $C^\infty(M)$-linear 2-tensors

$$\langle , \rangle, (,) : \Omega(A) \times \Omega(A) \to C^\infty(M)$$

defined as above point by point. Integrating along M we get $\mathbb{R}$-linear 2-tensors

$$\langle\langle , \rangle\rangle , ((,)) : \Omega(A) \times \Omega(A) \to \mathbb{R}.$$

The first $\langle\langle , \rangle\rangle$ is, clearly, given by (1)

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \langle \omega, \eta \rangle = \int_M \fint_A \omega \wedge \eta.$$

The $*$-Hodge operator $* : \Omega(A) \to \Omega(A)$ is defined point by point

$$*(\omega)(x) = *_x(\omega_x)$$

and we have

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta)).$$

2.2 Exterior coderivative

We define exterior coderivative $d_A^* : \Omega^k(M) \to \Omega^{m+n-k}(A)$ by the formula

$$d_A^*(\omega) = (-1)^{k(m+n-k)} (-1)^k * d_A * (\omega), \quad \omega \in \Omega^k(A),$$

where $*$ is the $*$-Hodge operator in $\Omega(A)$. We have

(a)

$$**(\omega) = (-1)^{k(m+n-k)} \cdot \omega, \quad \omega \in \Omega^k(A), \tag{2}$$

(b)

$$((d^*(\omega), \eta)) = ((\omega, d_A(\eta))),$$

i.e. d_A^* is adjoint to d_A with respect to the scalar product $((,))$ in $\Omega(A)$.

3 Laplacian and harmonic differential forms

It enables us to introduce the Laplacian

$$\Delta_A = (d_A + d_A^*)^2 = d_A d_A^* + d_A^* d_A.$$

Clearly

$$\ker \Delta_A = (\operatorname{Im} \Delta_A)^\perp$$

(with the respect to the scalar product $((,))$).

Proposition 2 *The Laplacian Δ_A is elliptic, self-adjoint and nonnegative operator. In consequence*

$$\Omega(A) = \ker \Delta_A \bigoplus \operatorname{Im} \Delta_A. \tag{3}$$

Proof. The first property follows from the ellipticity of the complex $\left\{d_A^k\right\}$ (namely: the symbol of the adjoint operator d_A^{*k} is equal to the minus of the adjoint symbol of d_A^k and to prove the ellipticity of the Laplacian we use Remark 6.34 from [W]), the next two properties are trivial consequence of the definition. The last property (3) can be proved in the same way as the Theorem 55 from [L-M] using extension $\mathrm{Sec}\bigwedge A^*$ to the Hilbert Sobolev spaces $H_s\left(\bigwedge A^*\right)$, extension Δ to continuous operator $\Delta_s : H_s\left(\bigwedge A^*\right) \to H_{s-2}\left(\bigwedge A^*\right)$ and the fact that $\ker\Delta_s$ consists only of smooth sections, $\ker\Delta_s = \ker\Delta$. ■

A A-differential form $\omega \in \Omega(A)$ is called *harmonic* if $d_A\omega = 0$ i $d_A^*\omega = 0$. Denote the space of harmonic A-differential forms by $\mathcal{H}(A)$ and harmonic k-A-differential forms by $\mathcal{H}^k(A)$. $\mathcal{H}(A)$ is a graded vector space

$$\mathcal{H}(A) = \bigoplus\nolimits_{k=0}^{m+n} \mathcal{H}^k(A)$$

and

$$\mathcal{H}(A) = \ker\Delta_A.$$

is the eigenspace of the operator Δ corresponding to the zero value of the eigenvalue.

Simple calculations assert that $\ker\Delta^k$ and $\mathrm{Im}\, d^{k-1}$ are orthogonal, therefore the inclusion

$$\mathcal{H}^k(W) = \ker\Delta^k \hookrightarrow \ker d^k$$

induce a monomorphism

$$\ker\Delta^k \rightarrowtail \mathbf{H}^k(W). \tag{4}$$

Since $\mathrm{Im}\,\Delta^k \subset \mathrm{Im}\, d_A^{k-1} + \mathrm{Im}\, d_A^{*(k+1)}$, the inclusion (3) yields $\Omega^k(A) = \ker\Delta_A^k + \mathrm{Im}\, d_A^{k-1} + \mathrm{Im}\, d_A^{*(k+1)}$ and easily we can notice that these three subspaces are orthogonal. Therefore

$$\Omega^k(A) = \ker\Delta_A^k \bigoplus \mathrm{Im}\, d_A^{k-1} \bigoplus \mathrm{Im}\, d_A^{*(k+1)}$$

and

$$\ker d^k = \ker\Delta^k \bigoplus \mathrm{Im}\, d^{k-1}$$

which implies the Hodge Theorem for Lie algebroids:

