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Signature of transitive Lie algebroids

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Lie algebroids appeared as infinitesimal objects of Lie groupoids, principal fibre bundles, vector bundles (Pradines, 1967), TC-foliations and nonclosed Lie subgroups (Molino, 1977), Poisson manifolds (Dazord, Coste, Weinstein, 1987), etc. Their algebraic equivalences are known as Lie pseudo-algebras (Herz 1953) called also further as Lie-Rinehart algebras (Huebschmann, 1990).

A *Lie algebroid* on a manifold M is a triple

$$A = (A, [\![\cdot,\cdot]\!], \#_A)$$

where A is a vector bundle on M, $(\mathrm{Sec}\, A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

$$\#_A : A \to TM$$

is a linear homomorphism (called the anchor) of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta, \quad f\in C^\infty(M),\ \xi,\eta\in \mathrm{Sec}\, A.$$

The anchor is bracket-preserving,

$$\#_A \circ [\![\xi,\eta]\!] = [\#_A\circ\xi, \#_A\circ\eta].$$

A Lie algebroid is called **transitive** if $\#_A$ is an epimorphism.

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0,$$

$\boldsymbol{g} := \ker \#_A$. The fiber $\boldsymbol{g}_x$ of the bundle $\boldsymbol{g}$ in the point $x \in M$ is the Lie algebra with the commutator operation being

$$[v, w] = [\![\xi, \eta]\!](x), \quad \xi, \eta \in \operatorname{Sec} A, \quad \xi(x) = v, \eta(x) = w, \quad v, w \in \boldsymbol{g}_x.$$

The Lie algebra $\boldsymbol{g}_x$ is called the isotropy Lie algebra of L at $x \in M$. The vector bundle $\boldsymbol{g}$ is a Lie Algebra Bundle (LAB in short), called the *adjoint* of A, the fibres are isomorphic Lie algebras.

TM is a Lie algebroid with $id : TM \to TM$ as the anchor,

$\mathfrak{g}$ -finitely dimensional Lia algebra - is a Lie algebroid over $M = \{*\}$.

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\Omega(A) = \mathrm{Sec} \bigwedge A^*, \quad \text{- the space of cross-sections of } \bigwedge A^*$$

$$d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

$$\begin{aligned}(d_A \omega)(\xi_0, ..., \xi_k) &= \sum_{j=0}^{k} (-1)^j \left(\#_A \circ \xi_j\right)(\omega(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega\left([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k\right),\end{aligned}$$

$\omega \in \Omega^k(A)$, $\xi_i \in \mathrm{Sec}\, A$.

The exterior derivative d_A introduces the cohomology algebra

$$\mathbf{H}(A) = \mathbf{H}(\Omega(A), d_A).$$

For the trivial Lie algebroid TM - the tangent bundle of the manifold M - the differential d_{TM} is the usual de-Rham differential d_M of differential forms on M whereas, for $L = \mathfrak{g}$ - a Lie algebra $\mathfrak{g}$ - the differential $d_{\mathfrak{g}}$ is the usual Chevalley-Eilenberg differential, $d_{\mathfrak{g}} = \delta_{\mathfrak{g}}$.

For each transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A \xrightarrow{\#_A} TM \to 0,$$

over **compact oriented** manifold M the following conditions are equivalent (Kubarski-Mishchenko, 2004) ($m = \dim M$, $n = \dim \boldsymbol{g}_{|x}$, i.e. $\operatorname{rank} A = m + n$)

(1) $\mathbf{H}^{m+n}(A) \neq 0$,

(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$,

(3) A is the so-called *invariantly oriented,* i.e. there exists a global non-singular cross-section $\boldsymbol{\varepsilon}$ of the vector bundle $\bigwedge^n \boldsymbol{g}$,

$$\begin{aligned} \varepsilon &\in \operatorname{Sec}\left(\bigwedge^n \boldsymbol{g}\right) \\ 0 &\neq \varepsilon_x \in \bigwedge^n \boldsymbol{g}_{|x} \end{aligned}$$

invariant with respect to the adjoint representation of A.

