Linear direct connections in Lie groupoids, curvature and characteristic classes

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The plan of the talk

1. Linear direct connections τ (called also linear quasi-connections) in tangent bundles and in vector bundles. The Teleman's theorem

2. Underlying a usual linear connection ∇^{τ} and a direct proof of this theorem, the curvature of τ versus connection of ∇^{τ}.

3. Groupoids point of view and groupoids generalizations.

1 Linear direct connections in vector bundles and Teleman's theorem

Nicola Teleman in the papers

N.Teleman, *Distance Function, Linear quasi-Connections and Chern Character,* June 2004, IHES/M/04/27

N.Teleman, *Direct Connections and Chern Character,* Proceedings of the International Conference in Honor of Jean-Paul Brasselet, Luminy, May 2005,

shows how the Chern character of the tangent bundle of a smooth manifold may be extracted from the geodesic distance function by means of cyclic homology.

The processing has the following steps:

1. Let M be a smooth Riemannian manifold and let

$$r : M \times M \to [0, \infty)$$

be the induced geodesic distance function.

The function r^2 is smooth on a neighbourhood of the diagonal.

2. Let χ be a cut-off smooth monotone decreasing real valued function, identically 1 on a neighbourhood of 0, having support on a sufficiently small interval, so that $\chi \circ r^2$ be well defined and smooth. For $x, y \in M$ a linear mapping

$$A(y, x) : T_x M \to T_y M$$

is given by the formula

$$A(y, x) \left(\sum_i \xi^i \frac{\partial}{\partial x^i} \right) = \sum_{i,j,k} \xi^i \frac{\partial^2 (\chi \circ r^2)(x, y)}{\partial x^i \partial y^j} g^{jk}(y) \frac{\partial}{\partial y^k}$$

($A(y, x)$ is independent of the local coordinates).

For sufficiently close points x, y,

— $A(y, x)$ is an isomorphism and

— $A(x, x)$ is the identity.

Therefore A is a linear direct connection (=linear quasi-connection), with respect to the definition below.

3. With the object A there is associated the function $\Phi_k : U_{k+1} \to \mathbb{R}$, where U_{k+1} is a neighbourhood of the diagonal in M^{k+1}

$$\Phi_k (x_0, x_1, ..., x_k) := Trace\ A(x_0, x_1) \circ A(x_1, x_2) \circ ... \circ A(x_{k-1}, x_k) \circ A(x_k, x_0).$$

4. Next, N.Teleman studies the function Φ_k in the context of cyclic homology:

— firstly, he notices that Φ_k, k =even, is a cyclic cycle over the algebra $\mathcal{A} = C^\infty(M)$,

— secondly, he uses the Connes' isomorphism which associates with Φ_k a closed differential form

$$\Omega(\Phi_k)(x) = \frac{1}{k!} \sum_{i_1, i_2, .., i_k} \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} ... \frac{\partial}{\partial x_k^{i_k}} \Phi_k (x_0, x_1, ..., x_k)_{x_0 = x_1 = ... = x_k = x} dx^{i_1} \wedge ... \wedge dx^{i_k},$$

(we use the same local coordinate system on each factor).

— thirdly, he proves

Theorem 1 *The top degree component of the cyclic homology class of Φ_k is equal to*

$$[\Omega(\Phi_{2k})] = c \cdot Ch_k(M)$$

where c is a constant and $Ch_k(M)$ is the k-component of the Chern character of the tangent bundle of M.

The object A is a particular case of the linear direct connection introduced by N.Teleman.

Definition 2 *Let E be a real or complex smooth vector bundle over the manifold M. A* ***linear direct connection*** *τ in E consists of assigning to any two points $x, y \in M$, sufficiently close one to each other, an isomorphism*

$$\tau(y, x) : E_{|x} \to E_{|y},$$

such that

$$\tau(x, x) = id,$$

and $\tau(y, x)$ depends smoothly on the pair x, y.

The parallel transport defined by a usual linear connection in E along the small geodesics of an affine connection in M induces a linear direct connection in E (see for example A.Connes and H.Moscovici, "*Cyclic cohomology, the Novikov conjecture and hyperbolic groups*", Topology 29, n 3 345-388, 1990).

-i) As for A with τ there is associated the function Φ_k by the formula

$$\Phi_k\left(x_0, x_1, ..., x_k\right) := Trace\ \tau\left(x_0, x_1\right) \circ \tau\left(x_1, x_2\right) \circ ... \circ \tau\left(x_{k-1}, x_k\right) \circ \tau\left(x_k, x_0\right).$$

The function

$$\Phi_2\left(x_0, x_1, x_2\right) = Trace\ \tau\left(x_0, x_1\right) \circ \tau\left(x_1, x_2\right) \circ \tau\left(x_2, x_0\right)$$

plays a role of the *curvature* of τ and the differential form $\Omega\left(\Phi_2\right)$ - the *curvature form* of τ.

-ii) Any two smooth linear direct connections in a smooth vector bundle are smoothly homotopic. The results above implay

Theorem 3 ***(N.Teleman)*** *For any smooth linear direct connection τ in the smooth vector bundle E over the manifold M,*

- *-i) Φ_k, $k =$even, is a cyclic cycle over the algebra $C^\infty(M)$,*
- *-ii) the cohomology class of $\Omega\left(\Phi_{2k}\right)$ is (up to a multiplicative constant) is the k-component of the Chern character of E.*

2 Underlying linear connection ∇^τ and a direct proof of this theorem

In the paper

J.Kubarski, N.Teleman, *Linear direct connections,* Banach Center Publications, 2007, in print,

we study the geometry of direct connections τ:

- we construct the "infinitesimal part" ∇^τ and show that ∇^τ is a usual linear connection. We next determine the curvature tensor R of ∇^τ and show that the **equality of differential forms** holds

$$\Omega\left(\Phi_{2k}\right) = c \cdot Tr\ R^k.$$

We intend to extract from a direct connection its infinitesimal part along the diagonal.

Definition 4 *Let X be a smooth tangent field over M and ϕ a smooth section in E. Let x_0 be an arbitrary point in M and let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ be an integral path of the field X with the initial condition $\gamma(0) = x_0$.*

We define

$$\nabla^{\tau}_{X(x_0)}(\phi) = \frac{d}{dt} \{\tau(\gamma(0), \gamma(t)) (\phi(\gamma(t)))\}_{|t=0} \in E_{|x_0}.$$

Theorem 5 *The right hand side of the above formula depends only on the value of X at x_0. The operator $\nabla^{\tau}_{X(x_0)}(\phi)$ is a usual linear connection in E.*

We intend to describe $\nabla^{\tau}_{X(x_0)}(\phi)$ locally.