Corollary 3 *The monomorphism (4) is an isomorphism*

$$\mathcal{H}^k(W) \cong \ker d^k / \mathrm{Im}\, d^{k-1} = \mathbf{H}^k(W).$$

It means that in each cohomology class $\alpha \in \mathbf{H}^k(A)$ there is exactly one harmonic A-differential form $\omega \in \mathcal{H}^k(W)$.

Let $\varepsilon_k = (-1)^{k(n+m-k)}$. Simple calculations yields the equality

$$*\Delta_A\omega = \varepsilon_{k-1}\varepsilon_k (-1)^{n+m+1}\Delta_A * \omega, \quad \omega \in \Omega^k(A),$$

therefore

$$*\left[\mathcal{H}^k(W)\right] \subset \mathcal{H}^{m+n-k}(W),$$

and (thanks (2))

$$*: \mathcal{H}^k(W) \to \mathcal{H}^{m+n-k}(W)$$

is an isomorphism. In consequence we obtain (independently on [K2]) the Duality Theorem

$$\mathbf{H}^k(A) \simeq \mathbf{H}^{m+n-k}(A).$$

We restrict the scalar product $((\cdot,\cdot)) : \Omega^k(A) \times \Omega^k(A) \to \mathbb{R}$ to the space of harmonic A-differential forms

$$((\cdot,\cdot)) : \mathcal{H}^k(A) \times \mathcal{H}^k(A) \to \mathbb{R},$$

and we restrict the tensor $\langle\langle\cdot,\cdot\rangle\rangle : \Omega^k(A) \times \Omega^{m+n-k}(A) \to \mathbb{R}$ to harmonic A-differential forms

$$\mathcal{B}^k = \langle\langle\cdot,\cdot\rangle\rangle : \mathcal{H}^k(A) \times \mathcal{H}^{m+n-k}(A) \to \mathbb{R}.$$

Using the isomorphism $\mathcal{H}^k(A) \cong \mathbf{H}^k(A)$ we see that

$$\mathcal{B}^k = \mathcal{P}^k_A,$$

therefore, if $m+n=4p$ then

$$\operatorname{Sign}(A) = \operatorname{Sign}\mathcal{B}^{2p}.$$

4 Hirzebruch signature operator

Assume $m+n=4p$. Considering the direct sum $\mathcal{H}^{2p}(A) = \mathcal{H}^{2p}_+(A) \bigoplus \mathcal{H}^{2p}_-(A),$ where

$$\mathcal{H}^{2p}_\pm(W) = \left\{\omega \in \mathcal{H}^{2p}(W);\ *\omega = \pm\omega\right\},$$

and noticing that $\mathcal{B}^{2p}$ is positive on $\mathcal{H}^{2p}_+(A)$, and is negative on $\mathcal{H}^{2p}_-(A)$ we see that

$$\operatorname{Sign}\left(\mathcal{B}^{2p}\right) = \dim_{\mathbb{R}}\mathcal{H}^{2p}_+(W) - \dim_{\mathbb{R}}\mathcal{H}^{2p}_-(W).$$

To construction the Hirzebruch signature operator the fundamental role is played by an auxiliary operator

$$\tau : \Omega(A) \to \Omega(A)$$

defined by

$$\tau^k\left(\omega^k\right) = \tilde{\varepsilon}_k * \left(\omega^k\right), \quad \tilde{\varepsilon}_k \in \{-1,+1\}, \quad \omega^k \in \Omega^k(A),$$

fulfilling the properties

i) $\tau \circ \tau = Id$,

ii) $d_A^* = -\tau \circ d_A \circ \tau$,

iii) $\tau^{2p} = *$.

Lemma 4 *The operator τ fulfils axioms i)-iii) if and only if* $\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} (-1)^p$.

Proof. Easy calculations. ■

We put

$$\begin{aligned} \Omega(A)_+ &= \{\omega \in \Omega(A);\ \tau\omega = +\omega\}, \\ \Omega(A)_- &= \{\omega \in \Omega(A);\ \tau\omega = -\omega\}, \end{aligned}$$

The spaces $\Omega(A)_+$ and $\Omega(A)_-$ are eigenspaces of τ corresponding to the eigenvalues $+1$ i -1 and are spaces of cross-sections of suitable vector bundles.

