Foliations in commutative algebras

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Plan:
– foliations and category of foliated manifolds
– Foliated commutative algebras
– Lie-Rinehart algebras - the important source of foliated commutative algebras
– Integrals and open problems
– Characteristic classes associated with foliations

1 Preliminaries: foliated manifolds

Assume $g : M \to M'$ is a smooth mapping between paracompact Hausdorff manifolds and let $\xi' \to M'$ be a vector bundle. Then the $C^\infty(M')$-module $\Gamma(\xi')$ is projective and finitelly generated.

(1) Take a pullback $g^*\xi' \to M$

$$\begin{array}{ccc} g^*\xi' & \to & \xi' \\ \downarrow & & \downarrow \\ g : M & \to & M' \end{array}$$

and recall that

$$\Gamma(g^*\xi') \underset{\cong}{\overset{\rho}{\longleftarrow}} C^\infty(M) \otimes_{C^\infty(M')} \Gamma(\xi')$$

$$\rho(h \otimes \nu')(p) = h(p) \cdot \nu'(g(p)).$$

(2) For any linear homomorphism of vector bundles $G : \xi \to \xi'$ over $g : M \to M'$

$$\begin{array}{ccc} \xi & \to & \xi' \\ \downarrow & & \downarrow \\ g : M & \to & M' \end{array}$$

there exists a strong linear homomorphism $\tilde{G} : \xi \to g^*\xi'$ (i.e. over the identity $id : M \to M$) such that for the canonical one $g^*\xi' \to \xi'$ we have the composition

$$\begin{array}{cccccc} G : & \xi & \overset{\tilde{G}}{\to} & g^*\xi' & \to & \xi' \\ & \downarrow & & \downarrow & & \downarrow \\ & M & = & M & \to & M'. \end{array}$$

Considering $\tilde{G}$ on cross-sections,

$$\tilde{G} : \Gamma(\xi) \to \Gamma(g^*\xi') = C^\infty(M) \otimes_{C^\infty(M')} \Gamma(\xi')$$

we see that for $\nu \in \Gamma(\xi)$ there exists some functions $a^i \in C^\infty(M)$ and cross-sections $\nu'^i \in \Gamma(\xi')$ such that

$$\tilde{G}(\nu) = \sum_i a^i \otimes \nu'^i.$$

(3) As the example consider the linear homomorphism: the differential of the mapping g

$$g_* = dg : TM \to TM',$$

$$dg(X_p) \to T_{g(p)}M',$$
$$dg(X_p)(\alpha') = X_p(\alpha' \circ g),$$

and the induced strong linear homomorphism

$$\widetilde{dg} : TM \to g^*(TM').$$

We recall also that a vector field $X \in \Gamma(TM) = \mathfrak{X}(M)$ is the same as a differential of the algebra $C^\infty(M)$

$$X : C^\infty(M) \to C^\infty(M), \quad \alpha \longmapsto X(\alpha),$$

and

$$\Gamma(TM) \to \mathrm{Der}(C^\infty(M))$$

is an isomorphism of $\mathbb{R}$-Lie algebras.

Analogously, a cross-section of $g^*(TM')$ is the same as a differential of the algebra $C^\infty(M')$ in $C^\infty(M)$,

$$\Gamma(g^*TM') \to \mathrm{Der}(C^\infty(M'), C^\infty(M)),$$
$$Y' \longmapsto (\alpha' \longmapsto Y'\alpha')$$

where for $Y' \in \Gamma(g^*TM')$ and $p \in M,\ Y'(p) \in T_{g(p)}M'$.

In conclusion,

$$\widetilde{dg}(X)_p = \widetilde{dg}(X_p) \in T_{g(p)}M'$$

and

$$\widetilde{dg}(X)(\alpha) = X(\alpha' \circ g).$$

(4) The classical Frobenius Theorem say:

— If $F \subset TM$ is a vector subbundle of the tangent bundle, then the conditions are equivalent:

— F is a foliation, i.e. through any point $p \in M$ pass an integral, i.e. immersed submanifold $L \hookrightarrow M$ such that $T_qL = F_q$, $q \in L$,

— the the $C^\infty(M)$-module of cross-section $\Gamma(F)$ is involutive, i.e. $[X, Y] \in \Gamma(F)$ for $X, Y \in \Gamma(F)$, in other words, $\Gamma(F) \subset \Gamma(TM) = \mathfrak{X}(M)$ is a Lie subalgebra of the Lie algebra of vector fields on M.

Clearly,

- $Q := TM/F$ is a vector bundle, therefore the quotient module

  $$\Gamma(Q) = \mathfrak{X}(M)/\Gamma(F)$$

  is projective and finitely generated.
- F is a a direct summand in TM, therefore the same holds for the module of tangent vector fields fo F, i.e.
- $\Gamma(F)$ is a direct summand in $\Gamma(TM) = \mathfrak{X}(M) = \mathrm{Der}(C^\infty(M))$.

Proposition 1.1 *Let $g : M \to M'$ be a smooth mapping, $dg : TM \to TM'$ and $\widetilde{dg} : TM \to g^*(TM')$ the induced linear homomorphims. Take regular foliations $F \subset TM$ and $F' \subset TM'$ on M and M' respectively. The conditions are equivalent:*

(a) $dg[F] \subset F'$ (we say that then g is a homomorphism of foliated manifolds and write $g : (M, F) \to (M', F')$),

(b) if $L \hookrightarrow M$ is an integral of F then $g[L]$ is contained in some integral of F',

(c) $\widetilde{dg}\left[F\right]\subset g^{*}\left[F'\right]$ *(i.e. g determines a strong linear homomorphism of vector bundles $\widetilde{dg}:F\rightarrow g^{*}\left[F'\right]$),*

(d) the strong linear homomorphism $\widetilde{dg}:TM\rightarrow g^{*}\left(TM'\right)$ *has the property: for $X\in\Gamma\left(F\right)$ the exists a natural number s and functions $a^{i}\in C^{\infty}\left(M\right)$ and vector fileds $X'^{i}\in\Gamma\left(F'\right)$, $i=1,2,...,s,$ such that*

$$\widetilde{dg}\left(X\right)=\sum_{i=1}^{s}a^{i}\otimes X'^{i},$$

(equivalently

$$X\left(g^{*}\alpha'\right)=\sum_{i=1}^{s}a^{i}\cdot g^{*}\left(X'^{i}\left(a'\right)\right)\qquad for\quad \alpha'\in C^{\infty}\left(M'\right),$$

where $g^{*}:C^{\infty}\left(M'\right)\rightarrow C^{\infty}\left(M\right),\ \alpha'\longmapsto\alpha'\circ g,$ is the induced homomorphism of algebras).

2 Foliated commutative algebras and examples

2.1 Foliated commutative algebras

The above yields the natural generalizations of a foliation in arbitrary commutative algebras.

