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Hirzebruch signature operator, algebraic aspects, applications to Lie algebroids

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Lie algebroids appeared as infinitesimal objects of Lie groupoids, principal fibre bundles, vector bundles (Pradines, 1967), TC-foliations and nonclosed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalences are known as Lie pseudo-algebras (Herz 1953) called also further as Lie-Rinehart algebras (Huebschmann, 1990).

1 Signature of Lie algebroids

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0.$$

The fiber $\boldsymbol{g}_x$ possesses a structure of a Lie algebra, called the isotropy Lie algebra of A at $x \in M$.

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \operatorname{Sec} \bigwedge A^*, \quad \text{- the space of cross-sections of } \bigwedge A^*$$

$$d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

$$\begin{aligned}(d_A\omega)(\xi_0, ..., \xi_k) = & \sum_{j=0}^{k} (-1)^j \left(\#_A \circ \xi_j\right)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ & + \sum_{i<j} (-1)^{i+j} \omega\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k\right),\end{aligned}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \operatorname{Sec} A$. The operators d_A^k satisfy

$$d_A(\omega \wedge \eta) = d_A\omega \wedge \eta + (-1)^k \omega \wedge d_A\eta,$$

so they are of first order and the symbol of d_A^k is equal to

$$S\left(d_A^k\right)_{(x,v)} : \bigwedge^k A_x^* \to \bigwedge^{k+1} A_x^*$$

$$S\left(d_A^k\right)_{(x,v)}(u) = \tilde{v} \wedge u,$$

where for $0 \neq v \in T_x^*M$

$$\tilde{v} : A_x \stackrel{\#_x}{\to} T_xM \stackrel{v}{\to} \mathbb{R}$$

In consequence the sequence of symbols

$$\bigwedge^k A_x^* \stackrel{S\left(d_A^k\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+1} A_x^* \stackrel{S\left(d_A^{k+1}\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+2} A_x^*$$

is exact which imply that

- the complex $\left\{d_A^k\right\}$ is an **elliptic complex.**

Theorem 1 (Kubarski and Mishchenko, CEJM, Vol. 2(5), 2004) *For each transitive Lie algebroid* $(A, [\![\cdot,\cdot]\!], \#_A)$ *over* ***compact oriented*** *manifold* M *the following conditions are equivalent* ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, *i.e.* $\operatorname{rank} A = m + n$)

(1) $\mathbf{H}^{m+n}(A) \neq 0$,

(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$,

(3) A is the so-called *invariantly oriented,* i.e. there exists a global non-singular cross-section $\boldsymbol{\varepsilon}$ of the vector bundle $\bigwedge^n \boldsymbol{g}$,

$$\begin{aligned} \varepsilon &\in \operatorname{Sec}\left(\bigwedge^n \boldsymbol{g}\right) \\ 0 &\neq \varepsilon_x \in \bigwedge^n \boldsymbol{g}_{|x} \end{aligned}$$

invariant with respect to the adjoint representation of A in the vector bundle $\bigwedge^n \boldsymbol{g}$.

(4) the vector bundle $\boldsymbol{g}$ is oriented and the modular class of A vanish, $\theta_A = 0$.

Assume that A is invarianty oriented. The scalar Poincaré product

$$\mathcal{P}_A^i : \mathbf{H}^i(A) \times \mathbf{H}^{m+n-i}(A) \to \mathbb{R},$$

$$([\omega], [\eta]) \longmapsto \int_A^{\#} \omega \wedge \eta \quad \left(:= \int_M \left(\fint_A^{\#} \omega \wedge \eta \right) \right)$$

where the so-called *fibre integral*

$$\fint_A : \Omega^{\bullet}(A) \to \Omega_{dR}^{\bullet - n}(M)$$

is defined by the substitution operator

$$\left(\fint_A \omega^k \right)_x (w_1, ..., w_{k-n}) = (-1)^{kn} \omega_x (\boldsymbol{\varepsilon}_x, \tilde{w}_1, ..., \tilde{w}_{k-n}), \quad \#_A(\tilde{w}_i) = w_i,$$

Thanks to the assumption that ε is invariant, the operator $\fint_A$ commutes with the differentials d_A and d_M (Kubarski, KLUWER ACADEMIC PUBLISHERS, Dordrecht, 1999) giving a homomorphism in cohomology

$$\fint_A^{\#} : \mathbf{H}^{\bullet}(A) \to \mathbf{H}_{dR}^{\bullet - n}(M).$$

In particular, we have

$$\int_A^{\#} : \mathbf{H}^{m+n}(A) \overset{\cong}{\to} \mathbf{H}_{dR}^m(M) = \mathbb{R}.$$

The scalar product $\mathcal{P}_A^i$ is nondegenerated and if

$$m + n = 4k$$

then

$$\mathcal{P}_A^{2k} : \mathbf{H}^{2k}(A) \times \mathbf{H}^{2k}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore (Kubarski, Topology Appl., Vol 121, 3, June 2002) its **signature** is defined and is called the signature of A, and is denoted by

$$\operatorname{Sign}(A).$$

The problem is:

- to calculate the signature $\operatorname{Sign}(A)$ and give some conditions to the equality $\operatorname{Sign}(A) = 0$. There are examples for which $\operatorname{Sign}(A) \neq 0$ (this is announced in the paper by Kubarski and Mishchenko, Doklady Mathematical Sciences, 68, 5/1, 2003).

2 Signature and spectral sequences

Now, I give a general mechanism of the calculation of the signature via spectral sequences (Kubarski and Mishchenko 2003) and use two of spectral sequences associated with Lie algebroids:

a) the **spectral sequence of the Čech-de Rham complex,**

b) the **Hochschild-Serre spectral sequence.**

The idea of applying spectral sequences to the signature comes from

- Chern, Hirzebruch and Serre *On the index of a fibered manifold,* Proc. AMS, 8 (1957),

where the Leray spectral sequence was used to calculation of the signature of the total space E of a fiber bundle $E \to M$ wih the typical fiber F:

Theorem 2 (Chern, Hirzebruch, Serre, Proc. AMS, 8 (1957)) *If the fundamental group $\pi_1(M)$ acts trivially on the cohomology ring $\mathbf{H}^*(F)$ of F then*

$$\operatorname{Sign}(E) = \operatorname{Sign}(F) \cdot \operatorname{Sign}(M).$$

The Chern-Hirzebruch-Serre arguments are purely algebraic and lead to the following general theorem (Kubarski and Mishchenko, Matem. Sbornik 194, No 7, 2003).

Theorem 3 *Let* $\left(\left(A,\langle,\rangle\right),A^{r},\cup,D,A_{j}\right)$ *be any DG-algebra with a decreasing regular filtration* A_j

$$A=A_{0}\supset\cdots\supset A_{j}\supset A_{j+1}\supset\cdots$$

and $(E_s^{p,q},d_s)$ *its spectral sequence. We assume that there exist natural numbers* m *and* n *with the following conditions:*

- $E_2^{p,q}=0$ *for* $p>m$ *and* $q>n$, $m+n=4k$,
- E_2 *is a Poincaré algebra with respect to the total gradation and the top group* $E_2^{(m+n)}=E_2^{m,n}=\mathbb{R}$.

Then all terms $\left(E_s^{(\bullet)},\cup,d_s\right)$ $2\leq s<\infty$, *and* $\left(E_\infty^{(\bullet)},\cup\right)$ *are Poincaré algebras and*

$$\operatorname{Sign}E_{2}=\operatorname{Sign}E_{3}=...=\operatorname{Sign}E_{\infty}=\operatorname{Sign}\mathbf{H}\left(A\right).$$

If m and n are odd then $\operatorname{Sign} E_2 = 0$,
if m and n are even then

$$\begin{aligned} \operatorname{Sign} E_2 &= \operatorname{Sign}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R} \right) \\ &= \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R} \right) \end{aligned}$$

a) **Using the spectral sequences for the Čech-de Rham complex of the Lie algebroid** A.

