Flat connections with isolated singularities in some transitive Lie algebroids

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8th Conference on Geometry and Topology of Manifolds
(Luxembourg-Poland-Ukraine conference,
Lie algebroids, dynamical systems and applications)

Przemyśl (Poland) – L'viv (Ukraine) 30.04.07 - 6.05.2007

1 Introduction

The index theorem of Euler-Poincaré-Hopf for sphere bundles is well-known (see, for example):

Theorem 1 *Let E be an n-sphere bundle with a connected compact oriented base manifold M of dimension $n+1$, such that E is given the local product orientation. Let σ be a cross-section of E with finitely many singularities $a_1, \ldots, a_k$. Then the index sum*

$$\sum_{v=1}^{k} j_{a_v}(\sigma),$$

where $j_{a_v}(\sigma)$ is the index of σ at a_v, is independent of the choice of the cross-section σ and the Euler class χ_E of E is given by $\chi_E = \sum_{v=1}^{k} j_{a_v}(\sigma) \cdot \omega_M$ where ω_M is the orientation class of M.

Example 2 *Assume M is 2-dimensional compact oriented manifold equipped with the structure*

$$(f, \Omega), \quad f \in C^{\infty}(M), \quad \Omega \in \Omega^2(M).$$

Let $\hat{\sigma} \in \Omega^1 (M \setminus \{a_1, ..., a_k\})$ be a 1 form such that

$$d\hat{\sigma} = \Omega - df \wedge \hat{\sigma},$$

i.e.

$$\Omega = d\hat{\sigma} + df \wedge \hat{\sigma} = d_{df}(\hat{\sigma})\,..$$

(d_ω is the Guedira-Lichnerowicz operator, for the short elementary proof of its properties see: Rybicki-Haller).

REMARK. From the triviality of $\mathbf{H}^2_{dR}(M \setminus \{a_1, ..., a_k\}) = 0$ follows that the form $\hat{\sigma}$ exist. Namely if

$$e^f \cdot \Omega = d\varphi \quad \text{on} \quad M \setminus \{a_1, ..., a_k\}$$

then

$$\hat{\sigma} = e^{-f} \cdot \varphi$$

$$\begin{aligned} d_{df}(\hat{\sigma}) &= d_{df}\left(e^{-f} \cdot \varphi\right) = d\left(e^{-f} \cdot \varphi\right) + df \wedge e^{-f} \cdot \varphi \\ &= -e^{-f} \cdot df \wedge \varphi + e^{-f} \cdot d\varphi + df \wedge e^{-f} \cdot \varphi \\ &= e^{-f} \cdot d\varphi = e^{-f} \cdot e^f \cdot \Omega \\ &= \Omega. \end{aligned}$$

By the index of $\hat{\sigma}$ at a_ν we define a number

$$j_{a_\nu}\hat{\sigma} := \int_{S^1_\nu} \left(e^f \cdot \hat{\sigma} + \eta_\nu\right)$$

where S^1_ν is here any small Jordan curve near a point a_ν and η_ν is arbitrary 1 form around a_ν such that

$$d\eta_\nu = -e^f \cdot \Omega.$$

The form $e^f \cdot \hat{\sigma} + \eta_\nu$ is closed

$$\begin{aligned}
& d\left(e^f \cdot \hat{\sigma} + \eta_\nu\right) \\
= & \ d\left(e^f \cdot \hat{\sigma}\right) + d\eta_\nu \\
= & \ e^f df \wedge \hat{\sigma} + e^f d\hat{\sigma} - e^f \cdot \Omega \\
= & \ e^f df \wedge \hat{\sigma} + e^f \cdot (\Omega - df \wedge \hat{\sigma}) - e^f \cdot \Omega \\
= & \ 0
\end{aligned}$$

therefore the index $j_{a_\nu}\hat{\sigma}$ is correctly defined (it is independent on the choice of S^1_ν and η_ν. As well as we have

$$d\left(e^f \cdot \hat{\sigma}\right) = -d\eta_\nu = e^f \cdot \Omega$$

Let $S^1_\nu = \partial\left(B_\nu\right)$, using the Stokes theorem we can easily to check following formula

$$\begin{aligned}
\sum_\nu j_{a_\nu}\hat{\sigma} &= \sum_\nu \int_{S^1_\nu} \left(e^f \cdot \hat{\sigma} + \eta_\nu\right) \\
&= \sum_\nu \int_{S^1_\nu} e^f \cdot \hat{\sigma} + \sum_\nu \int_{S^1_\nu} \eta_\nu \\
&= \int_{-\partial(M \setminus \cup B_\nu)} e^f \cdot \hat{\sigma} + \sum_\nu \int_{\partial B_\nu} \eta_\nu \\
&= -\int_{M \setminus \cup B_\nu} d\left(e^f \cdot \hat{\sigma}\right) + \sum_\nu \int_{\partial B_\nu} \eta_\nu \\
&= -\int_{M \setminus \cup B_\nu} e^f \cdot \Omega + \sum_\nu \int_{B_\nu} d\eta_\nu \\
&= -\int_{M \setminus \cup B_\nu} e^f \cdot \Omega - \sum_\nu \int_{B_\nu} e^f \cdot \Omega \\
&= -\int_M e^f \cdot \Omega = \int_M^{\#} \left[-e^f \cdot \Omega\right]
\end{aligned}$$

The above prove that the index sum $\sum_\nu j_{a_\nu}\hat{\sigma}$ is independent on the choice of the form $\hat{\sigma}$. The cohomology class $\left[-e^f \cdot \Omega\right]$ play a role of the Euler class and the forms $\hat{\sigma}$ play a role cross-sections of the sphere bundle.