Assume

- R is commutative unital ring (mainly a field),
- $(A, \cdot)$ is an unital associative commutative R-algebra, i.e. an R-module A together with a R-bilinear mapping (called *product*)

  $$\cdot : A \times A \to A,$$

  fulfilling the axiom of the associativity $(u \cdot v) \cdot w = u \cdot (v \cdot w)$ and there exists an unit $1_A \in A$ ($1_A \cdot u = u \cdot 1_A = u$). The set of derivations of the algebra A is denoted by $\mathrm{Der}\,(A)$. It is a A-module and an R-Lie algebra.

Definition 2.1 *By an [algebraic] foliation in A we mean a subspace*

$$F \subset \mathrm{Der}\,(A)$$

such that

— F is an A-submodule, and

— F is an R-Lie subalgebra.

The foliation is called regular if the quotient A-module $\mathrm{Der}\,(A)\,/F$ *is projective and finitely generated and F is a direct summand in* $\mathrm{Der}\,(A)$.

The pair (A, F) is then called a foliated commutative algebra.

To determine the category of commutative foliated algebras we must define a notion of the morphism of foliated algebras. Let A and A' be two R-algebras and $f : A' \to A$ any homomorphism of algebras. Then A can be considered usually as the A'-module with respect to the action

$$A' \times A \rightarrow A, (a', a) \mapsto f(a') \cdot a.$$

Clearly, A is a symmetric A'-bimodule.

Definition 2.2 *Let (A, F) and (A', F') be two foliated commutative algebras. By a homomorphism*

$$g : (A, F) \rightarrow (A', F')$$

between them we mean a homomorphism $g^* : A' \rightarrow A$ of algebras such that for each derivation $X \in F$ there exist $s \in \mathbb{N}$ and alements $a^i \in A$ and derivations $X'^i \in F'$, $i = 1, ..., s$, such that

$$X(g^*\alpha') = \sum\nolimits_{i=1}^{n} a^i \cdot g^* \left(X'^i(\alpha')\right), \quad \alpha' \in A'.$$

If $g : (A, F) \rightarrow (A', F')$ and $h : (A', F') \rightarrow (A'', F'')$ are homomorphisms of foliated algebras then their superposition $g \circ h : (A, F) \rightarrow (A'', F'')$,

$$(g \circ h)^* = g^* \circ h^* : A'' \xrightarrow{h^*} A' \xrightarrow{g^*} A$$

is also a homomorphism of foliated algebras.

2.2 Lie-Rinehart algebras - the important source of foliated commutative algebras

The fundamental source of foliated commutative algebras yields Lie-Rinehart algebras: the image of the so called anchor in this algebras is foliated commutative algebra.

Definition 2.3 *By an R-Lie Rinehart algebra [H], [R] over a commutative R-algebra A we mean a triple*

$$(L, [\![\cdot, \cdot]\!], \#_L)$$

such that the 2-R-linear tensor

$$[\![\cdot,\cdot]\!] : L \times L \to L$$

gives a structure of a R-Lie algebra in L, *and*

$$\#_L : L \to \operatorname{Der}(A)$$

is a A-linear homomorphism of R-Lie algebras fulfilling the Leibniz axiom:

$$[\![x, a \cdot y]\!] = a \cdot [\![x, y]\!] + \#_L(a) \cdot y, \quad a \in A, \quad x, y \in L.$$

Then, sometimes the pair $(L, [\![\cdot,\cdot]\!], \#_L)$ *is called a* (R, A)*-Lie-Rinehart algebra.*

If $(L, [\![\cdot,\cdot]\!], \#_L)$ is a Lie-Rinehart algebra, then

$$\ker \#_L \subset L$$

and

$$\operatorname{Im} \#_L \subset \operatorname{Der}(A)$$

possesses natural structure of Lie-Rinehart algebras and

$$0 \to \ker \#_L \to L \to \operatorname{Im} \#_L \to 0$$

is the short sequence of Lie-Rinehart algebras,

- $\ker \#_L \subset A$ is a A-Lie algebra with 0 as the anchor and,
- $\operatorname{Im} \#_L$ is a Lie-Rinehart algebra with the inclusion $\operatorname{Im} \#_L \hookrightarrow \operatorname{Der}(A)$ as the anchor.

Lie-Rinehart algebras are algebraic equivalence to differential object for manifolds, called Lie algebroids (Jean Pradines, 1966).

Definition 2.4 *By a Lie algebroid we mean a system* $(E, [\![\cdot,\cdot]\!], \#_E)$ *for which* E *is a vector bundle on* M, *the module* $(\Gamma(E), [\![\cdot,\cdot]\!])$

of global cross-sections of E *is an* $\mathbb{R}$*-Lie algebra,* $\#_E$: $E \to TM$ *is a linear homomorphism of vector bundles (called the anchor) fulfilling the Leibniz axiom*

$$[\![x, f \cdot y]\!] = f \cdot [\![x, y]\!] + \#_E(x)(f) \cdot y, \quad f \in C^\infty(M), \ x, y \in \Gamma(E).$$

If $(E, [\![\cdot,\cdot]\!], \#_E)$ is a Lie algebroid then the $C^\infty(M)$-module $\Gamma(E)$ of global cross-sections forms a Lie-Rinehart algebra.

There are many differential categories from which act the so-called "Lie Functor" to the category of Lie algebroids (and next to the category of Lie-Rinehart algebras)

– principal fibre bundles (Atiyah, Pradines),
– vector bundles,
– differential groupoids (Libermann, Pradines),
– transversely complete foliations (Molino),
– nonclosed Lie subgroups,
– Poisson manifolds (Coste, Dazord, Weinstein),
– Jacobi manifolds,
– etc.

Lie-Rinehart algebras appeared considerably earlier than Lie algebroids (see [M2, p. 100]),

- first in 1953 (Herz) under the name of *pseudo-algèbre de Lie*.

Next they appeared independently more then ten times under different names, for example:

- *regular restricted Lie algebra extension* (Hochschild, 1955),
- *Lie d-ring* (Palais, 1961),
- (R, C)*-Lie algebra* (Rinehart, 1963),
- *Lie algebra with an associated module structure* (Hermann, 1967),
- *Lie module* (Nelson, 1967),

- $(\mathcal{A}, \mathcal{C})$-*system* (Ne'eman, 1971; Kostant and Sternberg, 1990),
- *sheaf of twisted Lie algebras* (Kamber and Tondeur, 1971),
- *algèbre de Lie sur* C/R (Illusie, 1972),
- *Lie algebra extension* (Teleman, 1972),
- *Lie-Cartan pairs* (Kastler and Stora, 1985),
- *Atiyah algebras* (Beilinson and Schechtmann, 1988; Manin, 1988) and
- *differential Lie algebra* (Kosmann-Schwarzbach and Magri, 1990).