Given a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in J}$ of M, where J is a countable ordered index set (this means that all U_α and all finite intersections $\bigcap_i U_{\alpha_i}$ are diffeomorphic to an Euclidean space $\mathbb{R}^m$) we can form the double complex (of the Čech-de Rham type)

$$K^{p.q} = C^p\left(\mathfrak{U}, \Omega^q(A)\right) := \prod_{\alpha_0<\ldots<\alpha_p} \Omega^q\left(A_{|U_{\alpha_0\cdots\alpha_p}}\right)$$

$p, q \geq 0$, with the standard product structure $\cup$ and two boundary homomorphisms, d and δ. Now, consider the "horizontal" decreasing filtration K_j, the total differential operator $D = d + \delta$ and the spectral sequence of the graded differential algebras $(K = \oplus K^{p,q}, K_j, \cup, D)$

$$(E_s^{p,q}, d_s),$$

The filtration K_j is **regular,** $K_0 = K$, therefore we can use the general theorem on signature.

Theorem 4 *If the presheaf*

$$\mathcal{H}(A) = \left(U \longmapsto \mathbf{H}\left(A_{|U}\right)\right)$$

is constant on the good covering $\mathfrak{U}$ *(i.e. the monodromy representation* $\rho : \pi_1(M) = \pi_1(N(\mathfrak{U})) \to \mathrm{Aut}(\mathbf{H}(\mathfrak{g}))$ *is trivial) then*

$$E_2^{p,q} \cong \mathbf{H}_{dR}^p(M) \otimes \mathbf{H}^q(\mathfrak{g}).$$

All isomorphisms are canonical isomorphisms of bigraded algebras. E_2 *lives in the rectangle* $p \leq m$, $q \leq n$,

$$E_2^{(m+n)} = E_2^{m,n} = \mathbf{H}_{dR}^m(M) \otimes \mathbf{H}^n(\mathfrak{g}) = \mathbb{R}$$

and

$$\begin{aligned}
\mathrm{Sign}(A) &= \mathrm{Sign}\,\mathbf{H}(A) = \mathrm{Sign}\,E_2 \\
&= \mathrm{Sign}(\mathbf{H}_{dR}(M) \otimes \mathbf{H}(\mathfrak{g})) = \mathrm{Sign}\,\mathbf{H}_{dR}(M) \cdot \mathrm{Sign}\,\mathbf{H}(\mathfrak{g}) \\
&= \mathrm{Sign}\,\mathbf{H}_{dR}(M) \cdot 0 = 0,
\end{aligned}$$

since for a unimodular Lie algebra $\mathfrak{g}$ $\quad \mathrm{Sign}\,\mathbf{H}(\mathfrak{g}) = \mathrm{Sign}\bigwedge \mathfrak{g}^* = 0.$

Example 5 *The condition of the triviality of the monodromy holds if*

- M is simply connected,
- $\operatorname{Aut} G = \operatorname{Int} G$, where G is simply connected Lie group with the Lie algebra $\mathfrak{g}$, for example, if $\mathfrak{g}$ is a simple Lie algebra of type

$$B_l\ (= SO\,(2l+1)),\quad C_l\ (= Sp\,(2l)\ ),\ E_7,\ E_8,\ F_4,\ G_2.$$

- the adjoint Lie algebra bundle $\boldsymbol{g}$ is trivial in the category of flat bundles (the bundle $\mathbf{H}\,(\boldsymbol{g})$ of cohomology of isotropy Lie algebras with the typical fibre $\mathbf{H}\,(\mathfrak{g})$ possess canonical flat covariant derivative - which will be important for studying of the Hochshild-Serre spectral sequence). For example for the Lie algebroid $A\,(G;H)$ of the the TC-foliation of left cosets of a nonclosed Lie subgroup H in any Lie group G.

b) **Using the Hochshild-Serre spectral sequence.**

For a transitive Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \#_A)$ with the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \overset{\#_A}{\to} TM \to 0$ we will consider the pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g}, \mathfrak{k})$ where

$$\mathfrak{g} = \mathrm{Sec}\,(A)\,, \quad \mathfrak{k} = \mathrm{Sec}\,(\boldsymbol{g})\,.$$

Following Mackenzie (1987) (see also Itskov, Karashev, and Vorobjev (1998)), we will consider in the $C^\infty(M)$-module of A-differential forms $\Omega^k(A)$ the Hochschild-Serre decreasing filtration $\Omega_j \subset \Omega(A)$ as follows:

— $\Omega_j = \Omega(A)$ for $j \le 0$,

— if $j > 0$, $\Omega_j = \bigoplus_{k\ge j} \Omega_j^k$, $\Omega_j^k = \Omega_j \cap \Omega^k(A)$, where Ω_j^k consists of all those k-differential forms ω for which

$$\omega(\xi_1, ..., \xi_k) = 0$$

whenever $k - j + 1$ of the arguments ξ_i belongs to $\mathfrak{k}$. We obtain in this way a graded filtered differential space

$$\left(\Omega(A) = \bigoplus_k \Omega^k(A)\,, d_A, \Omega_j\right)$$

and its spectral sequence

$$\left(E_{A,s}^{p,q}, d_{A,s}\right).$$

Assume as above

$$m = \dim M,\ n = \dim \boldsymbol{g}_{|x}, \quad \text{i.e. } \operatorname{rank} A = m + n.$$

Theorem 6 *There is a flat covariant derivative* ∇^q *in the vector bundle* $\mathbf{H}^q(\boldsymbol{g})$ *such that*

$$E_{A,2}^{p,q} \cong \mathbf{H}_{\nabla^q}^p\left(M; \mathbf{H}^q(\boldsymbol{g})\right).$$

The flat covariant derivative ∇^q *is defined by the formula: for* $f \in \Omega^p\left(M; Z\left[\bigwedge^q \boldsymbol{g}^*\right]\right)$, $[f] \in \Omega^p\left(M; \mathbf{H}^q(\boldsymbol{g})\right)$

$$\nabla_X^q [f] = [\mathcal{L}_X f]$$

where $\left(\mathcal{L}_X f\right)(\sigma_1, ..., \sigma_q) = \partial_X\left(f(\sigma_1, ..., \sigma_q)\right) - \sum_{i=1}^q f\left(\sigma_1, ..., [\![\lambda X, \sigma_i]\!], ..., \sigma_q\right)$ *(where* $\lambda : TM \to A$ *is arbitrary auxiliary connection in* A*).*

For a detailed proof see: Kubarski and Mishchenko, CEJM, 2004.

Theorem 7 *If A is a transitive invariantly oriented Lie algebroid such that $m+n=4k$ ($m=\dim M$, $n=\dim \boldsymbol{g}_{|x}$) then*

a) if m and n are odd then $\operatorname{Sign} A=0$,

b) if m and n are even then

$$\begin{aligned} \operatorname{Sign} A &= \operatorname{Sign} E_2 = \operatorname{Sign}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \\ &= \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \end{aligned}$$

where

$$E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}_{\nabla^{\frac{n}{2}}}^{\frac{m}{2}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)$$

and

$$\mathbf{H}_{\nabla^{\frac{n}{2}}}^{\frac{m}{2}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}_{\nabla^{\frac{n}{2}}}^{\frac{m}{2}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbf{H}_{\nabla^{n}}^{m}\left(M;\mathbf{H}^{n}(\boldsymbol{g})\right) = \mathbb{R}$$

is defined via the usual multiplication of differential forms with respect to the multiplication of cohomology classes for isotropy Lie algebras.

$$\phi_x : \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}_x) \times \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}_x) \to \mathbf{H}^{n}(\boldsymbol{g}_x) = \mathbb{R}.$$

We notice that

- if $\frac{n}{2}$ is even then $\frac{m}{2}$ is even, $\dim M = m = 4p$ for some p,
  $\phi : \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \times \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \to \mathbb{R}$ is symmetric and nondegenerated,
- if $\frac{n}{2}$ is odd then $\frac{m}{2}$ is odd, $\dim M = 4p+2$ for some p,
  $\phi : \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \times \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \to \mathbb{R}$ is symplectic (i.e skew-symmetric nondeg.)