Is there any common relation with that two Euler classes?

This Euler-Poincaré-Hopf theorem for sphere bundles can be applied, in particular, to G-principal bundles P over manifolds M of dimension $\dim M = \dim G + 1$ for Lie groups G diffeomorphic to a sphere $G \cong S^n$, i.e. for $n = 1$ and $n = 3$:

(i) S^1-principal bundles over M^2,

(ii) $\mathrm{Spin}\,(3)$-principal bundles over M^4.

A locally defined cross-section $f : U \to P_{|U}$ of a principal bundle P determines (in an evident manner) a flat connection $H^f \subset T\left(P_{|U}\right)$ in $P_{|U}$ in such a way that $H^f\left(f\left(x\right)\right) = f_*\left[T_x M\right] \subset T_{f(x)}P$, but not conversely: there are more (in general) flat connections than cross-sections.

We will try to join these observations is on the ground of Lie algebroids.

The notion of a Lie algebroid comes from J.Pradines and was invented for the study of differential groupoids. Let F be a C^∞ constant dimensional and involutive distribution on a C^∞ Hausdorff paracompact connected manifold M. By a *regular Lie algebroid* over (M,F) we mean a system

$$(A, [\![\cdot,\cdot]\!], \#_A)$$

consisting of a vector bundle A over M and mappings

$$[\![\cdot,\cdot]\!] : \mathrm{Sec}A \times \mathrm{Sec}A \to \mathrm{Sec}A, \quad \#_A : A \to TM,$$

such that

– (i) $(\mathrm{Sec}\,A, [\![\cdot,\cdot]\!])$ is a real Lie algebra,

– (ii) $\#_A$, called an *anchor*, is a homomorphism of vector bundles, and $\mathrm{Im}\,\#_A = F$,

– (iii) $\mathrm{Sec}\,\#_A : \mathrm{Sec}\,A \to \mathfrak{X}(M)$, $\xi \mapsto \#_A \circ \xi$, is a homomorphism of Lie algebras,

– (iv) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\#_A\circ\xi)(f)\cdot\eta$, $\xi,\ \eta \in \mathrm{Sec}\,A$, $f \in C^\infty(M)$.

In the case when $F = TM$, i.e. when $\#_A : A \to TM$ is a surjective homomorphism, the algebroid is called a *transitive Lie algebroid*. Let A and A' be two regular Lie algebroids on a manifold M. A homomorphism $H : A \longrightarrow A'$ of vector bundles is said to be a *homomorphism of Lie algebroids* if $\#'_A \circ H = \#_A$ and $\mathrm{Sec}\,H : \mathrm{Sec}\,A \to \mathrm{Sec}\,A'$ is a homomorphism of Lie algebras.

The short exact sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} F \longrightarrow 0 \tag{1}$$

is called the *Atiyah sequence* of A. Each fibre $\boldsymbol{g}_{|x}$, $x \in M$, possesses some natural structure of a Lie algebra defined by $[v,w] := [\![\xi,\eta]\!](x)$ where $\xi,\eta \in \mathrm{Sec}\,A$ are arbitrarily taken cross-sections of A such that $\xi(x) = v$, $\nu(x) = w$. For transitive Lie algebroid A, $\boldsymbol{g}$ is a LAB.

The following are simple examples of transitive Lie algebroids:

Finitely dimensional real Lie algebra $\mathfrak{g}$.
Tangent bundle TM **to a manifold** M.
Trivial Lie algebroid $TM \times \mathfrak{g}$ where $\mathfrak{g}$ is a finitely dimensional Lie algebra.
Bundle of jets $J^k TM$.

The following are important examples of transitive Lie algebroids:

The Lie algebroid $A(P)$ **of a** G**-principal bundle** $P = P(M, G)$.