We prefer *Lie-Rinehart algebra* according to:

- J. Huebschmann [H], *Poisson cohomology and quantization,* J. für die Reine und Angew. Math. 408 (1990), 57-113. 9.

Remark 2.5 *Let* $(A, L, \#_L)$ *be a Lie-Rinehart algebra with the anchor*

$$\#_L : L \to \mathrm{Der}\,(A)\,.$$

Since by definition that the anchor $\#_L$ *ia an* A*-linear homomorphism of* R*-Lie algebras we obtain:*

– the image of the anchor

$$\mathrm{Im}\,\#_L \subset \mathrm{Der}\,(A)$$

is a foliation in A *called characteristic for the Lie-Rinehart algebra* L*. This foliation is irregular, in general.*

Now we give some very important algebraic categories from which act Lie Functor to the category of Lie-Rinehart algebras.

2.2.1 Covariant operators

As a preliminary take a vector bundle ξ over a manifold M and arbitrary covariant derivative $\nabla_X \nu$, $X \in \mathfrak{X}(M)$, $\nu \in \Gamma(\xi)$. The operator

$$\nabla_X : \Gamma(\xi) \to \Gamma(\xi)$$

is a covariant differential operator with the anchor X, i.e. for any smooth function $f \in C^\infty(M)$ and a cross-section $\nu \in \Gamma(\xi)$ we have

$$\nabla_X (f \cdot \nu) = f \cdot \nabla_X(\nu) + X(f) \cdot \nu.$$

Denote by $CDO(\xi)$ the space of all covariant differential operators.

- It is a Lie algebra with the canonical structure bracket

  $$[D, D'] = D \circ D' - D' \circ D$$

  of the differential operators,
- and let $\# : CDO(\xi) \to \mathfrak{X}(M)$ denote the mapping (homomorphism of $C^\infty(M)$-modules and $\mathbb{R}$-Lie algebras) whose assign the anchor to any covariant differential operator.

— Clearly, $CDO(\xi)$ forms a Lie-Rinehart algebra over the algebra $C^\infty(M)$.

— Additionally, there exists a vector bundle $A(\xi)$ such that

$$\Gamma(A(\xi)) = CDO(\xi),$$

the anchor determines a linear surjective homomorphism $\#_\xi : A(\xi) \to TM$,

— and $A(\xi)$ is then a Lie algebroid.

— A covariant derivative ∇ is exactly the same as the splitting of the sequence

$$0 \to \mathrm{End}\,(\xi) \to A\,(\xi) \underset{\overleftarrow{\nabla}}{\to} TM \to 0$$

The important generalization of the Lie-Rinehart algebra $CDO\,(\xi)$ is as follows:

Definition 2.6 *For arbitrary A-module M a R-linear operator $D : M \to M$ is called a covariant operator if there exists a differential $\delta \in \mathrm{Der}\,(A)$ such that*

$$D\,(a \cdot m) = a \cdot D\,(m) + \delta\,(a) \cdot m, \quad a \in A,\ m \in M.$$

— The differental δ is called the anchor of D.
— The set of all covariant operators is denoted by $CO\,(M)$.
— It is A-module and R-Lie algebra under the bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1,$$

$D_1,\ D_2 \in CO\,(M)$.
— If $D_1,\ D_2 \in CO\,(M)$ possesses δ_1 and δ_2 as anchors then $[D_1, D_2]$ is a covariant operator with $[\delta_1, \delta_2]$ as the anchor.

Under the assumption that the representation

$$\rho : A \to \mathrm{End}_R\,(M)\,,$$
$$a \longmapsto (m \longmapsto a \cdot m)$$

is faithfull, i.e. ρ is a monomorphism, than the anchor δ of any covariant operator D is uniquely determined. In this assumption we have a correctly defined homomorphism

$$\#_M : CO\,(M) \to \mathrm{Der}\,(A)\,,$$
$$\#_M\,(D) = \text{anchor of } D,$$

which is a homomorphism od R-Lie algebras and is A-linear. Clearly

$$\ker \#_M = \operatorname{End}_A(M) .$$

- The module $CO(M)$ becomes a Lie-Rinehart algebra over A.

By a covariant derivative in M we mean a R-linear operator $\nabla : \operatorname{Der}(A) \to CO(M)$ such that $\#_M \circ \nabla = id$, i.e. the splitting of the sequence

$$0 \to \operatorname{End}_A(M) \to CO(M) \underset{\overleftarrow{\nabla}}{\to} \operatorname{Der}(A) \to 0.$$

Theorem 2.7 *If M is a projective A-module then there exists a covariant derivative in M, in consequence, the sequence*

$$0 \to \operatorname{End}_A(M) \to CO(M) \longrightarrow \operatorname{Der}(A) \to 0$$

is exact.

2.2.2 Poisson algebra

Definition 2.8 *By a Poisson algebra we mean a pair* $(A, \{\cdot,\cdot\})$, *where* A *is an* R*-algebra equipped with an* R*-Lie algebra structure*

$$\{\cdot,\cdot\} : A \times A \to A$$

such that for any $a \in A$ *the mapping*

$$\{a,\cdot\} : A \to A$$

is a differential of the algebra,

$$\{a,\cdot\} \in \mathrm{Der}\,(A)\,,$$

i.e.

$$\{a, bc\} = \{a, b\}\, c + b\, \{a, c\}\,.$$

The differental $\{a,\cdot\}$ *is denoted by* X_a *and it is called the Hamiltonian of* a.

Remark 2.9 *The* A*-module generated by Hamiltonians forms a foliation in* A.

Example 2.10 *A manifold* M *equipped with the Poisson structure in the algebra* $C^{\infty}(M)$ *is called a Poisson manifold. In the space of 1-forms (i.e. cross sections of the cotangent bundle* $T^{*}M$ *) there exists a structure od Lie algebroid given by the Lie algebra structure*

$$[\![\cdot,\cdot]\!] : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$$

determined uniquely by demanding that

$$[\![df, dg]\!] = d\,\{f, g\}\,, \quad f, g \in C^{\infty}(M)$$

and the anchor

$$\# : T^{*}M \to TM$$

such that

$$\#\left(df\right) = X_f.$$

How can we define objects analogous to 1-forms and its bracket $[\![\cdot,\cdot]\!]$ for arbitrary Poisson algebra?

It is appeared that the module $\Omega^1_{A|R}$ of Kähler differentiations is suitable.

We recall that by the Kähler module of differentiations of an R-algebra A we mean an A-bimodule $\Omega^1_{A|R}$ generated by formal set

$$\{da;\ a \in A\}$$

quotient by elements

$$\begin{aligned} &d\left(ea+sb\right) - rda - sdb,\\ &d\left(ab\right) - a\left(db\right) - \left(da\right)b, \end{aligned}$$

$a, b \in A$, $r, s \in R$.

