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SMOOTH GROUPOIDS OVER FOLIATIONS AND THEIR ALGEBROIDS THE CONCEPT OF PRADINES-TYPE GROUPOIDS

ABSTRACT This is the first paper of the series of the author's works devoted to applications of the theory of differential spaces in the sense of Sikorski [19] to groupoids. Much more general than differential groupoid, the notion of a smooth groupoid over a C^∞-manifold is defined here. Next, J.Pradines' [15], [16] idea of constructing, for each differential groupoid, some vector bundle with natural algebraic structures (called an *algebroid* of this diff. groupoid) – playing the analogous role to that of a Lie algebra of a Lie group – is used for smooth groupoids. Namely, some object, more general than a vector bundle, is assigned to each smooth groupoid. A particularly interesting situation takes place in the case when, for a given smooth groupoid, this object is a vector bundle. The groupoid is then called a *groupoid of Pradines type*. There are much more groupoids of this type than differential ones. For example, the equivalence relation of any (not only simple) foliation is such a groupoid. A large class of other examples is given. The idea of Pradines-type groupoids with singularities, inspired by foliations with singularities in the sense of P.Stefan, and the Frobenius-Sussmann theorem, – is given. Some new facts from the theory of differential spaces we shall need are obtained.

1

Introduction

The differential groupoids introduced by Ehresmann [6] are a natural extension of Lie groups, firstly – by the nature of the definitions, secondly – by some geometric circumstance. They constitute an appropriate direction for the development of certain geometric theories such as connexions (also those of higher orders) and Lie pseudogroups; besides, they find an application in problems of the equivalence and deformation of structures, and even in some problems of partial differential equations. One of the first applications of these objects was a new approach to the theory of Cartan pseudogroups.

One of the important works concerning Lie pseudogroups was that written by Libermann [12] in 1959. The author found a natural relation between the vector bundle $J^k(TV)$ and the Lie groupoid $\Pi^k(V)$. It is a family of linear isomorphisms

$$\lambda_h : J^k(TV)_{|x} \to T_h\left(\Pi^k(V)_x\right), \quad h \in \Pi^k(V)_x, \quad x \in V,$$

where $\Pi^k(V)_x$ denotes the principal fibre bundle of k-jets with source x, which enables one to define a linear isomorphism between the C^∞-module of global sections of the bundle $J^k(TV)$ and the $C^\infty(V)$-module of right-invariant vector fields on $\Pi^k(V)$, and next, to define an $\mathbb{R}$-algebra structure in the space of global sections of $J^k(TV)$.

The works by J.Pradines of the years 1966-1967, mainly [15], [16], were the **landmark** in the theory of differential groupoids. The author defined, for each differential groupoid Φ, some vector bundle $T_0^\alpha\Phi \subset T\Phi$ consisting of all vectors tangent to the units of the groupoid Φ. Each smooth section of the bundle $T_0^\alpha\Phi$ extends uniquely to exactly one smooth right-invariant vector field on the groupoid Φ, which enables us to carry some $\mathbb{R}$-Lie algebra structure (with the Lie product denoted by $[\![\cdot,\cdot]\!]$ into the $C^\infty(V)$-module of global sections of $T_0^\alpha\Phi$. $\operatorname{Sec} T_0^\alpha\Phi \to \mathfrak{X}(V)$ is a homomorphism of Lie algebras, where $\beta : \Phi \to V$ is the target and $\tilde{\beta}_* : T_0^\alpha\Phi \to TV$ is the restriction of β_*. The constructed object

$$\left(T_0^\alpha\Phi, [\![\cdot,\cdot]\!], \tilde{\beta}_*\right)$$

was called by J.Pradines the *Lie algebroid* of the diff. groupoid Φ. In the case when $\Phi = \Pi^k(V)$, the bundle $T_0^\alpha\Phi$ is isomorphic to $J^k(TV)$ via the canonical isomorphism $\lambda : J^k(TV) \to T_0^\alpha\Pi^k(V)$ defined by the formula $\lambda_{|x} = \lambda_{u_x}$, u_x – the unit of $\Pi^k(V)$ over x. $\operatorname{Sec}\lambda$ is then an isomorphism of the Lie algebras $\operatorname{Sec} J^k(TV)$ and $\operatorname{Sec} T_0^\alpha\Pi^k(V)$. However, in the case when Φ is a Lie group, $T_0^\alpha\Phi$ is simply the Lie algebra of the Lie group. The functor introduced by J.Pradines (the so-called *Lie functor*) made possible to build a theory of diff. groupoids, similar to the general theory of Lie groups, and thereby, to formulate problems of the type of fundamental Lie theorems (partly solved by Pradines). The next stage was the defining of the exponential mapping and its applications, first, in a particular situation, by N.V.Que [17], and next – in general case – for all diff. groupoids by A.Kumpera [9], [10]. In the author's papers [K1]-[K3]

there are a development and applications of the exponential mapping method to the global (general) theory of Lie groupoids and algebroids. The proofs out in such a way that one makes in them no use of the classical theorems from the theory of Lie groups. In the author's opinion, the reasoning induced in these proofs can be helpful in a much more general case of nontransitivity groupoids in which the space is not a manifold. The theory of foliations is the source of such groupoids. For example, the equivalence relation R on a manifold V determined by a foliation is not, in general, regular in the sense of Godement or the subgroupoid Φ' of a Lie groupoid Φ, consisting of the elements for which the source and the target lie on some leaf of a given foliation $\mathcal{F}$, is not – in general – a submanifold. The latter situation is a description (in the language of groupoids) of the important geometrical objects consisting of a principal fibre bundle P and a foliation $\mathcal{F}$ on the base, studied, for example, by F.Kamber and Ph.Tondeur [1]. However, it is not difficult to notice that in the cases considered: the equivalence relation $R \subset V \times V$ and – more generally – the subgroupoid Φ' of the Lie groupoid Φ, one can always define on the sets R and Φ' some natural structures of *differential spaces* in the sense of Sikorski [19], as differential (proper) subspaces of the spaces $(V \times V, C^\infty (V \times V))$ and $(\Phi, C^\infty (\Phi))$, respectively. All operations in these groupoids are then smooth. This gives rise to the defining of groupoids in the category of differential spaces, and next, of smooth groupoids over manifolds.

The present author's observations show that, by following the idea of Pradines, one can construct, for each smooth groupoid, an object, analogous to the Lie algebroid of a diff. groupoid, not being – a general – a vector bundle. The two above examples of smooth groupoids (R and Φ') have the property that the constructed objects are vector bundles although R and Φ' are hardly ever differential groupoids. The importance of these examples is a sufficient reason for the author to call the groupoids for which this object is a vector bundle – *Pradines-type groupoids.* An especially important role will be played by those groupoids from among them which are the so-called smooth groupoids over foliations. They are – in the author's opinion – a proper generalization of principal fibre bundles (from the geometric point of view), for they enable one to build a sensible theory of connexions [K4] (let us add that these connexions in the groupoid Φ' defined above correspond to those partial connexions in the principal fibre bundle P which project onto the tangent bundle of the foliation $\mathcal{F}$).

In the end, it should be mentioned here that the elasticity of the language of groupoids and diff. spaces is so great that, without especial difficulties, one can describe and study smooth groupoids over foliations with singularities in the sense of P.Stefan [22], and construct characteristic classes.

