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Hirzebruch signature operator for transitive Lie algebroids

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Lie algebroids appeared as infinitesimal objects of Lie groupoids, principal fibre bundles, vector bundles (Pradines, 1967), TC-foliations and nonclosed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalences are known as Lie pseudo-algebras (Herz 1953) called also further as Lie-Rinehart algebras (Huebschmann, 1990).

A *Lie algebroid* on a manifold M is a triple

$$A = (A, [\![\cdot,\cdot]\!], \#_A)$$

where A is a vector bundle on M, $(\operatorname{Sec} A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

$$\#_A : A \to TM$$

is a linear homomorphism (called the anchor) of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta, \quad f\in C^\infty(M),\ \xi,\eta\in\operatorname{Sec} A.$$

The anchor is bracket-preserving,

$$\#_A\circ[\![\xi,\eta]\!] = [\#_A\circ\xi, \#_A\circ\eta].$$

A Lie algebroid is called **transitive** if $\#_A$ is an epimorphism.

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0,$$

$\boldsymbol{g} := \ker \#_A$. The fiber $\boldsymbol{g}_x$ of the bundle $\boldsymbol{g}$ in the point $x \in M$ is the Lie algebra with the commutator operation being

$$[v,w] = [\![\xi,\eta]\!](x), \quad \xi,\eta \in \operatorname{Sec} A, \quad \xi(x) = v, \eta(x) = w, \quad v,w \in \boldsymbol{g}_x.$$

The Lie algebra $\boldsymbol{g}_x$ is called the isotropy Lie algebra of L at $x \in M$. The vector bundle $\boldsymbol{g}$ is a Lie Algebra Bundle (LAB in short), called the *adjoint* of A, the fibres are isomorphic Lie algebras.

TM is a Lie algebroid with $id : TM \to TM$ as the anchor,

$\mathfrak{g}$ -finitely dimensional Lia algebra - is a Lie algebroid over $M = \{*\}$.

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \mathrm{Sec} \bigwedge A^*, \quad \text{- the space of cross-sections of } \bigwedge A^*$$

$$d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

$$\begin{aligned}(d_A\omega)(\xi_0, ..., \xi_k) = & \sum_{j=0}^{k} (-1)^j \left(\#_A \circ \xi_j\right)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ & + \sum_{i<j} (-1)^{i+j} \omega\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k\right),\end{aligned}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \mathrm{Sec}\, A$. The operators d_A^k satisfy

$$d_A(\omega \wedge \eta) = d_A\omega \wedge \eta + (-1)^k \omega \wedge d_A\eta,$$

so they are first order and the symbol of d_A^k is equal to

$$\begin{aligned} S\left(d_A^k\right)_{(x,v)} &: \bigwedge^k A_x^* \to \bigwedge^{k+1} A_x^* \\ S\left(d_A^k\right)_{(x,v)}(u) &= \left(v \circ (\#_A)_x\right) \wedge u, \quad 0 \neq v \in T_x^*M. \end{aligned}$$

In consequence the sequence of symbols

$$\bigwedge^{k} A_x^* \overset{S\left(d_A^k\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+1} A_x^* \overset{S\left(d_A^{k+1}\right)_{(x,v)}}{\longrightarrow} \bigwedge^{k+2} A_x^*$$

is exact which imply that the complex $\left\{d_A^k\right\}$ is an **elliptic complex.**

The exterior derivative d_A introduces the cohomology algebra

$$\mathbf{H}\left(A\right) = \mathbf{H}\left(\Omega\left(A\right), d_A\right).$$

For the trivial Lie algebroid TM - the tangent bundle of the manifold M - the differential d_{TM} is the usual de-Rham differential d_M of differential forms on M whereas, for $L = \mathfrak{g}$ - a Lie algebra $\mathfrak{g}$ - the differential $d_{\mathfrak{g}}$ is the usual Chevalley-Eilenberg differential, $d_{\mathfrak{g}} = \delta_{\mathfrak{g}}$.

For each transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A \xrightarrow{\#_A} TM \to 0,$$

over **compact oriented** manifold M the following conditions are equivalent (Kubarski-Mishchenko, 2004) ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, i.e. $\operatorname{rank} A = m + n$)

(1) $\mathbf{H}^{m+n}(A) \neq 0$,

(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$,

(3) A is the so-called *invariantly oriented,* i.e. there exists a global non-singular cross-section $\boldsymbol{\varepsilon}$ of the vector bundle $\bigwedge^n \boldsymbol{g}$,

$$\begin{aligned} \varepsilon &\in \operatorname{Sec}\left(\bigwedge^n \boldsymbol{g}\right) \\ 0 &\neq \varepsilon_x \in \bigwedge^n \boldsymbol{g}_{|x} \end{aligned}$$

invariant with respect to the adjoint representation of A in the vector bundle $\bigwedge^n \boldsymbol{g}$ (which is extending of the adjoint representation ad_A of A in $\boldsymbol{g}$ given by $(ad_A)(\xi) : \operatorname{Sec} \boldsymbol{g} \to \operatorname{Sec} \boldsymbol{g}$, $\nu \longmapsto [\![\xi, \nu]\!]$; in other words, ad_A is a homomorphism of A into the Lie algebroid $A(\boldsymbol{g})$ of the vector bundle $\boldsymbol{g}$, and next into $A(\bigwedge^n \boldsymbol{g})$.

Assume that A is invarianty oriented. The scalar Poincaré product

$$\mathcal{P}_A^i : \mathbf{H}^i(A) \times \mathbf{H}^{m+n-i}(A) \to \mathbb{R},$$

$$([\omega], [\eta]) \longmapsto \int_A^{\#} \omega \wedge \eta \quad \left(:= \int_M \left(\fint_A^{\#} \omega \wedge \eta \right) \right)$$

where the so-called *fibre integral*

$$\fint_A : \Omega^\bullet(A) \to \Omega_{dR}^{\bullet - n}(M)$$

is defined by the formula ($\boldsymbol{\varepsilon} \in \operatorname{Sec}(\bigwedge^n \boldsymbol{g})$ is nonzero)

$$\left(\fint_A \omega^k \right)_x (w_1, ..., w_{k-n}) = (-1)^{kn} \omega_x (\boldsymbol{\varepsilon}_x, \tilde{w}_1, ..., \tilde{w}_{k-n}), \quad \#_A(\tilde{w}_i) = w_i,$$

The operator $\fint_A$ commutes with the differentials d_A and d_M giving a homomorphism in cohomology

$$\fint_A^{\#} : \mathbf{H}^\bullet(A) \to \mathbf{H}_{dR}^{\bullet - n}(M).$$

In particular we have

$$\int_A^{\#} : \mathbf{H}^{m+n}(A) \overset{\cong}{\to} \mathbf{H}^{m}_{dR}(M) = \mathbb{R}.$$

The scalar product $\mathcal{P}^i_A$ is nondegenerated and if $m + n = 4k$ then

$$\mathcal{P}^{2k}_A : \mathbf{H}^{2k}(A) \times \mathbf{H}^{2k}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and is called the signature of A, and is denoted by

$$\mathrm{Sign}(A).$$

The problem is:

- to calculate the signature $\mathrm{Sign}(A)$ and give some conditions to the equality $\mathrm{Sign}(A) = 0$. There are examples for which $\mathrm{Sign}(A) \neq 0$.

My talk concerns this problem.

(I) Firstly, I give a general mechanism of the calculation of the signature via spectral sequences (Kubarski-Mishchenko 2003) and use to two kinds of spectral sequences associated with Lie algebroids:

a) the **spectral sequence of the Čech-de Rham complex,**

b) the **Hochschild-Serre spectral sequence.**

(II) Secondly, using the $*$-Hodge operator we construct **two** Hirzebruch operators to calculate the signature. For each of them the index is equal to the signature of A. Therefore the Atiyah-Singer formula for the index can be used to calculate of the signature.

AD(I)

(i) The general approach to the use of the spectral sequences.

The idea of applying spectral sequences to the signature comes from

- Chern-Hirzebruch-Serre *On the index of a fibered manifold,* Proc. AMS, 8 (1957), 587-596.

Via spectral sequences the authors proved

Theorem 1 *Let $E \to M$ be a fiber bundle, wih the typical fiber F, such that the following conditions are satisfied:*

(1) E, M, F are compact connected oriented manifolds;

(2) the fundamental group $\pi_1(M)$ acts trivially on the cohomology ring $H^*(F)$ of F.

Then, if E, M, F are oriented coherently, so that the orientation of E is induced by those of F and M, the index of E is the product of the indices of F and M, that

$$\mathrm{Sign}\,(E) = \mathrm{Sign}\,(F) \cdot \mathrm{Sign}\,(M)\,.$$

The authors consider the cohomology Leray spectral sequence $E_s^{p,q}$ of the bundle $E \to B$ with the real fields as the coefficients field. The term E_2 by hypothesis (2) is the bigraded algebra

$$E_2^{p,q} \cong \mathbf{H}^p\left(M; \mathbf{H}^q\left(F\right)\right) \cong \mathbf{H}^p\left(M\right) \otimes \mathbf{H}^q\left(F\right),$$

therefore

$$E_2^{p,q} = 0 \text{ for } \quad p > m \quad \text{or} \quad q > n.$$

Clearly, E_2 is a **Poincaré algebra** by hypothesis (1). Using the spectral sequence argument the authors noticed that

$$\left(E_s, d_s, \cdot\right) \;\; s \geq 2,$$

and

$$\left(E_\infty, \cdot\right)$$

are Poincaré algebras and

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = ... = \operatorname{Sign} E_\infty = \operatorname{Sign} \mathbf{H}\left(E\right).$$

It appears that the Chern-Hirzebruch-Serre arguments used to prove the above theorems on the signature of the total space of the bundle $E \to M$ are pure algebraic and lead to the following general theorems (Kubarski-Mishchenko 2003).

Theorem 2 *Let* $((A, \langle , \rangle), A^r, \cup, D, A_j)$ *be any DG-algebra with a decreasing regular filtration* A_j

$$A = A_0 \supset \cdots \supset A_j \supset A_{j+1} \supset \cdots$$

and $(E_s^{p,q}, d_s)$ *its spectral sequence. We assume that there exist natural numbers* m *and* n *with the following conditions:*

- $E_2^{p,q} = 0$ *for* $p > m$ *and* $q > n$, $m + n = 4k$,
- E_2 *is a Poincaré algebra with respect to the total gradation and the top group* $E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}$.