We can prove that

$$d : A \to \Omega^1_{A|R}, \quad a \longmapsto da,$$

is a universal differentiation, i.e. for any differential $\delta : A \to M$ of the algebra A in an A-bimodule N there exists exactly one A-linear mapping $\phi_\delta : \Omega^1_{A|R} \to N$ such that $\phi_\delta \circ \delta = d$

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega^1_{A|R} \\ & {}_{\delta}\searrow & \downarrow \\ & & N \end{array}$$

In conclusion (J.L. Loday [L], *Cyclic Homology*, 1992)

$$\mathrm{Der}\left(A, N\right) = \mathrm{Hom}_A\left(\Omega^1_{A|R}, N\right).$$

Using $N = A$ and knowing that projective and finitely generated module is reflexive, we see that

- if $\Omega^1_{A'|R}$ is projective and finitely generated module, than:

$$\mathrm{Der}\,(A)^* = \mathrm{Hom}_A(\mathrm{Hom}_A(\Omega^1_{A|R}, A), A) \cong \Omega^1_{A|R}.$$

- Therefore, $\Omega^1_{C^\infty(P)|\mathbb{R}}$ performs the same role as the module of 1-forms on a smooth paracompact manifold P because we have:

$$\Omega^1_{C^\infty(P)|\mathbb{R}} \cong \mathrm{Der}\,(C^\infty\,(P))^* = \Gamma\,(T^*P)\,.$$

For an arbitrary Poisson algebra $(A, \{\cdot,\cdot\})$ we define a 2-linear and alternating tensor

$$\pi : \Omega^1_{A|R} \times \Omega^1_{A|R} \to A,$$
$$\pi\,(a\,(du)\,, b\,(dv)) = ab\,\{u, v\}\,.$$

π gives the anchor

$$\pi^\# : \Omega^1_{A|R} \to \mathrm{Hom}_A(\Omega^1_{A|R}, A) = \mathrm{Der}\,(A)$$

whereas the mapping

$$[\cdot,\cdot] : \Omega^1_{A|R} \times \Omega^1_{A|R} \to \Omega^1_{A|R}$$

defined by

$$[a\,(du)\,, b\,(dv)] = a\,\{u, b\}\,dv + b\,\{a, v\}\,du + abd\,\{u, v\}\,,$$

$a, b, u, v \in A$, introduce a structure of R-Lie algebra in A-module $\Omega^1_{A|R}$.

Theorem 2.11 *J. Huebschmann [H], 1990. The system*

$$\left(\Omega^1_{A|R}, [\cdot,\cdot]\,, \pi^\#\right)$$

is a Lie-Rinehart algebra over A.

2.3 Nonstrong homomorphisms of Lie-Rinehart algebras

The characteristic homomorphisms defined below are natural with respect to nonstrong homomorphisms of Lie-Rinehart algebras. They generalize the notion of the homomorphisms between foliated commutative algebras. On the other hand, there is a simple notion of the homomorphism $\Phi : G \to G'$ between two differential groupoids over a mapping $f : M \to M'$ of the manifolds of units: Φ must commute with the source, the tangent and must keep the multiplication. Passing to the Lie algebroids (the associated infinitesimal obiects) we obtain a notion of a nonstrong homomorphism of Lie algebroids.

If $(L, [\![\cdot,\cdot]\!], \#_L)$ is a R-Lie-Rinehart algebra over a commutative R-algebra A then $\operatorname{Im}\#_L \subset \operatorname{Der}(A)$ is of course a foliation of the R-algebra A. By a strong homomorphism of (R, A)-Lie-Rinehart algebras $H : (L, \#_L) \to (L', \#_{L'})$ we mean an A-linear homomorphism $H : L \to L'$ such that F is a homomorphism of R-Lie algebras and commutes with the anchors $\#_{L'} \circ H = \#_L$. One of the sources of Lie-Rinehart algebras are Lie algebroids over foliated manifolds. For regular Lie algebroids there is an important notion of a nonstrong homomorphism [K] (see also [M1], [M2], [H-M1]): if $\mathbf{L}$ and $\mathbf{L}'$ are regular Lie algebroids over foliated manifolds (N, F) and (N', F') with anchors $\#_L : \mathbf{L} \to F$ and $\#_{L'} : \mathbf{L}' \to F'$, respectively, than the pair (H, f) when $f : (N, F) \to (N'F')$ is a morphism of foliated manifolds and $H : \mathbf{L} \to \mathbf{L}'$ is a linear homomorphism of vector bundles over f, is called a homomorphism of Lie algebroids if (1) $\#_{L'} \circ H = f_* \circ \#_L$, (2) for arbitrary cross-sections $\xi, \xi' \in \Gamma(\mathbf{L})$ with H-decomposition

$$H \circ \xi = \sum_i h^i \cdot \left(\eta'^i \circ f\right), \quad H \circ \eta = \sum_j g^j \cdot \left(\nu'^j \circ f\right),$$

$h^i, g^j \in C^\infty(N)$, $\eta'^i, \nu'^j \in \Gamma(\mathbf{L}')$ we have

$$H \circ [\![\xi, \xi']\!]$$
$$= \sum_{i,j} h^i \cdot g^j \cdot [\![\eta'^i, \nu'^j]\!] \circ f + \sum_i (\#_L \circ \xi)\left(g^j\right) \cdot \nu'^j \circ f$$
$$- \sum_j (\#_L \circ \xi')\left(h^i\right) \circ \eta'^i \circ f.$$

Nonstrong homomorphism of Lie algebroids over a morphism $f : (N, F) \to (N'F')$ of foliated manifolds can be equivalently defined as a strong homomorphism $\bar{F}$: $\mathbf{L} \to f^\wedge \mathbf{L}'$ of Lie algebroids over (N, F), where $f^\wedge \mathbf{L}'$ is the inverse image of $\mathbf{L}'$ via f. The Lie algebroid $f^\wedge \mathbf{L}'$ is a vector subbundle of $F \bigoplus f^* \mathbf{L}'$

$$f^\wedge \mathbf{L}' = \{(v, w)\,; f_*(v) = \#_{L'}(w)\}$$

with the projection onto the first factor as the anchor and the structure of a Lie algebra in $\Gamma(f^\wedge \mathbf{L}')$ defined in such a way that for $\tilde{\xi}_i = \sum_j g_i^j \cdot \xi_i^j \circ f \in \Gamma(f^* \mathbf{L}')$ and $X_i \in \Gamma(F)$ fulfilled conditions $f_*(X_i(x)) = \sum_j g_i^j(x) \cdot \gamma'(\xi_i^j(f(x)))$, $x \in N$, we have:

$$[\![(X_1, \tilde{\xi}_1), (X_2, \tilde{\xi}_2)]\!]$$
$$= \left([X_1, X_2], \sum_{i,j} g_1^i \cdot g_2^j \cdot [\![\xi_1'^i, \xi_2'^j]\!] \circ f + \sum_j X_1(g_2^j) \cdot \xi_2'^j \circ f\right.$$
$$\left. - \sum_i X_2(g_1^i) \cdot \xi_1'^i \cdot \circ f\right).$$

Now we complete the category of R-Lie-Rinehart algebras towards nonstrong homomorphisms over homomorphisms of foliated algebras. For slightly different notion of a *comorphism* (not using foliated algebras) see [H-M2].