Assume that A is invarianty oriented. The scalar Poincaré product

$$\mathcal{P}_A^i : \mathbf{H}^i(A) \times \mathbf{H}^{m+n-i}(A) \to \mathbb{R},$$

$$([\omega],[\eta]) \longmapsto \int_A^{\#} \omega \wedge \eta \quad \left(:= \int_M \left(\fint_A^{\#} \omega \wedge \eta \right) \right)$$

where the so-called *fibre integral*

$$\fint_A : \Omega^\bullet(A) \to \Omega_{dR}^{\bullet - n}(M)$$

is defined by the formula ($\boldsymbol{\varepsilon} \in \Gamma\left(\bigwedge^n \boldsymbol{g}\right)$ is nonzero)

$$\left(\fint_A \omega^k \right)_x (w_1, ..., w_{k-n}) = (-1)^{kn} \omega_x (\boldsymbol{\varepsilon}_x, \tilde{w}_1, ..., \tilde{w}_{k-n}), \quad \#_A(\tilde{w}_i) = w_i,$$

The operator $\fint_A$ commutes with the differentials d_A and d_M giving a homomorphism in cohomology

$$\fint_A^{\#} : \mathbf{H}^\bullet(A) \to \mathbf{H}_{dR}^{\bullet - n}(M).$$

In particular we have

$$\int_A^{\#} : \mathbf{H}^{m+n}(A) \xrightarrow{\cong} \mathbf{H}_{dR}^{m}(M) = \mathbb{R}.$$

The scalar product $\mathcal{P}_A^i$ is nondegenerated and if $m+n=4k$ then

$$\mathcal{P}_A^{2k} : \mathbf{H}^{2k}(A) \times \mathbf{H}^{2k}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and is called the signature of A, and is denoted by

$$\mathrm{Sign}(A).$$

The problem is:

- to calculate the signature $\mathrm{Sign}(A)$ and give some conditions to the equality $\mathrm{Sign}(A) = 0$. There are examples for which $\mathrm{Sign}(A) \neq 0$.

My talk concerns this problem.

(I) Firstly, I give a general mechanism of the calculation of the signature via spectral sequences (Kubarski-Mishchenko 2003) and use to two kinds of spectral sequences associated with Lie algebroids:

a) the **spectral sequence of the Čech-de Rham complex,**

b) the **Hochschild-Serre spectral sequence.**

(II) Secondly, using the $*$-Hodge operator we construct two Hirzebruch operators to calculate the signature. For each of them the index is equal to the signature of A. Therefore the Atiyah-Singer formula for the index can be used to calculate of the signature.

AD(I)

(i) The general approach to the use of the spectral sequences.

The idea of applying spectral sequences to the signature comes from

- Chern-Hirzebruch-Serre *On the index of a fibered manifold,* Proc. AMS, 8 (1957), 587-596.

Via spectral sequences the authors proved

Theorem 1 *Let $E \to M$ be a fiber bundle, wih the typical fiber F, such that the following conditions are satisfied:*

(1) E, M, F are compact connected oriented manifolds;

(2) the fundamental group $\pi_1(M)$ acts trivially on the cohomology ring $H^*(F)$ of F.

Then, if E, M, F are oriented coherently, so that the orientation of E is induced by those of F and M, the index of E is the product of the indices of F and M, that

$$\mathrm{Sign}\,(E) = \mathrm{Sign}\,(F) \cdot \mathrm{Sign}\,(M)\,.$$

The authors consider the cohomology Leray spectral sequence $E_s^{p,q}$ of the bundle $E \to B$ with the real fields as the coefficients field. The term E_2 by hypothesis (2) is the bigraded algebra

$$E_2^{p,q} \cong \mathbf{H}^p(M; \mathbf{H}^q(F)) \cong \mathbf{H}^p(M) \otimes \mathbf{H}^q(F),$$

therefore

$$E_2^{p,q} = 0 \text{ for } \quad p > m \quad \text{or} \quad q > n.$$

Clearly, E_2 is a **Poincaré algebra** by hypothesis (1). Using the spectral sequence argument the authors noticed that

$$(E_s, d_s, \cdot) \;\; s \geq 2,$$

and

$$(E_\infty, \cdot)$$

are Poincaré algebras and

$$\operatorname{Sign} E_2 = \operatorname{Sign} E_3 = ... = \operatorname{Sign} E_\infty = \operatorname{Sign} \mathbf{H}(E).$$