Then each term $\left(E_s^{(\bullet)}, \cup, d_s\right)$ $2 \leq s < \infty$, *and* $\left(E_\infty^{(\bullet)}, \cup\right)$ *are Poincaré algebra*

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = ... = \operatorname{Sign} E_\infty = \operatorname{Sign} \mathbf{H}(A).$$

If m and n are odd then $\operatorname{Sign} E_2 = 0$, *if m and n are even then*

$$\begin{aligned}
\operatorname{Sign} E_2 &= \operatorname{Sign}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \\
&= \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right)
\end{aligned}$$

(ii) Using the Leray spectral sequence (the spectral sequence of the Čech-de Rham complex).

We use this mechanism to

(a) the spectral sequence for the Čech-de Rham complex of the Lie algebroid A.

Let A be any transitive Lie algebroid on a manifold M with isotropy Lie algebras $\boldsymbol{g}_{|x}$ isomorphic to a given Lie algebra

$$\mathfrak{g}.$$

If $U \subset M$ is an open subset diffeomorphic to $\mathbb{R}^m$ then the restriction $A_{|U}$ is the Lie algebroid isomorphic to the trivial one $TU \times \mathfrak{g}$ and

$$\mathbf{H}\left(A_{|U}\right) \cong \mathbf{H}(U) \otimes \mathbf{H}\left(\mathfrak{g}\right) \cong \mathbf{H}\left(\mathfrak{g}\right).$$

Given a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of M, where J is a countable ordered index set (this means that all U_α and all finite intersections $\bigcap_i U_{\alpha_i}$ are diffeomorphic to an Euclidean space $\mathbb{R}^m$) we can form the double complex (of the Čech-de Rham type)

$$K^{p.q} = C^p\left(\mathfrak{U}, \Omega^q\left(A\right)\right) := \prod_{\alpha_0 < \ldots < \alpha_p} \Omega^q\left(A_{|U_{\alpha_0 \ldots \alpha_p}}\right)$$

$p, q \geq 0$, with the product structure

$$\cup : K^{p,q} \times K^{r,s} \to K^{p+r,q+s}$$
$$(\omega \cup \eta)_{\alpha_0 \ldots \alpha_{p+r}} = (-1)^{qr}\, \omega_{\alpha_0 \ldots \alpha_p} | U_{\alpha_0 \ldots \alpha_{p+r}} \wedge \eta_{\alpha_p \ldots \alpha_{p+r}} | U_{\alpha_0 \ldots \alpha_{p+r}}.$$

This complex has two boundary homomorphisms, d and δ.

The vertical homomorphism $d : C^p\left(\mathfrak{U}, \Omega^q\left(A\right)\right) \to C^p\left(\mathfrak{U}, \Omega^{q+1}\left(A\right)\right)$ acts as external differential of A-forms

$$d = (-1)^p\, d_A.$$

The horizontal homomorphism $\delta : C^p(\mathfrak{U}, \Omega^q(A)) \to C^{p+1}(\mathfrak{U}, \Omega^q(A))$ acts as a coboundary homomorphism

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \, \omega_{\alpha_0 \ldots \hat{\imath} \ldots \alpha_{p+1}} | U_{\alpha_0 \ldots \alpha_{p+1}}.$$

The horizontal and vertical homomorphisms δ and d are antiderivations of degree $+1$ therefore

$$(K, K^{p,q}, \cup, d, \delta)$$

is a double complex of the first quadrant with a product structure. Now, consider the "horizontal" decreasing filtration

$$K_j = \bigoplus_{\substack{p \geq j \\ q \geq 0}} K^{p,q}.$$

Due to the general construction of spectral sequences for the filtration above which is in accord with the multiplicative structure of the DG-algebra, $\left(K, K^{(r)}, \cup, D, K_j\right)$, one can construct the spectral sequence of the graded differential algebras

$$(E_s^{p,q}, d_s),$$

The filtration K_j is **regular,** $K_0 = K$, therefore the spectral sequence $\left(E_s^{p,q}, d_s\right)$ converge to $H\left(K, D\right)$.

Theorem 3 *(1) The zero term* $\left(E_0, d_0\right)$ *:*

$$\begin{aligned} E_0^p &= K_p/K_{p+1}, \quad E_0^{p,q} = K^{p,q}, \\ d_0 &= d : K^{p,q} = C^p\left(\mathfrak{U}, \Omega^q\left(A\right)\right) \longrightarrow K^{p,q+1} = C^p\left(\mathfrak{U}, \Omega^{q+1}\left(A\right)\right), \end{aligned}$$

(2) The first term $\left(E_1, d_1\right)$

$$E_1^{p,q} = \mathbf{H}^{p,q}\left(K, d\right) = C^p\left(\mathfrak{U}, \mathcal{H}^q\left(A\right)\right), \quad d_1 = \delta^{\#} : E_1^{p,q} \rightarrow E_1^{p+1,q},$$

where

$$\mathcal{H}^*\left(A\right) = \left(U \longmapsto \mathbf{H}^*\left(A_{|U}\right)\right)$$

is the Leray type presheaf of cohomology, locally constant on the good covering $\mathfrak{U}$*, with values in the cohomology algebra* $\mathbf{H}^*\left(\mathfrak{g}\right)$ *of the structural Lie algebra* $\mathfrak{g}$*.*

(3) The second term

$$E_2^{p,q} = \mathbf{H}^{p,q}\left(H\left(K, D\right), \delta^{\#}\right) = \mathbf{H}^p_{\delta^{\#}}\left(\mathfrak{U}, \mathcal{H}^q\left(A\right)\right).$$

We assume that

- the presheaf $\mathcal{H}(A)$ is constant on the good covering $\mathfrak{U}$.

The condition is equivalent to that the monodromy representation of the presheaf $\mathcal{H}(A)$

$$\rho : \pi_1(M) = \pi_1(N(\mathfrak{U})) \to \mathrm{Aut}(\mathbf{H}(\mathfrak{g}))$$

is trivial.

Example 4 *The condition of the triviality of the monodromy holds if*

- M is simply connected,
- $\mathrm{Aut}\, G = \mathrm{Int}\, G$, where G is simply connected Lie group with the Lie algebra $\mathfrak{g}$, for example, if $\mathfrak{g}$ is a simple Lie algebra of type

$$B_l\ (= SO(2l+1)),\quad C_l\ (= Sp(2l)\ ),\ E_7,\ E_8,\ F_4,\ G_2.$$

- the adjoint Lie algebra bundle $\boldsymbol{g}$ is trivial in the category of flat bundles (the bundle $\mathbf{H}(\boldsymbol{g})$ of cohomology of isotropy Lie algebras with the typical fibre $\mathbf{H}(\mathfrak{g})$ possess canonical flat covariant derivative - which will be important for studying of the Hochshild-Serre spectral sequence). For example for the Lie algebroid $A(G;H)$ of the the TC-foliation of left cosets of a nonclosed Lie subgroup H in any Lie group G.

If the monodromy representation of the presheaf $\mathcal{H}(A)$ is **trivial** then

$$\begin{aligned} E_2^{p,q} &= \mathbf{H}^p_{\delta\#}\left(\mathfrak{U}, \mathcal{H}^q(A)\right) \cong \mathbf{H}^p\left(\mathfrak{U}, \mathbf{H}^q(\mathfrak{g})\right) \\ &\cong \mathbf{H}^p(\mathfrak{U}, \mathbb{R}) \otimes \mathbf{H}^q(\mathfrak{g}) \\ &\cong \mathbf{H}^p_{dR}(M) \otimes \mathbf{H}^q(\mathfrak{g}). \end{aligned}$$

All isomorphisms are canonical isomorphisms of bigraded algebras. It means that E_2 lives in the rectangle $p \leq m$, $q \leq n$, and

$$E_2^{(m+n)} = E_2^{m,n} = \mathbf{H}^m_{dR}(M) \otimes \mathbf{H}^n(\mathfrak{g}) = \mathbb{R}$$

therefore we have:

Theorem 5 *If A is a transitive Lie algebroid on a compact oriented connected manifold M with unimodular isotropy Lie algebras $\mathfrak{g}_{|x} \cong \mathfrak{g}$ and monodromy representation of the presheaf $\mathcal{H}(A)$ is trivial than the terms $E_2, ..., E_\infty$ as well as the cohomomology algebra $\mathbf{H}(A)$ are Poincaré algebras and*

$$\begin{aligned} \operatorname{Sign}(A) &= \operatorname{Sign}\mathbf{H}(A) = \operatorname{Sign} E_2 \\ &= \operatorname{Sign}\left(\mathbf{H}_{dR}(M) \otimes \mathbf{H}(\mathfrak{g})\right) = \operatorname{Sign}\mathbf{H}_{dR}(M) \cdot \operatorname{Sign}\mathbf{H}(\mathfrak{g}) \\ &= \operatorname{Sign}\mathbf{H}_{dR}(M) \cdot 0 = 0 \end{aligned}$$

because for unimodular Lie algebra $\mathfrak{g}$

$$\operatorname{Sign}\mathbf{H}(\mathfrak{g}) = \operatorname{Sign}\bigwedge \mathfrak{g}^* = 0.$$

(iii) Using the Hochshild-Serre spectral sequences.