2 Groupoids - fundamental definitions, notations and examples

By a *groupoid (N.V.Que [16]) we mean the system*

$$(\Phi, \alpha, \beta, V, \cdot) \tag{1}$$

consisting of sets Φ and V and mappings $\alpha, \beta : \Phi \to V$, $\cdot : \Phi * \Phi \to \Phi$ where $\Phi * \Phi := \{(g,h) \in \Phi * \Phi;\ \alpha g = \beta h\}$ for $(g,h) \in \Phi * \Phi$, fulfilling the axioms

- $\alpha(gh) = \alpha h$ and $\beta(gh) = \beta g$ for $(g,h) \in \Phi * \Phi$,
- $(fg)h = f(gh)$ for $(f,g),\ (g,h) \in \Phi * \Phi$,
- for each point $x \in V$, there exists an element $u_x \in \Phi$ such that
  - $\alpha(u_x) = \beta(u_x) = x$,
  - $h \cdot u_x = h$ when $\alpha h = x$,
  - $u_x \cdot g = g$ when $\beta g = x$,
  - for each element $h \in \Phi$, there exists an element $h^{-1} \in \Phi$ such that
    - $\alpha\left(h^{-1}\right) = \beta h$, $\beta\left(h^{-1}\right) = \alpha h$,
    - $h \cdot h^{-1} = u_{\beta h}$,
    - $h^{-1} \cdot h = u_{\alpha h}$.

The mapping α is called a *source* and β - a *target*, as well as $\cdot : \Phi * \Phi \to \Phi$ - a (*partial*) *multiplication.* The elements u_x and h^{-1} are uniquely determined and called the *unity over the point* x and the *element inverse to* h, respectively. The mapping

$$^{-1} : \Phi \to \Phi, \quad h \longmapsto h^{-1},$$

is called an *inverse.* The *rule of reduction :*

$$gh_1 = gh_2 \implies h_1 = h_2$$

holds in each groupoid.

For a given groupoid (1), briefly denoted by Φ, we denote the following:

(a) $u := (V \ni x \longmapsto u_x \in \Phi)$,

(b) $G_x := \{h \in \Phi;\ \alpha h = \beta h = x\}$, $x \in V$. G_x with the operation induced from the groupoid is a group called the *isotropy group at the point,*

(c) $\Phi_x := \alpha^{-1}(x)$,

(d) $D_h : \Phi_{\beta h} \to \Phi_{\alpha h}, \quad g \longmapsto gh, \quad h \in \Phi$. We notice the relations

$$D_{gh} = D_h \circ D_g \quad \text{and} \quad D_{h^{-1}} = (D_h)^{-1}.$$

(e) $R_\Phi := \{(x,y) \in V \times V;\ \exists_{h\in\Phi}\,(\alpha h = x \ \text{ and } \ \beta h = y)\}$. R_Φ is an equivalence relation on V. The set

$$L_x := \beta\,[\Phi_x]$$

is the abstract class of R_Φ containing x,

(f) $\beta_x := \beta|\Phi_x : \Phi_x \to L_x$. Of course, $G_x = \beta_x^{-1}\,(x)$,

(g) for each point $x \in V$, the mapping

$$\cdot : \Phi_x \times G_x \to \Phi_x, \quad (h,a) \longmapsto ha, \tag{2}$$

is a right, free action of G_x on Φ_x and its orbits are equal to the fibres of the projection β_x.

Groupoids are in one-to-one correspondence to small categories in which every morphism is an isomorphism (the categorial definition of a groupoid stems from Ehresmann's work of 1957 [5], and the above-mentioned correspondence, for example, from the paper by Waliszewski [24]).

Groupoid (1) is called *transitive* if $R_\Phi = V \times V$ (the notion of a transitive groupoid corresponds to that of a groupoid in the sense of Brandt [1]).

Example 1 *A group G, or a little more generally, a set $V \times G \times V$ (for any set V) forms a groupoid if we put $\alpha\,(x,a,y) = x$, $\beta\,(x,a,y) = y$ and $(y,a,z)\cdot(x,b,y) = (x,ab,z)$. It is called a trivial groupoid. We notice that the canonical mapping $H : G \to G_x$, $a \longmapsto (x,a,x)$, is an isomorphism of groups.*

Example 2 *Any equivalence relation $R \subset V \times V$ determines a groupoid*

$$(R, \alpha, \beta, V, \cdot)$$

in which $\alpha\,(x,y) = x$, $\beta\,(x,y) = y$, $(y,z)\cdot(x,y) = (x,z)$ for (x,y), $(y,z) \in R$. It is called a groupoid of the equivalence relation R. Of course, $u_x = (x,x)$, $(x,y)^{-1} = (y,x)$, $G_x = \{(x,x)\}$, $R_x = \{x\} \times L_x$. Besides, $D_{(x,y)} : R_y \to R_x$, $(y,z) \longmapsto (x,z)$.

Example 3 (Ehresmann's groupoid PP^{-1} [3]) *Let $(P,\pi,V,G,\cdot)$ be a principal fibre bundle with the projection $\pi : P \to V$, the structural Lie group G and the right action $\cdot : P \times G \to P$. We define the right action of G on $P \times P$ by the formula*

$$(P \times P) \times G \to P \times P, \quad ((z,z_1)\,,a) \longmapsto (za, z_1 a)\,,$$

and denote by $[z;z_1]$ the orbit of this action including (z,z_1). According to Ehresmann, the set of orbits is denoted by PP^{-1}. The orbit $[z,z_1]$ may be interpreted as a diffeomorphism

$$\bar{z}_1 \circ \bar{z}^{-1} : P_{|x} \to P_{|x_1}, \quad x := \pi z,\ x_1 := \pi z_1$$

(where, for any $z \in P_{|x}$, the symbol $\bar{z}$ stands for the diffeomorphism $G \to P_{|x}$, $a \longmapsto za$). In the set PP^{-1} there exists exactly one structure of a groupoid in which $\alpha([z,z_1]) = \pi z$, $\beta([z,z_1]) = \pi z_1$ and $[z_1,z_2]\cdot[z,z_1] = [z,z_2]$. Of course, PP^{-1} is a transitive groupoid, $u_x = [z,z]$ for each $z \in P_{|x}$, and $[z,z_1]^{-1} = [z_1,z]$. After the identification of the orbits with the above diffeomorphisms, the multiplication formula is then an ordinary superposition. We notice that, for the trivial bundle $P = \{e\} \times V$, the multiplication formula - after the identification of the orbit $[z,z_1]$ with a couple (x,x_1) - is equal to that in Example 2 (for $R = V \times V$). Whereas, for the trivial principal fibre bundle $P = G \times V$ and the identification of the orbit $[(a,x),(b,y)]$, first of all, with the diffeomorphism $G \times \{x\} \to G \times \{y\}$, $(c,x) \longmapsto \left(ba^{-1}c, y\right)$, and next, with the trio $\left(x, ba^{-1}, y\right)$, we get the multiplication formula equal to that in Example 1.