Let $(L, \#_L)$ and $(L', \#_{L'})$ be two R-Lie-Rinehart algebras over R-algebras A and A'. respectively, and let

$f : (A, \operatorname{Im} \#_L) \to (A', \operatorname{Im} \#_{L'})$ be a homomorphism of foliated algebras.

We construct the inverse-image $f^\wedge L'$ as a A-submodule

$$f^\wedge L' \subset F \bigoplus (A \bigotimes_{A'} L')$$

with a suitable structures. In this purpose take a homomorphism of A-modules

$$\varphi' : A \bigotimes{}'_{A'} L \xrightarrow{id \otimes \#_{L'}} A \bigotimes_{A'} \operatorname{Im} \#_{L'} \xrightarrow{\varphi} \operatorname{Der}(A', A)$$

where φ is defined via

$$\varphi : A \bigotimes_{A'} \operatorname{Im} \#_{L'} \longrightarrow \operatorname{Der}(A', A), \quad h \otimes X' \longmapsto h \cdot f \circ X'.$$

We put

$$f^\wedge L' = \left\{ (X, \alpha) \in \operatorname{Im} \#_L \bigoplus (A \bigotimes_{A'} L');\ X \circ f = \varphi'(\alpha) \right\}.$$

Clearly, $f^\wedge L'$ is an A-submodule and the projection on the first factor is a surjection. The R-Lie bracket in $f^\wedge L'$ is introduced in the following way:

$$\begin{aligned} &[\![(X_1, \alpha_1), (X_2, \alpha_2)]\!] \\ &= ([X_1, X_2], \\ &\sum\nolimits_{i,j} h^i \cdot g^j \otimes [\![\alpha'^i, \beta'^j]\!] + \sum\nolimits_j X_1(g^j) \otimes \beta'^j - \sum\nolimits_i X_2(h^i) \otimes \alpha'^i) \end{aligned}$$

for $\alpha_1 = \sum h^i \otimes \alpha'^i$ and $\alpha_2 = \sum g^j \otimes \beta'^j$, $h^i, g^j \in A$ and $\alpha'^i, \beta'^j \in L'$. We check the correctness of the definition, i.e. the independence of the choice of representations of α_1 and α_2 , analogously to the situation considered in the previous section using the R-2-linear homomorphism

$$\begin{aligned} G' : A \times L' &\longrightarrow A \bigotimes_{A'} L', \\ (g, \beta') &\longmapsto \sum\nolimits_i h^i \cdot g \otimes [\![\alpha'^i, \beta]\!] + X_1(g) \otimes \beta. \end{aligned}$$

and checking the condition $G'(g\cdot f(a'),\beta)=G'(g,a'\cdot\beta)$ for $a'\in A'$ (this condition easily follows from the equality $X_1\circ f=\varphi'(\alpha_1)$, i.e. from $X_1(fa')=\sum h^i\cdot f\left(\omega'\left(\alpha'^i\right)(a')\right)$. Therefore. there exists an R-linear homomorphism $\tilde{G}'$: $A\bigotimes_{A'}L'\to A\bigotimes_{A'}L'$ such that $\tilde{G}'(g\otimes\beta')=G'(g,\beta)$ and

$$\tilde{G}'(\alpha_2)=\sum\nolimits_{i,j}h^i\cdot g^j\otimes[\![\alpha'^i,\beta'^j]\!]+\sum\nolimits_j X_1\left(g^j\right)\otimes\beta'^j.$$

The verification of the Jacobi identity is left to the reader. It remains to check the properties of the anchor defined as the projection onto the first factor, but its A-linearity and the fact that it is a homomorphism of R-Lie algebras is evident as well as the equality

$$[\![(X_1,\alpha_1),\alpha\cdot(X_2,\alpha_2)]\!]=a\cdot[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!]+X_1(a)\cdot(X_2,\alpha_2).$$

Definition 2.12 *By a homomorphism $H:(L,\#_L)\to(L',\#_{L'})$ of R-Lie-Rinehart algebras over a homomorphism of foliated algebras $f:(A,\operatorname{Im}\#_L)\to(A',\operatorname{Im}\#_{L'})$ we mean a strong homomorphism of (R,A)-Lie-Rinehart algebras $H:L\to f^\wedge L'$, i.e. A-linear homomorphism of R-Lie algebras such that $pr_1\circ H=\#_L$. This means, that if $H(\alpha)=\left(\#_L(\alpha),\sum h^i\otimes\alpha'^i\right)$ then $\#_L(\alpha)\circ f=\sum h^i\cdot f\circ\#_{L'}\left(\alpha'^i\right)$ and if additionally $H(\beta)=\left(\#_L(\beta),\sum g^j\otimes\beta'^j\right)$ than*

$$\begin{aligned}&H([\![\alpha,\beta]\!])\\&=([\#_L\alpha,\#_L\beta],\sum\nolimits_{i,j}h^ig^j\otimes[\![\alpha'^i,\beta'^j]\!]+\sum\nolimits_j\#_L(\alpha)\left(g^j\right)\otimes\beta'^j\\&\qquad-\sum\nolimits_i\#_L(\beta)\left(h^i\right)\otimes\alpha'^i).\end{aligned}$$

2.4 Integrals and open problems

Preliminary from classical differential geometry.

A smooth mapping $g : M \to M'$ between manifolds is called immersion if for each point $p \in M$ the differential

$$g_{*p} : T_pM \to T_{g(p)}M'$$

is a monomorphism.

- If $g : M \to M'$ is an immersion, then the pullback of the smooth functions $g^* : C^\infty(M') \to C^\infty(M)$ has the following property:

  — for
  $$X \in \mathfrak{X}(M) = \mathrm{Der}\left(C^\infty(M)\right)$$
  if
  $$X \circ g^* = 0 \quad \text{then} \quad X = 0.$$

  Indeed, let $X \circ g^* = 0$. Then for each point $p \in M$ and a smooth function $\alpha' \in M'$

  $$0 = (X \circ g^*)(\alpha')(p) = X(\alpha' \circ g)(p) = X_p(\alpha' \circ g) = g_{*p}(X_p)(\alpha'),$$
  $$0 = g_{*p}(X_p)$$

  therefore $X_p = 0$.

- The opposite does not hold; for example if $g : \mathbb{R} \to \mathbb{R}$ is given by the formula $g(t) = t^3$ then g is not an immersion but the above condition is fulfilled.

- g fulfiling the previous condition will be called weak immersion.