Following Hochschild-Serre (*Cohomology of Lie algebras,* Ann. Math. 57, 1953, 591-603) for a pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g}, \mathfrak{h})$ one can consider

– a graded cochain group of $\mathbb{R}$-linear alternating functions

$$A_{\mathbb{R}} = \bigoplus_{k \geq 0} A^k, \quad A^k = C^k(\mathfrak{g}),$$

– with the standard $\mathbb{R}$-differential operator d of degree 1

– and Hochschild-Serre decreasing filtration $A_j \subset A_{\mathbb{R}}$ as follows:

— $A_j = A_{\mathbb{R}}$ for $j \leq 0$,

— if $j > 0$, $A_j = \bigoplus_{k \geq j} A_j^k$, $A_j^k = A_j \cap A^k$, where A_j^k consists of all those k-cochains f for which $f(\gamma_1, ..., \gamma_k) = 0$ whenever $k - j + 1$ of the arguments γ_i belongs to $\mathfrak{k}$

$$A_j^k = \begin{cases} f \in A^k = C^k(\mathfrak{g}) \\ f(\gamma_1, ..., \gamma_k) = 0 \\ \forall \gamma_1, ..., \gamma_{k-j+1} \in \mathfrak{k}. \end{cases}$$

In this way, we have obtained a graded filtered differential $\mathbb{R}$-vector space

$$\left(A_{\mathbb{R}} = \bigoplus_{k \geq 0} A^k, d, A_j \right)$$

and we can use its spectral sequence

$$\left(E_s^{p,q}, d_s \right).$$

For a transitive Lie algebroid $A = (A, [\![\cdot, \cdot]\!], \#_A)$ with the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \overset{\#_A}{\to} TM \to 0$ we will consider the pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g}, \mathfrak{k})$

where

$$\mathfrak{g} = \mathrm{Sec}\,(A)\,, \quad \mathfrak{k} = \mathrm{Sec}\,(\boldsymbol{g})\,.$$

Following K.C.M.Mackenzie (1987) (see also V.Itskov, M.Karashev, and Y.Vorobjev (1998)), we will consider the $C^{\infty}(M)$-submodule of $C^{\infty}(M)$-linear altarnating cochains

$$\Omega^{k}(A) \subset C^{k}(\mathfrak{g})$$

and the induced filtration

$$\Omega_{j} = \Omega_{j}(A) = A_{j} \cap \Omega(A)$$

of $C^{\infty}(M)$-modules. We obtain in this way a graded filtered differential space

$$\left(\Omega(A) = \bigoplus_{k} \Omega^{k}(A)\,, d_{A}, \Omega_{j}\right)$$

and its spectral sequence

$$\left(E_{A,s}^{p,q}, d_{A,s}\right).$$

Assume as above

$$m = \dim M, \;\; n = \dim \boldsymbol{g}_{|x}, \quad \text{i.e. } \operatorname{rank} A = m + n.$$

Theorem 6 *There is a flat covariant derivative* ∇^q *in the vector bundle* $\mathbf{H}^q(\boldsymbol{g})$ *such that*

$$E_{A,2}^{p,q} \cong \mathbf{H}^p_{\nabla^q}\left(M;\mathbf{H}^q(\boldsymbol{g})\right).$$

The flat covariant derivative ∇^q *is defined by the formula: for* $f \in \Omega^p\left(M; Z\left[\bigwedge^q \boldsymbol{g}^*\right]\right)$, $[f] \in \Omega^p\left(M;\mathbf{H}^q(\boldsymbol{g})\right)$

$$\nabla^q_X [f] = [\mathcal{L}_X f]$$

where $(\mathcal{L}_X f)(\sigma_1,...,\sigma_q) = \partial_X\left(f(\sigma_1,...,\sigma_q)\right) - \sum_{i=1}^q f(\sigma_1,...,[\![\lambda X,\sigma_i]\!],...,\sigma_q)$ *(where* $\lambda : TM \to A$ *is arbitrary auxiliary connection in* A*).*

Theorem 7 *If* A *is a transitive invariantly oriented Lie algebroid such that* $m+n=4k$ *(*$m=\dim M,\quad n=\dim \boldsymbol{g}_{|x}$ *) then*

a) if m *and* n *are odd then* $\operatorname{Sign} A = 0$,

b) if m *and* n *are even then*

$$\begin{aligned}\operatorname{Sign} A &= \operatorname{Sign} E_2 = \operatorname{Sign}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right)\\ &= \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right)\end{aligned}$$

where

$$E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)$$

and

$$\mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\right)\times\mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\right)\rightarrow\mathbf{H}^{m}_{\nabla^{n}}\left(M;\mathbf{H}^{n}\left(\boldsymbol{g}\right)\right)=\mathbb{R}$$

is defined via the usual multiplication of differential forms with respect to the multiplication of cohomology class for Lie algebras.

$$\phi:\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\times\mathbf{H}^{\frac{n}{2}}\left(\boldsymbol{g}\right)\rightarrow\mathbf{H}^{n}\left(\boldsymbol{g}\right)=M\times\mathbb{R}.$$

We notice that

- if $\frac{n}{2}$ is even then $\frac{m}{2}$ is even, $\dim M = m = 4s$ for some s, ϕ is symmetric nondegenerated,
- if $\frac{n}{2}$ is odd then $\frac{m}{2}$ is odd, $\dim M = 4s + 2$ for some s, ϕ is symplectic.

However always $\mathbf{H}^k\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{m-k}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$ is strong nondegenerated

$$\begin{aligned}\mathbf{H}^k\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) &= \mathbf{H}^{m-k}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)^*, \\ \mathbf{H}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) &= \mathbf{H}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)^*, \\ \dim \mathbf{H}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) &< \infty,\end{aligned}$$

and

$$\mathbf{H}^{\frac{m}{2}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{\frac{m}{2}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$$

is symmetric nondegenerated.

AD(II) Hirzebruch operator and the signature.

(i) general approach.

Four foundamental examples

We describe a general approach to the following four fundamental examples of the spaces with gradation and differential operator $\left(W=\oplus_{k=0}^{N}W^{k},d\right)$. M is here compact oriented Riemann manifold.

$$W^{k}=\begin{cases}\Omega^{k}\left(M\right),\ d_{dR};\quad N=4r,\\ \Omega^{k}\left(A\right),\ d_{A};\quad N=m+n=4r,\quad A\text{ - a transitive inv. or. Lie algebroid}\\ \Omega^{k}\left(M;E\right),\quad d_{\nabla};\quad \left(E,\left(,\right)_{0}\right)\text{ flat vector bundle,}\\ \qquad\qquad\left(,\right)_{0}\text{ - symmetric nondegenerated parallel, }\ N=4r\\ \Omega^{k}\left(M;E\right),\quad d_{\nabla};\quad \left(E,\left\langle ,\right\rangle_{0}\right)\text{ flat vector bundle,}\\ \qquad\qquad\left\langle ,\right\rangle_{0}\text{ -symplectic parallel, }\ N=4r+2\end{cases}$$

In all cases the sequences of differentials $\left\{d_{dR}^{k}\right\}$, $\left\{d_{A}^{k}\right\}$, $\left\{d_{\nabla}^{k}\right\}$ are elliptic complexes, $\dim\mathbf{H}^{k}\left(W\right)<\infty$ and the pairing

$$\mathbf{H}^{k}\left(W\right)\times\mathbf{H}^{N-k}\left(W\right)\rightarrow\mathbb{R}$$

is defined, which in the middle degree $\frac{N}{2}$ is symmetric. Its signature is defined to be the signature of W, $\operatorname{Sign}\left(W\right)$.

Below, we give a common algebraic approach to calculate the signature $\operatorname{Sign}(W)$ via the Hirzebruch signature operator.

Algebraic point of view on the $*$-Hodge operator, Hodge Theorem and Hirzebruch operator

Definition 8 *By a* ***Hodge space*** *we mean the triple* $(W, \langle , \rangle, (,))$ *where* W *is a real vector space* ($\dim W$ *finite or infinite),* $\langle , \rangle$, $(,) : W \times W \to \mathbb{R}$ *are* 2*-linear tensors such that*

(1) $(,)$ *is symmetric positive (i.e. is an inner product),*

(2) there exists a linear homomorphism

$$*_W : W \to W$$

called $*$***-Hodge operator*** *fulfilling properties*

(i) $\langle v, w \rangle = (v, *_W(w))$, *for all* $v \in V$,

(ii) $(v, w) = (*_W v, *_W w)$, *i.e.* $*_W$ *is an isometry with respect to* $(,)$.

Clearly, the $*$-Hodge operator is uniquely determined (if exists). The 2-tensor $\langle , \rangle$ is nondegenerated.

A 2-linear homomorphism $f : W \times W \to \mathbb{R}$ is called *nondegenerated* if both null-spaces are zero, if

$$\begin{aligned} \langle v, \cdot \rangle &= 0 \implies v = 0, \\ \langle \cdot, v \rangle &= 0 \implies v = 0. \end{aligned}$$

Two 2-tensors $f : V \times V \to \mathbb{R}$ i $g : W \times W \to \mathbb{R}$ determine tensor product

$$f \otimes g : (V \otimes W) \times (V \otimes W) \to \mathbb{R}$$

which is 2-linear. The tensor $f \otimes g$ is nondegenerated if both f and g are nondegenerated and is symmetric positive if both are the same (the dimensions of V and W can be infinite).

Lemma 9 *If $(V, \langle\cdot,\cdot\rangle_V, (\cdot,\cdot)_V)$ and $(W, \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_W)$ are Hodge spaces then their tensor product*

$$(V \otimes W, \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_W, (\cdot,\cdot)_V \otimes (\cdot,\cdot)_W)$$

is a Hodge space and

$$*_{V\otimes W} = *_V \otimes *_W.$$

Example 10 (Classical) *Let (V,G) be a real n-dimensional oriented Euclidean space with inner product $G: V\times V\to\mathbb{R}$ and the volume tensor*

$$\varepsilon\in\bigwedge\nolimits^{n}V,\quad \varepsilon=e_1\wedge\ldots\wedge e_n$$

(where e_i is a positive ON basis). We identify $\bigwedge^n V=\mathbb{R}$ via the isomorphism

$$\rho:\bigwedge\nolimits^{n}V\xrightarrow{\cong}\mathbb{R},\quad s\cdot\varepsilon\longmapsto s.$$

Let

$$\begin{aligned}\langle\cdot,\cdot\rangle &: \bigwedge V\times\bigwedge V\to\mathbb{R},\\ \langle\cdot,\cdot\rangle^{k} &: \bigwedge\nolimits^{k}V\times\bigwedge\nolimits^{n-k}V\to\bigwedge\nolimits^{n}V=\mathbb{R},\\ \left\langle v^{k},v^{N-k}\right\rangle &= \rho\left(v^{k}\wedge v^{N-k}\right),\end{aligned}$$

be the usual duality (we put $\langle,\rangle=0$ outside the pairs $(k,n-k)$)

Then there exists a positive scalar product $(,)$ in $\bigwedge V$ such that $(\bigwedge V,\langle,\rangle,(,))$ is a Hodge space. Clearly

a) the subspaces $\bigwedge^k V$ are orthogonal and *b)*

$$(\cdot,\cdot)^k : \bigwedge^k V \times \bigwedge^k V \to \mathbb{R}$$
$$(v_1 \wedge ... \wedge v_k, w_1 \wedge ... \wedge w_k)^k = \det\left[G\left(v_i, w_k\right)\right].$$

The $*$-Hodge operator is determined via an ON base e_i by the formula

$$*\left(e_{i_1} \wedge ... \wedge e_{i_k}\right) = \varepsilon_{(j_1,...,j_{n-k})} \cdot e_{j_1} \wedge ... \wedge e_{j_{n-k}}$$

where $i_1 < ... < i_k$ and $j_1 < ... < j_{n-k}$ and the sequence $(j_1,...,j_{n-k})$ is complementary to $(i_1,...,i_k)$ and $\varepsilon_{(j_1,...,j_{n-k})} = \operatorname{sgn}\left(j_1,...,j_{n-k},i_1,...,i_k\right)$.