The Lie algebroid $CDO(\mathfrak{f})$ **of covariant differential operators on a vector bundle** $\mathfrak{f}$ **(equivalently, the Lie algebroid of the of the frame bundle of** $\mathfrak{f}$ **)**

The Lie algebroid $A(M, F)$ **of a transversally complete foliation** $(M, \mathcal{F})$ **of a connected Hausdorff paracompact manifold** M**.**

The Lie algebroid $A(G; H)$ **of a nonclosed Lie subgroup** H **of** G**.**

— The structure (f, Ω) on M^2 determine a Lie algebroid structure in $A = TM \times \mathbb{R}$ with the anchor $\#_A : TM \times \mathbb{R} \to \mathbb{R}$, $(v, t) \longmapsto t$, the Lie algebra structure $[\![\cdot, \cdot]\!]$ in $\Gamma(A)$ defined by

$$[\![(X, f), (Y, g)]\!] = ([X, Y], \nabla_X g - \nabla_Y f - \Omega(X, Y))$$

where ∇ is the flat covariant derivative in the trivial vector bundle $M \times \mathbb{R}$ defined

$$\nabla_X g = \partial_X g + \omega(X) \cdot g, \quad \omega = df.$$

My talk is a first step to Lie algebroids nature of the index of singular connections (in some cases only). The next steps should be concern

(a) nontransitive Lie algebroids and nonisolated singularities,

(b) bigger class of Lie algebroids, than considered here.

The obtained above Lie algebroids A have the property

–

$$\mathbf{H}^{top}(A) = \mathbb{R}.$$

– the cohomology of the Lie algebras $\mathbb{R}$ or $so(3)$ are equal to $\mathbf{H}(S^n) .n = 1, 3.$

Let A be a transitive Lie algebroid on a manifold M with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A \overset{\#_A}{\to} TM \to 0.$$

By a connection in A we mean a splitting $\lambda : TM \to A$, i.e. $\#_A \circ \lambda = id_A$.

By a local connection with singularity at a point $a \in M$ in A we mean a connection

$$\lambda : T\left(M_{|U\setminus\{a\}}\right) \to A_{|U\setminus\{a\}}, \quad U \subset M, \quad a \in U,$$

defined is some neighbourhood of a point a.

If λ is a homomorphism of Lie algebroids, i.e.

$$\lambda [X, Y] = [\![\lambda X, \lambda Y]\!]$$

then λ is called a flat connection and the pullback

$$\lambda^* : \Omega(A) \to \Omega(M)$$

commutes with differentials giving a homomorphism on cohomology

$$\lambda^{\#} : \mathbf{H}(A) \to \mathbf{H}_{dR}(M).$$

- In the sequel we are interested in flat connections

$$\lambda : T\left(M \setminus \{a_1, ..., a_k\}\right) \to A_{M\setminus\{a_1,...,a_k\}}$$

in A with singularities at finite number of points $a_1, ..., a_k$.

2 Final assumptions

Let $\dim M = m$ and $\dim \boldsymbol{g}_x = n$.

Finally we will assume the following conditions:

1) M is compact and oriented.
2) $\mathbf{H}^{top}(A) = \mathbf{H}^{m+n}A \neq 0$,
3) $\mathbf{H}^*(\mathfrak{g}) \cong \mathbf{H}^*_{dR}(S^n)$, $\mathfrak{g} \cong \boldsymbol{g}_x$ is the isotropy Lie algebra,
4) $m = n + 1$.

Under these rather strong conditions we can observe an interesting analogy of the theory of sphere bundles. Flat connections correspond to cross-sections of sphere bundles.

Theorem 3 *A n-dimensional Lie algebra $\mathfrak{g}$ for which $\mathbf{H}^*(\mathfrak{g}) \cong \mathbf{H}^*_{dR}(S^n)$, i.e. such that*

$$\mathbf{H}^k(\mathfrak{g}) = \begin{cases} \mathbb{R}, & k = 0, n \\ 0, & 1 < k < n-1 \end{cases}$$

is isomorphic to $\mathbb{R}$ or $sk(3)$ or $sl(2,\mathbb{R})$.

REMARKS: Below, a transitive Lie algebroid A with isotropy Lie algebras isomorphic to a given Lie algebra $\mathfrak{g}$ will be shortly denoted as $\mathfrak{g}$-Lie algebroid.

A) Not every $\mathbb{R}$-Lie algebroid is integrable,

B) Lie algebras $sk(3)$ and $sl(2,\mathbb{R})$ are semisimple, therefore any transitive Lie algebroid with such a isotropy Lie algebra is integrable. a form $\hat{\sigma}$. However, the obtained principal fibre bundle may not has **connected** structural Lie group.

Theorem 4 *(Kubarski-Mishchenko) The condition* $\mathbf{H}^{m+n}A \neq 0$ *is equivalent to this: the isotropy Lie algebra bundle* $\boldsymbol{g}$ *is orientable and there exists a volume* $\varepsilon \in \Gamma\left(\bigwedge^n \boldsymbol{g}\right)$ *(i.e.* $\varepsilon_x \neq 0$ *for all* $x \in M$ *) which is invariant under adjoint representation. [This also imply that the cohomology algebra* $\mathbf{H}(A)$ *fulfils the Poincaré duality.]*

Example 5 *(1) lcs structures.*

Consider on the manifold M *the following data*

$$(\omega, \Omega)$$

where

- $\omega \in \Omega^1(M)$ *is a closed 1-form,*
- $\Omega \in \Omega^2(M)$ *is a 2-form such that*

$$d\Omega = -\omega \wedge \Omega.$$

(REMARK: if (ω, Ω) *is an lcs structure then the above conditions are fulfilled [and also that* Ω *is nondegenerated])*

Take the trivial vector bundle $\boldsymbol{g} = M \times \mathbb{R}$ *and equip it with the flat covariant derivative*

$$\nabla_X f = \partial_X f + \omega(X) \cdot f.$$

Clearly, the condition $d\Omega = -\omega \wedge \Omega$ *is equivalent to* $\nabla\Omega = 0$.