Example 4 *Let (E,p,V) be any vector bundle. The set $\mathrm{GL}(E)$ of all linear isimorphisms between the fibres at this bundle becomes a groupoid if, for $h : E_{|x} \to E_{|y}$ and $g : E_{|y} \to E_{|z}$, we put $\alpha h = x$, $\beta h = y$ and $g \cdot h = g \circ h$.*

Example 5 *Any transitive groupoid (1) and any equivalence relation $R \subset V \times V$ determine a new groupoid $\left(\Phi', \alpha', \beta', V, \cdot'\right)$ in which $\Phi' = (\alpha,\beta)^{-1}[R]$, $\alpha' h = \alpha h$, $g \cdot' h \cdot = g \cdot h$ for $g, h \in \Phi'$.*

3 Review of the definitions of groupoids with differential structures

The first groupoids with differential structures were transitive ones: the Ehresmann groupoids PP^{-1} [3] and the groupoid $\Pi^k(V)$ of all k-jets of local diffeomorphisms of a manifold V [4]. Both PP^{-1} and $\Pi^k(V)$ are differential manifolds (of the class C^∞), whereas the mappings source and target are submersions. Next, Y.Matsuhima in 1955 [13], while examining G-structures and Lie transitive pseudogroupes of higher orders, defined a *Lie subgroupoid* $\Pi \subset \Pi^k(V)$ as a subgroupoid being a submanifold of $\Pi^k(V)$ for which the mappings: multiplication and inverse are smooth (i.e. C^∞-class), and $(\alpha,\beta) : \Pi \to V \times V$ is a submersion (onto). These axioms were used by N.V.Que in 1967 [17] to formulate the definition of a *Lie groupoid* as a transitive one (1) in which Φ and V are diff. manifolds, $\alpha, \beta : \Phi \to V$ are submersions and the mappings $u, \cdot, ^{-1}$ are smooth.

Another example is $\mathrm{GL}(E)$, where E is a vector bundle.

A more general definition of the so-called *differential groupoid* stems from Ehresmann's work of 1958 [6] (see also N.V.Que [17], A.Kumpera [9], [10]). Below, by this notion we shall mean any groupoid (1) which fulfils all the axioms of a Lie groupoid (above-mentioned) except transitivity. In the authors mentioned above we can find different deviations from this definition, concerning the assumptions about the mappings α and β.

In J.Pradines' paper of 1966 [14] there is another definition - quite different from the above one - of a *partly differential groupoid* as a couple (Φ, W)

consisting of an (algebraical) groupoid (1) and a differerential manifold W such that

- $u[V] \subset W \subset \Phi$,
- W generates Φ (as a groupoid),
- there exists some structure of a manifold on the set V, such that $\alpha : W \to V$ is a submersion and $u : V \to W$ - an immersion,
- the multiplication and inverse mappings are - with respect to W - C^∞-class.

The notion of a *nice-groupoid*, introduced by the author (for other reasons than in Pradines' paper) is similar to the above notion.

4 Some facts from the theory of differential spaces

By using only differential manifolds, there is no possibility of a global differential description of many important groupoids, for example, of the equivalence relations (except **regular** ones in the sense of Godement [18, Ch.III, §12]. It turns out that this is possible in the language of differential spaces - much more general objects. Now, we give a short introduction to this theory and some new facts we shall need further.

Let M be any set and C - a family of real functions defined on M. By τ_C [19], [20] we mean the weakest topology on M in which the functions from C are continuous. For $A \subset M$, we denote

- $C|A = \{g|A;\ g \in C\}$,
- C_A - the family of real functions on A which may be extended locally to functions on M, i.e. the set

$$\{h : A \to \mathbb{R};\ \forall_{x\in A}\exists_{U\in\tau_C}\exists_{g\in C}\,(x \in A,\ h|U \cap A = g|U \cap A)\},$$

- $\operatorname{sc} C = \{\varphi \circ (g_1, ..., g_m);\ m \in \mathbb{N},\ g_1, ..., g_m \in C,\ \varphi \in C^\infty(\mathbb{R}^m)\}$.

The equalities:

$$\tau_C|A = \tau_{C|A} = \tau_{C_A}, \qquad \tau_C = \tau_{\operatorname{sc} C} \quad \text{and} \quad (C|A)_A = C_A$$

hold, moreover for $B \subset A$

$$(C_A)_B = C_B.$$

The set $(\operatorname{sc} C)_M$ is the smallest of the sets C' of real functions on M such that $C \subset C'$ and $\operatorname{sc} C' = C' = C'_M$ [11], [26]. By a *differential space [1], [11], [19]* we mean each couple (M, C) (briefly denoted by M) consisting of a set M and a non-empty family C of real functions on M such that

$$C_M = C = \operatorname{sc} C, \tag{3}$$

or equivalently, such that if $f : M \to \mathbb{R}$ is any function which locally (i.e. with respect to the topology τ_C) equals $\varphi \circ (g_1, ..., g_m)$ for some $m \in \mathbb{N}$, $g_i \in C$, $i \leq m$, $\varphi \in C^\infty(\mathbb{R}^m)$, then $f \in C$. Every diff. space (M, C) is also considered a the topological space (M, τ_C). Any non-empty set C of real functions on M fulfilling (3) is called a *differential structure on* M. If $C = \left(\mathrm{sc}\,\hat{C}\right)_M$, then we say that the diff. structure C is *generated* by $\hat{C}$. If C is a diff. structure on M, then for any subset $A \subset M$, the family C_A is a diff. structure on A [19]. It is easy to check that if C is a diff. structure on M generated by $\hat{C}$, then, for any $A \subset M$, the diff. structure C_A is generated by $\hat{C}|M$.

Examples of diff. spaces are:

- $(\mathbb{R}, \mathrm{sc}\,(\{x \longmapsto |x|\})_{\mathbb{R}})$,
- $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$,
- $(M, C^\infty(M))$ where M is any C^∞-manifold and, more generally,
- $(A, C^\infty(M)_A)$ for any $A \subset M$.

Let (M, C) and (N, D) be any diff. spaces. The mapping $f : M \to N$ is called

(1) *smooth* [19] if $g \circ f \in C$ for $g \in D$. Then we write

$$f : (M, C) \to (N, D), \tag{4}$$

(2) a *diffeomorphism* [19] *if it is a bijection and* f and f^{-1} are smooth,

(3) an *embedding* if $f : (M, C) \to \left(f[M], D_{f[M]}\right)$ is a diffeomorphism.

It is easy to see [19] that if D is generated by $\hat{D}$ and $g \circ f \in C$ for $g \in \hat{D}$, then (4) is continuous if we investigate the topologies τ_C and τ_D in M and N, repsectively.

Let M be any C^∞-manifold. Then the topology of M is equal to $\tau_{C^\infty(M)}$ iff M is Hausdorff. In the Hausdorff case, for any open set $A \subset M$, the equality $C^\infty\left(M_{|U}\right) = C^\infty(M)_U$ holds. In this connection, we adopt the following definition [19]: a diff. space (M, C) is called an *n-dim. differential manifold* if each point of M has a neighbourhood diffeomorphic to an open subset of $\mathbb{R}^n$ (of course, with the diff. structure induced from $C^\infty(\mathbb{R}^n)$). The topology τ_C is then Hausdorff.

By the *product* of diff. spaces (M, C) and (N, D) [19] we mean the diff. space $(M \times N, C \times D)$ where $C \times D$ is the diff. structure generated by $\{g \circ \mathrm{pr}_1;\ g \in C\} \cup \{h \circ \mathrm{pr}_2;\ h \in D\}$ ($(M \times N, C \times D)$ is also denoted by $(M, C) \times (N, D)$). It is easy to prove that if $A \subset M$ and $B \subset N$, then

$$(C \times D)_{A \times B} = C_A \times D_B. \tag{5}$$

Besides, if M and N are manifolds then

$$C^{\infty}(M \times N) = C^{\infty}(M) \times C^{\infty}(N).$$

By a *tangent vector* to a diff. space (M, C) at a point $x \in M$ [19] we mean each linear mapping $v : C \to \mathbb{R}$ such that $v(f \cdot g) = v(f) \cdot g(x) + f(x) \cdot v(g)$ for $f, g \in C$. All tangent vectors at x form a vector space which is denoted by $T_x(M, C)$ and called a *tangent space* at x. By a *differential* at x of any smooth mapping (4) we mean the linear mapping $f_{*x} : T_x(M, C) \to T_{f(x)}(N, D)$ defined by the formula: $f_{*x}(v)(g) = v(g \circ f)$, $g \in D$.