The above $*$-Hodge operator differes from classical by the coefficient (then there is $\varepsilon_{(i_1,...,i_k)}$). If change the definition of the duality $\langle,\rangle$ by multiplication on $(-1)^{k(n-k)}$, i.e. if take $\langle,\rangle_0^k = (-1)^{k(n-k)}\langle,\rangle$, then the $*$-Hodge operator will agree with the classical one.

The obtained Hodge space

$$\left(\bigwedge V, \langle,\rangle, (,)\right)$$

*is called **classical.***

Example 11 *Let (V, G) be n-dimensional Eucliden spave and $(W, \langle\cdot,\cdot\rangle, (\cdot,\cdot))$ any Hodge space then there exists canonically defined Hodge spaces - the tensor products*

$$\bigwedge V \otimes W \quad \text{and} \quad \bigwedge V^* \otimes W.$$

Lemma 12 *Let $(W, \langle\cdot,\cdot\rangle, (\cdot,\cdot))$ be a finitely dimensional real vector space equipped with two 2-tensors $\langle\cdot,\cdot\rangle, (\cdot,\cdot)$ such that the second one $(,)$ is symmetric positive. Let e_i be an ON basis of $(W, (,))$. Then the $*$-Hodge operator $*_W : W \to W$ (i.e. linear homomorphism such that $\langle v, w\rangle = (v, *w)$) is an isometry if and only if the matrix*

$$h_{ij} = \langle e_i, e_j \rangle$$

of the form $\langle,\rangle$ is orthogonal.

Example 13 *Consider arbitrary Riemann oriented manifold M and finitely dimensional Hodge space $(W, \langle\cdot,\cdot\rangle, (\cdot,\cdot))$. Then for any point $x \in M$ there exists a Hodge space $\bigwedge T_x^* M \otimes W$. Taking an ON basis of $T_x M$ and arbitrary basis s_α of W the $*$-Hodge operator $*_x : \bigwedge T_x^* M \otimes W \to \bigwedge T_x^* M \otimes W$ is defined by*

$$* \left(e_{i_1} \wedge ... \wedge e_{i_k} \otimes s_\alpha\right) = \operatorname{sgn}\left(j_1, ..., j_{n-k}, i_1, ..., i_k,\right) e_{j_1} \wedge ... \wedge e_{j_{n-k}} \otimes \sum_\beta h_{\beta\alpha} s_\beta.$$

Assuming compactness of M we can define two 2-$\mathbb{R}$-linear tensors

$$((\alpha, \beta)),\ \langle\langle\alpha, \beta\rangle\rangle : \Omega(M; W) \times \Omega(M; W) \to \mathbb{R},$$

by the formulae

$$((\alpha, \beta)) = \int_M (\alpha_x, \beta_x)\, dM, \qquad \langle\langle\alpha, \beta\rangle\rangle = \int_M \langle\alpha_x, \beta_x\rangle\, dM = \int_M \alpha \wedge_\varphi \beta$$

where

$$\varphi_x = \langle\cdot,\cdot\rangle_x^k : \bigwedge^k T_x^* M \otimes W \times \bigwedge^{n-k} T_x^* M \otimes W \to \bigwedge^n T_x^* M = \mathbb{R}$$

is the $\langle\cdot,\cdot\rangle$-wedge product. The 2-form $((\cdot,\cdot))$ is symmetric positive and the triple

$$(\Omega\left(M;W\right),\langle\langle\alpha,\beta\rangle\rangle,((\alpha,\beta)))$$

is a Hodge space with the $*$-Hodge operator $\langle\langle\alpha,\beta\rangle\rangle=((\alpha,*\beta))$ defined point by point

$$\left(*\beta\right)_x=*\left(\beta_x\right).$$

The above example can be easily generalized taking arbitrary flat vector bundles instead of a single vector space W.

Definition 14 *By a* ***Hodge space with gradation and differential operator*** *we mean the system*

$$\left(W=\bigoplus\nolimits_{k=0}^{N}W^k,\langle\cdot,\cdot\rangle,(\cdot,\cdot),d\right)$$

where $(W,\langle\cdot,\cdot\rangle,(\cdot,\cdot))$ is a Hodge space and

(1) $\langle\cdot,\cdot\rangle^k:W^k\times W^{N-k}\to\mathbb{R}$ and $\langle,\rangle=0$ outside the pairs $(k,n-k)$,

(2) W^k are orthogonal with respect to $(\cdot,\cdot)$,

(3) d is homogeneous of degree $+1$, i.e. $d:W^k\to W^{k+1}$, and $d^2=0$,

(4) $\langle dw,u\rangle=(-1)^{k+1}\langle w,du\rangle$ for $w\in W^k$.

Clearly, a pairing

$$\langle , \rangle_{\mathbf{H}}^{k} : \mathbf{H}^{k}(W) \times \mathbf{H}^{N-k}(W) \to \mathbb{R},$$

$$\langle [u], [w] \rangle_{\mathbf{H}}^{k} = \langle u, w \rangle^{k},$$

is correctly defined.

For such a Hodge space $*\left[W^{k}\right] \subset W^{N-k}$ and

$$* : W^{k} \to W^{N-k}$$

is an isomorphism.

Let $d^{*} : W \to W$ be the adjoint operator, i.e. the one such that

$$\left(d^{*}(w_1), w_2\right) = \left(w_1, d(w_2)\right).$$

Lemma 15 *The operator (called the Laplacian)*

$$\Delta := \left(d + d^{*}\right)^{2} = dd^{*} + d^{*}d$$

is self-adjoint $(\Delta v, w) = (v, \Delta w)$, *nonnegative* $(\Delta v, v) \geq 0$ *and*

$$\{v \in W;\ (\Delta v, v) = 0\} = \{v \in W;\ dv = 0 = d^{*}v\}.$$

Definition 16 *A vector $v \in W$ is called* ***harmonic*** *if $dv = 0$ i $d^* v = 0$.*

$$\mathcal{H}(W) = \{v \in W;\ dv = 0,\ d^* v = 0\},$$
$$\mathcal{H}^k(W) = \left\{v \in W^k;\ dv = 0,\ d^* v = 0\right\}.$$

The harmonic vectors form a graded vector space

$$\mathcal{H}(W) = \bigoplus\nolimits_{k=0}^{N} \mathcal{H}^k(W).$$

Lemma 17 $\mathcal{H}^k(W) = \ker\left\{d + d^* : W^k \to W\right\} = \ker\left\{\Delta^k : W^k \to W^k\right\}$, *i.e.*

$$\mathcal{H}(W) = \ker \Delta.$$

is the eigenspace of the operator Δ corresponding to the zero value of the eigenvalue.

Lemma 18

$$\mathcal{H}(W) = \ker \Delta = (\operatorname{Im} \Delta)^{\perp}.$$

Remark 19 *If W is a Hilbert space and $Y \subset W$ is closed, then we have the direct sum*

$$\boxed{W = Y \bigoplus (Y)^{\perp}}$$

In applications to Riemannian vector bundles over Riemannian compact oriented manifolds, the space $W = \operatorname{Sec}(\xi)$ is not a Hilbert one (is not complete), but

- ***the equality** $W = \operatorname{Im}\Delta \bigoplus \ker \Delta$ **holds if** Δ **is self-adjoint elliptic operator.***

Theorem 20 *(a) $\ker \Delta^{k}$ and $\operatorname{Im} d^{k-1}$ are orthogonal, therefore the inclusion*

$$\mathcal{H}^{k}(W) = \ker \Delta^{k} \hookrightarrow \ker d^{k}$$

induce a monomorphism

$$\ker \Delta^{k} \rightarrowtail \mathbf{H}^{k}(W).$$

(b) If $\boxed{W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}}$ then

$$\ker d^{k} = \ker \Delta^{k} \bigoplus \operatorname{Im} d^{k-1}.$$

Therefore the above inclusion $\mathcal{H}^k(W) = \ker \Delta^k \hookrightarrow \ker d^k$ *induce the isomorphism*

$$\mathcal{H}^k(W) \cong \ker d^k / \operatorname{Im} d^{k-1} = \mathbf{H}^k(W).$$

It means that in each cohomology class there is exactly one harmonic vector.

Lemma 21 *Let* $\left(W = \bigoplus_{k=0}^{N} W^k, \langle \cdot, \cdot \rangle, (\cdot, \cdot), d\right)$ *be a Hodge space with gradation and differential operator. Let* $\varepsilon : \{0, 1, ..., N\} \to \{-1, 1\}$ *be arbitrary function. If 2-tensor* $\langle , \rangle$ *is* ε*-antycommutative, i.e.*

$$\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle$$

for $v^k \in W^k$, $v^{N-k} \in W^{N-k}$, *then*

(a) the adjoint operator d^* *exists and is given by*

$$d^*\left(w^k\right) = \varepsilon_k (-1)^k * d * \left(w^k\right), \quad w^k \in W^k,$$

where $*$ *is the* $*$*-Hodge operator in* W.

(b)

$$* * \left(w^k\right) = \varepsilon_k \cdot w^k,$$

(c) if $\varepsilon_{k-1} = \varepsilon_{k+1}$ *then* $*\Delta = \pm\Delta *$, *more detailed*

$$*\Delta w^k = \varepsilon_{k-1}\varepsilon_k \left(-1\right)^{N+1} \Delta * w^k,$$

therefore

$$*\left[\mathcal{H}^k \left(W\right)\right] \subset \mathcal{H}^{N-k} \left(W\right),$$

and

$$* : \mathcal{H}^k \left(W\right) \to \mathcal{H}^{N-k} \left(W\right)$$

is an isomorphism.

*(d) (**Duality Theorem**) If additionally* $\boxed{W = \operatorname{Im}\Delta \bigoplus \left(\operatorname{Im}\Delta\right)^{\perp}}$ *then*

$$\mathbf{H}^k \left(W\right) \simeq \mathbf{H}^{N-k} \left(W\right).$$

We cut the scalar positive product $(\cdot,\cdot) : W^k \times W^k \to \mathbb{R}$ to the space of harmonic vectors

$$(\cdot,\cdot) : \mathcal{H}^k \left(W\right) \times \mathcal{H}^k \left(W\right) \to \mathbb{R},$$

and we cut the tensor $\langle\cdot,\cdot\rangle : W^k \times W^{N-k} \to \mathbb{R}$ to harmonic vectors

$$\mathcal{B}^k = \langle\cdot,\cdot\rangle : \mathcal{H}^k \left(M\right) \times \mathcal{H}^{N-k} \left(M\right) \to \mathbb{R}.$$

The first remains a positive and the second - nondegenerate.