Lemma 2.13 *g is immersion if and only if g is a weak immersion and the image of the homomorphism $\tilde{g}_* : TM \to g^*(TM')$ is a subbundle, i.e. for the modules of cross* *sections the image of the mapping*

$$\begin{array}{cccc} \tilde{g}_* : & \mathfrak{X}(M) & \rightarrow & \Gamma(g^*TM') \\ & \parallel & & \parallel \\ & \mathrm{Der}(C^\infty(M)) & \rightarrow & \mathrm{Der}(C^\infty(M'), C^\infty(M)) \\ & X & \longmapsto & X \circ f \end{array}$$

is a direct summand in $\mathrm{Der}(C^\infty(M'), C^\infty(M))$.

Indeed, if the image of $\tilde{g}_* : TM \to g^*(TM')$ is a subbundle, then the kernel $\ker \tilde{g}_*$ is also a subbundle. Therefore, if $g_*(v) = 0$ then there exists a vector field X such that $X \in \ker g_*$, i.e $X \circ g^* = 0$, and $X_p = v$. From the assumptions $X = 0$ so $v = 0$.

Definition 2.14 *A homomorphism of algebras* $f : A \to L$ *is called weak immersion if for an arbitrary derivation* $X \in \mathrm{Der}(L)$ *the following property holds*

$$\textit{if } \; X \circ f = 0 \; \textit{ then } \; X = 0,$$

i.e. if the homomorphism of L*-modules*

$$f_* : \mathrm{Der}(L) \longrightarrow \mathrm{Der}(A, L), \; X \longmapsto X \circ f,$$

is a monomorphism. A homomorphism $f : A \to L$ *is called immersion if it is a weak immersion and the image* $\mathrm{Im}[f_*]$ *is a direct summand in* $\mathrm{Der}(A, L)$.

Proposition 2.15 *Let* $f : A' \to A$ *and* $g : A \to B$ *be two immersions. Then the superposition* $g \circ f : A' \to B$ *is an immersion provided that the modules of Kähler differentials* $\Omega^1_{A|R}$ *and* $\Omega^1_{A'|R}$ *are projective and finitely generated.*

Now I would like to give a concept of the notion of the integral of a foliation.

Definition 2.16 *Let* (A, F) *be a foliated algebra. A homomorphism* $f : A \to L$ *of algebras is called an integral of* F *if*

- *f is an immersion,*
- *$f : (L, \mathrm{Der}(L)) \to (A, F)$ is a homomorphism of foliated algebras.*

Let $X \in \mathrm{Der}(A)$. Then $F = Lin_A(X) := \{a \cdot X;\ a \in A\}$ is a foliation of A.

Definition 2.17 *Let $X \in \mathrm{Der}(A)$. By an integral algebra of X we mean each triple*

$$(A', X', c)$$

consisting of an commutative algebra A', a derivation $X' \in \mathrm{Der}(A')$ and an immersion $c : A \to A'$ such that

(1) $\mathrm{Der}(A')$ *is a free A'-module with one generator X',*

(2) $c : (A', Lin_{A'}(X')) \to (A, F)$ is a homomorphism of foliated algebras such that

$$X' \circ c = c \circ X.$$

Definition 2.18 *A' is called connected integral algebra if $x = 0$ or $x = 1_A$ is the unique solutions of the equation $x^2 = x$, $x \in A$.*

Problem 2.19 *Existing of integral algebras is open. Also any version of the Frobenius Theorem is not known.*

In the case of algebras of smooth functions on manifolds the condition (1) corresponds to 1-dimensional manifolds, whereas the condition (2) corresponds for a vector field X on M, for the condition $\dot{c} = X$. The connectedness of A' means precisely the connectedness of the suitable manifold.

The problem of characterizing the injectivity of a smooth mapping $g : M \to M'$ of manifolds in the terms of homomorphism of algebras

$$g^* : C^\infty(M') \to C^\infty(M')$$

can be done in terms of reals spectrum $\mathrm{Spec}_r\left(C^\infty\left(M\right)\right)$ of the algebras equipped with the Gelfand topology. Indeed, for any paracompact manifold M, we have an homeomorphism

$$M \cong \mathrm{Spec}_r\left(C^\infty\left(M\right)\right)$$

(see [N-G], Juan A. Navarro Gonzáles and Juan B. Sancho de Salas, . C^∞*-Differentiable Spaces,* Springer 2003).

Therefore, g is injective if and only if the induced homomorphism

$$g^* : \mathrm{Spec}_r\left(C^\infty\left(M'\right)\right) \to \mathrm{Spec}_r\left(C^\infty\left(M\right)\right)$$

is injective. Therefore we can post that for arbitrary R-algebras A and A' a homomorphism $c : A \to A'$ is called injective if the induced homomorphism of real spectrums $c^* : \mathrm{Spec}_r\left(A'\right) \to \mathrm{Spec}_r\left(A\right)$ is injective.

3 Characteristic classes associated with foliations

3.1 Primary characteristic classes

3.1.1 Lie-algebroid's preliminary and a piece of history

The ring of primary characteristic classes associated with Lie-Rinehart algebra over algebraic foliation in an R-algebra A generalizes the well known ring of primary characteristic classes of principal fibre bundles or vector bundles (generated by the Pontryagin or Chern classes). These rings are images by the so-called Chern-Weil characterisitic homomorphism.

In the language of Lie-Rinehart algebras it was firstly given by

— Nicolae Teleman, in 1972, *A characteristic ring of a Lie algebra extension*, Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **8** (1972), 498-506.

We must add that Teleman does not use the term "Lie-Rinehart algebra" but "extension of Lie algebras".

He associated with a short exact sequence

$$\mathbf{e}: \quad 0 \longrightarrow L' \longrightarrow L \xrightarrow{\pi} L'' \longrightarrow 0$$

of Lie-Rinehart algebras over R-algebra A, assuming that there exists a splitting of this sequence

$$\nabla : L'' \to L, \quad \pi \circ \nabla = id_{L''} \ ,$$

some characteristic homomorphism $h_{\mathbf{e}}$. In particular this concerns a sequence associative with one Lie-Rinehart algebra

$$\mathbf{e}_L : \quad 0 \to \ker \#_L \to L \to \operatorname{Im} \#_L \to 0$$

($\operatorname{Im} \#_L \subset \operatorname{Der}(A)$ is an algebraic foliation in A). Teleman notice, that if the last sequence is associated with the Lie algebroid of a principal fibre bundle P and

- the structure Lie group G of P is **connected** then the Chern-Weil homomorphism of $\mathbf{e}_L$ and of P are equivalent.

In the paper by J.K.

The Chern-Weil homomorphism of regular Lie algebroids, Publ. Dep. Math. Univ. Lyon 1, 1991.

it is proved that this hold **without any** assumptions on structural Lie group G (we must only assume that P is connected). The same results was repeated independently by I. Belko in 1994.