It appears that the Chern-Hirzebruch-Serre arguments used to prove the above theorems on the signature of the total space of the bundle $E \to M$ are pure algebraic and lead to the following general theorems (Kubarski-Mishchenko 2003).

Theorem 2 *Let* $((A, \langle , \rangle), A^r, \cup, D, A_j)$ *be any DG-algebra with a decreasing regular filtration* A_j

$$A = A_0 \supset \cdots \supset A_j \supset A_{j+1} \supset \cdots$$

and $(E_s^{p,q}, d_s)$ *its spectral sequence. We assume that there exist natural numbers* m *and* n *with the following conditions:*

- $E_2^{p,q} = 0$ *for* $p > m$ *and* $q > n$, $m + n = 4k$,
- E_2 *is a Poincaré algebra with respect to the total gradation and the top group* $E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}$.

Then each term $\left(E_s^{(\bullet)}, \cup, d_s\right)$ $2 \leq s < \infty$, *and* $\left(E_\infty^{(\bullet)}, \cup\right)$ *are Poincaré algebra*

$$\mathrm{Sign}\, E_2 = \mathrm{Sign}\, E_3 = ... = \mathrm{Sign}\, E_\infty = \mathrm{Sign}\, \mathbf{H}(A).$$

If m *and* n *are odd then* $\operatorname{Sign} E_2 = 0,$ *if* m *and* n *are even then*

$$\begin{aligned} \operatorname{Sign} E_2 &= \operatorname{Sign}\left(E_2^{(2k)} \times E_2^{(2k)} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \\ &= \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \to E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \end{aligned}$$

(ii) Using the Leray spectral sequence (the spectral sequence of the Čech-de Rham complex).

We use this mechanism to

(a) the spectral sequence for the Čech-de Rham complex of the Lie algebroid A.

Let A be any transitive Lie algebroid on a manifold M with isotropy Lie algebras $\boldsymbol{g}_{|x}$ isomorphic to a given Lie algebra

$$\mathfrak{g}.$$

If $U \subset M$ is an open subset diffeomorphic to $\mathbb{R}^m$ then the restriction $A_{|U}$ is the Lie algebroid isomorphic to the trivial one $TU \times \mathfrak{g}$ and

$$\mathbf{H}\left(A_{|U}\right) \cong \mathbf{H}(U) \otimes \mathbf{H}\left(\mathfrak{g}\right) \cong \mathbf{H}\left(\mathfrak{g}\right).$$

Given a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of M, where J is a countable ordered index set (this means that all U_α and all finite intersections $\bigcap_i U_{\alpha_i}$ are diffeomorphic to an Euclidean space $\mathbb{R}^m$) we can form the double complex (of the Čech-de Rham type)

$$K^{p.q} = C^p\left(\mathfrak{U}, \Omega^q\left(A\right)\right) := \prod_{\alpha_0 < \ldots < \alpha_p} \Omega^q\left(A_{|U_{\alpha_0 \ldots \alpha_p}}\right)$$

$p, q \geq 0$, with the product structure

$$\begin{gathered} \cup : K^{p,q} \times K^{r,s} \rightarrow K^{p+r,q+s} \\ \left(\omega \cup \eta\right)_{\alpha_0 \ldots \alpha_{p+r}} = \left(-1\right)^{qr} \omega_{\alpha_0 \ldots \alpha_p} | U_{\alpha_0 \ldots \alpha_{p+r}} \wedge \eta_{\alpha_p \ldots \alpha_{p+r}} | U_{\alpha_0 \ldots \alpha_{p+r}}. \end{gathered}$$

This complex has two boundary homomorphisms, d and δ.