Let $(x^1, x^2, ..., x^m)$ $(dimM = m)$ be a local coordinate system on an open neighborhood $\mathcal{V}$ of a point x_0. Using the same local coordinate system on both factors of the direct product $M \times M$, any point $(x, y) \in \mathcal{V} \times \mathcal{V}$ will be given by local coordinates $(x^1, x^2, .., x^m | y^1, y^2, ..., y^m)$.

Theorem 6 *Let $\{e_1, e_2, ..., e_n\}$ be a local frame in E over $\mathcal{V}$. Let $\tau(x|y)$ be the matrix describing locally the direct connection τ :*

$$\tau(x|y) = \|\tau_i^j(x|y)\| \in M_{n,n}(\mathbb{K}),$$

$$\tau(x, y)\,(e_i(y)) = \sum_j \tau_i^j(x|y) \cdot e_j(x), \qquad \tau_i^j(x|x) = \delta_i^j.$$

Then the coefficients $\Gamma_{i,\alpha}^j$ of the connection ∇^τ are given locally by

$$\nabla^\tau_{\frac{\partial}{\partial x^\alpha}} e_i = \sum_j \Gamma_{i,\alpha}^j\, e_j,$$

where

$$\Gamma_{i,\alpha}^j(x) = \frac{\partial}{\partial y^\alpha} \tau_i^j(x^1, x^2, .., x^m | y^1, y^2, ..., y^m)_{y=x}.$$

In conclusion, representing the tangent field X locally

$$X(x)=\sum_{\alpha} X^{\alpha}(x)\cdot\frac{\partial}{\partial x^{\alpha}},$$

one has the formula

$$\nabla^{\tau}_{X(x_0)}(\sum_i \phi^i\, e_i)=\sum_{\alpha=1}^{m}\{\,\sum_{i,j}\Gamma^{j}_{i,\alpha}(x_0)\cdot X^{\alpha}(x_0)\cdot\phi^i(x_0)\, e_j(x_0)\,\}+\sum_i(d\phi^i)(X)(x_0)\, e_i(x_0).$$

Remark 7 *The above formula also show that ∇^{τ} is a linear connection in the vector bundle E. The linear connection ∇^{τ} will be called* associated, or underlying, linear connection to the direct connection τ.

Proposition 8 *Let $R = (\nabla^\tau)^2$ be the curvature tensor of the connection ∇^τ. The components of the curvature R are*

$$\begin{aligned} R^j_{i\alpha\beta}(x) &= \frac{\partial}{\partial x^\alpha}\Gamma^j_{i\beta}(x) - \frac{\partial}{\partial x^\beta}\Gamma^j_{i\alpha}(x) + \Gamma^j_{k\alpha}(x)\cdot\Gamma^k_{i\beta}(x) - \Gamma^j_{k\beta}(x)\cdot\Gamma^k_{i\alpha}(x) \\ &= \frac{\partial^2}{\partial x^\alpha \partial y^\beta}\tau^j_i(x|y)_{y=x} - \frac{\partial^2}{\partial x^\beta \partial y^\alpha}\tau^j_i(x|y)_{y=x} + \\ &+ \frac{\partial}{\partial y^\alpha}\tau^j_k(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\beta}\tau^k_i(x|y)_{y=x} - \frac{\partial}{\partial y^\beta}\tau^j_k(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\alpha}\tau^k_i(x|y)_{y=x}. \end{aligned}$$

Corollary 9 *The curvature form R of the underlying linear connection ∇^τ, associated to the direct connection τ, is given by*

$$\begin{aligned} R = &\Big(\frac{\partial^2}{\partial x^\alpha \partial y^\beta}\tau^j_i(x|y)_{y=x} - \frac{\partial^2}{\partial x^\beta \partial y^\alpha}\tau^j_i(x|y)_{y=x} + \\ &+ \frac{\partial}{\partial y^\alpha}\tau^j_k(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\beta}\tau^k_i(x|y)_{y=x} - \frac{\partial}{\partial y^\beta}\tau^j_k(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\alpha}\tau^k_i(x|y)_{y=x}\Big)dx^\alpha \wedge dx^\beta. \end{aligned}$$

Although, $\tau(x,y) = (\tau(y,x))^{-1}$ is not true in general, it is true, however, that it holds infinitesimally. In fact, we have the

Proposition 10 *For any direct connection τ, its matrix components satisfy the identities*

-i)

$$\frac{\partial}{\partial x^\alpha}\tau^j_i(x|y)_{y=x} + \frac{\partial}{\partial y^\alpha}\tau^j_i(x|y)_{y=x} = 0.$$

-ii)

$$\frac{\partial}{\partial x^\alpha}\{\tau(x|y)\circ\tau(y|x)\}_{y=x} = 0 = \frac{\partial}{\partial y^\alpha}\{\tau(x|y)\circ\tau(y|x)\}_{y=x}.$$

As $\tau(x|x) = Id.$, we get that the directional derivative $(\frac{\partial}{\partial x^\alpha}+\frac{\partial}{\partial y^\alpha})$ of τ along the diagonal **vanishes**. This proves -i). The second identity is a consequence of the first.

The above properties of any direct connection are fundamental for comparing the curvature tensor R to the differential form $\Omega\left(\Phi^\tau_{2k}\right)$.

We obtain an important explicit link between $\Omega\left(\Phi_{2k}\right)$ and the classical Chern-Weil forms, at the level of **differential forms** rather than **cohomology classes.**

Theorem 11 *Let τ be a direct connection and let ∇^{τ} be its underlying linear connection. Then*

$$\Omega(\Phi_2^{\tau}) = \frac{1}{4} \cdot Tr\ R,$$

and more generally,

$$\Omega(\Phi_{2k}^{\tau}) = \frac{1}{(2k)!} \cdot \frac{1}{2^k} \cdot Tr\ R^k,$$

where $R = (\nabla^{\tau})^2$ is the curvature of the underlying linear connection ∇^{τ}.

In consequence, the mentioned above Teleman's theorem follows from this directly.

3 Groupoids point of view and groupoids generalizations

N.Teleman in yours papers said:

"*The arguments discussed here may be extended to the language of groupoids*".

My further talk is the first step in this direction.