The next step was to construction of the Chern-Weil

$$h_{(L,A)} : I(A) \longrightarrow H(L)$$

of the pair of of Lie algebroids (L, A) assuming that A is regular over a foliation.

B. Balcerzak, J. Kubarski, W. Walas, *Primary characteristic homomorphism of pairs of Lie algebroids and Mackenzie algebroid*, Lie Algebroids and Related Topics in Differential Geometry, Banach Center Publications, Volume **54**, 71-97, IMPAN Warszawa 2001.

The Chern-Weil homomorphism $h_{(L,A)}$ of a pair (L, A)

$$h_{(L,A)} : I(A) \longrightarrow H(L)$$

is constructed by use the so-called L-connection in A, i.e a linear homomorphism

$$\nabla : L \to A$$

commuting with the anchors

$$\#_A \circ \nabla = \#_L.$$

The algebra $I(A)$ (=domain of $h_{(L,A)}$) and the cohomology algebra $H(L)$ will be defined below in more general algebraic context. We add only that for a regular foliation $F \subset TM$ (which is of course a Lie algebroid) the cohomology algebra $H(F)$ is the cohomology of the tangential differential forms.

The definition of L-connection in $\mathbf{A}$ covers

— usual and partial covariant derivatives in vector bundles,

— usual and partial connections in principal bundles,

— connections in transitive or regular Lie algebroids,

— connections in extensions

$$\mathbf{e}: 0 \to L' \to A \to L \to 0$$

of Lie algebroids, in particular, complete differentials of higher order understood as the splittings of the jet-bundles

$$0 \to S^k(TM, TM) \to J^k(TM) \to J^{k-1}(TM) \to 0,$$

— transversal connections in extensions of principal fibre bundles,

— known in Poisson geometry covariant and contravariant connections (the last are important also for Poisson algebras).

The Chern-Weil homomorphism $h_{(L,A)}$ is trivial if there exists a flat L-connection $\nabla : L \to A$ in A, i.e. such that ∇ is a homomorphism of Lie algebras.

This approach generalizes the well known constructions

— by Teleman 1972,

— Mackenzie 1988,

— Kubarski 1991,

— Vaisman 1994,

— Belko 1997,

— Moore and Schochet 1988,

— Huebschmann 1999,

— Itskov, Karasev and Vorobjev 1999,

— Fernandes (preprints 2000),

— Crainic (preprint 2001).

The Chern-Weil homomorphism $h_{(L,A)}$ for a pair is compared with the other Chern-Weil homomorphisms h_L, h_A, and $h_{\mathbf{e}}$ in the case of an extension

$$\mathbf{e}\,(\pi) : 0 \longrightarrow L' \to L \xrightarrow{\pi} A \longrightarrow 0$$

$$\begin{array}{ccccc}
I(\mathbf{L}) & \xrightarrow{\pi^{+*}} & I(\mathbf{A}) & \xrightarrow{j^{+*}} & I(\mathbf{e}(\pi)) \\
 & \searrow^{h_L} & \downarrow{h_A} & \searrow^{h_{L,A}} & \downarrow{h_{\mathbf{e}(\pi)}} \\
 & & H(F) & \xrightarrow[\#_L^{\sharp}]{} & H(L).
\end{array} \qquad (3.1)$$

3.1.2 Primary characteristic classes for a pair of Lie-Rinehart algebras

This part of my talk is based on the results by Witold Walas (in printing) which is a generalization of the method by Nicola Teleman (1972).

We fix two Lie-Rinehart algebras $(L, [\![\cdot,\cdot]\!], \#_L)$, $(K, [\![\cdot,\cdot]\!], \#_K)$ over an R-algebra A. Let

$$\mathfrak{g} = \ker \#_L.$$

The A-module $\mathfrak{g}$ is also an A-Lie algebra.

A A-linear mapping

$$\nabla : K \to L$$

compatible with the anchors

$$\#_L \circ \nabla = \#_K$$

is called a K-connection in L. By the curvature of ∇ we mean an alternating 2-A-linear tensor $\Omega^\nabla : K \times K \to L$, $\Omega^\nabla \in \mathrm{Alt}_A^2(K, \mathfrak{g})$ defined by

$$\Omega^\nabla(x, y) = \nabla[\![x, y]\!] - [\![\nabla x, \nabla y]\!].$$

To any K-connection in L we associate standardly an operator

$$d^\nabla : \mathrm{Alt}_A^n(K, \mathfrak{g}) \to \mathrm{Alt}_A^n(K, \mathfrak{g})$$

by the formula

$$\begin{aligned}\left(d^\nabla\varphi\right)(x_0, ..., x_n) = &\sum_{i=0}^{n} (-1)^i [\![\nabla x_i, \varphi(x_0, ..., \hat{x}_i, ..., x_n)]\!] \\ &+ \sum_{i<j} (-1)^{i+j} \varphi([\![x_i, x_j]\!], x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n),\end{aligned}$$

$\varphi \in \mathrm{Alt}_A^n(K, \mathfrak{g})$, $x_i \in K$.

Theorem 3.1 ***Bianchi identity***

$$d^{\nabla}\left(\Omega^{\nabla}\right)=0.$$

Remark 3.2 *Since* $\left(d^{\nabla}\right)^2\varphi=(-1)^n\,\Omega^{\nabla}\wedge\varphi$, *then* $\left(d^{\nabla}\right)^2\neq 0$, *in general.*

Example 3.3 *If* ∇ *is flat,* $\Omega^{\nabla}=0$, *then* $\left(d^{\nabla}\right)^2=0$ *and the cohomology* $H^{\nabla}(K;L)$ *can be defined. In particular, taking arbitrary Lie-Rinehart algebra* K *and*

$$\nabla=\#_K:K\to L:=\operatorname{Im}\#_K\subset\operatorname{Der}(A),$$

we have flatness of ∇ *because the anchor* $\#_K$ *is a homomorphism of Lie algebras. In this way, we have the usual algebra of cohomology* $H(K)$ *of a Lie-Rinehart algebra* K.

Considering $CO(M)$ instead of L for any A-module M we have a notion of K-covariant derivative

$$\nabla:K\to CO(M)$$

assuming the axiom: the anchor of $x\in K$ is equal to the anchor of ∇x. (We recall that for a covariant operator $D\in CO(M)$ its anchor is not uniquely determined unless the assumption that the natural representation $\rho:A\to\operatorname{End}_R(M)$ is faithfull.

If ∇ is here a homomorphism of Lie algebras, ∇ is called a representation, and the differential and cohomology $H(K;M)$ are defined.

An element $m\in M$ is called ∇-invariant if for each $x\in K$

$$\nabla_x(m)=0$$

If K is projective then $H(K;M)$ can be done via the functor Ext (G.Rinehart 1963).