However, for each k $\mathbf{H}^{k}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{m-k}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$ is strongly nondegenerated

$$\begin{aligned} \mathbf{H}^{k}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) &= \mathbf{H}^{m-k}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)^{*}, \\ \mathbf{H}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) &= \mathbf{H}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)^{*}, \\ \dim \mathbf{H}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) &< \infty, \end{aligned}$$

and

$$\mathbf{H}_{\nabla}^{\frac{m}{2}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}_{\nabla}^{\frac{m}{2}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$$

is symmetric and nondegenerated. We want to calculate the signature of this pairing.

3 Algebraic Hirzebruch operator and the signature.

3.1 Four foundamental examples

We describe a general approach to the following four fundamental examples of the spaces with gradation and differential operator $\left(W=\oplus_{k=0}^{N}W^{k},d\right)$. In what follows, M is a compact oriented Riemann manifold.

$$W^{k}=\begin{cases}\left(\Omega^{k}\left(M\right),\ d_{dR}\right); \quad N=4p, \quad [\textbf{classical example}]\\ \left(\Omega^{k}\left(A\right),\ d_{A}\right); \quad N=m+n=4p, \quad \begin{array}{l}A\text{ - a transitive inv. or. Lie algebroid}\\ [\textbf{Lie algebroids example}]\end{array}\\ \left(\Omega^{k}\left(M;E\right),\ d_{\nabla}\right); \quad \left(E,(,)_{0}\right)\text{ flat vector bundle,}\\ \qquad (,)_{0}\text{ - symmetric nondegenerated parallel, } N=4p\\ \qquad [\textbf{Lusztig example}]\\ \left(\Omega^{k}\left(M;E\right),\ d_{\nabla}\right); \quad \left(E,\langle,\rangle_{0}\right)\text{ flat vector bundle,}\\ \qquad \langle,\rangle_{0}\text{ -symplectic parallel, } N=4p+2\\ \qquad [\textbf{Gromov example}]\end{cases}$$

In all cases the sequences of differentials $\left\{d_{dR}^k\right\}$, $\left\{d_A^k\right\}$, $\left\{d_\nabla^k\right\}$ are elliptic complexes, $\dim \mathbf{H}^k(W) < \infty$ and the pairing

$$\mathbf{H}^k(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R}$$

is defined, which in the middle degree $\frac{N}{2}$ is symmetric. Its signature is defined to be the signature of W, $\operatorname{Sign}(W)$.

Below, we give a common algebraic approach to calculate the signature $\operatorname{Sign}(W)$ via the Hirzebruch signature operator.

3.2 General approach

We give **algebraic point of view on the $*$-Hodge operator, Hodge Theorem and Hirzebruch operator**

Definition 8 *By a* ***Hodge space*** *we mean the triple* $(W, \langle , \rangle, (,))$ *where* W *is a real vector space* ($\dim W$ *finite or infinite*), $\langle , \rangle$, $(,) : W \times W \to \mathbb{R}$ *are* 2*-linear tensors such that*

(1) $(,)$ *is symmetric and positive definite (i.e. is an inner product),*

(2) there exists a linear homomorphism

$$*_W : W \to W$$

called $*$***-Hodge operator*** *fulfilling properties*

(i) for all $v \in V$,

$$\langle v, w \rangle = (v, *_W (w)),$$

(ii) $*_W$ *is an isometry with respect to* $(,)$, *i.e.*

$$(v, w) = (*_W v, *_W w),$$

Clearly, the $*$-Hodge operator is uniquely determined (if exists).

Lemma 9 *Let $(W, \langle\cdot,\cdot\rangle)$ be a finite dimensional real vector space equipped with a 2-tensor $\langle\cdot,\cdot\rangle$. Then there exists an inner product $(\cdot,\cdot)$ such that the system $(W, \langle,\rangle, (,))$ is a Hodge space if and only if in some basis of W the matrix of $\langle,\rangle$ is orthogonal.*

Lemma 10 *If $(V, \langle\cdot,\cdot\rangle_V, (\cdot,\cdot)_V)$ and $(W, \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_W)$ are Hodge spaces then their tensor product*

$$(V \otimes W, \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_V \otimes (\cdot,\cdot)_W)$$

is a Hodge space and

$$*_{V\otimes W} = *_V \otimes *_W.$$

Remark: Two 2-tensors $f : V \times V \to \mathbb{R}$ i $g : W \times W \to \mathbb{R}$ determine tensor product

$$f \otimes g : (V \otimes W) \times (V \otimes W) \to \mathbb{R}$$

which is 2-linear. The tensor $f \otimes g$ is symmetric and positive definited if both are the same (the dimensions of V and W can be infinite).

Now we give a number of finite domensional Hodge spaces.

Example 11 (Classical) *Let* (V, G) *be a real* N*-dimensional oriented Euclidean space with inner product* $G : V \times V \to \mathbb{R}$ *and the volume tensor* $\varepsilon = e_1 \wedge ... \wedge e_N \in \bigwedge^N V$, *(where* e_i *is a positive ON basis). We identify* $\bigwedge^N V = \mathbb{R}$ *via the isomorphism*

$$\rho : \bigwedge^N V \xrightarrow{\cong} \mathbb{R}, \quad s \cdot \varepsilon \longmapsto s.$$

We have the classical Hodge space

$$\left(\bigwedge V, \langle , \rangle , (,)\right)$$

where $\langle \cdot, \cdot \rangle : \bigwedge V \times \bigwedge V \to \mathbb{R}$,

$$\begin{aligned} \langle \cdot, \cdot \rangle^k &: \bigwedge^k V \times \bigwedge^{N-k} V \to \bigwedge^N V = \mathbb{R}, \\ \left\langle v^k, v^{N-k} \right\rangle &= \rho\left(v^k \wedge v^{N-k}\right), \end{aligned}$$

be the usual duality (we put $\langle , \rangle = 0$ *outside the pairs* $(k, N-k)$ *),*

$$(\cdot, \cdot)^k : \bigwedge^k V \times \bigwedge^k V \to \mathbb{R}, \qquad (v_1 \wedge ... \wedge v_k, w_1 \wedge ... \wedge w_k)^k = \det\left[G\left(v_i, w_k\right)\right],$$

the subspaces $\bigwedge^k V,\ k=0,1,...,N$ *are orthogonal (by definition).*

The $*$*-Hodge operator exists and it is determined via an ON base* e_i *by the formula*

$$*\left(e_{i_1}\wedge...\wedge e_{i_k}\right)=\varepsilon_{(j_1,...,j_{n-k})}\cdot e_{j_1}\wedge...\wedge e_{j_{n-k}}$$

where $i_1<...<i_k$ *and* $j_1<...<j_{n-k}$ *and the sequence* $(j_1,...,j_{n-k})$ *is complementary to* $(i_1,...,i_k)$ *and* $\varepsilon_{(j_1,...,j_{n-k})}=\operatorname{sgn}\left(j_1,...,j_{n-k},i_1,...,i_k\right).$

The above can be used

- *for* $V=T_xM$ *or* $V=T_x^*M$ *where* M *is a Riemnann manifold (tensor metryczny* G_x *indukuje iloczyn skalarny w przestrzeni dualnej* T_x^*M*).*
- *for* $V=A_x$ *where* A *is a transitive invariantly oriented Lie algebroid over a Riemann manifold (see below).*

Example 12 (Lusztig example, 1972) *[L]Let* $(,)_0 : E \times E \to \mathbb{R}$ *be a symmetric (indefinite) nondegenerated tensor on a finite dimensional vector space* E. *We fix some positive definite scalar product* $(,)'$ *on* E. *Then we take a unique splitting* $E = E_+ \oplus E_-$ *which is both* $(,)_0$ *and* $(,)'$ *orthogonal and such that* $(,)_0$ *on* E_+ *is positive and* $(,)_0$ *on* E_- *is negative. We denote by* τ *the involution* $\tau : E \to E$ $(\tau^2 = id)$ *such that* $\tau|E_+ = id$, $\tau|E_- = -id$. *Then, the quadratic form*

$$(v, w) = (v, \tau w)_0$$

is symmetric positive definite. The involution τ *is the* $*$*-Hodge operator in* $(E, (,)_0, (,))$, *i.e.*

$$(v, w)_0 = (v, \tau w),$$

and is an isometry

$$(\tau v, \tau w) = \left(\tau v, \tau^2 w\right)_0 = (\tau v, w)_0 = (w, \tau v)_0 = (w, v).$$

Therefore $(E, (,)_0, (,))$ *is a Hodge-space.*

Example 13 (Gromov example, 1995) *[G]Let* $\langle,\rangle_0 : E \times E \to \mathbb{R}$, *be a skew-symmetric nondegerated tensor on a finite dimensional vector space* E.