3.1 Direct connections and the Lie groupoid $GL(E)$

Let E be a real or complex smooth vector bundle over the manifold M. Consider the **transitive Lie groupoid**

$$\Phi = GL(E)$$

of all linear fibre isomorphisms $h : E_{|x} \to E_{|y}$ of the vector bundle E, with the source α, $\alpha(h) = x$, and the target β, $\beta(h) = y$, and the unit $u_y = \mathrm{id}_{E_{|y}}$. The mappings

$$\alpha, \beta : \Phi \to M, \quad (\alpha, \beta) : \Phi \to M \times M$$

are submersions, the injection

$$u : M \to \Phi,\ y \to u_y,$$

is smooth, and the partial multiplication

$$\cdot : \Phi \times_{(\alpha,\beta)} \Phi \to \Phi, \quad (g, h) \longmapsto gh,$$

is also smooth. and $GL(E)$ be a Lie groupoid of linear fibre isomorphisms.

Remark 12 *A linear direct connection in a vector bundle E is equivalently a smooth mapping*

$$\tau : U \to GL(E)$$

where $U \subset M \times M$ is an open neighborhood of the diagonal $\Delta = \{(x,x)\,;\ x \in M\}$, such that

$$\tau(x,y) : E_{|y} \to E_{|x}$$

i.e.

$$\alpha \circ \tau(x,y) = y, \quad \beta \circ \tau(y,x) = x,$$

and

$$\tau(x,x) = \mathrm{id} : E_{|x} \to E_{|x}.$$

3.2 Lie Groupoids and point of view of linear direct connections and the using of the Lie algebroids

According to the Pradines definition, the **Lie algebroid** of an arbitrary transitive Lie groupoid Φ is equal to the vector bundle

$$A(\Phi) = u^*(T^\alpha \Phi)$$

where $u : M \to \Phi$, $y \to u_y$, and $T^\alpha \Phi = \ker \alpha_*$, equipped with the suitable structures: the bracket of cross-sections $[\![\xi, \eta]\!]$, $\xi, \eta \in SecA(\Phi)$ is defined in the following way. The cross-sections ξ, η can be extended to right invariant vector fields ξ', η' on Φ, their usual bracket $[\xi', \eta']$ is invariant too, so it determines a cross-section of $u^*(T^\alpha \Phi)$ denoting by $[\![\xi, \eta]\!]$. The anchor is defined as the restriction of β_*.

We recall the definition of a Lie algebroid.

Definition 13 *By a Lie algebroid on a manifold M we mean a system*

$$A = (A, [\![\cdot, \cdot]\!], \gamma) \tag{1}$$

consisting of a vector bundle A (over M) and mappings

$$[\![\cdot, \cdot]\!] : \operatorname{Sec} A \times \operatorname{Sec} A \longrightarrow \operatorname{Sec} A, \quad \gamma : A \longrightarrow TM,$$

such that

(i) $(\operatorname{Sec} A, [\![\cdot, \cdot]\!])$ *is an* $\mathbb{R}$*-Lie algebra,*

(ii) γ*, called an A nchor, is a homomorphism of vector bundles,*

(iii) $\operatorname{Sec} \gamma : \operatorname{Sec} \gamma \longrightarrow \mathfrak{X}(M)$, $\xi \longmapsto \gamma \circ \xi$*, is a homomorphism of Lie algebras,*

(iv) $[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + (\gamma \circ \xi)(f) \cdot \eta$ *for* $f \in C^\infty(M)$, $\xi, \eta \in \operatorname{Sec} A$.

Lie algebroid (1) is called*transitive* if γ is an epimorphism. $\boldsymbol{g} = \ker \gamma$ is a vector bundle, called the adjoint of (1), and the short exact sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} E \longrightarrow 0 \tag{2}$$

is called the Atiyah sequence of (1).

Example 14 The following are simple fundamental examples of transitive Lie algebroids:

(1⁰) Finitely dimensional Lie algebra.

(2⁰) Tangent bundle TM to a manifold M with the bracket $[\cdot,\cdot]$ of vector fields and id_{TM} as an anchor.

(3⁰) Trivial Lie algebroid $TM \times \mathfrak{g}$ (Ngo-Van-Que) where $\mathfrak{g}$ is as in (1^o). The bracket is defined by the formula,

$$[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - \mathcal{L}_Y\sigma + [\sigma,\eta]),$$

$X, Y \in \mathcal{X}(M),\ \ \sigma, \eta : M \to \mathfrak{g}$, and the anchor is the projection $TM \times \mathfrak{g} \to TM$.

(4⁰) Bundle of jets J^kTM (P.Libermann).

(5⁰) General form (K.Mackenzie, J.Kubarski). Let a system $(\boldsymbol{g}, \nabla, \Omega_b)$ be given, consisting of a Lie algebra bundle $\boldsymbol{g}$ on a manifold M, a covariant derivative ∇ in $\boldsymbol{g}$ and a 2-form $\Omega_b \in \Omega^2(M, \boldsymbol{g})$ on M with values in $\boldsymbol{g}$, fulfilling the conditions:

(i) $\nabla^2 \sigma = -[\Omega_b, \sigma], \quad \sigma \in \operatorname{Sec} \boldsymbol{g},$

(ii) $\nabla_X[\sigma, \eta] = [\nabla_X \sigma, \eta] + [\sigma, \nabla_X \eta], \quad X \in X(M), \quad \sigma, \eta \in \operatorname{Sec} \boldsymbol{g},$

(iii) $\nabla \Omega_b = 0.$

Then $TM \oplus \boldsymbol{g}$ forms a transitive Lie algebroid with the bracket defined by

$$[\![(X, \sigma), (Y, \eta)]\!] = ([X, Y], -\Omega_b(X, Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma, \eta]),$$

the anchor being the projection onto the first component.

Every transitive Lie algebroid is — up to an isomorphism — of this form.

Example 15 The following are important examples of transitive Lie algebroids:

(6⁰) The Lie algebroid $A(P) = TP/G$ of a G-principal bundle P (K.Mackenzie, J.Kubarski).

(7⁰) The Lie algebroid CDO(E) of covariant differential operators on a vector bundle E (K.Mackenzie). Another isomorphic construction of this object is the Lie algebroid $A(E)$ of a vector bundle E (J.Kubarski), here the fibre $A(E)_{|x}$ is the space of linear homomorphisms $l : \mathrm{Sec}\, E \to E_{|x}$ such that there exists a vector $u \in T_xM$ for which $l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu(x)$, $f \in C^\infty(M)$, $\nu \in \mathrm{Sec}(E)$.

(8⁰) The Lie algebroid $A(\Phi) := i^*T^\alpha\Phi$ of a Lie groupoid Φ (J.Pradines). (Remark: if $\Phi = GL(E)$ is the Lie algebroid of a all linear fibre isomorphisms of fibres of E then $A(E) = A(\Phi)$).