The vertical homomorphism $d : C^p\left(\mathfrak{U}, \Omega^q\left(A\right)\right) \rightarrow C^p\left(\mathfrak{U}, \Omega^{q+1}\left(A\right)\right)$ acts as external differential of A-forms

$$d = \left(-1\right)^p d_A.$$

The horizontal homomorphism $\delta : C^p(\mathfrak{U}, \Omega^q(A)) \to C^{p+1}(\mathfrak{U}, \Omega^q(A))$ acts as a coboundary homomorphism

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \, \omega_{\alpha_0 \ldots \hat{\imath} \ldots \alpha_{p+1}} | U_{\alpha_0 \ldots \alpha_{p+1}}.$$

The horizontal and vertical homomorphisms δ and d are antiderivations of degree $+1$ therefore

$$(K, K^{p,q}, \cup, d, \delta)$$

is a double complex of the first quadrant with a product structure. Now, consider the "horizontal" decreasing filtration

$$K_j = \bigoplus_{\substack{p \geq j \\ q \geq 0}} K^{p,q}.$$

Due to the general construction of spectral sequences for the filtration above which is in accord with the multiplicative structure of the DG-algebra, $\left(K, K^{(r)}, \cup, D, K_j\right)$, one can construct the spectral sequence of the graded differential algebras

$$(E_s^{p,q}, d_s),$$

The filtration K_j is **regular,** $K_0 = K$, therefore the spectral sequence $(E_s^{p,q}, d_s)$ converge to $H(K, D)$.

Theorem 3 *(1) The zero term* (E_0, d_0) *:*

$$\begin{aligned} E_0^p &= K_p/K_{p+1}, \quad E_0^{p,q} = K^{p,q}, \\ d_0 &= d : K^{p,q} = C^p(\mathfrak{U}, \Omega^q(A)) \longrightarrow K^{p,q+1} = C^p\left(\mathfrak{U}, \Omega^{q+1}(A)\right), \end{aligned}$$

(2) The first term (E_1, d_1)

$$E_1^{p,q} = \mathbf{H}^{p,q}(K, d) = C^p(\mathfrak{U}, \mathcal{H}^q(A)), \quad d_1 = \delta^{\#} : E_1^{p,q} \to E_1^{p+1,q},$$

where

$$\mathcal{H}^*(A) = \left(U \longmapsto \mathbf{H}^*\left(A_{|U}\right)\right)$$

is the Leray type presheaf of cohomology, locally constant on the good covering $\mathfrak{U}$*, with values in the cohomology algebra* $\mathbf{H}^*(\mathfrak{g})$ *of the structural Lie algebra* $\mathfrak{g}$.

(3) The second term

$$E_2^{p,q} = \mathbf{H}^{p,q}\left(H(K, D), \delta^{\#}\right) = \mathbf{H}_{\delta^{\#}}^p(\mathfrak{U}, \mathcal{H}^q(A)).$$

We assume that

- the presheaf $\mathcal{H}(A)$ is constant on the good covering $\mathfrak{U}$.

The condition is equivalent to that the monodromy representation of the presheaf $\mathcal{H}(A)$

$$\rho : \pi_1(M) = \pi_1(N(\mathfrak{U})) \to \mathrm{Aut}(\mathbf{H}(\mathfrak{g}))$$

is trivial.

Example 4 *The condition of the triviality of the monodromy holds if*

- M is simply connected,

- $\mathrm{Aut}\, G = \mathrm{Int}\, G$, where G is simply connected Lie group with the Lie algebra $\mathfrak{g}$, for example, if $\mathfrak{g}$ is a simple Lie algebra of type $B_l, C_l, E_7, E_8, F_4, G_2$.