To construction of the Hirzebruch signature operator the fundamental role is played by an operator

$$\tau : W \otimes \mathbb{C} \to W \otimes \mathbb{C}$$

such that

$$\tau\left(w^k\right) = \tilde{\varepsilon}_k * \left(w^k\right), \ \tilde{\varepsilon}_k \in \mathbb{C}, \quad |\tilde{\varepsilon}_k| = 1,$$

and

- i) $\tau^2 = Id$,
- ii) $d^* = -\tau \circ d \circ \tau$.

We check the existence of τ.

Proposition 22 *(1) If*

$$\varepsilon_k = c \cdot (-1)^{k(N-k)}$$

for all k*, where* $c \in \{-1, 1\}$,

then the operator τ *exists and* $\tilde{\varepsilon}_k = (-1)^{\frac{2N-k-1}{2}k}\, \tilde{\varepsilon}_0$ *for* $\tilde{\varepsilon}_0 \in \{1, -1, i, -i\}$ *such that* $(\tilde{\varepsilon}_0)^2 = c \cdot (-1)^{\frac{N(N-1)}{2}}$.

(2) (in opposite) If $d^k \neq 0$ *for all* k *and the operator* τ *exists then* $\varepsilon_k = c \cdot (-1)^{k(N-k)}$ *for* $c \in \{-1, 1\}$.

Remark 23 *If* $\varepsilon_k = c \cdot (-1)^{k(N-k)}$ *then* $\varepsilon_{k-1} = \varepsilon_{k+1}$.

Assuming. From the point of view of the signature we need to consider that N is even,

$$N = 2n \quad \text{and} \quad \varepsilon_n = +1,$$

then

$$\langle , \rangle^n : W^n \times W^n \to \mathbb{R}$$

and

$$\mathcal{B}^n : \mathcal{H}^n (W) \times \mathcal{H}^n (W) \to \mathbb{R}$$

are symmetric. Therefore in cohomology, the tensor

$$\langle , \rangle^n_{\mathbf{H}} : \mathbf{H}^n (W) \times \mathbf{H}^n (W) \to \mathbb{R}$$

is also symmetric. If

$$\dim \mathbf{H}^n (W) < \infty$$

we define the signature of W as the signature of $\langle , \rangle^n_{\mathbf{H}}$

$$\operatorname{Sig} (W) := \operatorname{Sig} \langle , \rangle^n_{\mathbf{H}} .$$

- **Under the assumption** $\boxed{W = \operatorname{Im} \Delta \bigoplus (\operatorname{Im} \Delta)^{\perp}}$ we have $\mathcal{H}^n (W) \cong \mathbf{H}^n (W)$ and $\mathcal{B}^n = \langle , \rangle^n_{\mathbf{H}}$. Therefore if $\dim \mathbf{H}^n (W) < \infty$ then

$$\operatorname{Sig} (W) = \operatorname{Sig} \langle , \rangle^n_{\mathbf{H}} = \operatorname{Sig} \mathcal{B}^n .$$

Lemma 24 $\mathcal{H}^n(W) = V_1 \bigoplus V_2$ *dla*

$$\mathcal{H}^n_+(W) = V_1 = \{\alpha \in \mathcal{H}^n(W);\ *\alpha = \alpha\},$$
$$\mathcal{H}^n_-(W) = V_2 = \{\alpha \in \mathcal{H}^n(W);\ *\alpha = -\alpha\}.$$

The 2-tensor $\mathcal{B}^n$ on V_1 is positive, on V_2 is negative and if $\dim \mathcal{H}^n(W) < \infty$ *then*

$$\mathrm{Sig}(\mathcal{B}^n) = \dim_{\mathbb{R}} \mathcal{H}^n_+(W) - \dim_{\mathbb{R}} \mathcal{H}^n_-(W).$$

Assume $N = 2n$, $\varepsilon_n = +1$, and the operator $\tau : W \to W$ such that $\tau\left(w^k\right) = \tilde{\varepsilon}_k * \left(w^k\right)$, $\tilde{\varepsilon}_k \in \mathbb{C}$, $|\tilde{\varepsilon}_k| = 1$, and $\tau^2 = Id$, $d^* = -\tau \circ d \circ \tau$ exists. In this situation must be $\varepsilon_k = c \cdot (-1)^{k(2N-k)} = c \cdot (-1)^k$, but $\varepsilon_n = +1$, so

$$\varepsilon_k = (-1)^n (-1)^k,$$

and

$$\tilde{\varepsilon}_k = (-1)^{\frac{k(k+1)}{2}} \tilde{\varepsilon}_0$$

for $\tilde{\varepsilon}_0 \in \{-1, 1\}$. Therefore in this case the operator τ **is real**, $\tau : W \to W$. We choose $\tilde{\varepsilon}_0$ in such a way that in the middle degree n

$$\tilde{\varepsilon}_n = +1,$$

i.e.

$$\tau\left(w^{n}\right)=*\left(w^{n}\right),$$

therefore $\tilde{\varepsilon}_0=(-1)^{\frac{n(n+1)}{2}}$. Then

$$\tilde{\varepsilon}_k=(-1)^{\frac{k(k+1)}{2}}(-1)^{\frac{n(n+1)}{2}}.$$

We put

$$W_{\pm}=\{w\in W;\ \tau w=\pm w\},$$

they are eigenspaces corresponding to the eigenvalues $+1$ i -1 of the operator τ.

We notice that

$$\left(d+d^{*}\right)\left[W_{+}\right]\subset W_{-}.$$

Definition 25 *The operator*

$$D_{+}=d+d^{*}:W_{+}\rightarrow W_{-}$$

is called the ***Hirzebruch operator*** *(or the signature operator).*

Theorem 26 *If* $\dim\mathbf{H}(W)<\infty$ *and* $\boxed{W=\operatorname{Im}\Delta\bigoplus(\operatorname{Im}\Delta)^{\perp}}$ *then*

$$\operatorname{Sig}W=\operatorname{Ind}D_{+}=\dim_{\mathbb{R}}\ker\left(D_{+}\right)-\dim_{\mathbb{R}}\ker\left(D_{+}^{*}\right).$$

General setting of the above four examples

Example 27 *Consider a graded vector bundle* $\xi = \bigoplus_{k=0}^{N} \xi^k$*, over a compact oriented Riemann manifold* M*, with suitable structures*

$$\left(\xi = \bigoplus\nolimits_{k=0}^{N} \xi^k, \langle , \rangle , (,) , d\right),$$

1) $\langle , \rangle , (,)$ *are field of 2-tensors in* ξ

$$\langle , \rangle_x , \quad (,)_x : \xi_x \times \xi_x \to \mathbb{R},$$

such that

1a) $\left\langle \xi^k, \xi^r \right\rangle = 0$ *if* $k + r \neq N$*, and the axiom* ε*-anticommutativity holds*

$$\left\langle v^k, v^{N-k} \right\rangle = \varepsilon_k \left\langle v^{N-k}, v^k \right\rangle , \qquad v^k \in \mathrm{Sec}\left(\xi^k\right),\ v^{N-k} \in \mathrm{Sec}\left(\xi^{N-k}\right),$$

where

$$\varepsilon_k \in \{-1, +1\},$$

1c) $(,)$ *is symmetric positive tensor such that the subbundles* ξ^r *are orthogonal,* *2) there exists an isometry* $*:\xi\rightarrow\xi$ *(with respect to* $(,)$, $(v,w)=(*v,*w)$ *)*

$$*_x:\xi_x\longrightarrow\xi_x$$

(called $*$*-Hodge operator) such that*

$$\langle v,w\rangle_x=(v,*_x w)_x\,,\quad v,w\in\xi_x,\quad x\in M.$$

By integration along M *we define 2-linear tensors*

$$\begin{gathered}\langle\langle,\rangle\rangle\,,((,)):\mathrm{Sec}\,(\xi)\times\mathrm{Sec}\,(\xi)\longrightarrow\mathbb{R},\\ \langle\langle\alpha,\beta\rangle\rangle:=\int_M\langle\alpha_x,\beta_x\rangle\,dM\\ ((\alpha,\beta)):=\int_M(\alpha_x,\beta_x)\,dM.\end{gathered}$$

Then $((,))$ *is an positive scalar product in* $\mathrm{Sec}\,(\xi)$, *the* $*$*-Hodge operator is still isometry* $((\alpha,\beta))=((*\alpha,*\beta))$ *and*

$$\langle\langle\alpha,\beta\rangle\rangle=((\alpha,*\beta))\,.$$

3) d is a differential in $\text{Sec}(\xi)$, $d^2 = 0$, *of the degree* $+1$, $d^k : \text{Sec}\left(\xi^k\right) \to \text{Sec}\left(\xi^{k+1}\right)$, *such that, by definition*

3a) d^k *are first order,*

3b)

$$\langle\langle dw, u\rangle\rangle = (-1)^{k+1} \langle\langle w, du\rangle\rangle$$

for $w \in \text{Sec}\left(\xi^k\right)$, $u \in \text{Sec}(\xi)$.

In conclusion

$$\left(\text{Sec}(\xi) = \bigoplus\nolimits_{k=0}^{N} \text{Sec}(\xi)^k, \langle\langle \cdot, \cdot\rangle\rangle, ((\cdot, \cdot)), d\right)$$

is a Hodge space with gradation and differential operator.