(9⁰) The Lie algebroid $A(M, \mathcal{F})$ of a transversally complete foliation $(M, \mathcal{F})$ (P.Molino); in particular,

(10⁰) the Lie algebroid $A(G; H)$ of the foliation of left cosets of a Lie group G by a nonclosed connected Lie subgroup $H \subset G$ (for the construction independent of the theory of transversally complete foliations, see J.Kubarski).

There are many sources of nontransitive Lie algebroids: Lie equations, Differential groupoids, Poisson manifolds, etc.

Let $\Phi = GL(E)$ be the Lie groupoid of all linear fibre isomorphisms of fibres of E.

For $y \in M$ the submanifold $\Phi_y = GL(E)_y \subset GL(E)$ of all elements $u \in GL(E)$ for which $\alpha(u) = y$,

$$GL(E)_y = \alpha^{-1}(y),$$

is a $GL(E_y)$-principal fibre bundle.

- Lie algebroid of the Lie groupoid is the infinitesimal object and play analogous role to that of Lie algebras for Lie groups.

- The space [Lie algebra] of global cross-sections $Sec(A(\Phi))$, $\Phi = GL(E)$ where E is a vector bundle, is naturally isomorphic to the Lie algebra of all Covariant Derivative Operators, i.e. to the space of differential operators of the rank ≤ 1

  $$\mathfrak{L} : SecE \rightarrow SecE$$

  such that $\mathfrak{L}(f \cdot \xi) = f \cdot \mathfrak{L}(\xi) + X(f) \cdot \xi$, for a vector field X called the anchor of $\mathfrak{L}$, $f \in C^{\infty}(M)$, $\xi \in SecE$.

Let $\tau : (M \times M)_{|U} \to GL(E)$ (where $U \subset M \times M$ is an open neighbourhood of the diagonal $\Delta = \{(x,x)\,;\ x \in M\}$) be a linear quasi-connection,

$$\tau_{(x,y)} : E_{|y} \to E_{|x},$$

so $\alpha\left(\tau_{(x,y)}\right) = y$ and $\beta\left(\tau_{(x,y)}\right) = x$ and let ∇^{τ} be the **underlying linear connection of τ in E.**

Now, we fix y and take

$$\tau(\cdot, y) : M \to GL(E)_y\,, \quad x \longmapsto \tau(x,y)\,.$$

It is a smooth mapping such that $\beta \circ \tau(\cdot, y) = \mathrm{id}$. Therefore the composition of the differential

$$\tau(\cdot, y)_{*x} : T_x M \to T_{\tau(x,y)}\left(GL(E)_y\right)$$

with the differential of $\beta | GL(E)_y \to M$ is identity

$$\mathrm{id} : T_x M \overset{\tau(\cdot,y)_{*x}}{\longrightarrow} T_{\tau(x,y)}\left(GL(E)_y\right) \overset{\beta_*}{\longrightarrow} T_x M.$$

Taking $x = y$ and using the fact $\tau(y,y) = u_y = \mathrm{id}_{E_y}$ we see that

$$\tau(\cdot, y)_{*y} : T_y M \to T_{u_y}\left(GL(E)_y\right).$$

Therefore τ determines a usual connection

$$\begin{aligned} \bar{\nabla}^{\tau} &: TM \rightarrow u^{*}\left(T^{\alpha}\Phi\right) \\ \bar{\nabla}^{\tau}\left(v_{y}\right) &= \tau\left(\cdot, y\right)_{*y}\left(v_{y}\right). \end{aligned}$$

in the Lie algebroid $u^{*}\left(T^{\alpha}\Phi\right)$ $(\Phi = GL\left(E\right))$, i.e. a splitting of the Atiyah sequence

$$0 \rightarrow \boldsymbol{g} \rightarrow A\left(\Phi\right) \underset{\nabla^{\tau}}{\overset{\beta_{*}}{\rightleftarrows}} TM \rightarrow 0.$$

$\bar{\nabla}^{\tau}$ is the "usual covariant derivative" since the anchor of the Covariant Derivative Operator $\bar{\nabla}^{\tau}\left(X\right) : SecE \rightarrow SecE$ is just equal to X, therefore noticing $\bar{\nabla}^{\tau}\left(X\right)\left(\xi\right)$ in the form

$$\bar{\nabla}_{X}^{\tau}\left(\xi\right)$$

the usual axioms for covariant derivative are fulfilled.

Theorem 16 $\bar{\nabla}^{\tau} = \nabla^{\tau}$, *i.e. the connection* $\bar{\nabla}^{\tau}$ *is equal to the* ***underlying linear connection of*** τ ***in*** E.

Proof. (a sketch) Since we need to prove it at any point $y \in M$ so we can prove it locally for $E = M \times \mathbb{R}^n$ and $M = \mathbb{R}^m$. Then $GL(E) = M \times GL(\mathbb{R}^n) \times M$, $\alpha^{-1}(y) = GL(E)_y = M \times GL(\mathbb{R}^n) \times \{y\}$. Let $\{e_i\}_{i=1}^n$ be a trivial local basis of E, then the induced linear connection ∇^τ is determined by

$$\nabla^\tau_{\frac{\partial}{\partial x^k}|y} e_i = \frac{\partial \tau_i^j}{\partial x^{m+k}}(y, y) \cdot e_j.$$

We can obtain the same results for $\bar{\nabla}^\tau$. ■

3.3 Groupoids generalization

The above consideration has "groupoids sense" so we can it generalize to any transitive Lie groupoids.

Let Φ be an arbitrary transitive Lie groupoid with the anchor α and the target β. We denote by u_y the unit of Φ at y.

Definition 17 *By a linear direct connection in Φ we mean a mapping*

$$\tau : (M \times M)_{|U} \to \Phi,$$

such that

$$\alpha \circ \tau (x, y) = y, \quad \beta \circ \tau (x, y) = x,$$

and

$$\tau (x, x) = u_x.$$

For y the submanifold $\Phi_y \subset \Phi$ of all elements $h \in \Phi$ for which $\alpha (h) = y$ ($\Phi_y = \alpha^{-1} (y)$) is a Φ_y^y-principal fibre bundle where

$$\Phi_y^y = \{h \in \Phi; \ \alpha (h) = \beta (h) = y\}$$

is the isotropy Lie algebra of Φ at y. Now, we fix y and take

$$\tau (\cdot, y) : M \to \Phi_y, \quad x \longmapsto \tau (x, y) .$$

It is a smooth mapping such that

$$\beta \circ \tau(\cdot, y) = \mathrm{id}\,.$$

Taking the differential

$$\tau(\cdot, y)_{*x} : T_x M \to T_{\tau(x,y)}(\Phi_y)$$

such that the composition with the differential of $\beta|\Phi_y \to M$ is identity

$$\mathrm{id} : T_x M \overset{\tau(\cdot,y)_{*x}}{\longrightarrow} T_{\tau(x,y)}(\Phi_y) \overset{\beta_*}{\longrightarrow} T_x M$$

and taking $x = y$ and using the fact $\tau(y, y) = u_y$ we see that

$$\tau(\cdot, y)_{*y} : T_y M \to T_{u_y}(\Phi_y) = A(\Phi)_{|y}\,,$$

where $A(\Phi)$ is the Lie algebroid of the Lie groupoid Φ.