There exists an anti-involution τ in E, $\tau^2 = -id$ (i.e. a complex structure) such that

$$\langle \tau v, \tau w \rangle_0 = \langle v, w \rangle_0 , v, w \in E,$$

$$\langle v, \tau v \rangle_0 > 0 \qquad dla \qquad v \neq 0.$$

Then the tensor

$$\begin{aligned} (,) &: \quad E \times E \to \mathbb{R} \\ (v, w) &: \quad = \langle v, \tau w \rangle_0 \end{aligned}$$

is symmetric and positive defined and

$$(\tau v, \tau w) = (v, w) ,$$

i.e. τ preserves both forms $\langle , \rangle_0$ and $(,)$. The system $(E, \langle , \rangle_0 , (,))$ is a Hodge-space since the operator $-\tau$ is the $*$-Hodge operator

$$\langle v, w \rangle_0 = \left\langle v, -\tau^2 w \right\rangle_0 = \langle v, \tau (-\tau w) \rangle_0 = (v, -\tau w) ,$$

and $-\tau$ is an isometry $(-\tau v, -\tau w) = (\tau v, \tau w) = (v, w)$.

Example 14 (Important example) *Consider an arbitrary Riemann oriented manifold M of dimension N and a vector bundle ξ of Hodge spaces*

$$\left(\xi_x, \langle , \rangle_x , (,)_x\right)$$

where $\langle , \rangle$ and $(,)$ are $C^\infty(M)$ 2-tensor fields. Then for any point $x \in M$ we take the tensor product of Hodge spaces $\bigwedge T_x^* M \otimes \xi_x$. Assuming compactness of M we can define two 2-$\mathbb{R}$-linear tensors

$$((\alpha, \beta)), \ \langle\langle \alpha, \beta \rangle\rangle : \Omega(M; \xi) \times \Omega(M; \xi) \to \mathbb{R},$$

by integrating along the Riemannian manifold

$$((\alpha, \beta)) = \int_M (\alpha_x, \beta_x) \, dM, \qquad \langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha_x, \beta_x \rangle \, dM = \int_M \alpha \wedge_\varphi \beta$$

where

$$\varphi_x = \langle \cdot, \cdot \rangle_x^k : \bigwedge^k T_x^* M \otimes \xi_x \times \bigwedge^{N-k} T_x^* M \otimes \xi_x \to \bigwedge^N T_x^* M = \mathbb{R}$$

is the wedge product with respect to the multiplication $\langle , \rangle_x$ of the values. The 2-form $((\cdot, \cdot))$ is symmetric and positive definite and the triple

$$\left(\Omega(M; W), \langle\langle \alpha, \beta \rangle\rangle, ((\alpha, \beta))\right)$$

is a Hodge space with the $*$-Hodge operator $\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$ defined point by point

$$(*\beta)_x = *_x (\beta_x)$$

gdzie $*_x$ jest iloczynem tensorowym zwykłej gwiazdki Hodge w $\bigwedge T_x^* M$ i gwiazdki w ξ_x.

Definition 15 *By a **Hodge space with gradation and differential operator** we mean the system*

$$\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), d \right)$$

where $(W, \langle \cdot, \cdot \rangle, (\cdot, \cdot))$ is a Hodge space and

(1) $\langle \cdot, \cdot \rangle^k : W^k \times W^{N-k} \to \mathbb{R}$ and $\langle , \rangle = 0$ outside the pairs $(k, N-k)$,

(2) W^k are orthogonal with respect to $(\cdot, \cdot)$,

(3) d is homogeneous of degree $+1$, i.e. $d : W^k \to W^{k+1}$, and $d^2 = 0$,

(4) $\langle dw, u \rangle = (-1)^{k+1} \langle w, du \rangle$ for $w \in W^k$.

Clearly,

a) the pairing

$$\langle , \rangle_{\mathbf{H}}^{k} : \mathbf{H}^{k}(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R},$$
$$\langle [u], [w] \rangle_{\mathbf{H}}^{k} = \langle u, w \rangle^{k},$$

is correctly defined,

b) $* \left[W^k \right] \subset W^{N-k}$, and $\quad * : W^k \to W^{N-k} \quad$ is an isomorphism.

Assume that W is a **Hodge space with gradation and differential operator,** and let $d^* : W \to W$ be the adjoint operator with respect to $(,)$, i.e. the one such that

$$(d^*(w_1), w_2) = (w_1, d(w_2)).$$

Assume that d^* exists.

Lemma 16 *The operator (called the Laplacian)*

$$\Delta := (d + d^*)^2 = dd^* + d^* d$$

is self-adjoint $(\Delta v, w) = (v, \Delta w)$, *nonnegative* $(\Delta v, v) \geq 0$, *and*

$$\{v \in W;\ (\Delta v, v) = 0\} = \{v \in W;\ dv = 0 = d^* v\}.$$

Definition 17 *A vector $v \in W$ is called **harmonic** if $dv = 0$ i $d^*v = 0$.*

$$\mathcal{H}(W) = \{v \in W;\ dv = 0,\ d^*v = 0\},$$
$$\mathcal{H}^k(W) = \left\{v \in W^k;\ dv = 0,\ d^*v = 0\right\}.$$

The harmonic vectors form a graded vector space

$$\mathcal{H}(W) = \bigoplus_{k=0}^{N} \mathcal{H}^k(W).$$

Lemma 18 $\mathcal{H}^k(W) = \ker\left\{d + d^* : W^k \to W\right\} = \ker\left\{\Delta^k : W^k \to W^k\right\}$, *i.e.*

$$\mathcal{H}(W) = \ker \Delta.$$

is the eigenspace of the operator Δ corresponding to the zero value of the eigenvalue.

Lemma 19

$$\mathcal{H}(W) = \ker \Delta = (\operatorname{Im} \Delta)^{\perp}.$$

Proof. Simple calculations. ∎

Remark 20 *If* W *is a Hilbert space and* $Y \subset W$ *is closed, then we have the direct sum* $\boxed{W = Y \bigoplus (Y)^{\perp}.}$

For a Riemannian vector bundle ξ *over a Riemannian manifold, the space* $W = \mathrm{Sec}\,(\xi)$ *is not a Hilbert one (is not complete).*

But we have the well known

Theorem 21 *Let ξ be a Riemannian vector bundle over a compact oriented Riemann manifold M. If $\Delta : \operatorname{Sec}\xi \to \operatorname{Sec}\xi$ is a* ***self-adjoint elliptic operator*** *then* $\ker\Delta = (\operatorname{Im}\Delta)^{\perp}$ *and*

$$\operatorname{Sec}\xi = \operatorname{Im}\Delta \bigoplus \ker\Delta = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}.$$

In the sequel W is an arbitrary Hodge space with gradation and differential operator d.

Theorem 22 *(a)* $\ker \Delta^k$ *and* $\operatorname{Im} d^{k-1}$ *are orthogonal, therefore the inclusion*

$$\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$$

induce a monomorphism

$$\ker \Delta^k \rightarrowtail \mathbf{H}^k(W).$$

(b) ***Hodge Theorem.*** *If* $\boxed{W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}}$ *then*

$$\ker d^k = \ker \Delta^k \bigoplus \operatorname{Im} d^{k-1}.$$

Therefore the above inclusion $\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$ *induces the isomorphism*

$$\mathcal{H}^k(W) \cong \ker d^k / \operatorname{Im} d^{k-1} = \mathbf{H}^k(W).$$

It means that in each cohomology class there is exactly one harmonic vector.