Theorem. *The vector bundle* $A = TM \times \mathbb{R}$ *forms a transitive Lie algebroid with*

- *the anchor* $\#_A : TM \times \mathbb{R} \to \mathbb{R}$, $(v, t) \longmapsto t$,
- *the Lie algebra structure* $[\![\cdot,\cdot]\!]$ *in* $\Gamma(A)$ *defined by*

$$[\![(X, f), (Y, g)]\!] = \left([X, Y], \nabla_X g - \nabla_Y f - \Omega(X, Y)\right).$$

REMARK: *Each transitive Lie algebroid with the associated Lie algebra bundle* $\boldsymbol{g} = M \times \mathbb{R}$ *is of the above form.*

Theorem 6 *The above Lie algebroid* $A = TM \times \mathbb{R}$ *defined by the data* (ω, Ω) *on the compact oriented manifold* M *fulfils the condition* $\mathbf{H}^3(A) \neq 0$ *if and only if* ω *is exact. It means, if* A *comes from an lcs structure than the condition* $\mathbf{H}^3(A) \neq 0$ *is equivalent to that: lcs structure* (ω, Ω) *is globally conformal symplectic structure.*

If ω *is exact,* $\omega = df$*, then a positive function* $\varepsilon \in C^\infty(M) = \Gamma(M \times \mathbb{R})$ *is invariant under adjoint representation if and only if*

$$\varepsilon = c \cdot e^{-f}, \quad c \in \mathbb{R}.$$

3 Fibre integral

Theorem 7 *Assume* $\boxed{\mathbf{H}^{m+n} A \neq 0}$ *and let* $\varepsilon \in \Gamma\left(\bigwedge^n \boldsymbol{g}\right)$ *be an invariant volume. Then there exists an operator of the fibre integral*

$$\int\!\!\!\!\!\!-_{A} : \Omega^\star(A) \longrightarrow \Omega^{\star - n}(M)$$

defined by the formula

$$\left(\int\!\!\!\!\!\!-_{A} \phi\right)_x (w_1, ..., w_k) = (-1)^{nk} \phi_x(\varepsilon_x, \tilde{w}_1, ..., \tilde{w}_k)$$

where $\tilde{w}_i \in A_x$ *and* $\#_A(\tilde{w}_i) = w_i$*. The operator* $\int\!\!\!\!-_{A}$ *has the following properties*

- $\int\!\!\!\!-_{A} \circ \#_A = 0$, *i.e.* $\operatorname{Im} \#_A \subset \ker \int\!\!\!\!-_{A}$,
- $\int\!\!\!\!-_{A}$ *is an epimorphism and commutes with differentials giving a homomorphism on cohomology*

$$\int\!\!\!\!\!\!-_{A}^{\#} : \mathbf{H}^*(A) \to \mathbf{H}^*(M).$$

- $\ker \fint_A$ *is* d_A*-stable subspace of* $\Omega(A)$.

4 Euler class and difference class of two flat connections

Now we examine the short sequence of graded differential spaces

$$0 \to \ker \int_A \xrightarrow{\iota} \Omega(A) \xrightarrow{\fint_A} \Omega(M) \to 0$$

and corresponding canonical long exact sequence in cohomology

$$.... \to \mathbf{H}^{*+n}(A) \xrightarrow{\fint_A^{\#}} \mathbf{H}^{*}(M) \xrightarrow{\partial} \mathbf{H}^{*+n+1}\left(\ker \int_A\right) \xrightarrow{\iota^{\#}} \mathbf{H}^{*+n+1}(A) \to ...$$

with the connecting homomorphism of the degree $n+1$.

Theorem 8 *If additionally the condition* $\boxed{\mathbf{H}^*(\mathfrak{g}) \cong \mathbf{H}^*_{dR}(S^n)}$, $\mathfrak{g} \cong \boldsymbol{g}_x$ *holds, then*

$$(\#_A)^{\#} : \mathbf{H}_{dR}(M) \xrightarrow{\cong} \mathbf{H}\left(\ker \int_A\right)$$

is an isomorphism.