Therefore τ determines a splitting of the Atiyah sequence of Φ

$$0 \to \mathbf{g} \to A(\Phi) \underset{\nabla^{\tau}}{\overset{\beta_*}{\rightleftarrows}} TM \to 0,$$

i.e. a usual connection in the Lie algebroid $A(\Phi) = u^*(T^{\alpha}\Phi)$,

$$\nabla^{\tau} : TM \to u^*(T^{\alpha}\Phi) = A(\Phi)$$

$$\boxed{\nabla^{\tau}(v_y) = \tau(\cdot, y)_{*y}(v_y)}$$

The connection ∇^{τ} will be called the **underlying linear connection of the linear direct connection** τ. .

Now we can ask on a very important question:

- How can we reconstruct the curvature tensor of ∇ from the linear direct connection in the Lie groupoid Φ? And next how can we reconstruct the Chern-Weil homomorphism of Lie groupoids Φ (i.e. equivalently of the principal bundle Φ_y) from arbitrary taken linear direct connection τ?

3.4 Curvature tensor of the linear direct connection in transitive Lie groupoids

Take any transitive Lie groupoid Φ and its Lie algebroid $A(\Phi)$ with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A(\Phi) \longrightarrow TM \to 0.$$

The fibre of $\boldsymbol{g}$ at x

$$\boldsymbol{g}_{|x} = T_{u_x}\Phi_x^x$$

is the **right** Lie algebra of the structural Lie group Φ_x^x. For a linear direct connection τ in Φ denote by

$$\nabla^\tau : TM \to A(\Phi)$$

the underlying linear connection in the Lie algebroid $A(\Phi)$ induced by τ. Consider the curvature tensor $\Omega^\tau \in \Omega^2(M; \boldsymbol{g})$ of ∇^τ

$$\Omega^\tau(X,Y) = [\![\nabla_X^\tau, \nabla_Y^\tau]\!] - \nabla_{[X,Y]}^\tau.$$

The linear direct connection τ determines the mapping

$$\Psi_k^\tau : \left(\underbrace{M \times ... \times M}_{k+1} \right)_{|U} \rightarrow \Phi,$$

$$\Psi_k^\tau \left(x_0, x_1, ..., x_k \right) = \tau \left(x_0, x_1 \right) \cdot \tau \left(x_1, x_2 \right) \cdot ... \cdot \tau \left(x_{k-1}, x_k \right) \cdot \tau \left(x_k, x_0 \right)$$

having the values in the associated **Lie group bundle,**

$$\Psi_k^\tau \left(x_0, x_1, ..., x_k \right) \in \Phi_{x_0}^{x_0}.$$

For example, for $k = 2$, the function

$$\Psi_2^\tau : \left(M \times M \times M \right)_{|U} \rightarrow \Phi,$$

$$\Psi_2^\tau \left(x_0, x_1, x_2 \right) = \tau \left(x_0, x_1 \right) \cdot \tau \left(x_1, x_2 \right) \cdot \tau \left(x_2, x_0 \right)$$

is called the *curvature* of τ.

Analogously to the previous cases we can associate some **differential form** to the function Ψ_k. Namely, fixing a point x_0 we define

$$\Psi_k^\tau \left(x_0 \right) : \left(\underbrace{M \times ... \times M}_{k} \right)_{|U} \rightarrow \Phi_{x_0}^{x_0},$$

$$\left(x_1, ..., x_k \right) \longmapsto \Psi_k^\tau \left(x_0, x_1, ..., x_k \right) \in \Phi_{x_0}^{x_0}.$$

Next, we take a coordinate system $(x^1,...,x^m)$ $(dim M = m)$ on an open neighborhood $\mathcal{V}$ of the point x_0. Using the same local coordinate system on each factors of the direct product $M \times \cdots \times M$ we take for $(x_1,...,x_k) \in \mathcal{V} \times ... \times \mathcal{V}$

$$\frac{\partial}{\partial x_k^j} \Psi_k^\tau (x_0) \in T_{\Psi_k(x_0,x_1,...,x_k)} \Phi_{x_0}^{x_0}.$$

This vector we can translate via **right** translation to the unit. Let $r_h : \Phi_{x_0}^{x_0} \to \Phi_{x_0}^{x_0}$ denote the right translation on the element h, $r_h(z) = z \cdot h$.

$$\frac{\partial}{\partial \tilde{x}_k^j} \Psi_k^\tau (x_0) := \left(r_{(\Psi_k(x_0,x_1,...,x_k))^{-1}} \right)_{*\Psi_k(x_0,x_1,...,x_k)} \left(\frac{\partial}{\partial x_k^j} \Psi_k^\tau (x_0) \right) \in T_{u_{x_0}} \left(\Phi_{x_0}^{x_0} \right) = \boldsymbol{g}_{|x_0}.$$

The function obtained

$$(x_1,...,x_{k-1},x_k) \longmapsto \frac{\partial}{\partial \tilde{x}_k^j} \Psi_k^\tau (x_0) \in \boldsymbol{g}_{|x_0}$$

can be differentiated usually as a vector valued function.

$$(x_1,...,x_{k-1},x_k) \longmapsto \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} ... \frac{\partial}{\partial x_{k-1}^{i_{k-1}}} \frac{\partial}{\partial \tilde{x}_k^{i_k}} \Psi_k^\tau (x_0,x_1,...,x_k) \in \boldsymbol{g}_{|x_0}.$$

We put

$$\boxed{\Omega \left(\Psi_k^\tau \right)(x) = \frac{1}{k!} \sum_{i_1,i_2,..,i_k} \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} \frac{\partial}{\partial \tilde{x}_k^{i_k}} \Psi_k^\tau (x_0,x_1,...,x_k)_{x_0=x_1=...=x_k=x} dx^{i_1} \wedge ... \wedge dx^{i_k}.}$$

It is a k-form on M with values in the vector bundle $\boldsymbol{g}$

$$\Omega\left(\Psi_k^\tau\right) \in \Omega^k\left(M; \boldsymbol{g}\right)$$

Considering $k = 2$ we obtain a 2-form with values in $\boldsymbol{g}$,

$$\Omega\left(\Psi_2^\tau\right) \in \Omega^2\left(M; \boldsymbol{g}\right),$$

called the **curvature form** of τ.