Lemma 23 *Let* $\left(W=\bigoplus_{k=0}^{N}W^{k},\langle\cdot,\cdot\rangle,(\cdot,\cdot),d\right)$ *be a Hodge space with gradation and differential operator. Let* $\varepsilon:\{0,1,...,N\}\rightarrow\{-1,1\}$ *be an arbitrary function. If the 2-tensor* $\langle,\rangle$ *is* ε*-antycommutative, i.e.*

$$\left\langle v^{k},v^{N-k}\right\rangle=\varepsilon_{k}\left\langle v^{N-k},v^{k}\right\rangle$$

for $v^{k}\in W^{k}$, $v^{N-k}\in W^{N-k}$, *then*

(a) the adjoint operator d^{*} *exists and is given by*

$$d^{*}\left(w^{k}\right)=\varepsilon_{k}\left(-1\right)^{k}*d*\left(w^{k}\right),\quad w^{k}\in W^{k},$$

where $*$ *is the* $*$*-Hodge operator in* W.

(b)

$$**\left(w^{k}\right)=\varepsilon_{k}\cdot w^{k},$$

(c) if $\varepsilon_{k-1}=\varepsilon_{k+1}$ *then* $*\Delta=\pm\Delta*$, *to be precise*

$$*\Delta w^{k}=\varepsilon_{k-1}\varepsilon_{k}\left(-1\right)^{N+1}\Delta*w^{k},$$

therefore

$$*\left[\mathcal{H}^{k}\left(W\right)\right]\subset\mathcal{H}^{N-k}\left(W\right),$$

and

$$*: \mathcal{H}^k(W) \to \mathcal{H}^{N-k}(W)$$

is an isomorphism.

As a corollary we have

*(d) (**Duality Theorem**) If additionally* $\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}}$ *then*

$$\mathbf{H}^k(W) \simeq \mathbf{H}^{N-k}(W).$$

We restrict the scalar positive product $(\cdot,\cdot): W^k \times W^k \to \mathbb{R}$ to the space of harmonic vectors

$$(\cdot,\cdot): \mathcal{H}^k(W) \times \mathcal{H}^k(W) \to \mathbb{R},$$

and we restrict the tensor $\langle\cdot,\cdot\rangle: W^k \times W^{N-k} \to \mathbb{R}$ to harmonic vectors

$$\mathcal{B}^k = \langle\cdot,\cdot\rangle: \mathcal{H}^k(M) \times \mathcal{H}^{N-k}(M) \to \mathbb{R}.$$

Assumption. From the point of view of the signature we need to consider that N is even,

$$N = 2n \quad \text{and} \quad \varepsilon_n = +1,$$

then

$$\langle , \rangle^n : W^n \times W^n \to \mathbb{R}$$

and

$$\mathcal{B}^n : \mathcal{H}^n (W) \times \mathcal{H}^n (W) \to \mathbb{R}$$

are symmetric. Therefore in cohomology, the tensor

$$\langle , \rangle^n_{\mathbf{H}} : \mathbf{H}^n (W) \times \mathbf{H}^n (W) \to \mathbb{R}$$

is also symmetric. If

$$\dim \mathbf{H}^n (W) < \infty$$

we define the signature of W as the signature of $\langle , \rangle^n_{\mathbf{H}}$

$$\operatorname{Sign} (W) := \operatorname{Sign} \langle , \rangle^n_{\mathbf{H}} .$$

- **Under the assumption** $\boxed{W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}}$ we have $\mathcal{H}^n (W) \cong \mathbf{H}^n (W)$ and

$$\mathcal{B}^n = \langle , \rangle^n_{\mathbf{H}} .$$

Therefore if $\dim \mathbf{H}^n(W) < \infty$ then

$$\operatorname{Sign}(W) = \operatorname{Sign}\langle,\rangle^n_{\mathbf{H}} = \operatorname{Sign}\mathcal{B}^n.$$

In the construction of the Hirzebruch signature operator [for $N = 2n$] the fundamental role is played by an operator (small modification of the $*$-Hodge operator)

$$\begin{aligned} \tau &: W \to W, \\ \tau^k &: W^k \to W^{N-k}, \\ \tau^k(w) &= \tilde{\varepsilon}_k \cdot *w, \quad \tilde{\varepsilon}_k \in \{-1, 1\}. \end{aligned}$$

such that

i) $\tau \circ \tau = Id$,

ii) $d^* = -\tau \circ d \circ \tau$,

iii) $\tau^n = *$, i.e. $\tilde{\varepsilon}_n = 1$, $\quad (N = 2n)$.

We check the existence of τ.

Theorem 24 *If $N = 2n$ and*

$$\varepsilon_k = (-1)^n (-1)^{k(N-k)} = (-1)^n (-1)^k ,$$

then the operator τ exists and it is given by

$$\tau^k (w) = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{n(n+1)}{2}} \cdot *w.$$

(Conversely, if $d^k \neq 0$ for all $k = 0, 1, ..., N-1$ and τ exists then ε_k must be given by the above formula).

Assume $N = 2n,\ \varepsilon_k = (-1)^n (-1)^k$, and take the operator τ. We put

$$W_\pm = \{w \in W;\ \tau w = \pm w\} ,$$

the eigenspaces corresponding to the eigenvalues $+1$ i -1 of the operator τ.

- We notice that

$$(d + d^*) [W_+] \subset W_-.$$

Definition 25 *The operator*

$$D_+ = d + d^* : W_+ \to W_-$$

is called the ***Hirzebruch operator*** *(or the signature operator).*

Remark 26 *If* $\dim \mathbf{H}(W) < \infty$ *then the index*

$$\operatorname{Ind} D_+ = \dim_{\mathbb{R}} \ker(D_+) - \dim_{\mathbb{R}} \ker\left(D_+^*\right)$$

is correctly defined (the dimensions are finite)

$$\ker(D_+) = W_+ \cap \mathcal{H}(W)$$

and analogously for the adjoint operator $(D_+)^* = D_- : W_- \to W_+$

$$\ker(D_-) = W_- \cap \mathcal{H}(W).$$

Theorem 27 (Hirzebruch Theorem on signature) *If* $\dim \mathbf{H}(W) < \infty$ *then*

$$\operatorname{Ind} D_+ = \operatorname{Sign}(\mathcal{B}^n : \mathcal{H}^n(W) \times \mathcal{H}^n(W) \to \mathbb{R}).$$

If additionally $\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^\perp}$ *then* $\operatorname{Ind} D_+ = \operatorname{Sign} W$.

3.3 General setting of the above four examples

We give some applications of the above theory and theorems to vector bundles over manifolds. The other applications to more general objects than manifolds are available.

Example 28 *Consider a graded vector bundle* $\xi = \bigoplus_{k=0}^{N} \xi^k$ *of Hodge spaces over a compact oriented Riemann manifold* M*,*

$$\left(\xi = \bigoplus_{k=0}^{N} \xi^k, \langle , \rangle , (,) , d\right),$$

1) $\langle , \rangle , (,)$ *are fields of smooth 2-tensors in* ξ *such that*

$$(\xi_x, \langle , \rangle_x , (,)_x)$$

is a Hodge space, $x \in M$*, with a* $*$*-Hodge operator* $*_x : \xi_x \to \xi_x$*, and assume that*

$\langle v, w \rangle = 0$ *if* $v \in \xi^r$*,* $w \in \xi^s$*,* $r + s \neq N$*, and that*

subbundles ξ^k *are orthogonal,*

2) the axiom ε-anticommutativity holds

$$\left\langle v^{k},v^{N-k}\right\rangle =\varepsilon_{k}\left\langle v^{N-k},v^{k}\right\rangle ,$$

where

$$\varepsilon_{k}\in\{-1,+1\}.$$

By integration along M we define 2-linear tensors

$$\begin{gathered}\langle\langle,\rangle\rangle,((,)):\mathrm{Sec}(\xi)\times\mathrm{Sec}(\xi)\rightarrow\mathbb{R},\\ \langle\langle\alpha,\beta\rangle\rangle:=\int_{M}\langle\alpha_{x},\beta_{x}\rangle\,dM\\ ((\alpha,\beta)):=\int_{M}(\alpha_{x},\beta_{x})\,dM.\end{gathered}$$

Then $((,))$ is an positive definite scalar product in $\mathrm{Sec}(\xi)$, the $*$-Hodge operator is an isometry

$$((\alpha,\beta))=((*\alpha,*\beta))$$

and

$$\langle\langle\alpha,\beta\rangle\rangle=((\alpha,*\beta)).$$

3) d *is a differential in* $\mathrm{Sec}(\xi)$, $d^2 = 0$, *of the degree* $+1$,

$$d^k : \mathrm{Sec}\left(\xi^k\right) \to \mathrm{Sec}\left(\xi^{k+1}\right),$$

such that, by definition

3a) d^k *are differential operators of first order,*

3b) $\langle\langle dw, u\rangle\rangle = (-1)^{k+1}\langle\langle w, du\rangle\rangle$ *for* $w \in \mathrm{Sec}\left(\xi^k\right)$, $u \in \mathrm{Sec}(\xi)$.