Definition 9 *We define Gysin homomorphism (degree* $n+1$*)*

$$D : \mathbf{H}(M) \overset{(-1)^{\deg +1}}{\longrightarrow} \mathbf{H}(M) \xrightarrow{\partial} \mathbf{H}\left(\ker \int_A\right) \overset{\left((\#_A)^{\#}\right)^{-1}}{\longrightarrow} \mathbf{H}(M)$$

and the Euler class

$$\chi_A := D(1) = \left((\#_A)^{\#}\right)^{-1}(\partial(-1)) \in \mathbf{H}^{n+1}(M).$$

D is a homomorphism of degree $n+1$, giving the exactness of the triangle

$$\begin{array}{ccc} H\left(M\right) & & \\ & \searrow^{\gamma^{\#}} & \\ D\uparrow & & H_A\left(M\right) \; . \\ & \swarrow_{\int_A^{\#}} & \\ H\left(M\right) & & \end{array}$$

We modify the long exact sequence in cohomology

$$\to \mathbf{H}^*\left(M\right) \stackrel{D}{\to} \mathbf{H}^{*+n+1}\left(M\right) \stackrel{\left(\#_A\right)^{\#}}{\to} \mathbf{H}^{*+n+1}\left(A\right) \stackrel{\int_A^{\#}}{\to} \mathbf{H}^{*+1}\left(M\right) \stackrel{D}{\to} \mathbf{H}^{*+n+2}\left(M\right) ...$$

to obtain the so called *Gysin sequence* of A.

Remark 10 *1) The Euler class can be calculate via the Chern-Weil homomorphism h_A of A.*

a) the case $n = 1$, i.e. $\mathfrak{g} = \mathbb{R}$. Assume that $\varepsilon \in \Gamma\left(\boldsymbol{g}\right)$ is invariant. Take $\varepsilon^* \in \Gamma\boldsymbol{g}^*$ such that

$$i_\varepsilon \varepsilon^* = 1.$$

Then ε^* is invariant, therefore it is in the domain of the Chern-Weil homomorphism h_A of A.

Theorem 11 *Under the above assumptions*

$$\chi_A = h_A\left(\varepsilon^*\right).$$

More detailed: the considered Lie algebroid is defined via the data (ω, Ω), with exact form $\omega = df$. and let $\varepsilon = e^{-f}$. Therefore $\varepsilon^* = e^f$ and

$$\chi_A = \left[\langle e^f, \Omega^\lambda \rangle\right]$$

where Ω^λ is the curvature form of any connection in A. Taking the connection $\lambda : TM \to A = TM \times \mathbb{R}$ defined by $\lambda(v) = (v, 0)$ we have: $\Omega^\lambda = -\Omega$ which implies

$$\chi_A = \left[-e^f \cdot \Omega\right].$$

b) $n = 3$

2) There exists nonintegrable Lie algebroid with isotropy Lie algebras $\mathbb{R}$ such that $\chi_A \neq 0$.

Example 12 *Let G be an arbitrary compact connected semisimple Lie group and $H \subset G$ a nonclosed Lie subgroup such that $\dim \bar{H} - \dim H = 1$. Then the Lie algebroid of the TC-foliation of left coset of G by H is nonintegrable and has nonzero Euler class.*

Theorem 13 *(a) $D(\alpha) = \alpha \wedge \chi_A$.*

(b) If A is flat then $\chi_A = 0$, so $D = 0$. Therefore the Gysin sequence reduce to exact sequences

$$0 \to \mathbf{H}^*(M) \overset{(\#_A)^\#}{\to} \mathbf{H}^*(A) \overset{\int_A^\#}{\to} \mathbf{H}^{*-n}(M) \to 0$$

and if $\lambda : TM \to A$ is a flat connection then

$$\mathbf{H}^*(A) = \ker \int_A^\# \oplus \ker \lambda^\#$$

so

$$\int_A^{\#} : \ker \lambda^{\#} \xrightarrow{\cong} \mathbf{H}(M)$$

is an isomorphism.

Definition 14 *The unique cohomology class $\omega_\lambda \in \mathbf{H}^n(A)$ such that*

$$\int_A^{\#} (\omega_\lambda) = 1 \in \mathbf{H}^0(M)$$

is called the cohomology class of the flat connection λ.

For arbitrary two flat connections λ, σ in A there exists exactly one cohomology class $[\lambda, \sigma] \in \mathbf{H}^n(M)$ such that

$$\omega_\lambda - \sigma_\lambda = (\#_A)^{\#}([\lambda, \sigma])$$

called the difference class of two flat connections λ and σ.