The fundamental role is playing by the following

Theorem 18 *For an arbitrary linear direct connection $\tau : (M \times M)_{|U} \to \Phi$ in the Lie groupoid Φ the curvature form of τ and the curvature form of the underlying connection in $A(\Phi)$ are differs on a constant*

$$\Omega\left(\Psi_2^\tau\right) = \frac{1}{4} \cdot \Omega^\tau.$$

Proof. (a sketch) three steps of the proof:

-1) Of course, we need to prove the equality point by point, so we can look at this locally. Assume that Φ is a trivial Lie groupoid $\Phi = M \times G \times M$ with the source pr_3 and the target pr_1 and the partial multiplication

$$(z, a, y) \cdot (y, b, x) = (z, ab, x).$$

Then $A(\Phi)_{|x} = T_x M \times \mathfrak{g}$ where $\mathfrak{g}$ is the **right** ! Lie algebra of G,

$$A(\Phi) = TM \times \mathfrak{g}.$$

The bracket in $\mathfrak{g}$ will be denoted by $[v,w]^R$. The linear direct connection τ is given by

$$\begin{aligned} \tau &: (M\times M)_{|U} \to M\times G\times M \\ \tau(x,y) &= (x, \hat{\tau}(x,y), y), \\ \hat{\tau} &: (M\times M)_{|U} \to G, \quad \hat{\tau}(x,x) = e. \end{aligned}$$

Therefore

$$\frac{\partial \hat{\tau}}{\partial x^i}_{|(x,x)} + \frac{\partial \hat{\tau}}{\partial x^{m+i}}_{|(x,x)} = 0$$

and

$$\Psi_2^\tau(x_0,x_1,x_2) = \left(x_0,\ \hat{\tau}(x_0,x_1)\cdot\hat{\tau}(x_1,x_2)\cdot\hat{\tau}(x_2,x_0),\ x_0\right).$$

The induced linear connection $\nabla^\tau : TM \to TM\times\mathfrak{g}$ is equal to

$$\nabla^\tau_{|y}(v) = \nabla^\tau_v = \tau(\cdot,y)_{*y}(v) = \left(v, \hat{\tau}(\cdot,y)_{*y}(v)\right).$$

We calculate the curvature tensor Ω^τ of ∇^τ.

$$\Omega^\tau_{|x_0} = 2\sum_{i<j}\left(\frac{\partial^2\hat{\tau}}{\partial x^j\partial x^{m+i}} - \frac{\partial^2\hat{\tau}}{\partial x^i\partial x^{m+j}} + \left[\frac{\partial\hat{\tau}}{\partial x^i}, \frac{\partial\hat{\tau}}{\partial x^j}\right]^R\right)_{|(x_0,x_0)} dx^i\wedge dx^j.$$

-ii) Now we calculate the form $\Omega\left(\Psi_2\right)$:

$$\Omega\left(\Psi_2\right)\left(x_0\right)=\frac{1}{2!}\sum_{i,j}\frac{\partial^2\Psi_2\left(x_0,x_1,x_2\right)}{\partial x_1^i\partial\tilde{x}_2^j}_{|x_0=x_1=x_2}dx^i\wedge dx^j$$

for $\Psi_2\left(x_0,x_1,x_2\right)=\hat{\tau}\left(x_0,x_1\right)\cdot\hat{\tau}\left(x_1,x_2\right)\cdot\hat{\tau}\left(x_2,x_0\right).$

-a) Firstly we calculate it for **the Lie group** G **of matrices,**

$$G\subset GL\left(V\right)\subset\mathbb{R}^{(\dim V)^2}$$

for some finitely dimensional vector space V, using the fact that the differential of the left and the right translations, l_g and r_g, are exactly respectively the left and the right multiplication by matrices,

$$\left(l_g\right)_{*h}=gh,\quad\left(r_g\right)_{*h}=hg.$$

After calculations we obtain the equality

$$\Omega\left(\Psi_2\right)=\frac{1}{4}\cdot\Omega^{\tau}.$$

-b) For an arbitrary Lie group G we use the following **"IMPORTANT LOCAL TRICK"**:

We need to consider only local Lie group structure near the unit. But every Lie algebra is isomorphic to a Lie algebra of matrices (because there exists a faithful representation in some finitely dimensional vector space V) and a Lie algebra of matrices is a Lie algebra of a Lie subgroup of the Lie group $GL(V)$. Therefore the above result concerning $G \subset GL(V)$ is valid for arbitrary Lie group G ! ■

3.5 Characteristic classes

The last theorem gives that we can extract the Chern-Weil homomorphism of Φ via any local direct connection τ on the level of differential forms. The Chern-Weil homomorphism of Φ is really the Chern-Weil homomorphism of the Lie algebroid $A(\Phi)$ of Φ.

We recall the construction of the Chern-Weil homomorphism for Lie algebroids

- Jan Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids*, UNIVERSITE CLAUDE BERNARD – LYON 1, Publications du Département de Mathématiques, nouvelle série, 1991, 1-70.

Consider a transitive Lie algebroid A with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A \to TM \to 0$$

with the adjoint bundle of Lie algebras $\boldsymbol{g}$.

The Chern-Weil homomorphism for transitive Lie algebroid A is defined as follows:

$$\begin{aligned} h_A : \bigoplus^{k\geq 0} \left(\operatorname{Sec} \bigvee^{k} \boldsymbol{g}^* \right)_{I^0} &\longrightarrow \mathbf{H}_{dR}(M) \\ \Gamma \longmapsto & \left[\frac{1}{k!} \langle \Gamma, \Omega \vee .... \vee \Omega \rangle \right] \end{aligned}$$

where $\Omega \in \Omega^2_E(M; \boldsymbol{g})$ is the curvature tensor of any connection in A, whereas $\left(\operatorname{Sec} \bigvee^k \boldsymbol{g}^*\right)_{I^0}$ is the space of invariant cross-sections of $\bigvee^k \boldsymbol{g}^*$ with respect to the adjoint representation of A on $\bigvee^k \boldsymbol{g}^*$, i.e. $\Gamma \in \left(\operatorname{Sec} \bigvee^k \boldsymbol{g}^*\right)_{I^0}$ if and only if

$$\forall_{\xi \in \operatorname{Sec} A} \forall_{\sigma_1, \ldots, \sigma_k \in \operatorname{Sec} \boldsymbol{g}} \left((\gamma \circ \xi) \langle \Gamma, \sigma_1 \vee ... \vee \sigma_k \rangle = \sum_{i=1}^{k} \langle \Gamma, \sigma_1 \vee ... \vee [\![\xi, \sigma_i]\!] \vee .... \vee \sigma_k \rangle \right)$$

The nontriviality of h_A means, of course, that in A there is no flat connection.