Summing up,

$$\left(\mathrm{Sec}(\xi) = \bigoplus_{k=0}^{N} \mathrm{Sec}(\xi)^k, \langle\langle\cdot,\cdot\rangle\rangle, ((\cdot,\cdot)), d\right)$$

is a Hodge space with gradation and differential operator.

Then the adjoint operator $d^* : \mathrm{Sec}(\xi) \to \mathrm{Sec}(\xi)$

$$((\alpha, d^*\beta)) = ((d\alpha, \beta))$$

exists and $d^*\left(\alpha^k\right) = \varepsilon_k (-1)^k * d * \left(\alpha^k\right)$.

Question. When the Laplacian $\Delta = (d + d^*)^2 = dd^* + d^*d$ is elliptic?

Theorem 29 *If $\{d^k\}$ is an elliptic complex, i.e., is exact on the level of symbols ($0 \neq v \in T_x^* M$)*

$$\xi_x^k \xrightarrow{S(d^k)_{(x,v)}} \xi_x^{k+1} \xrightarrow{S(d^{k+1})_{(x,v)}} \xi_x^{k+2}$$

then the Laplacian Δ is elliptic. In consequence

$$\mathrm{Sec}\,(\xi) = \mathrm{Im}\,\Delta \bigoplus (\mathrm{Im}\,\Delta)^{\perp}$$

and if we assume that $N = 2n$ and $\varepsilon_k = (-1)^n (-1)^k$ and

$$\dim \mathbf{H}^{\bullet} (\mathrm{Sec}\,(\xi)\,, d) < \infty$$

we get the Hirzebruch operator $D_+ = d + d^* : \mathrm{Sec}\,(\xi)_+ \to \mathrm{Sec}\,(\xi)_-$ and the equality

$$\mathrm{Sig}\,\langle\langle,\rangle\rangle_{\mathbf{H}}^n = \mathrm{Ind}\, D_+.$$

In all four above examples the complexes of differentials, $\{d^k\}$, $\{d_A^k\}$, $\{d_\nabla^k\}$ are elliptic, since the sequnces of symbols are exact.

APPLICATIONS.

We describe four fundamental examples of the Hodge space with gradation and differential operator. The fundamental idea is as follows: we have a 2-tensor $\langle , \rangle$ and we want to find a scalar positive tensor $(,)$ such that the $*$-Hodge operator exists and is an isometry.

Example 30 *1. (standard) M is compact oriented Riemannian manifold,*

$$\dim M = 4p.$$

$W^k = \Omega^k(M) = \mathrm{Sec}\left(\bigwedge^k T^*M\right),$

$\langle\langle , \rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta.$

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge \beta = (-1)^{k(N-k)} \int_M \beta \wedge \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}.$$

In the middle degree $2p$, the tensor $\langle\langle , \rangle\rangle$ is symmetric.

$d : W^k \to W^{k+1}$ is the differntiation of differential forms.

$\langle\langle d\alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$ for $\alpha \in W^k$, $\beta \in W^{N-(k+1)}$ (which follows from $\int_M d(\alpha \wedge \beta) = 0$).

With respect to the standard defined inner product in $\bigwedge T_x^* M$ *we have a finite dimensional Hodge-space*

$$\left(\bigwedge T_x^* M, \langle , \rangle_x , (,)_x\right).$$

By integrating along the Riemannian manifold M *we obtain* 2*-linear tensors*

$$\langle\langle , \rangle\rangle \, , \ ((,)) : \Omega\left(M\right) \times \Omega\left(M\right) \to \mathbb{R},$$

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \, dM = \int_M \alpha \wedge \beta, \quad ((\alpha, \beta)) = \int_M (\alpha, \beta) \, dM$$

and the equality

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$$

holds giving a graded Hodge-space with differentaial operator $\left(\Omega\left(M\right), \langle\langle , \rangle\rangle , ((,)) , d\right).$ *The signature* $\mathrm{Sign}\, M = \mathrm{Sign}\, \langle\langle , \rangle\rangle_{\mathbf{H}}^{2p}$ *can be calculated as the index of the Hirzebruch operator.*

$$D_+ = d_{dR} + d_{dR}^* : \Omega\left(M\right)_+ \to \Omega\left(M\right)_-$$

(d_{dR}^* *is the adjoint operator to* d_{dR} *with respect to the scalar product* $((,))$*).*

Example 31 *2. (NEW) Let A be a transitive Lie algebroid over a compact oriented manifold M and let*

$$\operatorname{rank} A = N = 4p = m + n, \qquad m = \dim M, \quad n = \dim \mathbf{g}_{|x}.$$

We assume that A is invariantly oriented via a volume tensor

$$\boldsymbol{\varepsilon} \in \operatorname{Sec}\left(\bigwedge^{n} \mathbf{g}\right)$$

invariant with respect to the adjoint representation Ad_A.

$W^k = \Omega^k(A) = \operatorname{Sec}\left(\bigwedge^k A^*\right)$,

$\langle\langle , \rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \fint_A \alpha \wedge \beta.$

$$\langle\langle \alpha, \beta\rangle\rangle^k = \int_M \fint_A \alpha \wedge \beta = (-1)^{k(N-k)} \int_M \fint_A \beta \wedge \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha\rangle\rangle^{N-k}.$$

This tensor is symmetric in the middle degree

$$\langle\langle , \rangle\rangle^{2p} : W^{2p} \times W^{2p} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \fint_A \alpha \wedge \beta.$$

$$\langle\langle \alpha, \beta\rangle\rangle^{2p} = (-1)^{2p} \langle\langle \beta, \alpha\rangle\rangle^{2p} = \langle\langle \beta, \alpha\rangle\rangle^{2p}.$$

$d_A : W^k \to W^{k+1}$ is the differentiation of A-differential forms,
$\langle\langle d_A\alpha, \beta\rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d_A\beta\rangle\rangle$ for $\alpha \in W^k$, $\beta \in W^{N-(k+1)}$.
Now we find a scalar product $((,))$ in $W = \Omega(A)$ such that the

$$(\Omega(A), \langle\langle,\rangle\rangle, ((,)))$$

is a graded Hodge space with differential. To this aim consider

— any Riemannian tensor G_1 in the LAB $\boldsymbol{g} = \ker \#_A$ for which $\boldsymbol{\varepsilon}$ is the volume tensor (such a Riemannian tensor exists).

— any Riemannian tensor G_2 on M.