Proposition 6 (see also [26]) *If $(M, C) \times (N, D)$ is a connected diff. manifold, then the diff. spaces (M, C) and (N, D) are diff. manifolds, too.*

Proof. Let the product $(M, C) \times (N, D)$ be a connected diff. manifold, say - of dimension k. Take $x_0 \in M$ and $y_0 \in N$ and put $m := \dim T_{x_0}(M, C)$, $n := \dim T_{y_0}(N, D)$. Of course $k = m + n$. There exist some neighbourhoods $U \in \tau_C$ and $V \in \tau_D$ of x_0 and y_0, respectively, and a diffeomorphism $\varphi : (U, C_U) \times (V, D_V) \to \left(\Omega, C^{\infty}\left(\mathbb{R}^k\right)_{\Omega}\right)$ for some open subset $\Omega \subset \mathbb{R}^k$. We put $U_1 := \varphi[U \times \{y_0\}]$ and $V_1 := \varphi[\{x_0\} \times V]$ and take the diffeomorphisms

$$\begin{aligned} \varphi_1 &: = \varphi(\cdot, y_0) : (U, C_U) \to \left(U_1, C^{\infty}\left(\mathbb{R}^k\right)_{U_1}\right), \\ \varphi_2 &: = \varphi(x_0, \cdot) : (V, D_V) \to \left(V_1, C^{\infty}\left(\mathbb{R}^k\right)_{V_1}\right). \end{aligned}$$

From the main theorem of paper [7], we infer - diminishing U and V if necessary - that there exists some diffeomorphisms

$$\begin{aligned} \psi_1 &: \left(U_1, C^{\infty}\left(\mathbb{R}^k\right)_{U_1}\right) \to \left(\Omega_1, C^{\infty}\left(\mathbb{R}^m\right)_{\Omega_1}\right), \quad \Omega_1 \subset \mathbb{R}^m, \\ \psi_2 &: \left(V_1, C^{\infty}\left(\mathbb{R}^k\right)_{V_1}\right) \to \left(\Omega_2, C^{\infty}\left(\mathbb{R}^n\right)_{\Omega_2}\right), \quad \Omega_2 \subset \mathbb{R}^n. \end{aligned}$$

Hence, by (5), the superposition

$$(\psi_1 \times \psi_2) \circ (\varphi_1 \times \varphi_2) \circ \varphi^{-1} : \left(\Omega, C^{\infty}\left(\mathbb{R}^k\right)_{\Omega}\right) \to \left(\Omega_1 \times \Omega_2, C^{\infty}\left(\mathbb{R}^k\right)_{\Omega_1 \times \Omega_2}\right)$$

is a diffeomorphism. Therefore $\Omega_1 \times \Omega_2$ is open in $\mathbb{R}^k$, so Ω_1 and Ω_2 are open in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, which finishes the proof. ■

It is easy to check that, for any Hausdorff C^{∞}-manifold M, the ring of smooth functions on the tangent bundle TM is the diff. structure generated by $\{g \circ \pi;\ g \in C^{\infty}(M)\} \cup \{dg;\ g \in C^{\infty}(M)\}$ where $\pi : TM \to M$ is the canonical projection and $dg(v) = v(g)$, $v \in TM$. Following A.Kowalczyk [8], we accept

Definition 7 *Let (M, C) be any diff. space. We put*

Case 8 •

(a) $T(M, C) = \bigsqcup_{x \in M} T_x(M, C)$ (disjoint union),

(b) $\pi : T(M,C) \to M$ *- the canonical projection,*

(c) $TC = \left(\mathrm{sc}\,\hat{C}\right)_{T(M,C)}$ *where* $\hat{C} = \{g \circ \pi;\ g \in C\} \cup \{dg;\ g \in C\}$ *and* $dg(v) = v(g)$ *for* $v \in T(M,C)$.

The diff. space $(T(M,C), TC)$ (denoted briefly by $T(M,C)$) is called the *tangent bundle over* (M,C).

Proposition 9 *The tangent bundle* $T(M,C)$ *over a diff. space* (M,C) *has the properties:*

(1) the mappings

$$\begin{aligned} + &: T(M,C) \oplus T(M,C) \to T(M,C), \quad (v,w) \longmapsto v+w, \\ \cdot &: \mathbb{R} \times T(M,C) \to T(M,C), \quad (r,v) \longmapsto r \cdot v, \end{aligned}$$

are smooth, where $T(M,C) \oplus T(M,C)$ *is the proper diff. subspace of* $T(M,C) \times T(M,C)$ *consisting of all pairs* (v,w) *for which* $\pi(v) = \pi(w)$,

(2) for any smooth vector fields $X_1, ..., X_m \in \mathfrak{X}(M,C)$ *and* $U \in \tau_C$ *such that the vectors* $X_1(x), ..., X_m(x)$ *are linearly independt for* $x \in U$*, the mapping*

$$\varphi : U \times \mathbb{R}^m \to T(M,C), \quad (x,a) \longmapsto \sum_{i=1}^{m} a^i \cdot X_i(x),$$

is a diffeomorphism onto its image.

Proof. (2) *The smoothness of* φ^{-1} (the rest is easy). Put $\mathrm{pr}_1 : U \times \mathbb{R}^m \to U$, $(x,a) \longmapsto x$, and $p^s : U \times \mathbb{R}^m \to \mathbb{R}$, $\left(x, \left(a^1, ..., a^m\right)\right) \longmapsto a^s$, $s \leq m$. Of course,

$$\mathrm{pr}_1 \circ \varphi^{-1} = \pi | \varphi[U \times \mathbb{R}^m] \in (TC)_{\varphi[U \times \mathbb{R}^m]}.$$

It suffices to show that

$$p^s \circ \varphi^{-1} \in (TC)_{\varphi[U \times \mathbb{R}^m]}.$$

For this purpose, we notice that $p^s \circ \varphi^{-1}\left(\sum_{i=1}^{m} a^i \cdot X_i(x)\right) = a^s$ and we take any point $x_0 \in U$ and functions $f^1, ..., f^m \in C$ such that [20] $X_i\left(f^j\right)(x_0) = \delta_i^j$. Then, for some neighbourhood $\tilde{U}$ of x_0, we have $\det\left[X_i\left(f^j\right)(x)\right] \neq 0$, $x \in \tilde{U}$, and we can define the mapping $\psi : \tilde{U} \to \mathrm{GL}(n,\mathbb{R})$ by the formula $x \longmapsto \left[X_i\left(f^j\right)(x)\right]$. Let ${}^{-1} \circ \psi(x) = \left[c_j^k(x)\right]$, $x \in \tilde{U}$. For

$$F : \mathbb{R}^{2m} \to \mathbb{R}, \quad \left(x^1, ..., x^m, y^1, ..., y^m\right) \longmapsto \sum_{j=1}^{m} y_j \cdot x_j,$$

we have

(1) $F\left(c_1^s \circ \pi, ..., c_m^s \circ \pi, df^1, ..., df^m\right) \in (TC)_{\pi^{-1}[\tilde{U}]}$,

(2) if $v \in \pi^{-1}\left[\tilde{U}\right] \cap \varphi\left[U \times \mathbb{R}^m\right]$ is equal to $\sum_{i=1}^m a^i \cdot X_i(x)$, then

$$\begin{aligned} & F\left(c_1^s \circ \pi, ..., c_m^s \circ \pi, df^1, ..., df^m\right)(v) \\ = & \sum_{j=1}^m v\left(f^j\right) \cdot c_j^s(x) = \sum_{i=1}^m a^i \sum_{j=1}^m X_i\left(f^j\right)(x) \cdot c_j^s(x) = a^s \\ = & p^s \circ \varphi^{-1}(v), \end{aligned}$$

which ends the proof.