We explain also that $\Omega\vee...\vee\Omega$ is the usual skew multiplication of differential forms with values multiplying symmetrically

$$\Omega\vee...\vee\Omega\left(x;v_{1},...,v_{2k}\right)=\sum_{\sigma\in\Sigma^{2k}}sgn\,\sigma\cdot\Omega\left(x;v_{\sigma_{1}},v_{\sigma_{2}}\right)\vee...\vee\Omega\left(x;v_{\sigma_{2k-1}},v_{gs_{2k}}\right)\in\bigvee^{k}\boldsymbol{g}_{x}.$$

For example: for the Lie algebroid $A(P)$ of a principal fibre bundle P (P is assumed to be connected) and equivalently for Lie algebroid of the Ehresmann Lie groupoid $\Phi=PP^{-1}$, there is a natural isomorphism of algebras ν such that the diagram commutes

$$\begin{array}{ccc}\bigoplus^{k\geq0}\left(\mathrm{Sec}\bigvee^{k}\boldsymbol{g}^{*}\right)_{I^{0}} & & \\ & \searrow^{h_{A(P)}} & \\ \cong\uparrow\nu & & H_{dR}\left(M\right) \\ & \nearrow_{h_{P}} & \\ \left(\bigvee\mathfrak{g}^{*}\right)_{I} & & \end{array}$$

which means that the Chern-Weil homomorphism of a Lie algebroid is some generalization of this notion known on the ground of principal bundles. On the other hand, this also means that the Chern-Weil homomorphism of a principal bundle is a characteristic feature of its Lie algebroid (for connected principal bundles).

In addition, we must point out two things:

1) A Lie algebroid is - in some sense - a simpler structure than a principal bundle. Namely, nonisomorphic principal bundles can possess isomorphic Lie algebroids. For example, there exists a nontrivial principal bundle for which the Lie algebroid is trivial (the nontrivial Spin (3)-structure of the trivial principal bundle $\mathbb{R}P(5) \times \mathrm{SO}(3)$).

2) There exist other sources of Lie algebroids than principal bundles, for example, transversally complete foliations, nonclosed Lie subgroups, Poisson manifolds and other.

Example 19 *Let* $\Phi = GL(E)$ *be a Lie groupoid of all linear fibre isomorphisms. Then the Atiyah sequence of* $A(\Phi)$ *is*

$$0 \to End(E) \to A(\Phi) \to TM \to 0.$$

Consider the Chern-Weil homomorphism

$$h : \bigoplus^{k \geq 0} \left(\mathrm{Sec} \bigvee^{k} End(E)^{*} \right)_{I^0} \to H_{dR}(M), \Gamma \longmapsto \left[\frac{1}{k!} \langle \Gamma, \Omega \vee .... \vee \Omega \rangle \right]$$

where $\Omega \in \Omega^2(M; End(E))$ *is the curvature tensor af any connection in* Φ.

Pontryagin classes. *Take the* ***invariant*** *cross section* $C_k \in \Gamma\left(\mathrm{Sec}\bigvee^k End(E)^*\right)$ *by*

$$C_{k|x} = tr\left(\varphi_1 \Box ... \Box \varphi_k\right)$$

where for $\varphi_i \in End\left(E_{|x}\right)$ *the linear mapping* $\varphi_1 \Box ... \Box \varphi_k : \bigwedge^k E_{|x} \to \bigwedge^k E_{|x}$ *is defined [Greub-Halperin-Vanstone] by*

$$\varphi_1 \Box ... \Box \varphi_k \left(v_1 \wedge ... \wedge v_k\right) = \sum_{\sigma \in \Sigma^k} sgn\sigma \cdot \varphi_1\left(v_{\sigma_1}\right) \wedge ... \wedge \varphi_k\left(v_{\sigma_k}\right).$$

Then the Ponryagin class is equal to

$$p_k(E) = p_k(\Phi) = h\left(C_{2k}\right) = \frac{1}{(2k)!}\left[\langle C_{2k}, \underbrace{\Omega \vee .... \vee \Omega}_{2k \ \ times}\rangle\right]_{dR}.$$

The class $p_k(E)$ *is represented by the differential form*

$$c \cdot tr\left(\Omega \Box ... \Box \Omega\right).$$

According to the notation of Greub-Halperin-Vanstone, the forms $\Omega \vee .... \vee \Omega$ *and* $\Omega \Box ... \Box \Omega$ *are the usual skew multiplication of differential forms for which the values are multiplicated by the suitable mappings*

$$\begin{aligned} \vee &: End(E) \times ... \times End(E) \to \bigvee^{2k} End(E). \\ \Box &: End(E) \times ... \times End(E) \to End\left(\bigwedge^{2k} E\right). \end{aligned}$$

Trace classes. *Take the* ***invariant*** *cross section* $Tr_k \in \Gamma\left(\mathrm{Sec}\bigvee^k End\left(E\right)^*\right)$ *by*

$$Tr_k\left(\varphi_1, ..., \varphi_k\right) = \sum_{\sigma\in\Sigma^k} tr\left(\varphi_{\sigma_1}\circ ...\circ \varphi_{\sigma_k}\right).$$

Then the trace class is equal to

$$tr_k\left(E\right) = tr_k\left(\Phi\right) = h\left(Tr_{2k}\right) = \frac{1}{(2k)!}\left[\langle Tr_{2k}, \underbrace{\Omega\vee ....\vee\Omega}_{2k\ \ times}\rangle\right]_{dR}.$$

The class $tr_k\left(E\right)$ *is represented by the differential form*

$$c\cdot tr\left(\Omega\circ ...\circ\Omega\right)$$

(the values of the skew multiplication of $\Omega\circ...\circ\Omega$ *are multilied by the composing of the linear mapping.*