5 Index of a flat connection at an singularity

Consider an open coordinate neighbourhood $U \subset M$ of a point $a \in U$ and a flat local connection with singularity at

$$\sigma : T(U \setminus \{a\}) \to A_{|U\setminus\{a\}}.$$

Taking arbitrary flat connection $\lambda : TU \to A_{|U}$ (always exists since U is difeomorphic to Euclidean space) we can use the difference class

$$\left[\lambda_{|U\setminus\{a\}}, \sigma\right] \in \mathbf{H}^n(U \setminus \{a\}).$$

Assume now $m = n + 1$, i.e. $\boxed{\dim M = n + 1}$. Then we have the canonical map (isomorphism)

$$\alpha_U : \mathbf{H}^n(U \setminus \{a\}) \xrightarrow{\cong} \mathbb{R}$$

(on U we have orientation induced by M). The mapping α_U is defined by the formula: choose a smooth function f on U so that $f = 0$ in a neighbourhood of a and $f = 1$ outside a compact set. Then

$$\begin{aligned} \alpha_U : \mathbf{H}^n(U \backslash \{a\}) &\overset{\cong}{\to} \mathbb{R} \\ [\phi] &\longmapsto \int_U df \wedge \phi. \end{aligned}$$

Definition 15 *The number*

$$\alpha_U\left(\left[\lambda_{|U \backslash \{a\}}, \sigma\right]\right) \in \mathbb{R}$$

is called the index of σ at a point a and is denoted by

$$j_a(\sigma).$$

(The number $j_a(\sigma)$ is independent of the auxiliary flat connection λ and a coordinate neighbourhood U).

The main purpose of my talk is to introduce the theorem joining the index sum

$$\sum_\nu j_{a_\nu}\sigma$$

of any flat connection on A with a finite number of singularities $\{a_1, ..., a_k\}$ to the so called Euler class of A.

Theorem 16 (The Euler-Poincaré-Hopf theorem for flat connections) *Let A be a Lie algebroid fulfilling the conditions*

- $\boxed{\mathbf{H}^{m+n} A \neq 0}$ *with an invariant volume* $\varepsilon \in \Gamma\left(\bigwedge^n \boldsymbol{g}\right)$,
- $\boxed{\mathbf{H}^*(\mathfrak{g}) \cong \mathbf{H}^*_{dR}(S^n)}$, $\mathfrak{g} \cong \boldsymbol{g}_x$, $\dim \mathfrak{g} = n$.
- *M is compact oriented,* $m = \dim M = n + 1$.

  If $\sigma : T(M \setminus \{a_1, \ldots, a_k\}) \to A$ *is a flat connection with singularities at points* $a_1, \ldots, a_k$, *.then the Euler class* $\chi_A \in H^{n+1}(M)$ *is given by the formula*

$$\chi_A = \left(\sum_{v=1}^{k} j_{a_v}(\sigma) \right) \cdot \omega_M$$

  where ω_M *is the orientation class of* M; *equivalently,*

$$\int_M^{\#} \chi_A = \sum_{v=1}^{k} j_{a_v}(\sigma) .$$

  In particular, the index sum

$$j_A := \sum_{v=1}^{k} j_{a_v}(\sigma)$$

  is independent of the choice of the connection. j_A *is called Euler number of* A.

The crucial role in the proof fulfils the following fact:

Theorem 17 *Let* $\{U,V\}$ *be an open covering of* M *and let* $\lambda_U : TU \to A_{|U}$ *and* $\sigma_V : TV \to A_{|V}$ *be flat connections in* A *over* U *and* V*, respectively (*U*,* V *need not be connected). Consider the Mayer-Vietoris sequence of the triad* $\{M,U,V\}$ *for the usual real de Rham cohomology and let* $\partial : H(U\cap V) \to H(M)$ *be the connecting homomorphism. Then*

$$\chi_A = \partial[\lambda,\sigma]$$

where $\lambda = \lambda_U|_{U\cap V}$ *and* $\sigma = \sigma_V|_{U\cap V}$ *.*

Example 18 *For a Lie algebroid* $A = TM\times\mathbb{R}$ *on two-dimensional compact oriented manifold* M *defined by* (f,Ω) *[see the first example] the Euler number* j_A *is egual to*

$$j_A = \int_M^{\#}\left[-e^f\cdot\Omega\right]$$

Remark 19 *The Euler number* j_A *is not – in general – an invariant of the cohomology algebra of* A*. Indeed, we have:*

Let A *and* A' *be two Lie algebroids on* M *fulfilling assumptions of the above E-P-H theorem with non zero Euler classes* $\chi_A \neq 0$*,* $\chi_{A'} \neq 0$*. Then there exists an isomorphism of cohomology algebras* $\mathbf{H}(A) \cong \mathbf{H}(A')$ *.*

Remark 20 *The Euler number* j_A *has nothing in common with the usual defined Euler-Poincaré characteristic of* A *equal to* $\sum(-1)^\nu \dim\mathbf{H}^\nu(A)$ *. The last sum when* $n+m$ *is odd is* 0 *(i.e. for the cases when* $n=1$*,* $m=2$ *and when* $n=3$ *and* $m=4$ *).*

6 Local formula for the index

Each Lie algebroid A is locally isomorphic to a trivial one $A_U \cong TU\times\mathfrak{g}$. If $\mathbf{H}^{top}(A)\neq 0$ and $\varepsilon\in\Gamma\left(\bigwedge^n \boldsymbol{g}\right)$ is invariant then the coresponding - via the isomorphism $A_U\cong TU\times\mathfrak{g}$ - the cross section ε' of the bundle $M\times\bigwedge^n\mathfrak{g}$ is invariant too.