■

The notion of a subspace of a diff. space can be found in [19] and [20]. It turns out that it is too strong for us. We adopt

Definition 10 *A diff. space (N', D') is called a differential subspace of a diff. space (N, D) if $N' \subset N$ and, for each $y \in N'$, there exists a neighbourhood $U \in \tau_{D'}$ of y such that $D'_U = D_U$. Then we write*

$$(N', D') \hookrightarrow (N, D).$$

By means of the differential of the inclusion $i : (N', D') \hookrightarrow (N, D)$ we shall systematically identify the vector space $T_y(N', D')$ with the vector subspace $\mathrm{Im}\,(i_*)_y$ of $T_y(N, D)$, $y \in N'$. As an example we take any C^∞-manifold M and any immersion submanifold $L \hookrightarrow M$. Then $(L, C^\infty(L)) \hookrightarrow (M, C^\infty(M))$.

A diff. subspace (N', D') is called a *proper subspace* of (N, D) id $D' = D_N$ (equivalently, the inclusion i is an embedding), The following proposition will be fundamental for constructing many new subspaces.

Proposition 11 *For any smooth mapping (4) and any diff. subspace $(N', D') \hookrightarrow (N, D)$, we have: on the set $M' := f^{-1}[N']$ there exists exactly one diff. structure C' which fulfils the condition*

$$\text{if } A \in \tau_{D'} \text{ and } D'_A = D_A, \quad \text{then } f^{-1}[A] \in \tau_{C'} \quad \text{and } C'_{f^{-1}[A]} = C_{f^{-1}[A]}. \quad (6)$$

Besides, $f|M' : (M', C') \to (N', D')$ is a smooth mapping.

Proof. *Uniqueness.* Let C' and C'' be two diff. structures on M' fulfilling (6). We take any $g \in C'$, any point $x \in M'$ and a set $A \in \tau_{D'}$, such that $f(x) \in A$ and $D'_A = D_A$. Thus $f^{-1}[A] \in \tau_{C''}$ and $g|f^{-1}[A] \in C'_{f^{-1}[A]} = C_{f^{-1}[A]} = C''_{f^{-1}[A]}$, so $g \in C''$.

Existence. Put $\mathcal{A} := \{A \in \tau_D;\ D'_A = D_A\}$ and

$$C' := \left\{g : M' \to \mathbb{R};\ \ \forall_{A \in \mathcal{A}}\left(g|f^{-1}[A] \in C_{f^{-1}[A]}\right)\right\}.$$

We shall prove that C' fulfils (6). Of course, $C|M' \subset C'$, which proves the non-emptiness of C'. First, we shall show that C' is a diff. structure (i.e. $\mathrm{sc}\, C' \subset C'$

and $C'_M \subset C'$). For the purpose, let us take $g_1, ..., g_m \in C'$, $\varphi \in C^\infty(\mathbb{R}^m)$ and $A \in \mathcal{A}$, we then have

$$\begin{aligned} \varphi \circ (g_1, ..., g_m) | f^{-1}[A] &= \varphi \circ \left(g_1 | f^{-1}[A], ..., g_m | f^{-1}[A]\right) \in \mathrm{sc}\left(C' | f^{-1}[A]\right) \subset \\ &\subset \mathrm{sc}\left(C_{f^{-1}[A]}\right) = C_{f^{-1}[A]}, \end{aligned}$$

while, for $g \in C'_M$ and $A \in \mathcal{A}$,

$$g | f^{-1}[A] \in (C'_M)_{f^{-1}[A]} = C'_{f^{-1}[A]} = C_{f^{-1}[A]}.$$

Now, we prove the smoothness of $f|M' : (M', C') \to (N', D')$. In order to do this, we take any $g \in D'$, $A \in \mathcal{A}$ and $x \in f^{-1}[A]$. Since $g|A \in D'_A = D_A$, there exist a neighbourhood $U \in \tau_D$ and a function $h \in D$ such that $f(x) \in U$ and $g|A \cap U = h|A \cap U$. But $f^{-1}[U] \in \tau_C$ and $h \circ f \in C$, so

$$\begin{aligned} & g \circ (f|M') | f^{-1}[A] \cap f^{-1}[U] \\ = \; & g|A \cap U \circ \left(f | f^{-1}[A] \cap f^{-1}[U]\right) = h|A \cap U \circ \left(f | f^{-1}[A] \cap f^{-1}[U]\right) \\ = \; & (h \circ f) | f^{-1}[A] \cap f^{-1}[U] \in C_{f^{-1}[A] \cap f^{-1}[U]}, \end{aligned}$$

which proves that

$$g \circ (f \circ |M') | f^{-1}[A] \in C_{f^{-1}[A]},$$

thus $g \circ (f|M') \in C'$. From the above it follows that $f^{-1}[A] \in \tau_{C'}$ for $A \in \mathcal{A}$. It remains to show the equality

$$C'_{f^{-1}[A]} = C_{f^{-1}[A]}.$$

Seeing that $C|M' \subset C'$, we have $C_{f^{-1}[A]} \subset C'_{f^{-1}[A]}$. On the other hand, directly from the definition of C' we have: $C'|f^{-1}[A] \subset C_{f^{-1}[A]}$, so $C'_{f^{-1}[A]} = \left(C'|f^{-1}[A]\right)_{f^{-1}[A]} \subset C_{f^{-1}[A]}$, which completes the proof. ■

In paper [25] there exists some generalizations of the notion of a coregular mapping to the case of maps between diff. spaces. However, a considerably stronger notion will be more useful to our purposes.

Definition 12 *Smooth surjective mapping (4) is called strong-coregular if for natural number n and each point $x \in M$, there exist neighbourhoods $U \in \tau_C$ and $V \in \tau_D$ of x and $f(x)$, respectively, and a diffeomorphism*

$$\psi : (U, C_U) \to (V, D_V) \times (\mathbb{R}^n, C^\infty(\mathbb{R}^n)),$$

such that

$$f|U = \mathrm{pr}_1 \circ \psi.$$

According to Proposition 6 and [18, Ch.III,§10], we see that, in the case when (M, C) is a manifold, smooth surjective mapping (4) is strong-coregular iff (a) (N, D) is a manifold, (b) the differential f_{*x} is an epimorphism for each $x \in M$.