- the adjoint Lie algebra bundle $\boldsymbol{g}$ is trivial in the category of flat bundles (the bundle $H(\boldsymbol{g})$ of cohomology of isotropy Lie algebras with the typical fibre $H(\mathfrak{g})$ possess canonical flat covariant derivative - which will be important for studying of the Hochshild-Serre spectral sequence). For example for the Lie algebroid $A(G; H)$ of the the TC-foliation of left cosets of a nonclosed Lie subgroup H in any Lie group G.

If the monodromy representation of the presheaf $\mathcal{H}(A)$ is **trivial** then

$$\begin{aligned} E_2^{p,q} &= \mathbf{H}^p_{\delta\#}\left(\mathfrak{U}, \mathcal{H}^q(A)\right) \cong \mathbf{H}^p\left(\mathfrak{U}, \mathbf{H}^q(\mathfrak{g})\right) \\ &\cong \mathbf{H}^p(\mathfrak{U}, \mathbb{R}) \otimes \mathbf{H}^q(\mathfrak{g}) \\ &\cong \mathbf{H}^p_{dR}(M) \otimes \mathbf{H}^q(\mathfrak{g}). \end{aligned}$$

All isomorphisms are canonical isomorphisms of bigraded algebras. It means that E_2 lives in the rectangle $p \leq m$, $q \leq n$, and

$$E_2^{(m+n)} = E_2^{m,n} = \mathbf{H}^m_{dR}(M) \otimes \mathbf{H}^n(\mathfrak{g}) = \mathbb{R}$$

therefore we have:

Theorem 5 *If A is a transitive Lie algebroid on a compact oriented connected manifold M with unimodular isotropy Lie algebras $\mathfrak{g}_{|x} \cong \mathfrak{g}$ and monodromy representation of the presheaf $\mathcal{H}(A)$ is trivial than the terms $E_2, ..., E_\infty$ as well as the cohomomology algebra $\mathbf{H}(A)$ are Poincaré algebras and*

$$\begin{aligned} \operatorname{Sign}(A) &= \operatorname{Sign}\mathbf{H}(A) = \operatorname{Sign} E_2 \\ &= \operatorname{Sign}\left(\mathbf{H}_{dR}(M) \otimes \mathbf{H}(\mathfrak{g})\right) = \operatorname{Sign}\mathbf{H}_{dR}(M) \cdot \operatorname{Sign}\mathbf{H}(\mathfrak{g}) \\ &= \operatorname{Sign}\mathbf{H}_{dR}(M) \cdot 0 = 0 \end{aligned}$$

because for unimodular Lie algebra $\mathfrak{g}$

$$\operatorname{Sign} \mathbf{H}(\mathfrak{g}) = \operatorname{Sign} \bigwedge \mathfrak{g}^* = 0.$$

(iii) Using the Hochshild-Serre spectral sequences.

Following Hochschild-Serre (*Cohomology of Lie algebras,* Ann. Math. 57, 1953, 591-603) for a pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g}, \mathfrak{h})$ one can consider

– a graded cochain group of $\mathbb{R}$-linear alternating functions

$$A_{\mathbb{R}} = \bigoplus_{k \geq 0} A^k, \quad A^k = C^k(\mathfrak{g}),$$

– with the standard $\mathbb{R}$-differential operator d of degree 1

– and Hochschild-Serre decreasing filtration $A_j \subset A_{\mathbb{R}}$ as follows:

— $A_j = A_{\mathbb{R}}$ for $j \leq 0$,

— if $j > 0$, $A_j = \bigoplus_{k \geq j} A_j^k$, $A_j^k = A_j \cap A^k$, where A_j^k consists of all those k-cochains f for which $f(\gamma_1, ..., \gamma_k) = 0$ whenever $k - j + 1$ of the arguments γ_i belongs to $\mathfrak{k}$

$$A_j^k = \begin{cases} f \in A^k = C^k(\mathfrak{g}) \\ f(\gamma_1, ..., \gamma_k) = 0 \\ \forall \gamma_1, ..., \gamma_{k-j+1} \in \mathfrak{k}. \end{cases}$$