Lemma 21 *Let $TU \times \mathfrak{g}$ be a trivial Lie algebroid. A cross-section ε' of the isotropy Lie algebras bundle $M \times \bigwedge^n \mathfrak{g}$ is invariant if and only if ε' is a constant one.*

Therefore to local formula of the index we can take consider

- the trivial Lie algebroid $A = T\mathbb{R}^{n+1} \times \mathfrak{g}$ where $\dim \mathfrak{g} = n$ and $\boxed{\mathbf{H}^* (\mathfrak{g}) \cong \mathbf{H}^*_{dR} (S^n)}$,
- a constant volume $\varepsilon_x = \varepsilon \in \bigwedge^n \mathfrak{g}$.

Let $\varphi \in \bigwedge^n \mathfrak{g}^*$ be the tensor such that $\langle \varepsilon, \varphi \rangle = 1$.

Theorem 22 *For the flat singular connection*

$$\sigma : T\left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to A_{|\mathbb{R}^{n+1} \setminus \{0\}} = T\left(\mathbb{R}^{n+1} \setminus \{0\}\right) \times \mathfrak{g},$$

$$\sigma (v) = (v, \check{\sigma} (v))$$

where

$$\check{\sigma} \in \Omega^1 \left(\mathbb{R}^{n+1} \setminus \{0\} ; \mathfrak{g}\right)$$

is a 1-form with values in $\mathfrak{g}$, the index of σ is equal to

$$\begin{aligned} j_0 \sigma &= \frac{1}{n!} \langle \int_{S^n} (\check{\sigma} \wedge .... \wedge \check{\sigma}) , \varphi \rangle \\ &= \int_{S^n} \sigma_S^* (\varphi) \end{aligned}$$

where σ_S is a nonstrong homomorphism of Lie algebroids

$$\sigma_S : TS^n \hookrightarrow T\left(\mathbb{R}^{n+1} \setminus \{0\}\right) \xrightarrow{\check{\sigma}} \mathfrak{g}.$$

In particular, for $n = 1$ and $\varepsilon = 1$ we have

$$j_0 \sigma = \int_{S^1} \check{\sigma}.$$

The trivial Lie algebroid A is integrable,

$$A = A(P)$$

for the principal fibre bundle $P = \mathbb{R}^{n+1} \times G$ where G is an arbitrary connected Lie group with the Lie algebra $\mathfrak{g}$.

Take a singular flat connection

$$\sigma : T\left(\mathbb{R}^{n+1} \backslash \{0\}\right) \to A_{|\mathbb{R}^{n+1} \backslash \{0\}}.$$

σ induces a usual connection $H \subset T\left(\left(\mathbb{R}^{n+1} \backslash \{0\}\right) \times G\right)$ in the principal bundle $\dot{P} = \left(\mathbb{R}^{n+1} \backslash \{0\}\right) \times G$. Flatness of σ means the integrability of H.

6.1 Assumption $n \geq 2$.

Then $\mathbb{R}^{n+1} \backslash \{0\}$ is simple connected. Therefore via the reduction theorem each leaf L of H is the graph of some function $f : \mathbb{R}^{n+1} \backslash \{0\} \to G$.

Theorem 23 *Local formula for the index*

$$j_0 \sigma = \int_{S^n} \left(f_{|S^n}\right)^* \Delta_R$$

where Δ_R is the right invariant n form on G such that $(\Delta_R)_e = \varphi$ and $\langle \varphi, \varepsilon \rangle = 1$.

Problem 24 *Can we use the above formula for a generalization of the notion of index for other Lie algebras?*

(A) G is noncompact. Then $\Delta_R = d(\Theta)$ for some Θ therefore

$$\int_{S^n} (f|S^n)^* \Delta_R = \int_{S^n} d\left(f^*_{|S^n} \Theta\right) = 0$$

(the case is noniteresting),

(B) G is compact. Then

$$\int_{S^n} \left(f_{|S^n}\right)^* \Delta_R = \deg\left(f_{|S^n}\right) \cdot \int_G \Delta_R.$$

When the number $\int_{S^n} \left(f_{|S^n}\right)^* \Delta_R$ can be nonzero? Namely, if $\deg\left(f_{|S^n}\right) \neq 0$. *But, by the degree theorem, the relation* $\deg\left(f_{|S^n}\right) \neq 0$ *implies the injectivity*

$$\left(f_{|S^n}\right)^{\#} : \mathbf{H}(G) \to \mathbf{H}(S^n),$$

so G is spherical ($\mathbf{H}(G) \cong \mathbf{H}(S^n)$), i.e. $n = 3$ and $\mathfrak{g}$ is equal to $so(3)$ (the case $sl(2,\mathbb{R})$ gives noncompact case $G = SL(2,\mathbb{R})$). Therefore for other compact Lie groups we have noninterested case $\int_{S^n} \left(f_{|S^n}\right)^* \Delta_R = 0$.