Next, taking an arbitrary connection

$$\lambda : TM \to A \qquad (\#_A \circ \lambda = id_{TM})$$

i.e. a splitting of the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \underset{\lambda}{\overset{\#_A}{\longrightarrow}} TM \longrightarrow 0,$$

and the horizontal space

$$\begin{aligned} H &= \operatorname{Im}\lambda, \\ A &= \boldsymbol{g} \bigoplus H \end{aligned}$$

we define a Riemannian tensor G *on* $A = \boldsymbol{g} \bigoplus H$ *such that* $\boldsymbol{g}$ *and* H *are orthogonal, on* $\boldsymbol{g}$ *we have* G_1 *but on* H *we have the pullback* $\lambda^* G_2$*. The vector bundle* A *is oriented (since* $\boldsymbol{g}$ *and* M *are oriented).*

At each point $x \in M$ *we consider the defined above scalar product* G_x *on* $A_{|x}$ *and the multiplication of tensors*

$$\langle , \rangle_x^k : \bigwedge^k A_x^* \times \bigwedge^{N-k} A_x^* \to \bigwedge^N A_x^* \stackrel{\rho_x}{\to} \mathbb{R}$$

where ρ_x *is defined via the volume form for* G_x*.*

We can notice that ρ_x *is the composition*

$$\begin{array}{llll} \bigwedge^N A_x^* & & \stackrel{\rho_x}{\to} & \mathbb{R} \\ \downarrow (-1)^{Nn} i_{\boldsymbol{\varepsilon}_x} & \int_{A_p} \searrow & & \nearrow \rho_{G_2 x} \\ \bigwedge^m A_x^* & \stackrel{\rho_{1x}}{\longrightarrow} & \bigwedge^m T_x^* M & \end{array}$$

The scalar product G_x *in* A_x *we extend to a scalar product in* $\bigwedge A_x^*$ *and we can notice that we obtain the classical finite dimensional Hodge-space*

$$\left(\bigwedge A_x^*, \langle , \rangle_x , (,)_x \right).$$

We obtain two $C^\infty(M)$*-tensors*

$$\langle , \rangle , (,) : \Omega(A) \times \Omega(A) \to C^\infty(M)$$

defined as above point by point. Integrating along M *we get a graded Hodge-space with differential operator*

$$\left(\Omega(M), \langle\langle , \rangle\rangle, ((,)), d_A\right)$$

and

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle = \int_M \not\!\!\int_A \alpha \wedge \beta.$$

The tensor $\langle\langle , \rangle\rangle$ *induces a 2-tensor in cohomology*

$$\langle\langle , \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$$

which in the middle degree

$$\langle\langle , \rangle\rangle_{\mathbf{H}}^{2p} : \mathbf{H}^{2p}(M) \times \mathbf{H}^{2p}(M) \to \mathbb{R}$$

is symmetric. The dimension $\dim \mathbf{H}(A)$ *is finite (Kubarski, Mishchenko, 2003).*

Therefore, the signature of $\langle\langle , \rangle\rangle_{\mathbf{H}}^{2r}$ *can be calculated as the index of the Hirzebruch operator*

$$D_{+} = d_A + d_A^{*} : \Omega(A)_{+} \to \Omega(A)_{-}$$

where d_A^{*} *is adjoint to* d_A *with respect to the scalar product* $((,))$.

By the facts given at the beginning - concerning the calculation of the signature of a Lie algebroid via **Hochschild-Serre spectral sequence** of A - we can calculate the signature of a Lie algebroid using a different Hirzebruch operator following the Lusztig and Gromov examples.

Example 32 ***Lusztig (1971), Gromov (1995). Signature for flat bundles.*** *Let M be a compact oriented $N = 4p$-dimensional manifold and $E \to M$ a flat bundle equipped with a flat covariant derivative ∇ and nondegenerated indefinite symmetric tensor*

$$G_0 = (,)_0 : E \times E \to M \times \mathbb{R}, \quad (,)_{0x} : E_x \times E_x \to \mathbb{R},$$

constant for ∇, i.e. satisfying $\partial_X (\sigma, \eta) = (\nabla_X \sigma, \eta) + (\sigma, \nabla_X \eta)$.

$W^k = \Omega^k (M; E)$,

the differential operator $d_\nabla : W^k \to W^{k+1}$ defined standartly via ∇. From $\nabla G_0 = 0$ we have

$$d (\alpha \wedge_{G_0} \beta) = d_\nabla \alpha \wedge_{G_0} \beta + (-1)^{|\alpha|} (\alpha \wedge_{G_0} d_\nabla \beta)$$

therefore if $|\alpha| + |\beta| = N - 1$ then

$$\int_M (d_\nabla \alpha) \wedge_{G_0} \beta = - (-1)^{|\alpha|} \int_M \alpha \wedge_{G_0} d_\nabla \beta. \tag{1}$$

Define the duality

$$\begin{aligned}\langle\langle\alpha,\beta\rangle\rangle^k &: W^k \times W^{N-k} \to \mathbb{R} \\ \langle\langle\alpha,\beta\rangle\rangle^k &= \int_M \alpha \wedge_{G_0} \beta.\end{aligned}$$

and we see that

$$\langle\langle d_\nabla \alpha, \beta\rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta\rangle\rangle$$

is fulfilled. Since G is symmetric we have

$$\alpha \wedge_{G_0} \beta = (-1)^{k(N-k)} \beta \wedge_{G_0} \alpha$$

and

$$\langle\langle\alpha,\beta\rangle\rangle^k = \int_M \alpha \wedge_G \beta = (-1)^{k(N-k)} \int_M \beta \wedge_G \alpha = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle\beta,\alpha\rangle\rangle^{N-k}.$$

The tensor is symmetric in the middle degree

$$\langle\langle,\rangle\rangle^{2p} : W^{2p} \times W^{2p} \to \mathbb{R}, \quad (\alpha,\beta) \longmapsto \int_M \alpha \wedge_{G_0} \beta.$$

$$\langle\langle\alpha,\beta\rangle\rangle^{2p} = (-1)^{2p} \langle\langle\beta,\alpha\rangle\rangle^{2p} = \langle\langle\beta,\alpha\rangle\rangle^{2p}.$$

We find a scalar product $((,))$ *in* W^k *for which* $(W, \langle\langle,\rangle\rangle, ((,)))$ *is a Hodge space. To this aim we fix some positive definite scalar product* $(,)'$ *on the vector bundle* E*. Then we take a unique splitting* $E = E_+ \oplus E_-$ *which is both* $(,)_0$ *and* $(,)'$ *orthogonal and such that* $(,)_0$ *on* E_+ *is positive and* $(,)_0$ *on* E_- *is negative. We denote by* τ *the involution* $\tau : E \to E$ $(\tau^2 = id\ )$ *such that* $\tau|E_+ = id$, $\tau|E_- = -id$*. Then, the quadratic form*

$$(v, w) = (v, \tau w)_0$$

is symmetric positive definite. The involution τ *is the* $*$*-Hodge operator in* $(E_x, (,)_{0x}, (,)_x)$, *i.e.*

$$(v, w)_0 = (v, \tau w)\,,$$

and is an isometry

$$(\tau v, \tau w) = \left(\tau v, \tau^2 w\right)_0 = (\tau v, w)_0 = (w, \tau v)_0 = (w, v)\,.$$

Therefore $(E_x, (,)_{0x}, (,)_x)$ *is a Hodge-space.*

In each fibre $\bigwedge T_x^* M \bigotimes E_x$ *we introduce the tensor product of Hodge-spaces: the classical one* $\bigwedge T_x^* M$ *and the above* E_x*.*

Point by point we obtain tensors

$$\begin{aligned}
\langle , \rangle &: \Omega(M;E) \times \Omega(M;E) \to C^\infty(M), \\
(,) &: \Omega(M;E) \times \Omega(M;E) \to C^\infty(M), \\
* &: \Omega(M;E) \to \Omega(M;E) \\
\langle \alpha, \beta \rangle &= (\alpha, *\beta)
\end{aligned}$$

and integrating along M we obtain a Hodge-space $(\Omega(M;E), \langle\langle , \rangle\rangle, ((,)))$ where

$$\begin{aligned}
\langle\langle \alpha, \beta \rangle\rangle &= \int_M \langle \alpha, \beta \rangle \, dM = \int_M \alpha \wedge_G \beta, \\
((\alpha, \beta)) &= \int_M (\alpha, \beta) \, dM
\end{aligned}$$

and

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta)).$$

Let d_∇^* be the adjoint operator to d_∇ with respect to $((,))$. The tensor $\langle\langle , \rangle\rangle$ induce a 2-tensor in cohomology $\langle\langle , \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$ which in the middle degree

$$\langle\langle , \rangle\rangle_{\mathbf{H}}^{2p} : \mathbf{H}^{2p}(M;E) \times \mathbf{H}^{2p}(M;E) \to \mathbb{R}$$

is symmetric and the signature of it is the index of the Hirzebruch operator

$$D_+ = d_\nabla + d_\nabla^* : W_+ \to W_-$$

where $W_\pm = \{\alpha \in \Omega(M;E);\ \tau\alpha = \pm\alpha\}$ *for*

$$\tau\left(\alpha^k\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^p * \left(\alpha^k\right).$$

Example 33 ***Gromov (1995) [G].*** *Let M be a compact oriented manifold M of dimension $\dim M = N = 4p+2$ and let $E \to M$ be a symplectic vector bundle equipped with a flat covatiant derivative ∇ and parallel symplectic structure [i.e. skew symmetric nondegenerated] $S = \langle , \rangle : E \times E \to M \times \mathbb{R}$, $\langle , \rangle_x : E_x \times E_x \to \mathbb{R}$, $\nabla S = 0$.*

$W^k = \Omega^k(M;E)$,

$d_\nabla : W^k \to W^{k+1}$ *- the differential operator defined via* ∇.