Proposition 13 *If mapping (4) is strong-coregular,* $(N', D') \hookrightarrow (N, D)$ *and* C' *is the diff. structure on* $M' = f^{-1}[N']$ *fulfilling (6), then the mapping* $f|M' : (M', C') \to (N', D')$ *is strong-coregular, too. What is more, if* (N', D') *is a diff. manifold, then* (M', C') *is a diff. manifold, too.*

Proof. Let $x \in M'$. We take any neighbourhood $U \in \tau_C$ and $V \in \tau_D$ of the points x and $f(x)$, respectively, a number n and a diffeomorphism $\psi : (U, C_U) \to (V, D_V) \times (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ such that $f|U = \mathrm{pr}_1 \circ \psi$. Besides, we take any set $A \subset V$ such that $f(x) \in A$, $A \in \tau_{D'}$ and $D'_A = D_A$. Since $x \in U \cap f^{-1}[A] = \psi^{-1}\left[\mathrm{pr}_1^{-1}[A]\right] \in \tau_{C'}$ and $C_{f^{-1}[A]\cap U} = C'_{f^{-1}[A]\cap U}$, we have the commuting diagram

$$\begin{array}{ccc} \left(f^{-1}[A] \cap U, C'_{f^{-1}[A]\cap U}\right) & \overset{\psi|f^{-1}[A]\cap U}{\longrightarrow} & (A, D'_A) \times ((\mathbb{R}^n, C^\infty(\mathbb{R}^n))) \\ & \searrow_{f|f^{-1}[A]\cap U} & \downarrow \\ & & (A, D'_A) \end{array}$$

which proves the proposition. ■

Definition 14 *By a (k-) leaf of a diff. space* (M, C) *we mean a subset* $L \subset M$ *if there exists a diff. structure* D *on* L *such that*

(1) (L, D) *is a diff. manifold (of dimension k),*

(2) (L, D) *is a diff. subspace of* (M, C) *(see def. 5),*

(3) for each locally arcwise connected topological space X *and a continuous mapping* $f : X \to (M, C)$ *such that* $f[X] \subset L$*, the mapping* $f : X \to (L, D)$*, defined by the same formula, is continuous, too.*

We notice that the diff. structure D is uniquely determined. Besides, if (X, E) is any diff. space whose topology τ_E is locally arcwise connected, then, for each smooth mapping $f : (X, E) \to (M, C)$ such that $f[X] \subset L$, the mapping $f : (X, E) \to (L, D)$ is also smooth.

Sometimes, the manifold (L, D) is called a *leaf* of (M, C).

As an example we take any (k-dim.) foliation $\mathcal{F}$ on a C^∞-manifold M. Then, each element $L \in \mathcal{F}$ is a (k-) leaf of $(M, C^\infty(M))$.

5 Smooth groupoids over differential manifolds

Having differential spaces at our disposal, we are able to give the following

Definition 15 *By a groupoid in the category of differential spaces we mean a groupoid*

$$\mathbf{\Phi} = (\Phi, \alpha, \beta, V, \cdot) \tag{7}$$

in which Φ *and* V *are diff. spaces and the mappings* $\alpha, \beta : \Phi \to V$, $u : V \to \Phi$, $^{-1} : \Phi \to \Phi$ *and* $\cdot : \Phi * \Phi \to \Phi$ *(where* $\Phi * \Phi$ *denotes the ptoper subspace of* $\Phi \times \Phi$*) are smooth.*

We notice that $u : V \to \Phi$ is an embedding.

Example 16 *Let R be any equivalence relation on a C^∞-manifold V. Then the system (see Example 2)*

$$\mathbf{R} = \left(\left(R, C^\infty \left(V \times V \right)_R \right), \alpha, \beta, V, \cdot \right) \tag{8}$$

is a groupoid in the category of diff. spaces.

Example 17 *Let Γ be any pseudogroup of smath transformation on a C^∞-manifold V. Then, for each $k = 1, 2, ...,$ the set of jests*

$$\left\{ j_x^k f;\ f \in \Gamma,\ x \in D_f \right\} \subset J^k \left(V, V \right),$$

with the diff. structure induced from $J^k \left(V, V \right)$, forms a groupoid in the category of diff. spaces.

Definition 18 *By a smooth groupoid over a differential manifold we mean a groupoid in the category of diff. spaces (7) in which V is a diff. manifold and for each point $x \in V$, the set $\alpha^{-1}(x)$ is a leaf of the diff. space Φ.*

The set $\alpha^{-1}(x)$ equipped with the suitable diff. manifold structure is called the *leaf of the groupoid* $\mathbf{\Phi}$ *over* x and denoted by Φ_x.

Remark 19 *(1) Connected components of the leaf Φ_x are equal to arcwise connected components of the proper diff. subspace $\alpha^{-1}(x)$ of Φ,*

(2) the mapping $D_h : \Phi_{\beta h} \to \Phi_{\alpha h},\ h \in \Phi$, is a diffeomorphism.

Example 20 *Let R be any equivalence relation on a diff. manifold V for which the family of all abstract classes is a foliation $\mathcal{F}$ on V. We denote by L_x the leaf of $\mathcal{F}$ through x, equipped with the natural structure of an immerse submanifold of V. Then the system (8) is a smooth groupoid over V called a groupoid of the foliation $\mathcal{F}$. The manifold R_x for which the mapping*

$$\gamma_x : L_x \to R_x,\ y \longmapsto (x, y), \tag{9}$$

is a diffeomorphism is the leaf of this groupoid over x.

Note that a smooth groupoid over V may also be obtained for a more general equivalence relation. Namely, it may be an equivalence reelation for which every abstract class L is a leaf of V, for example, the equivalence relation of a foliation with singularities in the sense of P.Stefan [21], [22].

Example 21 *Differential groupoid (7), see section 3, is, of course, a smooth groupoid over V, besides, the proper submanifold $\alpha^{-1}(x)$ of Φ is the leaf over x.*

Definition 22 *A groupoid in the category of diff. spaces (7) is called strong-coregular if the mapping*

$$(\alpha, \beta) : \Phi \to R_\Phi \tag{10}$$

is strong-coregular (where R_Φ is the proper subspace of $V \times V$).

Proposition 23 *If (7) is a strong-coregular diff. groupoid, then the equivalence relation R_Φ is regular (in the sense of Godement [18, Ch.III, §12]), in particular, the family of all connected components of all equivalence classes is a simple foliation.*

Proof. One should prove that (a) R_Φ is a proper submanifold of $V \times V$, (b) $\mathrm{pr}_1 : R_\Phi \to V$ is a submersion. We see that (a) results from Proposition 3 and the assumption of the strong coregularity of (10), while (b) - from the equality $\alpha = \mathrm{pr}_1 \circ (\alpha, \beta)$, which ends the proof. ■

The theorem below gives a large class of examples of smooth groupoids over V.

Theorem 24 *Let (7) be any Lie groupoid and $R \subset V \times V$ - any equivalence relation such that each equivalence class is a leaf of V. Then the groupoid $\Phi' = (\alpha, \beta)^{-1}[R]$ defined in example 5, equipped with the diff. structure of a proper diff. subspace of Φ, turns out to be a smooth groupoid over V. What is more, this is a strong-coregular groupoid for which $\beta'_x : \Phi'_x \to L_x$ is a submersion. The number $\dim \Phi'_x - \dim L_x$ does not depend on x and R.*

Proof. Of course, the system $\left(\Phi', \alpha', \beta', V, \cdot'\right)$ is a groupoid in the category of diff. spaces. The strong coregularity of this groupoid results from Prop. 13. Now, we consider the submersion $\beta_x : \Phi_x \to V$. On the set Φ'_x - which is equal to $\beta_x^{-1}[L_x]$ - we define the diff. structure according to Prop. 11. By Prop. 13, we see that (a) Φ'_x is then a diff. manifold and (b) $\beta'_x : \Phi'_x \to L_x$ is a submersion. It remains to show that Φ'_x is a leaf of Φ' but this follows from the fact (which is easy to see) that Φ'_x is a leaf of the manifold Φ_x. In the end we notice that $\dim \Phi'_x - \dim L_x = \dim \Phi - 2 \cdot \dim V$ which ends the proof. ■

Example 25 *A foliation $\mathcal{F}$ of V determines some smooth groupoid over V, for example, the groupoids of all linear isomorphisms*