In this way, we have obtained a graded filtered differential $\mathbb{R}$-vector space

$$\Big(A_{\mathbb{R}} = \bigoplus_{k\geq 0} A^k, d, A_j\Big)$$

and we can use its spectral sequence

$$\big(E_s^{p,q}, d_s\big).$$

For a transitive Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \#_A)$ with the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \overset{\#_A}{\to} TM \to 0$ we will consider the pair of $\mathbb{R}$-Lie algebras $(\mathfrak{g}, \mathfrak{k})$ where

$$\mathfrak{g} = \mathrm{Sec}\,(A)\,, \quad \mathfrak{k} = \mathrm{Sec}\,(\boldsymbol{g})\,.$$

Following K.C.M.Mackenzie (1987) (see also V.Itskov, M.Karashev, and Y.Vorobjev (1998)), we will consider the $C^\infty(M)$-submodule of $C^\infty(M)$-linear altarnating cochains

$$\Omega^k(A) \subset C^k(\mathfrak{g})$$

and the induced filtration

$$\Omega_j = \Omega_j(A) = A_j \cap \Omega(A)$$

of $C^\infty(M)$-modules. We obtain in this way a graded filtered differential space

$$\left(\Omega(A)=\bigoplus_k \Omega^k(A), d_A, \Omega_j\right)$$

and its spectral sequence

$$\left(E_{A,s}^{p,q}, d_{A,s}\right).$$

Assume as above

$$m=\dim M,\ n=\dim \boldsymbol{g}_{|x}, \quad \text{i.e. } \operatorname{rank} A=m+n.$$

Theorem 6 *There is a flat covariant derivative ∇^q in the vector bundle $\mathbf{H}^q(\boldsymbol{g})$ such that*

$$E_{A,2}^{p,q}\cong \mathbf{H}_{\nabla^q}^p\left(M;\mathbf{H}^q(\boldsymbol{g})\right).$$

The flat covariant derivative ∇^q is defined by the formula: for $f\in\Omega^p\left(M;Z\left[\bigwedge^q \boldsymbol{g}^*\right]\right)$, $[f]\in\Omega^p\left(M;\mathbf{H}^q(\boldsymbol{g})\right)$

$$\nabla_X^q[f]=[\mathcal{L}_X f]$$

where $(\mathcal{L}_X f)(\sigma_1,...,\sigma_q)=\partial_X\left(f(\sigma_1,...,\sigma_q)\right)-\sum_{i=1}^q f\left(\sigma_1,...,[\![\lambda X,\sigma_i]\!],...,\sigma_q\right)$ (where $\lambda: TM\to A$ is arbitrary auxiliary connection in A).

Theorem 7 *If A is a transitive invariantly oriented Lie algebroid such that $m+n=4k$ ($m=\dim M,\quad n=\dim \boldsymbol{g}_{|x}$) then*

a) if m and n are odd then $\operatorname{Sign} A=0$,

b) if m and n are even then

$$\begin{aligned} \operatorname{Sign} A &= \operatorname{Sign} E_2 = \operatorname{Sign}\left(E_2^{(2k)} \times E_2^{(2k)} \rightarrow E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \\ &= \operatorname{Sign}\left(E_2^{\frac{m}{2},\frac{n}{2}} \times E_2^{\frac{m}{2},\frac{n}{2}} \rightarrow E_2^{(m+n)} = E_2^{m,n} = \mathbb{R}\right) \end{aligned}$$

where

$$E_2^{\frac{m}{2},\frac{n}{2}} = \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right)$$

and

$$\mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \times \mathbf{H}^{\frac{m}{2}}_{\nabla^{\frac{n}{2}}}\left(M;\mathbf{H}^{\frac{n}{2}}(\boldsymbol{g})\right) \rightarrow \mathbf{H}^{m}_{\nabla^{n}}\left(M;\mathbf{H}^{n}(\boldsymbol{g})\right) = \mathbb{R}$$

is defined via the usual multiplication of differential forms with respect to the multiplication of cohomology class for Lie algebras.

