THE EULER-POINCARÉ-HOPF THEOREM FOR FLAT CONNECTIONS IN SOME TRANSITIVE LIE ALGEBROIDS

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Abstract. This paper is a continuation of [19], [21], [22]. We study flat connections with isolated singularities in some transitive Lie algebroids for which either $\mathbb{R}$ or $\mathrm{sl}(2,\mathbb{R})$ or $\mathrm{so}(3)$ are isotropy Lie algebras. Under the assumption that the dimension of the isotropy Lie algebra is equal to $n+1$, where n is the dimension of the base manifold, we assign to any such isolated singularity a real number called an index. For $\mathbb{R}$-Lie algebroids, this index cannot be an integer. We prove the index theorem (the Euler-Poincaré-Hopf theorem for flat connections) saying that the index sum is independent of the choice of a connection. Multiplying this index sum by the orientation class of M, we get the Euler class of this Lie algebroid. Some integral formulae for indices are given.

Keywords: Lie algebroid, Euler class, index theorem, integration over the fibre, flat connection with singularitity

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1. Introduction

1.1. Motivations

1.1.1. Index theorem of Euler-Poincaré-Hopf for sphere bundles. The index theorem of Euler-Poincaré-Hopf for sphere bundles is well-known (see, for example, [8, Vol. I]):

Theorem 1.1. *Let E be an n-sphere bundle with a connected compact oriented base manifold M of dimension $n+1$, such that E is given the local product orientation. Let σ be a cross-section of E with finitely many singularities $a_1,\dots,a_k$.*

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Then the index sum $\sum_{v=1}^{k} j_{a_v}(\sigma)$*, where* $j_{a_v}(\sigma)$ *is the index of* σ *at* a_v*, is independent of the choice of the cross-section* σ *and the Euler class* χ_E *of* E *is given by* $\chi_E = \sum_{v=1}^{k} j_{a_v}(\sigma) \cdot \omega_M$ *where* ω_M *is the orientation class of* M*.*

This theorem can be applied, in particular, to G-principal bundles P over manifolds M of dimension $\dim G + 1$ for Lie groups G diffeomorphic to a sphere:
(i) S^1-principal bundles over M^2,
(ii) Spin(3)-principal bundles over M^4.

A locally defined cross-section $f\colon U \to P|_U$ of a principal bundle P determines (in an evident manner) a flat connection $H^f \subset T(P|_U)$ in $P|_U$ in such a way that $H^f(f(x)) = f_*[T_xM] \subset T_{f(x)}P$, but not conversely: there are more (in general) flat connections than cross-sections (see below).

The theory of Lie algebroids is a convenient theory which allows us to treat connections as linear homomorphisms. Each principal bundle $P(M,G)$ possesses a transitive Lie algebroid $A(P) = TP/G$ [1], [29], [14], and connections in $P(M,G)$ correspond to splittings of the Atiyah sequence of $A(P)$

$$0 \longrightarrow P \times_{\mathrm{Ad}} \mathfrak{g} \longrightarrow TP/G \rightleftarrows TM \longleftarrow 0.$$

In this paper we present an index theory of flat connections with isolated singularities for some class of transitive Lie algebroids including Lie algebroids of the principal bundles (i) and (ii) mentioned above as well as ones coming from other sources (such as some TC-foliations). The index theorem obtained (Theorem 4.3) in application to principal bundles generalizes Theorem 1.1 from cross-sections to flat connections.

1.1.2. Analogy: sphere bundle—Lie algebroid. Roughly speaking, on the ground of transitive Lie algebroids, we observe an interesting analogy of the theory of sphere bundles, namely, it turns out that, in some sense, flat connections in Lie algebroids correspond to cross-sections of sphere bundles. The common ideas are the index at an anisolated singularity and the theorem of Euler-Poincaré-Hopf type as well as some technical methods. This analogy was first noticed for regular Lie algebroids with 1-dimensional isotropy Lie algebras [20], [24]. The main purpose of our paper is to study this phenomenon more generally in the domain of transitive Lie algebroids without a restriction of the dimensions of isotropy Lie algebras.

We point out that Lie algebroids arise in many subjects of differential geometry and play a role analogous to that of Lie algebras for Lie groups (i.e. compose an infinitesimal invariant). For example, they arise in the theory of differential groupoids and principal bundles [25], [26], [27], [28], [34], [35], [36], [14], [29], vector bundles [7], [25], [29], [35], [15], transversally complete and transversally parallelizable foliations [31], [32], nonclosed Lie subgroups [16], Poisson manifolds [3], [4], [6], [37], [38], [10], [11] and others (see the famous survey article by K. Mackenzie [30]).

Most probably, the "transitive" and "isolated" theory of indexes of flat connections, presented below, should have its equivalence—at least in some cases—for nontransitive algebroids over foliations with isotropy Lie algebras of the dimension greater than one (as we observe for dimension one for $\mathbb{R}$-Lie algebroids of Poisson manifolds ([20], [24]).