(a) $h : E_{|x} \to E_{|y}, \quad x, y \in L,\ L \in \mathcal{F}$, *where* $E_{|x} = T_x(L_x)$,

(b) $h : T_xV/E_{|x} \to T_yV/E_{|y}$, *$x$ and y as above.*

These groupoids are obtained according to Theorem 24 with the help of some Lie groupoid $\mathrm{GL}(E)$ *and the equivalence relation R of $\mathcal{F}$; in case (a) - $E = T\mathcal{F}$, while in (b) - $E = TV/T\mathcal{F}$.*

At the end of this section we discuss briefly the relation between Lie groupoids and principal fibre bundles (over the same manifold V). First of all, we see that $G_x = \beta_x^{-1}(x)$ is a Lie group, and the system

$$(\Phi_x, \beta_x, V, G_x, \cdot)$$

is a principal fibre bundle (briefly, p.f.b.), where "$\cdot$" means action 2 (see [17]). For two points x and y, the p.f.b.'s Φ_y and Φ_x are isomorphic; indeed, for any element $h \in \Phi$ such that $\alpha h = x$ and $\beta h = y$, the pair of mappings (D_h, H) where $H : G_y \to G_x$, $a \longmapsto h^{-1}ah$, is an isomorphism of Φ_y onto Φ_x.

On the other hand, the p.f.b. $(P, \pi, V, G, \cdot)$ determines the Ehresmann groupoid PP^{-1} according to Example 3. This is a Lie groupoid. The manifold PP^{-1} is the only one for which $r : P \times P \to PP^{-1}$ $((z, z') \longmapsto [z, z'])$ is a submersion and it is characterized by the property:

— for any two sections $\varphi_i : U_i \to P$, $U_i \subset V$, $i = 1, 2$, of the projection π, the mapping

$$(U_1 \times G \times U_2) \ni (x, a, y) \longmapsto [\varphi_1(x), \varphi_2(y) \cdot a] \in (\alpha, \beta)^{-1} [U_1 \times U_2] \subset PP^{-1}$$

is a diffeomorphism.

In the end, we notice (fixing the a $x \in V$) that

(1) the p.f.b.'s P and $\left(PP^{-1}\right)_x$ are isomorphic (noncanonically!). Namely, for any $z_0 \in P_{|x}$, the pair (φ, H), where $\varphi : P \to \left(PP^{-1}\right)_x$, $z \longmapsto [z_0, z]$, and $H : G \to G_x$, $a \longmapsto [z_0, z_0 \cdot a]$, establishes an isomorphism between them;

(2) the Lie groupoid $\Phi_x \Phi_x^{-1}$ and Φ are isomorphic in the sense that there exists a diffeomorphism $F : \Phi_x \Phi_x^{-1} \to \Phi$ such that

   (a) $\alpha(Fz) = z$,

   (b) $\beta(Fz) = \beta z$,

   (c) $F(z \cdot z') = F(z) \cdot F(z')$.

   Of course, we ought to put $F([z, z']) = z' \cdot z^{-1}$;

(3) the naturally appearing functors Ψ and Θ between the categories of p.f.b.'s - $\mathcal{P}$ and Lie groupoids - $\mathcal{G}$ (both over V), defined by the formulae

   (a) $\Psi : \mathcal{P} \to \mathcal{G}$; $P \longmapsto PP^{-1}$, $(\varphi : P \to P') \longmapsto \left(\Psi(\varphi) : PP^{-1} \to P'P'^{-1}\right)$ where $\Psi(\varphi)([z, z']) = [\varphi z, \varphi z']$;

   (b) $\Theta : \mathcal{G} \to \mathcal{P}$; $\Phi \longmapsto \Phi_x$, $(F : \Phi \to \Phi') \longmapsto \left(F_{|x} : \Phi_x \to \Phi'_x\right)$ where $F_{|x} := F|\Phi_x$,

   have the following properties:

   (i) the functors $\Psi \circ \Theta$ and $\mathrm{id}_{\mathcal{G}}$ are naturally equivalent (the family of isomorphisms from remark (2) above is a natural equivalence),

   (ii) the functors $\Theta \circ \Psi$ and $\mathrm{id}_{\mathcal{P}}$ are not naturally equivalent because the mapping

$$\mathrm{Hom}(P, P') \to \mathrm{Hom}\left(\left(PP^{-1}\right)_x, \left(P'P'^{-1}\right)_x\right), \quad \varphi \longmapsto \Psi(\varphi)_{|x},$$

   is not, in general, a bijection (for example, $\Psi(\varphi \circ R_a) = \Psi(\varphi)$ where $R_a : P \to P'$, $z \longmapsto z \cdot a$).

The above properties cause that the functors Ψ and Θ do not establish the quasi-isomorphy of the categories considered.

6 The Pradines-type groupoids

P.Libermann [12] discovered a relationship between the vector bundle $J^k(TV)$ and the Lie groupoid $\Pi^k(V)$. It is the family of linear isomorphisms

$$J^k(TV)_{|x} \to T_h\left(\Pi^k(V)_x\right), \quad h \in \Pi^k(V)_x, \quad x \in V,$$

which enables one to define a linear isomorphism between the $C^\infty(V)$-module of global sections of the bundle $J^k(TV)$ and the $C^\infty(V)$-module of right-invariant vector fields on $\Pi^k(V)$.

J.Pradines following P.Libermann constructed (in papers [15], [16]), for each differential groupoid, some vector bundle which fulfils the analogous property (for details, see A.Kumpera [9], [10]). That bundle (with natural algebraic structures) was called by Pradines the *Lie algebroid* of a given diff. groupoid. It plays the analogous role to that of a Lie algebra of a Lie group.

The present author's observations show that by following the idea of Pradines and making inescapabable use of *differential spaces in the sense of Sikorski*, one can construct an object analogous to groupoids from a much wider class, namely to smooth groupoids. The algebraic aspect of these objects will be dealt with in the fourth paper of this series. Now, we shall only define the objects. In general they are not vector bundles (among other things, for lack of the equality of dimensions of fibres). A particularly interesting situation will take place in the case when these objects are vector bundles.

First of all, any smooth groupoid (7) determines

(1) the diff. space

$$\left(A(\Phi), (TC)_{A(\Phi)}\right) \tag{11}$$

where

$$A(\Phi) = \bigsqcup_{x \in V} T_{u_x}\Phi_x \subset T\Phi$$

and C is the diff. structure of Φ,

(2) the projection

$$p : A(\Phi) \to V, \quad p(v) = x \iff v \in T_{u_x}\Phi_x.$$

Diff. space (11), which will be briefly denoted by $A(\Phi)$, is (by definition) a proper diff. subsapce of the tangent bundle $T\Phi$ (see def. (4). Of course, p is smooth because

$p = \pi | A(\Phi)$ where $\pi : T\Phi \to \Phi$ is the natural projection. The structure of a vector space is defined in each fibre of p. Below it will be shown that system

$$(A(\Phi), p, V) \tag{12}$$

possesses some other structures of algebraic nature with which it will be called the *algebroid of this groupoid* Φ.

Proposition 26 *System (12) has the properties:*

(1)

$$\begin{aligned} + &: A(\Phi) \oplus A(\Phi) \to A(\Phi), \quad (v,w) \longmapsto v+w, \\ \cdot &: \mathbb{R} \times A(\Phi) \to A(\Phi), \ (r,v) \longmapsto r \cdot v, \end{aligned}$$

are smooth, where $A(\Phi) \oplus A(\Phi)$ is the proper diff. subspace of $A(\Phi) \times A(\Phi)$ consisting of all pairs (v,w) for which $p(v) = p(w)$,

(2) for any sections $\xi_1, ..., \xi_m$ of the projection p and an open set $U \subset V$ such that the vectors $\xi_1(x), ..., \xi_m(x)$ are linearly independent for $x \in U$, the mapping

$$\varphi : U \times \mathbb{R}^m \to A(\Phi), \quad (x,a) \longmapsto \sum_{i=1}^{m} a^i \cdot \xi_i(x),$$

is a diffeomorphism onto its image.