Pffafian class for oriented $2k$***-dimensional vector bundle*** E. *Take the* ***invariant*** *cross-section* $pf \in \Gamma\left(\mathrm{Sec} \bigvee^{k} Sk\left(E\right)^{*}\right)$

$$pf^{F}\left(\varphi_{1},...,\varphi_{k}\right)=\left\langle e,\beta^{-1}\left(\varphi_{1}\right)\wedge...\wedge\beta^{-1}\left(\varphi_{k}\right)\right\rangle$$

where $\beta:\bigwedge^{2}\left(F\right)\xrightarrow{\cong}Sk_{F}$, $\beta\left(x\wedge y\right)\left(z\right)=\left\langle x,z\right\rangle y-\left\langle y,z\right\rangle x$ *and* $e\in\bigwedge^{k}F$ *determine the orientation and* $\left|\left\langle e,e\right\rangle\right|=1$. *Then the Atiyah sequence of* $A\left(IsoE\right)$ *is*

$$0\rightarrow Sk\left(E\right)\rightarrow A\left(IsoE\right)\rightarrow TM\rightarrow 0.$$

and the Pfaffian class is equal to $i^{k}\cdot h\left(pf\right)$ *and it is represented by*

$$c\cdot\left\langle\Delta,\left(\beta^{-1}\Omega\right)\wedge...\wedge\left(\beta^{-1}\Omega\right)\right\rangle.$$

Take a local direct connection τ in Φ and consider once again the curvature form $\Omega\left(\Psi_2^\tau\right) \in \Omega^2\left(M; \boldsymbol{g}\right)$,

$$\Omega^\tau_{|x_0} = 2 \sum_{i<j} \left(\frac{\partial^2 \hat{\tau}}{\partial x^j \partial x^{m+i}} - \frac{\partial^2 \hat{\tau}}{\partial x^i \partial x^{m+j}} + \left[\frac{\partial \hat{\tau}}{\partial x^i}, \frac{\partial \hat{\tau}}{\partial x^j} \right]^R \right)_{|(x_0,x_0)} dx^i \wedge dx^j.$$

We known that τ induces a usual connection ∇^τ in $A\left(\Phi\right)$ and that the curvature of it is a constant time the form $\Omega\left(\Psi_2^\tau\right)$,

$$\Omega^\tau = 4 \cdot \Omega\left(\Psi_2^\tau\right).$$

In conclusion, the Chern-Weil homomorphism of Φ (i.e. of the $A\left(\Phi\right)$) can be extracted via τ on the **level of differential forms** by

$$\langle \Gamma, \Omega \vee ... \vee \Omega \rangle = 4^k \langle \Gamma, \Omega\left(\Psi_2^\tau\right) \vee ... \vee \Omega\left(\Psi_2^\tau\right) \rangle.$$

Problem 20 *How can we express the form*

$$\Omega\left(\Psi_2^\tau\right) \vee ... \vee \Omega\left(\Psi_2^\tau\right) \in \Omega^{2k}\left(M; \bigvee \boldsymbol{g}\right)$$

with the help of $\Omega\left(\Psi_{2k}^\tau\right)$*?* *and the form* $\langle \Gamma, \Omega\left(\Psi_2^\tau\right) \vee ... \vee \Omega\left(\Psi_2^\tau\right) \rangle$ *for an invariant cross-section* $\Gamma \in \left(\operatorname{Sec} \bigvee^k \boldsymbol{g}^*\right)_{I^0}$ *with the help* Γ *and* $\Omega\left(\Psi_{2k}^\tau\right)$ *?*

We know that for $\Phi = GL(E)$ we have the adjoint Lie algebra bundle $\boldsymbol{g}$ is equal to the vector bundle of linear homomorphisms $Aut(E)$. Therefore the Atiyah sequence of $A(\Phi)$ equals

$$0 \rightarrow Aut(E) \rightarrow A(\Phi) \rightarrow TM \rightarrow 0.$$

Using the composition of linear homomorphisms

$$\circ : Aut(E) \times ... \times Aut(E) \rightarrow Aut(E)$$

we can obtain the following theorem.

Theorem 21 *The equality holds*

$$\Omega(\Psi^{\tau}_{2k}) = c \cdot \Omega(\Psi^{\tau}_{2}) \circ ... \circ \Omega(\Psi^{\tau}_{2}) \quad (\ k \ \textit{times})$$

or equivalently

$$\Omega(\Psi^{\tau}_{2k}) = c_1 \cdot \Omega^{\tau} \circ .... \circ \Omega^{\tau} \quad (\ k \ \textit{times}).$$

THE END

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Remark concerning the above Problem (20): For an arbitrary Lie groupoid Φ we consider

$$\Psi_{2k}^{\tau}(x_0, x_1, ..., x_{2k}) = \tau(x_0, x_1) \cdot \tau(x_1, x_2) \cdot ... \cdot \tau(x_{2k-1}, x_{2k}) \cdot \tau(x_{2k}, x_0) \in \Phi_{x_0}^{x_0}.$$

We define additionally

$$\begin{aligned} & \tilde{\Psi}_{2k}^{\tau}(x_0, x_1, ..., x_{2k}) \\ = \; & \tau(x_0, x_1) \cdot \tau(x_1, x_2) \cdot [\tau(x_2, x_0) \cdot \tau(x_0, x_3)] \cdot \tau(x_3, x_4) \cdot [\tau(x_4, x_0) \cdot \tau(x_0, x_5)] \cdot ... \\ & ... \cdot [\tau(x_{2k-2}, x_{2k}) \cdot \tau(x_0, x_{2k-1})] \cdot \tau(x_{2k-1}, x_{2k}) \cdot \tau(x_{2k}, x_0) \\ = \; & [\tau(x_0, x_1) \cdot \tau(x_1, x_2) \cdot \tau(x_2, x_0)] \cdot [\tau(x_0, x_3) \cdot \tau(x_3, x_4) \cdot \tau(x_4, x_0)] \cdot ... \\ & ... \cdot [\tau(x_0, x_{2k-1}) \cdot \tau(x_{2k-1}, x_{2k}) \cdot \tau(x_{2k}, x_0)] \end{aligned}$$

i.e.

$$\tilde{\Psi}_{2k}^{\tau}(x_0, x_1, ..., x_{2k}) = \Psi_2^{\tau}(x_0, x_1, x_2) \cdot \Psi_2^{\tau}(x_0, x_3, x_4) \cdot ... \cdot \Psi_2^{\tau}(x_0, x_{2k-1}, x_{2k})$$

Clearly $\Psi_{2k}^{\tau} \neq \tilde{\Psi}_{2k}^{\tau}$ in general, but (I think) the equality holds infinitesimally

$$\Omega(\Psi_{2k}^{\tau})(x) = \Omega\left(\tilde{\Psi}_{2k}^{\tau}\right)(x). \tag{3}$$

Next, using (3), we can try to calculate $\Omega(\Psi_{2k}^{\tau})(x)$ via $\Omega(\Psi_2^{\tau})(x)$.