We notice that

$$(d_A + d_A^*)\left[\Omega(A)_+\right] \subset \Omega(A)_- .$$

Definition 5 *The operator*

$$(D_A)_+ = d_A + d_A^* : \Omega(A)_+ \to \Omega(A)_-$$

is called the Hirzebruch operator *(or the* signature operator*) for the Lie algebroid A.*

Clearly

$$(D_A)_+^* = d_A + d_A^* : \Omega(A)_- \to \Omega(A)_+ .$$

Theorem 6

$$\operatorname{Sign} A = \operatorname{Ind}(D_A)_+ = \dim_{\mathbb{R}} \ker\left((D_A)_+\right) - \dim_{\mathbb{R}} \ker\left((D_A)_+^*\right).$$

Proof. It is sufficient to prove that $\operatorname{Ind}(D_A)_+ = \operatorname{Sign}\left(\mathcal{B}^{2p}\right)$. The proof is analogous to the classical case [Y]. Firstly, we notice that subspaces $\mathcal{H}^s(A) + \mathcal{H}^{m+n-s}(A)$ are τ-stable and for $s = 0, 1, ..., 2p-1$

$$\begin{aligned} \varphi_\pm : \mathcal{H}^s(A) &\to \left(\mathcal{H}^s(A) + \mathcal{H}^{m+n-s}(A)\right)_\pm \\ \omega &\longmapsto \frac{1}{2}(\omega \pm \tau\omega) \end{aligned}$$

is an isomorphism of real vector spaces. Secondly $\Omega^{2p}(A)_\pm \cap \mathcal{H}^{2p}(A) = \mathcal{H}^{2p}_\pm(A)$, the space $\Omega^s(A) + \Omega^{m+n-s}(A)$ is τ-stable and

$$\Omega(A) = \bigoplus_{s=0}^{2p-1} \left(\Omega^s(A) + \Omega^{m+n-s}(A)\right) \bigoplus \Omega^{2p}(A),$$

therefore

$$\Omega(A)_+ = \bigoplus_{s=0}^{2p-1} \left(\Omega^s(A) + \Omega^{m+n-s}(A)\right)_+ \bigoplus \Omega^{2p}(A)_+ .$$

Thirdly, (a)

$$
\begin{aligned}
&\ker\left(D_A\right)_+\\
&=\Omega\left(A\right)_+\cap\ker\left(d_A+d_A^*:\Omega\left(A\right)\rightarrow\Omega\left(A\right)\right)\\
&=\Omega\left(A\right)_+\cap\mathcal{H}\left(A\right)\\
&=\bigoplus\nolimits_{s=0}^{2p-1}\left(\Omega^s\left(A\right)+\Omega^{m+n-s}\left(A\right)\right)_+\bigoplus\Omega^{2p}\left(A\right)_+\\
&\cap\bigoplus\nolimits_{s=0}^{2p-1}\left(\mathcal{H}^s\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)\oplus\mathcal{H}^n\left(A\right)\\
&=\bigoplus\nolimits_{s=0}^{2p-1}\left(\mathcal{H}^s\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)_+\bigoplus\mathcal{H}^{2p}\left(A\right)_+
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\dim\ker\left(D_A\right)_+-\dim\ker\left(D_A\right)_+^*\\
&=\sum_{s=0}^{2p-1}\dim_{\mathbb{R}}\left(\mathcal{H}^s\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)_++\dim_{\mathbb{R}}\mathcal{H}_+^{2p}\left(A\right)\\
&-\sum_{s=0}^{2p-1}\dim_{\mathbb{R}}\left(\mathcal{H}^s\left(A\right)+\mathcal{H}^{m+n-s}\left(A\right)\right)_--\dim_{\mathbb{R}}\mathcal{H}_-^{2p}\left(A\right)\\
&=\dim_{\mathbb{R}}\mathcal{H}_+^{2p}\left(A\right)-\dim_{\mathbb{R}}\mathcal{H}_-^{2p}\left(A\right)\\
&=\operatorname{Sign}\left(\mathcal{B}^n\right).
\end{aligned}
$$

■

Thanks the above Theorem, we can use the Atiyah-Singer formula for calculating the signature of A.

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