Then, the adjoint operator $d^* : \text{Sec}(\xi) \to \text{Sec}(\xi)$

$$((\alpha, d^*\beta)) = ((d\alpha, \beta))$$

exists and

$$d^*\left(\alpha^k\right) = \varepsilon_k (-1)^k * d * \left(\alpha^k\right)$$

Problem 28 *When the laplacian* $\Delta = (d + d^*)^2 = dd^* + d^*d$ *is elliptic?*

Theorem 29 *If the sequence of the differentials* $\left\{d^k\right\}$

$$\cdots \to \mathrm{Sec}\left(\xi^k\right) \to \mathrm{Sec}\left(\xi^{k+1}\right) \to \mathrm{Sec}\left(\xi^{k+1}\right) \to \cdots$$

is elliptic, i.e. is exact on the level of symbols ($0 \neq v \in T_x^* M$ *)*

$$\xi_x^k \overset{S\left(d^k\right)_{(x,v)}}{\longrightarrow} \xi_x^{k+1} \overset{S\left(d^{k+1}\right)_{(x,v)}}{\longrightarrow} \xi_x^{k+2}$$

then the Laplacian Δ *is elliptic. In consequence* $\mathrm{Sec}(\xi) = \mathrm{Im}\,\Delta \bigoplus (\mathrm{Im}\,\Delta)^\perp$ *and for asumptions* $N = 2n$ *and* $\varepsilon_k = (-1)^n (-1)^k$ *and* $\dim \mathbf{H}_d^\bullet(\xi) < \infty$ *we can obtain a Hirzebruch operator* $D_+ = d + d^* : W_+ \to W_-$ *and the equality* $\mathrm{Sig}\,\langle\langle,\rangle\rangle_{\mathbf{H}}^n = \mathrm{Ind}\, D_+$.

Ellipticity of the Δ follows trivially from the following facts:

(i) the symbol $S\left(D^*\right)_{(x,v)}$ of the dual operator to arbitrary first order differential operator $D : \mathrm{Sec}(E) \to \mathrm{Sec}(F)$ is equal to the minus dual to the symbol,

$$S\left(D^*\right)_{(x,v)} = -S\left(D\right)^*_{(x,v)},$$

(ii) the symbol of the composition $S\left(D_1 D_2\right)$ is the composition of the symbols $S\left(D_1 D_2\right)_{(x,v)} = S\left(D_1\right)_{(x,v)} \circ S\left(D_2\right)_{(x,v)}$,

(iii) **Lemma:** Let U, V, W be finite dimensional inner product spaces, and suppose that $U \xrightarrow{A} V \xrightarrow{B} W$ is a complex (i.e. $BA = 0$). Let $A^* : V \to U$ and $B^* : W \to V$ be the adjoints of A and B respectively. Then the complex is exact if and only if

$$B^* B + AA^*$$

is an isomorphism.

From the above and ellipticity of the complex $\left\{d^k\right\}$ we have

$$\begin{aligned}
S\left(\Delta^k\right)_{(x,v)} &= S\left(d^{k-1}d^{(k-1)*} + d^{k*}d^k\right)_{(x,v)} \\
&= S\left(d^{k-1}\right)_{(x,v)} \circ S\left(d^{(k-1)*}\right)_{(x,v)} + S\left(d^{k*}\right)_{(x,v)} \circ S\left(d^k\right)_{(x,v)} \\
&= S\left(d^{k-1}\right)_{(x,v)} \circ \left(-S\left(d^{k-1}\right)^*_{(x,v)}\right) + \left(-S\left(d^k\right)^*_{(x,v)} \circ S\left(d^k\right)_{(x,v)}\right) \\
&= -\left(S\left(d^{k-1}\right)_{(x,v)} \circ S\left(d^{k-1}\right)^*_{(x,v)} + S\left(d^k\right)^*_{(x,v)} \circ S\left(d^k\right)_{(x,v)}\right)
\end{aligned}$$

is an isomorphism.

In all four above examples the complex of differentials, $\left\{d^k\right\}, \left\{d^k_A\right\}, \left\{d^k_\nabla\right\}$ are elliptic, since the sequnces of symbols are exact. It follows from the fact that the sequences of symbols look as follows:

– we have finitely dimensional vector spaces V and E and a vector $0 \neq v \in V$ and the sequence

$$\bigwedge^{k-1} V \bigotimes E \to \bigwedge^{k} V \bigotimes E \to \bigwedge^{k+1} V \bigotimes E$$

of the homomorphisms giving by $u \otimes e \longmapsto (v \wedge u) \otimes e$.
****

APPLICATIONS.

We describe four fundamental examples of the Hodge space with gradation and differential operator. The fundamental idea is as follows: we have a 2-tensor $\langle , \rangle$ and we want to find a scalar positive tensor $(,)$ such that the $*$-Hodge operator exists and is an isometry.

Example 30 *1. (standard) M is compact oriented manifold,*

$$\dim M = N = 4p.$$

$W^k = \Omega^k (M) = \mathrm{Sec} \left(\bigwedge^k T^* M \right),$

$\langle\langle , \rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta.$

$$\begin{aligned}\langle\langle \alpha, \beta \rangle\rangle^k &= \int_M \alpha \wedge \beta = (-1)^{k(N-k)} \int_M \beta \wedge \alpha \\ &= (-1)^{k(N-k)} \langle\langle \beta, \alpha \rangle\rangle^{N-k} = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}.\end{aligned}$$

In the middle degree, the tensor $\langle\langle , \rangle\rangle$ *is symmetric*

$$\langle\langle , \rangle\rangle^{2r} : W^{2r} \times W^{2r} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta.$$

$$\langle\langle \alpha, \beta \rangle\rangle^{2r} = (-1)^{2r} \langle\langle \beta, \alpha \rangle\rangle^{2r} = \langle\langle \beta, \alpha \rangle\rangle^{2r}.$$

$d : W^k \to W^{k+1}$ *is defined to be the differntiation of differential forms.*

$\langle\langle d\alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$ *for* $\alpha \in W^k$, $\beta \in W^{N-(k+1)}$ *(which follows from* $\int_M d(\alpha \wedge \beta) = 0$*).*

We find a scalar product $((,))$ *in* W^k *for which the* $*$*-Hodge operator for* $(W, \langle\langle , \rangle\rangle, ((,)))$ *is an isometry. To this aim take arbitrary Riemannian tensor* G *in* M*, and for each point* $x \in M$ *take the standard scalar product*

$$G_x : \bigwedge^k T_x^* M \times \bigwedge^k T_x^* M \to \mathbb{R}$$

G_x in $\bigwedge^k T_x^* M$ and multiplication $\langle , \rangle_x$ of tensors tensors

$$\langle , \rangle_x^k : \bigwedge^k T_x^* M \times \bigwedge^{N-k} T_x^* M \to \bigwedge^N T_x^* M \xrightarrow{\rho} \mathbb{R}$$

where ρ is defined by the volume tensor $\boldsymbol{\varepsilon} = e_1^* \wedge ... \wedge e_n^*$ (for ON positive frame e_i).

Consider the classical finite dimensional Hodge-space

$$\left(\bigwedge T_x^* M, \langle , \rangle_x , (,)_x \right).$$

We notice that the $*$-Hodge operator $*_x$ is defined (via ON positive basis e_i of $T_x M$) by

$$*_p \left(e_{i_1}^* \wedge ... \wedge e_{i_k}^* \right) = \operatorname{sgn} \left(j_1, ..., j_{N-k}, i_1, .., i_k \right) \cdot e_{j_1}^* \wedge ... \wedge e_{j_{N-k}}^*,$$

and is an isometry (the sequence $(j_1, ..., j_{N-k})$ is increasing and complementary to $(i_1, ..., i_k)$

Now we take all points $x \in M$ and consider the volume differential forms Ω, $\Omega_x = e_1^* \wedge ... \wedge e_n^*$ for positive ON frames $e_i \in T_x M$. We define auxiliarily $C^\infty (M)$ linear 2-tensors

$$\begin{aligned} \langle , \rangle &: \Omega (M) \times \Omega (M) \to C^\infty (M) , \\ (,) &: \Omega (M) \times \Omega (M) \to C^\infty (M) , \end{aligned}$$

by

$$\begin{aligned} \langle \alpha, \beta \rangle (x) &= \langle \alpha_x, \beta_x \rangle_x, \qquad \langle \alpha, \beta \rangle \cdot \Omega = \alpha \wedge \beta, \\ (\alpha, \beta)(x) &= (\alpha_x, \beta_x)_x . \end{aligned}$$

Let the operator

$$* : \Omega(M) \to \Omega(M)$$

be defined point by point

$$*(\alpha)(x) = *_x(\alpha_x).$$

Then for $\alpha, \beta \in \Omega(M)$

$$\langle \alpha, \beta \rangle = (\alpha, *\beta)$$

By integrating along the Riemannian manifold M *we obtaind* 2*-linear tensors*

$$\begin{aligned} \langle\langle , \rangle\rangle &: \Omega(M) \times \Omega(M) \to \mathbb{R}, \\ \langle\langle \alpha, \beta \rangle\rangle &= \int_M \langle \alpha, \beta \rangle \, dM = \int_M \alpha \wedge \beta \end{aligned}$$

$$\begin{aligned} ((,)) &: \Omega(M) \times \Omega(M) \to \mathbb{R}, \\ ((\alpha, \beta)) &= \int_M (\alpha, \beta) \, dM \end{aligned}$$

and the equality

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta))$$

holds giving a Hodge-space $(\Omega(M), \langle\langle , \rangle\rangle, ((,)))$.,

The tensor $\langle\langle , \rangle\rangle$ *induces 2-tensor on cohomology*

$$\langle\langle , \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$$

which in the middle degree

$$\langle\langle , \rangle\rangle_{\mathbf{H}}^{2r} : \mathbf{H}^{2r}(M) \times \mathbf{H}^{2r}(M) \to \mathbb{R}$$

is symmetric and its signature $\operatorname{Sign} M = \operatorname{Sign} \langle\langle , \rangle\rangle_{\mathbf{H}}^{2r}$ *can be calculated as the index of the Hirzebruch operator.*$D_+ = d_{dR} + d_{dR}^* : W_+ \to W_-$ *(*d_{dR}^* *is the adjoint operator to* d_{dR} *with respect to the scalar product* $((,))$. *We recall that* $W_\pm = \{\alpha \in \Omega(M);\ \tau\alpha = \pm\alpha\}$ *for*

$$\tau\left(\alpha^k\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^r \cdot *\alpha^k.$$

Example 31 *2. (NEW) Let A be a transitive Lie algebroid over a compact oriented manifold M and let*

$$\operatorname{rank} A = N = 4p = m + n, \qquad m = \dim M, \quad n = \dim \mathbf{g}_{|x}.$$

We assume that A is invariantly oriented via a volume tensor

$$\boldsymbol{\varepsilon} \in \operatorname{Sec}\left(\bigwedge^{n} \mathbf{g}\right)$$

invariant with respect to the adjoint representation Ad_A.