Proof. This is an immediate consequence of Proposition 5. ■

Definition 27 *By a groupoid of Pradines type we shall mean each smooth groupoid (7) for which the system (12) is a vector bundle.*

Example 28 (J.Pradines [15], [16]) *Differential groupoid (7) (in particular, Lie algebroid) is of Pradines type. Indeed, $A(\Phi) \cong u^* T^\alpha \Phi$ where $T^\alpha \Phi := \{v \in T\Phi;\ \alpha_* v = 0\}$.*

Example 29 *Groupoid (8) of any regular foliation $\mathcal{F}$ (see Example 20) is of Pradines type. Indeed, we define the mapping*

$$\varkappa : T\mathcal{F} \to A(R)$$

by the formula $\varkappa_{|x} = (\gamma_x)_{*x}$ (see (9)). $\varkappa_{|x}$ is, of course, a linear isomorphism, so to notice that $A(R)$ is a vector bundle, it suffices to show that $\varkappa$ is a diffeomorphism. The smoothness of $\varkappa$ follows from the fact that it is an appropriate restriction of the monomorphism $TV \ni x \longmapsto (0,v) \in T(V \times V)$ (over $V \ni x \longmapsto (x,x) \in V \times V$), while the smoothness of $\varkappa^{-1}$ - of the mapping $(\mathrm{pr}_2)_*$.

Example 30 *The smooth groupoid Φ', defined in Theorem 24 with the help of the equivalence relation R of a foliation $\mathcal{F}$, is a groupoid of Pradines type. In fact, since $\tilde{\beta}_* := \beta_* | A(\Phi) : A(\Phi) \to TV$ is an epimorphism and $\Phi'_x = \beta_x^{-1}[L_x]$, therefore*

$$A(\Phi') = \tilde{\beta}_*^{-1}[T\mathcal{F}]$$

is a vector subbundle of the bundle $A(\Phi)$.

Proposition 26 immediately implies

Proposition 31 *A smooth groupoid (7) is a groupoid of Pradines type if and only if*

(1) for each vector $v \in A(\Phi)$, *there exists a section* ξ *of* p *such that* $\xi(p(v)) = v$,

(2) the function $V \ni x \longmapsto \dim A(\Phi)_{|x}$ *is constant.*

Some other conditions (in terms of the elements of Φ) characterizing the Pradines-type property will be looked below.

Now, we give a generalization of the definition of groupoids of Pradines type, inspired by foliations with singularities in the sense of P.Stefan [21] and by the Frobenius-Sussmann theorem [23].

Definition 32 *By a Pradines-type groupoid with singularities we shall mean any smooth groupoid (7) for which system (12) has the property:*

- *for each vector* $w \in A(\Phi)$, *there exists a smooth section* $\xi : V \to A(\Phi)$ *of the projection* p, *such that* $\xi(pw) = w$.

Example 33 *Groupoid (8) of any foliation* $\mathcal{F}$ *with singularities is of Pradines type with singularities. Indeed, we ought to notice that the distribution* $T\mathcal{F} = \bigcup_{x \in V} T_x(L_x) \subset TV$ *(*$x \in L_x \in \mathcal{F}$*) is the so-called smooth distribution in the sense of Sussmann [23], i.e. for each vector* $v \in T\mathcal{F}$, *there exists a smooth vector field* $X \in \mathfrak{X}(T\mathcal{F})$ *such that* $X(\pi v) = v$.

Example 34 *The smooth groupoid* Φ', *defined in Theorem 24 with the help of the equivalence relation* R *of a foliation* $\mathcal{F}$ *with singularities, is a groupoid of Pradines type with singularities. Indeed, first of all,* $A(\Phi') = \tilde{\beta}_*^{-1}[T\mathcal{F}]$. *Next, we take* $w \in A(\Phi')$, $v := \tilde{\beta}_*(w)$ *and* $X \in \mathfrak{X}(T\mathcal{F})$ *such that* $X(\pi v) = v$. *Because of the surjectivity of* $\operatorname{Sec}\tilde{\beta}_* : \operatorname{Sec} A(\Phi) \to \mathfrak{X}(M)$, *we find a section* $\xi \in \operatorname{Sec} A(\Phi)$ *such that* $\tilde{\beta}_* \xi = X$ *and* $\xi(p(w)) = w$. *Of course,* $\xi \in \operatorname{Sec} A(\Phi')$.

Proposition 11 and the above examples justify the adoption of the followng generalization of the notion of a vector bundle:

Definition 35 *By a vector bungle with singularities we mean the system*

$$(A, p, V)$$

consisting of diff. spaces A *and* V *and a smooth mapping* $p : A \to V$ *in whose fibres there are structures of vector spaces, and we assume the axioms:*

(1)

$$\begin{aligned} + &: A \oplus A \to A, \quad (v, w) \longrightarrow v + w, \\ \cdot &: \mathbb{R} \times A \to A, \quad (r, v) \longmapsto r \cdot v, \end{aligned}$$

are smooth, where $A \oplus A$ *is the proper diff. subspace of* $A \times A$ *consisting of all pairs* (v, w) *for which* $p(v) = p(w)$,

(2) if $\xi_1, ..., \xi_m$ are smooth sections of p and U is open in V, such that $\xi_1(x), ..., \xi_m(x)$ are linearly independent for $x \in U$, then the function

$$\varphi : U \times \mathbb{R}^m \to A, \quad (x, a) \longmapsto \sum_i a^i \cdot \xi_i(x),$$

is a diffeomorphism onto its image,

(3) for each $v \in A$, there exists a section ξ of p such that $\xi(p(v)) = v$.

Example 36 *If (M, C) is any diff.space, then the system $(T'(M, C), \pi', (M, C))$ where (a) $T'(M, C)$ is the proper diff. subspace of the tangent bundle over (M, C), equal to $\bigsqcup_{x \in M} T'_x(M, C)$ where $T'_x(M, C)$ is the vector space consisting of those vectors v for which there exists a smooth vector field $X \in \mathfrak{X}(M, C)$ such that $X(x) = v$, (b) $\pi' : T'(M, C) \to (M, C)$ is the restriction of π.*

Proposition 37 *Smooth groupoid (7) is a Pradines-type groupoid with singularities if and only if system (12) is a vector bundle with singularities.*

7 Smooth groupoids over foliations

Definition 38 *By a smooth groupoid over a foliation $\mathcal{F}$ on a diff. manifold V we mean a smooth groupoid (7) for which*

(1) the family of abstract classes of R_Φ is equal to $\mathcal{F}$,

(2) $\beta_x : \Phi_x \to L_x$ is a submersion for each $x \in V$.

Axiom (2) enables one to equip the isotropy group $G_x = \beta_x^{-1}(x)$ with the structure of the proper submanifold of Φ_x. Of course, G_x is a diff. subspace of Φ (see Def. 6).

Smooth groupoids over foliations are examples of the so-called β-regular Lie groupoids, according to terminology of A.Kumpera [10, p.41].

Example 39 *By Theorem 24 the groupoid Φ' from Example 30 is a smooth groupoid over the foliation $\mathcal{F}$.*

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