Example 3.4 ***Adjoint representation.*** *Let* $\mathfrak{g} = \ker \#_K$, *then we have the adjoint representation*

$$ad_K : K \to CO(\mathfrak{g})$$

by

$$(ad_K)_x = [\![x, \cdot]\!].$$

A given representation $\nabla : K \to CO(M)$ determines new one

$$\mathrm{Hom}^k \nabla : K \to CO\left(\mathrm{Hom}_A^k (M, A)\right)$$

in the associated module $\mathrm{Hom}_A^k (M; A)$ by standard formula

$$\begin{aligned} &\left(\mathrm{Hom}^k \nabla\right)_x (\varphi)(m_1, ..., m_k) \\ &= (\#_K)(x)(\varphi(m_1, ..., m_k)) - \sum_{i=1}^{k} \varphi(m_1, ..., \nabla_x m_i, ..., m_k), \end{aligned}$$

$x \in K,\ m_i \in M,\ \varphi \in \mathrm{Hom}_A^k (M, A)$.

Let $\mathrm{Hom}_A^k (M, A)_I$ be the space of invariant tensors. Their direct sum forms an algebra.

Now taking $M := \mathfrak{g} = \ker \#_K$ we consider symmetric invariant tensors

$$I^k(\mathfrak{g}) := \left(\mathrm{Sym}_A^k (\mathfrak{g}; A)\right)_I.$$

The direct sum

$$I(\mathfrak{g}) = \bigcup\nolimits^k I^k(\mathfrak{g})$$

forms an algebra.

Let $\nabla : K \to L$ be any K-connection in L. We define a homomorphism of algebras

$$\chi_\nabla : I(\mathfrak{g}) \to \mathrm{Alt}_A (K; A)$$
$$\chi_\nabla(\varphi) = \varphi\left(\Omega^\nabla, ..., \Omega^\nabla\right)$$

where $\varphi\left(\Omega^{\nabla}, ..., \Omega^{\nabla}\right)$ is the usual alternating multiplication of forms by using multilinear homomorphism φ.

Theorem 3.5 *(Witold Walas) The differential forms from the image of χ_{∇} are closed and the induced homomorphism in cohomology*

$$h : I(\mathfrak{g}) \to H(K; A), \qquad \varphi \longmapsto \left[\varphi\left(\Omega^{\nabla}, ..., \Omega^{\nabla}\right)\right]$$

is independent on the chooice of the connection ∇.

The method of the independence on the connection is analogous to the methods by Teleman but with some strong modification. The cause is as follows: the K-connection in L do not possees a connection form which was uses by Teleman for the connection in extensions.

3.1.3 Exotic characteristic classes (sketch only)

The exotic (or secondary) characteristic classes compare two differential structures on a manifolds. For example: a given flat or partially flat connection in a principal fibre bundle and a given reduction i.e. some its subbundle. Among applications there are characteristic classes of a foliation. There are many different algebraic generalizations of these characteristic classes. Mainly in the language of Lie algebroids (Kubarski, Fernandes, Crainic) and lastly for Lie-Rinehart algebras (B.Balcerzak).

The "flat" classes for principal fibre bundles were considered extensively by Kamber and Tondeur in 1973-76. They have defined the characteristic homomorphism

$$\Delta_{\#\, P,P',\omega} = \Delta_{\#} : H^{*}(\mathfrak{g}, H) \longrightarrow H_{dR}(M) \tag{3.2}$$

for a G-principal fibre bundle P, a flat connection ω in P and an H-reduction $P' \subset P$ ($H \subset G$ is a closed Lie subgroup of G). The domain $H^{*}(\mathfrak{g}, H)$ is the *relative Lie algebra cohomology.*

The algebroids' generalization was done firstly by Kubarski

a) for flat classes in:

— *Algebroid nature of the characteristic classes of flat bundles*, in: Homotopy and Geometry, Banach Center Publications, Volume 45, Institute of Mathematics, Polish Academy of Science, Warszawa 1998, pp. 199–224.

b) for partially flat in:

— *The Weil algebra and the secondary characteristic homomorphism of regular Lie algebroids*, in: Lie Algebroids and Related Topics in Differential Geometry, Banach Center Publications, Volume 54, Institute of Mathematics, Polish Academy of Science, Warszawa 2001, pp. 135–173.

Next, the generalization on Lie-Rinehart algebras was given by B.Balcerzak (the paper is in preparation).

(A) In the context of "partially flat" characteristic classes a triple

$$(K, L, \nabla)$$

is considered, where
— $L \subset K$ are regular Lie algebroids over the same foliated manifold, and
— ∇ is a usual connection in K partially flat over some subfoliation.

The classical first example is as follows:

- for arbitrary regular foliation $F \subset M$ there exists the Bott connection in the normal bundle $Q = TM/F$ which is flat over F.
- Now considering a Riemannian metric on M, we have take the $O(n)$ reduction of the frame bundle and
- post the question: Does this Bott connection is Riemannian?, i.e. belongs to the geometry of this reduction?
- The exotic characteristic classes are measuring the independence of these two differential structures and called characteristic classes of a foliation.

In algebraic context of an algebraic foliation $F \subset \mathrm{Der}(A)$ we can also define a Bott flat partial connection ∇ in $M := \mathrm{Der}(A)/F$ by the formula

$$\nabla_X \bar{Y} = \overline{[X, Y]}, \quad X \in F, \quad Y \in \mathrm{Der}(A)$$

and $\bar{Y} \in \mathrm{Der}(A)/F$ is corresponding elements in the quotient. The construction of charactersistic classes in this context is open.

(B) In the context of "flat" characteristic classes there is constructed the secondary charactersistic homomorphism

$$\Delta_{\# K, L, \nabla}$$

of the triple

$$(K, L, \nabla) \ ,$$

where

- $L \subset K$ are regular Lie algebroids over the same foliated manifold and
- $\nabla : S \to K$ is a flat S-connection in K, S is an arbitrary irregular (in general) Lie algebroid.

Remark 3.6 *The particular case $S = K$ and $\nabla = \mathrm{id}_K$ gives rise to secondary characteristic homomorphism $\Delta_{\#K,L}$ of Lie subalgebroid, $L \subset K$.*

The characteristic homomorphism $\Delta_{\#K,L,\nabla}$ is factorized by $\Delta_{\#K,L}$, i.e.

$$\Delta_{\#K,L,\nabla} = \nabla^{\#} \circ \Delta_{\#K,L},$$

i.e. $\Delta_{\#K,L}$ fulfils a fundamental universal role.

These exotic characteristic homomorphism is constructed in the category of Lie-Rinehart algebras by Balcerzak.

This last can be used for a pair of Lie algebras $(\mathfrak{h}, \mathfrak{g})$, $\mathfrak{h} \subset \mathfrak{g}$ giving the well known Koszul homomorphism, as well as for reductions of principal fibre bundles $P' \subset P$ giving a new characteristic homomorphism.

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