Conclusion 25 *In conclusion, the unique interesting case (assuming the definition of the degree $j_0\sigma = \int_{S^n} (f|S^n)^* \Delta_R$) for $n \geq 2$ it is obtained for $\mathfrak{g} = \mathfrak{so}(3)$.*

Problem 26 *We also notice that in the case $\mathfrak{g} = \mathfrak{so}(3)$ ($m = 4$) the set of real numbers which are the indexes at a given point of singular local flat connections is discrete.*

Theorem 27 *For $sl(2,\mathbb{R})$ Lie algebroid A on M^4: if there exists a flat connection with finite number of isolated singularities then there exist a flat connection without any singularities (i.e. A is flat).*

Indeed, we can remove each isolated singularities: the problem is local. We can consider a flat connection

$$\sigma; T\left(\mathbb{R}^4 \setminus \{0\}\right) \to sl(2,\mathbb{R}).$$

It is given by a function $f : \mathbb{R}^4 \setminus \{0\} \to SL(2,\mathbb{R})$. The fact that $\pi_3(SL(2,\mathbb{R})) = 0$ implies: we can find a function $\bar{f} : \mathbb{R}^4 \to SL(2,\mathbb{R})$ such that $f(x) = \bar{f}(x)$ for $\|x\| \geq 1$. This implies that we may remove the singularity at 0.

6.2 Assumption $n = 1$.

It means we consider $\mathfrak{g}$ $=\mathbb{R}$, $m = 2$. Let $A = T\mathbb{R}^2 \times \mathbb{R}$ be a Lie algebroid and take the constant volume $\varepsilon_x = \varepsilon = 1 \in \bigwedge^1 \mathbb{R}$.

Theorem 28 *Each real number can be the index of a local flat singular connection.*

Proof. If $\sigma \in \Omega^1 \left(\mathbb{R}^2 \backslash \{0\}\right)$ be a closed 1 form. Then

$$\begin{array}{rcl} \hat{\sigma} & : & T\left(\mathbb{R}^2 \backslash \{0\}\right) \to A_{\mathbb{R}^2 \backslash \{0\}}, \\ v & \longmapsto & (v, \sigma(v)) \end{array}$$

is a flat singular connection. Taking

$$\sigma = \frac{k}{2\pi} \left(\frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \right)$$

we obtain a flat connection $\hat{\sigma}$ for which $j_0(\hat{\sigma}) = k$. ■

Theorem 29 *If* $\mathbf{H}^2(M) = 0$ *then there exists a global flat connection in* A *(assuming* $\mathfrak{g} = \mathbb{R}$ *and* $\mathbf{H}^3(A) \neq 0$ *). In particular for compact manifold* M *(*$\mathbf{H}^2(M \backslash \{a\}) = 0$ *) there exists a flat connection with one singularity.*

Example 30 *Consider Hopf* S^1*-bundle* $S^3 \to S^2$*. For arbitrary* $k \in \mathbb{R}$ *and two points* $p, q \in S^2$ *there exists a flat connection* σ *with singularities at* $\{p, q\}$ *such that* $j_p\sigma = p$ *and* $j_q\sigma = 1 - k$ *(assuming the "normalization"* $\int_{S^1} \Delta_R = 1$ *).*

Example 31 *We come back to the example* (M, f, Ω), $\dim M = 2$ *and* M *is compact oriented manifold. For the Lie algebroid* $A = T\mathbb{R}^2 \times \mathbb{R}$ *we have* $\mathbf{H}^3(A) \neq 0$ *and the Euler class of* A *is equal to*

$$\chi_A = \left[-e^f \cdot \Omega\right].$$

Let $\sigma : T(M\setminus\{a_1,...,a_k\}) \to A_{|M\setminus\{a_1,...,a_k\}}$ *be a flat connection.*

$$\sigma(v) = (v, \hat{\sigma}(v))$$

for 1-form $\hat{\sigma} \in \Omega^1(M\setminus\{a_1,...,a_k\})$. *The flatness of* σ *is equivalent to*

$$d\hat{\sigma} = \Omega - df \wedge \hat{\sigma}.$$

For a neighbourhood $U \subset M$ *diffeomorphic to* $\mathbb{R}^2$ *we take a 1-form* η *such that* $d\eta = -e^f \cdot \Omega$. *Then* $A_{|U}$ *is isomorphic to the trivial Lie algebroid* $TU \times \mathbb{R}$ *via the isomorphism*

$$H : A_{|U} \to TU \times \mathbb{R}$$

$$H(X, g) = \left(X, \eta(X) + e^f \cdot g\right)..$$

Therefore according to the naturality of the index we obtain

$$j_{a_i}\sigma = \int_{S^1} \left(e^f \cdot \hat{\sigma} + \eta\right)$$

where S^1 *is here any small Jordan curve near a point* a_i. *So it agree with the definition at the beginning.*