$W^k = \Omega^k(A) = \operatorname{Sec}\left(\bigwedge^k A^*\right)$,

$\langle\langle , \rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \fint_A \alpha \wedge \beta.$

$$\begin{aligned} \langle\langle \alpha, \beta \rangle\rangle^k &= \int_M \fint_A \alpha \wedge \beta = (-1)^{k(N-k)} \int_M \fint_A \beta \wedge \alpha \\ &= (-1)^{k(N-k)} \langle\langle \beta, \alpha \rangle\rangle^{N-k} = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k}. \end{aligned}$$

This tensor is symmetric in the middle degree

$$\langle\langle,\rangle\rangle^{2r} : W^{2r} \times W^{2r} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \int_A \alpha \wedge \beta.$$

$$\langle\langle \alpha, \beta \rangle\rangle^{2r} = (-1)^{2r} \langle\langle \beta, \alpha \rangle\rangle^{2r} = \langle\langle \beta, \alpha \rangle\rangle^{2r}.$$

$d_A : W^k \to W^{k+1}$ *is the differentiation of A-differential forms,*

$\langle\langle d_A\alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d_A\beta \rangle\rangle$ *for* $\alpha \in W^k$, $\beta \in W^{N-(k+1)}$.

Now we find a scalar product $((,))$ *in* W *such that the* $*$*-Hodge operator in* $(W, \langle\langle,\rangle\rangle, ((,)))$ *is an isometry. To this aim consider*

— *any Riemannian tensor* G_1 *in the anchor* $\boldsymbol{g} = \ker \#_A$ *for which* $\boldsymbol{\varepsilon}$ *is the volume tensor (such a tensor exists).*

— *any Riemannian tensor* G_2 *on* M.

Next, taking arbitrary connection

$$\lambda : TM \to A \qquad (\#_A \circ \lambda = id_{TM})$$

i.e. a splitting of the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \underset{\lambda}{\xrightarrow{\#_A}} TM \longrightarrow 0,$$

and the horizontal space

$$\begin{aligned} H &= \operatorname{Im}\lambda, \\ A &= \boldsymbol{g} \bigoplus H \end{aligned}$$

we define a Riemannian tensor G on $A = \boldsymbol{g} \bigoplus H$ such that $\boldsymbol{g}$ and H are orthogonal, on $\boldsymbol{g}$ we have G_1 but on H we have the pullback $\lambda^* G_2$. The vector bundle A is oriented (since $\boldsymbol{g}$ and M are oriented).

At each point $x \in M$ we consider the scalar product G_x on $A_{|x}$ and the multiplication of tensors

$$\langle , \rangle_x^k : \bigwedge^k A_x^* \times \bigwedge^{N-k} A_x^* \to \bigwedge^N A_x^* \xrightarrow{\rho_x} \mathbb{R}$$

where ρ_x is defined via the volume form for G_x.

We can notice that ρ_x is the composition

$$\begin{tikzcd}
\bigwedge^N A_x^* \arrow[rr, "\rho_x"] \arrow[d, "(-1)^{Nn} i_{\boldsymbol{\varepsilon}_x}"] \arrow[dr, "\int_{A_p}"] & & \mathbb{R} \\
\bigwedge^m A_x^* \arrow[r, "\rho_{1x}"] & \bigwedge^m T_x^* M \arrow[ur, "\rho_{G_2 x}"'] &
\end{tikzcd}$$

Defined a scalar product G_x in A_x we extend to a scalar product in $\bigwedge A_x^*$ and we consider the classical finitely dimensional Hodge-space

$$\left(\bigwedge A_x^*, \langle , \rangle_x , (,)_x\right).$$

The $*$-Hodge operator $*_x$ such that by definition

$$\langle \alpha_x, \beta_x \rangle = (\alpha_x, *_x \beta_x)$$

is given by

$$*_x \left(e_{i_1}^* \wedge ... \wedge e_{i_k}^*\right) = \operatorname{sgn}(j_1, ..., j_{N-k}, i_1, .., i_k)\, e_{j_1}^* \wedge ... \wedge e_{j_{N-k}}^*$$

(for ON positive frame e_i). Using all points $x \in M$ we obtain two $C^\infty(M)$-tensors

$$\langle , \rangle , (,) : \Omega(A) \times \Omega(A) \to C^\infty(M)$$

defined as above point by point. Integrating along M we get a Hodge-space

$$(\Omega(M), \langle\langle , \rangle\rangle, ((,)))$$

and

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle = \int_M \fint_A \alpha \wedge \beta.$$

The $*$-Hodge operator $* : \Omega(M) \to \Omega(M)$ is defined point by point

$$*(\alpha)(x) = *_x(\alpha_x)$$

and we have

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta)).$$

The tensor $\langle\langle , \rangle\rangle$ induces a 2-tensor in cohomology

$$\langle\langle , \rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$$

which in the middle degree

$$\langle\langle , \rangle\rangle_{\mathbf{H}}^{2r} : \mathbf{H}^{2r}(M) \times \mathbf{H}^{2r}(M) \to \mathbb{R}$$

is symmetric. The dimension $\dim \mathbf{H}(A)$ *is finite (Kubarski, Mishchenko, 2003). The signature of $\langle\langle , \rangle\rangle_{\mathbf{H}}^{2r}$ can be calculated as the index of the Hirzebruch operator*

$$D_+ = d_A + d_A^* : W_+ \to W_-$$

where d_A^* is adjoint to d_A with respect to the scalar product $((,))$. We recall that $W_\pm = \{\alpha \in \Omega(A);\ \tau\alpha = \pm\alpha\}$, for

$$\tau\left(\alpha^k\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^r * \left(u^k\right)$$

Example 32 ***Lusztig (1971) [L], Gromov (1995) [G] . Signature for flat bundles.*** *Let M be a compact oriented $N = 4p$-dimensional manifold and $E \to M$ a flat bundle equipped with a flat covariant derivative ∇ and nondegenerated indefinite symmetric tensor*

$$G = (,)_0 : E \times E \to M \times \mathbb{R}, \quad (,)_x : E_x \times E_x \to \mathbb{R},$$

constant for ∇, i.e. satisfying $\partial_X (\sigma, \eta)_0 = (\nabla_X \sigma, \eta)_0 + (\sigma, \nabla_X \eta)_0$.

$W^k = \Omega^k (M; E)$,

the differential operator $d_\nabla : W^k \to W^{k+1}$ defined standartly via ∇. From $\nabla G = 0$ we have

$$d (\alpha \wedge_G \beta) = d_\nabla \alpha \wedge_G \beta + (-1)^{|\alpha|} (\alpha \wedge_G d_\nabla \beta)$$

therefore if $|\alpha| + |\beta| = N - 1$ then

$$\int_M (d_\nabla \alpha) \wedge_G \beta = - (-1)^{|\alpha|} \int_M \alpha \wedge_G d_\nabla \beta. \tag{1}$$

Define the duality

$$\begin{aligned} \langle\langle \alpha, \beta \rangle\rangle^k &: W^k \times W^{N-k} \to \mathbb{R} \\ \langle\langle \alpha, \beta \rangle\rangle^k &= \int_M \alpha \wedge_G \beta. \end{aligned}$$

and we see that

$$\langle\langle d_\nabla \alpha, \beta \rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta \rangle\rangle$$

is fulfilled. Since G is symmetric we have

$$\alpha \wedge_G \beta = (-1)^{k(N-k)} \beta \wedge_G \alpha$$

and

$$\begin{aligned} \langle\langle \alpha, \beta \rangle\rangle^k &= \int_M \alpha \wedge_G \beta = (-1)^{k(N-k)} \int_M \beta \wedge_G \alpha \\ &= (-1)^{k(N-k)} \langle\langle \beta, \alpha \rangle\rangle^{N-k} = \underbrace{(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha \rangle\rangle^{N-k} . \end{aligned}$$

The tensor is symmetric in the middle degree

$$\begin{gathered} \langle\langle , \rangle\rangle^{2p} : W^{2p} \times W^{2p} \to \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \alpha \wedge_G \beta. \\ \langle\langle \alpha, \beta \rangle\rangle^{2p} = (-1)^{2p} \langle\langle \beta, \alpha \rangle\rangle^{2p} = \langle\langle \beta, \alpha \rangle\rangle^{2p} . \end{gathered}$$

We find a scalar product $((,))$ in W^k for which the $*$-Hodge operator for $(W, \langle\langle , \rangle\rangle, ((,)))$ is an isometry. To this aim we fix some positive definite scalar *product* $(,)'$ *on* E. *Then we take a unique splitting* $E = E_+ \oplus E_-$ *which is both* $(,)_0$ *and* $(,)'$ *orthogonal and such that* $(,)_0$ *on* E_+ *is positive and* $(,)_0$ *on* E_- *is negative. We denote by* τ *the involution* $\tau : E \to E$ $(\tau^2 = id\ )$ *such that* $\tau|E_+ = id$, $\tau|E_- = -id$. *Then, the quadratic form*

$$(v, w) = (v, \tau w)_0$$

is symmetric positive definite. The involution τ *is the* $*$*-Hodge operator in* $(E_x, (,)_{0x}, (,)_x)$, *i.e.*

$$(v, w)_0 = (v, \tau w)\,,$$

and is an isometry

$$(\tau v, \tau w) = \left(\tau v, \tau^2 w\right)_0 = (\tau v, w)_0 = (w, \tau v)_0 = (w, v)\,.$$

Therefore $(E_x, (,)_{0x}, (,)_x)$ *is a Hodge-space.*

In each fibre $\bigwedge T_x^* M \bigotimes E_x$ *we introduce the tensor product of Hodge-spaces: the classical one* $\bigwedge T_x^* M$ *and the above* E_x.