The condition

$$\int_M (d_\nabla \alpha) \wedge_S \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge_S d_\nabla \beta. \tag{2}$$

holds for $|\alpha| + |\beta| = N - 1$.

$\langle\langle\alpha,\beta\rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}$ is defined by

$$\langle\langle\alpha,\beta\rangle\rangle^k = \int_M \alpha \wedge_S \beta$$

and $\langle\langle d_\nabla \alpha,\beta\rangle\rangle = (-1)^{k+1} \langle\langle\alpha, d\beta\rangle\rangle$ is fulfilled. Since S is skewsymmetric, then

$$\alpha \wedge_S \beta = -(-1)^{k(N-k)} \beta \wedge_S \alpha$$

and

$$\langle\langle\alpha,\beta\rangle\rangle^k = \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle\beta,\alpha\rangle\rangle^{N-k}$$

$$= (-1)^k (-1)^{\frac{N}{2}} \langle\langle\beta,\alpha\rangle\rangle^{N-k} \quad / \frac{N}{2} = 2p+1 \text{ is odd}$$

We find a scalar product $((,))$ in W^k for which $(W, \langle\langle,\rangle\rangle, ((,)))$ is an Hodge space..

- *There exists an anti-involution τ in E, $\tau^2 = -\tau$ (i.e. a complex structure)such that*

(1) $\langle \tau v, \tau w \rangle = \langle v, w \rangle$, $v, w \in E_x$,

(2) $\langle v, \tau v \rangle > 0$ *dla* $v \neq 0$.

Then the tensor $(v, w) := \langle v, \tau w \rangle$ *is symmetric and positive defined and* $(\tau v, \tau w) = (v, w)$, *i.e.* τ *preserves both forms* $\langle , \rangle$ *and* $(,)$. *The operator* $-\tau$ *is the* $*$*-Hodge operator in* $(E_x, \langle , \rangle_x , (,)_x)$. *In consequence, the system* $(E_x, \langle , \rangle_x , (,)_x)$ *is a Hodge-space.*

At each point $x \in M$ *we take the tensor product* $\bigwedge T_x^* M \bigotimes E_x$ *of the classical Hodge space* $\bigwedge T_x^* M$ *and the above* E_x. *The remaining procedure as in the above example to obtain a graded Hodge-space* $(\Omega (M; E) , \langle\langle , \rangle\rangle , ((,)) , d)$ *with differential (where the* $*$*-Hodge operator is defined point by point* $* : \Omega (M) \to \Omega (M)$, $* (\alpha) (x) = *_x (\alpha_x)$ *) We obtain in cohomology* $\langle\langle , \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k (M) \times \mathbf{H}^{N-k} (M) \to \mathbb{R}$,

$$\langle\langle \alpha, \beta \rangle\rangle^k = \int_M \alpha \wedge_G \beta = - (-1)^{k(N-k)} \int_M \beta \wedge_S \alpha = \underbrace{- (-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}$$

which in the middle degree $2p + 1$ *is symmetric (thanks to the fact that* $\langle , \rangle$ *is skewsymmetric)*

$$\langle\langle , \rangle\rangle_{\mathbf{H}}^{2p+1} : \mathbf{H}^{2p+1} (M; E) \times \mathbf{H}^{2p+1} (M; E) \to \mathbb{R}$$

$$\langle\langle\alpha,\beta\rangle\rangle^{2p+1} = -\left(-1\right)^{2p+1}\langle\langle\beta,\alpha\rangle\rangle^{2p+1} = \langle\langle\beta,\alpha\rangle\rangle^{2p+1}.$$

We can calculate the signature of $\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p+1}$ *as the index of the Hirzebruch operator* $D_+ = d_\nabla + d_\nabla^* : \Omega\left(M;E\right)_+ \to \Omega\left(M;E\right)_-$.

Example 34 *In consequence, for a transitive invariantly oriented Lie algebroid* A *over a compact oriented manifold* M *and the Atiyah sequence*

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0,$$

$$m = \dim M, \quad n = \operatorname{rank}\mathbf{g} = \dim \mathbf{g}_x,$$

and under the assumption $\mathbf{H}^{m+n}\left(A\right) \neq 0$ *and*

$$m + n = 4p$$

we have two signature Hirzebruch operators.

(I) The first one. $D_+ = d_A + d_A^* : \Omega\left(A\right)_+ \to \Omega\left(A\right)_-$ *where* d_A^* *is adjoint to* d_A *with respect to the scalar product* $((\alpha,\beta)) = \int_M (\alpha,\beta)$ *defined in the example 2 above, and* $W_\pm = \{\alpha \in \Omega\left(A\right);\ \tau\alpha = \pm\alpha\}$, *for* $\tau\left(\alpha^k\right) = \left(-1\right)^{\frac{k(k+1)}{2}}\left(-1\right)^p \cdot * \left(\alpha^k\right)$.

(II) The second one. We use the equality

$$\operatorname{Sign}\mathbf{H}(A) = \operatorname{Sign}E_2$$

for the second term E_2 *of the Hochschild-Serr spectral sequence*

$$E_2^{p,q} = \mathbf{H}^p_{\nabla^q}\left(M; \mathbf{H}^q(\boldsymbol{g})\right).$$

The flat covariant derivative ∇^q *in the cohomology vector bundle* $\mathbf{H}^q(\boldsymbol{g})$ *depends on the Lie algebroid* A.

Let $m+n=4p$. *The signature* $\operatorname{Sign}E_2$ *is equal to the signature of the quadratic form*

$$E_2^{2p} \times E_2^{2p} \longrightarrow E_2^{m+n} = \mathbb{R},$$

and

a) if n *is odd then* $\operatorname{Sign}E_2 = 0$,

b) if n *is even then*

$$\operatorname{Sign}E_2 = \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \longrightarrow E_2^{m+n} = E_2^{m,n} = \mathbb{R}\right)$$

$$E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right).$$

Consider the form $\langle\langle,\rangle\rangle : \mathbf{H}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$,

$$\langle\langle,\rangle\rangle^k : \mathbf{H}^k_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{m-k}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbf{H}^m_{\nabla^n}\left(M;\mathbf{H}^n(\boldsymbol{g})\right) = \mathbb{R},$$

which is symmetric in the middle degree $k = \frac{m}{2}$ *and its signature is equal to the signature of* A. *For* $k = n$, *the bundle* $\mathbf{H}^n(\boldsymbol{g})$ *is trivial,* $\mathbf{H}^n(\boldsymbol{g}) \cong M \times \mathbb{R}$, *the connection* ∇^n *is equal to* ∂, *and the multiplication of values is with respect to* $\langle,\rangle : \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$.

We have $\frac{m}{2} + \frac{n}{2} = 2p$. *We need to consider two different cases:*

(a) $\frac{m}{2}$ *and* $\frac{n}{2}$ *even, then the form*

$$\mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$$

is symmetric and we can use Example 3 above to give a Hirzebruch signature operator $D_+ = d_{\nabla^{\frac{n}{2}}} + d^*_{\nabla^{\frac{n}{2}}} : \Omega_+\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right) \to \Omega_-\left(M;\mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right)$,

(b) $\frac{m}{2}$ *and* $\frac{n}{2}$ *are odd, then the form* $\mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$ *is symplectic and we can use Example 4 to give a Hirzebruch signature operator.*

THE END

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