$$\phi : \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \times \mathbf{H}^{\frac{n}{2}}(\boldsymbol{g}) \rightarrow \mathbf{H}^{n}(\boldsymbol{g}) = M \times \mathbb{R}.$$

We notice that

- if $\frac{n}{2}$ is even then $\frac{m}{2}$ is even, $\dim M = m = 4s$ for some s, ϕ is symmetric nondegenerated,
- if $\frac{n}{2}$ is odd then $\frac{m}{2}$ is odd, $\dim M = 4s + 2$ for some s, ϕ is symplectic.

However always $\mathbf{H}^k \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) \times \mathbf{H}^{m-k} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) \to \mathbb{R}$ is strong nondegenerated

$$\begin{aligned} \mathbf{H}^k \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) &= \mathbf{H}^{m-k} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right)^*, \\ \mathbf{H} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) &= \mathbf{H} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right)^*, \\ \dim \mathbf{H} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) &< \infty, \end{aligned}$$

and

$$\mathbf{H}^{\frac{m}{2}} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) \times \mathbf{H}^{\frac{m}{2}} \left(M; \mathbf{H}^{\frac{n}{2}} (\boldsymbol{g}) \right) \to \mathbb{R}$$

is symmetric nondegenerated.

AD(II) Hirzebruch operator and the signature.

(i) general approach.

We describe a general approach to the following four fundamental examples of the spaces with gradation and differential operator $\left(W=\oplus_{k=0}^{N}W^{k},d\right)$. M is here compact oriented manifold.

$$W^{k}=\begin{cases}\Omega^{k}\left(M\right),\ d_{dR}; & N=4r,\\ \Omega^{k}\left(A\right),\ d_{A}; & N=m+n=4r,\quad A\text{ - Lie algebroid}\\ \Omega^{k}\left(M;E\right),\quad d_{\nabla};\quad \left(E,\left(,\right)_{0}\right)\text{ flat vector bundle,} & \\ \qquad\qquad \left(,\right)_{0}\text{ - symmetric nondegenerated parallel,}\quad N=4r & \\ \Omega^{k}\left(M;E\right),\quad d_{\nabla};\quad \left(E,\left\langle ,\right\rangle_{0}\right)\text{ flat vector bundle,} & \\ \qquad\qquad \left\langle ,\right\rangle_{0}\text{ -symplectic parallel,}\quad N=4r+2 & \end{cases}$$

In all cases, using multiplication of differential forms and a suitable integration along M, we have a quadratic form

$$\left\langle ,\right\rangle^{k}:W^{k}\times W^{N-k}\to\mathbb{R}$$

In the middle degree $\left\langle ,\right\rangle^{\frac{N}{2}}:W^{\frac{N}{2}}\times W^{\frac{N}{2}}\to\mathbb{R}$ is symmetric.

In all cases we find a symmetric positive 2-tensor

$$(,) : W \times W \to \mathbb{R}$$

such that the $*$-Hodge operator i.e. the one $* : W \to W$ such that

$$\langle \cdot, w \rangle = (\cdot, *w)$$

is isometry with respect to the scalar positive symmetric tensor $(,)$. Then considering Laplacian operator $\Delta = (d + d^*)^2$, where d^* is the adjoint operator, we can obtain easily the Hirzebruch operator Δ_+ and prove easily that

$$\operatorname{Ind} \Delta_+ = \operatorname{Sig} \langle , \rangle^{\frac{N}{2}}_{|\mathcal{H}} \quad \text{-the cutting to the harmonic tensors}$$

If the equality

$$W = \operatorname{Im} \Delta + (\operatorname{Im} \Delta)^{\perp} \tag{1}$$

holds then $\mathcal{H}^k = \mathbf{H}^k(W)$ and

$$\operatorname{Ind} \Delta_+ = \operatorname{Sig} W.$$

The equality (1) is not trivial, it follows (from example) from that in all four cases Δ is an elliptic operator.