Point by point we obtain tensors

$$\begin{aligned}\langle , \rangle &: \Omega(M;E) \times \Omega(M;E) \to C^{\infty}(M),\\ (,) &: \Omega(M;E) \times \Omega(M;E) \to C^{\infty}(M),\\ * &: \Omega(M;E) \to \Omega(M;E)\\ \langle \alpha, \beta \rangle &= (\alpha, *\beta)\end{aligned}$$

and integrating along M *we obtain a Hodge-space* $(\Omega(M;E), \langle\langle , \rangle\rangle, ((,)))$ *where*

$$\begin{aligned}\langle\langle \alpha, \beta \rangle\rangle &= \int_M \langle \alpha, \beta \rangle \, dM = \int_M \alpha \wedge_G \beta,\\ ((\alpha, \beta)) &= \int_M (\alpha, \beta) \, dM\end{aligned}$$

and

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta)).$$

Let d_{∇}^{*} *be the adjoint operator to* d_{∇} *with respect to* $((,))$. *The tensor* $\langle\langle , \rangle\rangle$ *induce a 2-tensor in cohomology* $\langle\langle , \rangle\rangle_{\mathbf{H}} : \mathbf{H}^{k}(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$ *which in the middle degree*

$$\langle\langle , \rangle\rangle_{\mathbf{H}}^{2p} : \mathbf{H}^{2p}(M;E) \times \mathbf{H}^{2p}(M;E) \to \mathbb{R}$$

is symmetric and the signature of it is the index of the Hirzebruch operator

$$D_+ = d_\nabla + d_\nabla^* : W_+ \to W_-$$

where $W_\pm = \{\alpha \in \Omega(M;E);\ \tau\alpha = \pm\alpha\}$ *for*

$$\tau\left(\alpha^k\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^p * \left(\alpha^k\right).$$

Example 33 ***Gromov (1995) [G].*** *Let* M *be a compact oriented manifold* M *of the dimension* $\dim M = N = 4p+2$ *and let* $E \to M$ *be a symplectic vector bundle equipped with a flat covatiant derivative* ∇ *and parallel symplectic structure* $S = \langle,\rangle : E \times E \to M \times \mathbb{R}$, $\langle,\rangle_x : E_x \times E_x \to \mathbb{R}$, $\nabla G = 0$.

$W^k = \Omega^k(M;E)$,

$d_\nabla : W^k \to W^{k+1}$ *- the differential operator defined via* ∇.

The condition

$$\int_M (d_\nabla\alpha) \wedge_S \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge_S d_\nabla\beta. \tag{2}$$

holds for $|\alpha| + |\beta| = N - 1$.

$\langle\langle \alpha, \beta\rangle\rangle^k : W^k \times W^{N-k} \to \mathbb{R}$ *is defined by*

$$\langle\langle \alpha, \beta\rangle\rangle^k = \int_M \alpha \wedge_S \beta$$

and $\langle\langle d_\nabla \alpha, \beta\rangle\rangle = (-1)^{k+1} \langle\langle \alpha, d\beta\rangle\rangle$ *is fulfilled. Since* S *is skewsymmetric, then*

$$\alpha \wedge_S \beta = -(-1)^{k(N-k)} \beta \wedge_S \alpha$$

and

$$\begin{aligned}
\langle\langle \alpha, \beta\rangle\rangle^k &= \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha \\
&= -(-1)^{k(N-k)} \langle\langle \beta, \alpha\rangle\rangle^{N-k} = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha\rangle\rangle^{N-k} \\
&= (-1)^k (-1)^{\frac{N}{2}} \langle\langle \beta, \alpha\rangle\rangle^{N-k} \quad / \frac{N}{2} = 2p+1 \text{ is odd}
\end{aligned}$$

We find a scalar product $((,))$ *in* W^k *for which* $(W, \langle\langle, \rangle\rangle, ((,)))$ *is a Hodge space.*

- *There exists an anti-involution τ in E, $\tau^2 = -\tau$ (i.e. a complex structure)such that*

  (1) $\langle \tau v, \tau w \rangle = \langle v, w \rangle$, $v, w \in E_x$,

  (2) $\langle v, \tau v \rangle > 0$ *dla* $v \neq 0$.

Then the tensor $(v, w) := \langle v, \tau w \rangle$ *is symmetric and positive defined and* $(\tau v, \tau w) = (v, w)$, *i.e.* τ *preserves both forms* $\langle , \rangle$ *and* $(,)$. *The operator* $-\tau$ *is the* $*$*-Hodge operator in* $(E_x, \langle , \rangle_x, (,)_x)$ *since*

$$\langle v, w \rangle = \left\langle v, -\tau^2 w \right\rangle = \langle v, \tau(-\tau w) \rangle = (v, -\tau w),$$

and $-\tau$ *is an isometry* $(\tau v, \tau w) = (v, w)$ *to* $(-\tau v, -\tau w) = (v, w)$. *In consequence, the system* $(E_x, \langle , \rangle_x, (,)_x)$ *is a Hodge-space.*

At each point $x \in M$ *we take the tensor product* $\bigwedge T_x^* M \bigotimes E_x$ *of the classical Hodge space* $\bigwedge T_x^* M$ *and the above* E_x. *The remaining procedure as in the above example to obtain a Hodge-space* $(\Omega(M; E), \langle\langle , \rangle\rangle, ((,)))$, *with the* $*$*-Hodge operator* $* : \Omega(M) \to \Omega(M)$, $*(\alpha)(x) = *_x(\alpha_x)$, *and*

$$\langle\langle \alpha, \beta \rangle\rangle = ((\alpha, *\beta)).$$

We obtain in cohomology $\langle\langle,\rangle\rangle_{\mathbf{H}} : \mathbf{H}^k(M) \times \mathbf{H}^{N-k}(M) \to \mathbb{R}$,

$$\begin{aligned}\langle\langle \alpha, \beta\rangle\rangle^k &= \int_M \alpha \wedge_G \beta = -(-1)^{k(N-k)} \int_M \beta \wedge_S \alpha \\ &= -(-1)^{k(N-k)} \langle\langle \beta, \alpha\rangle\rangle^{N-k} = \underbrace{-(-1)^k}_{\varepsilon_k} \langle\langle \beta, \alpha\rangle\rangle^{N-k}\end{aligned}$$

which in the middle degree $2p+1$ *is symmetric (thanks to the fact that* $\langle,\rangle$ *is skewsymmetric)*

$$\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p+1} : \mathbf{H}^{2p+1}(M;E) \times \mathbf{H}^{2p+1}(M;E) \to \mathbb{R}$$

$$\langle\langle \alpha, \beta\rangle\rangle^{2p+1} = -(-1)^{2p+1} \langle\langle \beta, \alpha\rangle\rangle^{2p+1} = \langle\langle \beta, \alpha\rangle\rangle^{2p+1}.$$

We can calculate the signature of $\langle\langle,\rangle\rangle_{\mathbf{H}}^{2p+1}$ *as the index of the Hirzebruch operator* $D_+ = d_\nabla + d_\nabla^* : W_+ \to W_-$ *where* $W_\pm = \{\alpha \in \Omega(M;E);\ \tau\alpha = \pm\alpha\}$, *for*

$$\tau\left(\alpha^k\right) = (-1)^{\frac{k(k+1)}{2}} (-1)^{p+1} \cdot * \left(\alpha^k\right).$$

Example 34 *In consequence, for a transitive invariantly oriented Lie algebroid A over a compact oriented manifold M and the Atiyah sequence*

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0,$$

$$m = \dim M, \quad n = \operatorname{rank} \mathbf{g} = \dim \mathbf{g}_x,$$

and under the assumption $\mathbf{H}^{m+n}(A) \neq 0$ and

$$m + n = 4p$$

we have two signature Hirzebruch operators.

(I) The first one. $D_+ = d_A + d_A^* : \Omega(A)_+ \to \Omega(A)_-$ where d_A^* is adjoint to d_A with respect to the scalar product $((\alpha, \beta)) = \int_M (\alpha, \beta)$ defined in the example 2 above, and $W_\pm = \{\alpha \in \Omega(A);\ \tau\alpha = \pm\alpha\}$, for $\tau(\alpha^k) = (-1)^{\frac{k(k+1)}{2}} (-1)^p \cdot * (\alpha^k)$.

(II) The second one. We use the equality

$$\operatorname{Sign} \mathbf{H}(A) = \operatorname{Sign} E_2$$

for the second term E_2 of the Hochschild-Serr spectral sequence

$$E_2^{p,q} = \mathbf{H}^p_{\nabla^q}(M; \mathbf{H}^q(\boldsymbol{g})).$$

The flat covariant derivative ∇^q in the cohomology vector bundle $\mathbf{H}^q(\boldsymbol{g})$ depends on the Lie algebroid A.

Let $m+n=4p$. The signature $\operatorname{Sign} E_2$ *is equal to the signature of the quadratic form*

$$E_2^{2p} \times E_2^{2p} \to E_2^{m+n} = \mathbb{R},$$

and

a) if n is odd then $\operatorname{Sign} E_2 = 0$,

b) if n is even then

$$\operatorname{Sign} E_2 = \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{m+n} = E_2^{m,n} = \mathbb{R}\right)$$

$$E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right).$$

Consider the form $\langle\langle,\rangle\rangle : \mathbf{H}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbb{R}$,

$$\langle\langle,\rangle\rangle^k : \mathbf{H}^k_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{m-k}_{\nabla^{\frac{n}{2}}}\left(M; \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \to \mathbf{H}^m_{\nabla^n}\left(M; \mathbf{H}^n(\boldsymbol{g})\right) = \mathbb{R},$$

which is symmetric in the middle degree $k=\frac{m}{2}$ and its signature is equal to the signature of A. For $k=n$, the bundle $\mathbf{H}^n(\boldsymbol{g})$ is trivial, $\mathbf{H}^n(\boldsymbol{g}) \cong M \times \mathbb{R}$, *the connection* ∇^n *is equal to* ∂, *and the multiplication of values is with respect to* $\langle , \rangle : \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$.

We have $\frac{m}{2} + \frac{n}{2} = 2p$. *We need to consider two different cases:*

(a) $\frac{m}{2}$ *and* $\frac{n}{2}$ *even, then the form*

$$\mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$$

is symmetric and we can use the Example 3 above to give a Hirzebruch signature operator $D_+ = d_{\nabla^{\frac{n}{2}}} + d^*_{\nabla^{\frac{n}{2}}} : \Omega_+\left(M; \mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right) \to \Omega_-\left(M; \mathbf{H}^{\frac{n}{2}}(\mathbf{g})\right)$,

(b) $\frac{m}{2}$ *and* $\frac{n}{2}$ *are odd, then the form* $\mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \times \mathbf{H}^{\frac{n}{2}}(\mathbf{g}) \to \mathbf{H}^n(\mathbf{g}) = M \times \mathbb{R}$ *is symplectic and we can use the Example 4 to give a Hirzebruch signature operator.*

Remark 35 *In all four examples* $W = \operatorname{Im}\Delta \bigoplus (\operatorname{Im}\Delta)^{\perp}$ *for* $\Delta = (d + d^*)^2$, *thanks to this*

$$\ker\Delta = \ker D = \mathcal{H} \cong H(W)$$

and

$$\operatorname{Ind} D_+ = \operatorname{Sig} W.$$

It follows, for example, from the fact that $\Delta = dd^* + d^*d$ *are self-dual elliptic operators.*

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