1.1.3. Restriction to a spherical case. The index theory of flat connections with isolated singularities (in the transitive case, or for the regular case but with isolated singularities at each leaf) must be restricted to isotropy Lie algebras $\mathfrak{g}$ such that $H^*(\mathfrak{g}) = H^*_{dR}(S^{\dim \mathfrak{g}})$, i.e. the following conditions are fulfilled:

$$H^k(\mathfrak{g}) = \begin{cases} \mathbb{R} & \text{for } k = 0,\ \dim \mathfrak{g}, \\ 0 & \text{for } 1 \leqslant k \leqslant \dim \mathfrak{g} - 1. \end{cases} \tag{1.1}$$

(such Lie algebras are called *spherical*). According to [21, Th. 2.1], the Lie algebras $\mathbb{R}$, so(3) (the Lie algebra of real 3×3 skew symmetric matrices) and sl$(2, \mathbb{R})$ (the Lie algebra of real 2×2-matrices with trace zero) are the only ones. In Remark 4.10 below we briefly explain this limitation from the point of view of degree argument.

1.2. Preliminaries

1.2.1. Lie algebroids, definition. This paper deals with Lie algebroids and is based on papers [19], [23], [21]. By a *Lie algebroid on a manifold* M [36] we mean a system $A = (A, [\![\cdot,\cdot]\!], \gamma)$ consisting of a vector bundle A on M and mappings $[\![\cdot,\cdot]\!]\colon \operatorname{Sec} A \times \operatorname{Sec} A \to \operatorname{Sec} A$, $\gamma\colon A \to TM$, such that (1) $(\operatorname{Sec} A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, (2) γ, called the *anchor*, is a homomorphism of vector bundles, (3) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$, $f \in C^\infty(M)$. The anchor is bracket-preserving, [9], [2]. A Lie algebroid A is said to be *transitive* if γ is an epimorphism of vector bundles, and *regular* if γ is of constant rank. Next, we adopt the notions and the notation from [36], [29], [15], among them, the adjoint bundle of Lie algebras $\boldsymbol{g} := \operatorname{Ker}\gamma$, the Atiyah sequence $0 \to \boldsymbol{g} \to A \to F \to 0$, the notion of a connection and a (strong and non-strong) homomorphism of Lie algebroids. We recall that by an A -differential form of degree p we mean a cross-section $\Phi \in \Omega^p_A(M) := \operatorname{Sec} \bigwedge^p A^*$. In the space $\Omega_A(M) = \operatorname{Sec} \bigwedge A^*$ the exterior derivative d_A (first defined by L. Maxim-Raileanu in 1976, [33], see also [13], [29]) is defined by the formula

$$(d_A\Phi)(\xi_1,\dots,\xi_p) = \sum_{i=0}^{p}(-1)^p(\gamma\circ\xi_i)(\Phi(\xi_0,\dots\hat{\imath}\dots\xi_p)) + \sum_{i<j}(-1)^{i+j}\Phi([\![\xi_i,\xi_j]\!],\xi_0,\dots\hat{\imath}\dots\hat{\jmath}\dots,\xi_p),$$

$\xi_i\in\operatorname{Sec}A$. In particular, for the Lie algebroid $A=TM$, the tangent bundle of M, the exterior derivative d_{TM} is the usual exterior derivative of differential forms $d_{TM}=d_M$, so $H_{TM}(M)=H_{dR}(M)$.

1.2.2. Fibre integral. In [19] we introduced the notion of a *vertically oriented Lie algebroid* as a pair (A,ε) consisting of a regular Lie algebroid A and a volume form of $\boldsymbol{g}$, it means, a non-singular cross-section ε of $\bigwedge^n\boldsymbol{g}$, $n=\operatorname{rank}\boldsymbol{g}$. By a *homomorphism of vertically oriented Lie algebroids* $(T,t)\colon (A,\varepsilon)\to(A',\varepsilon')$, $\operatorname{rank}\boldsymbol{g}=\operatorname{rank}\boldsymbol{g}'=n$, we mean a non-strong, in general, homomorphism $T\colon A\to A'$ inducing $t\colon M\to M'$ of Lie algebroids such that $(\bigwedge^n T^+)(\varepsilon_x)=\varepsilon'_{tx}$, $x\in M$ (where $T^+\colon \boldsymbol{g}\to\boldsymbol{g}'$ is the restriction of T to adjoint bundles).

In [19] the operator of the fibre integral $⨍_A$ in a *vertically oriented Lie algebroid* (A,ε) is introduced. For a transitive Lie algebroid (the case considered in this paper), the fibre integral $⨍_A\colon \Omega_A^*(M)\to\Omega^{*-n}(M)$ is defined in the following way: $⨍_A\Phi=0$ if $\deg\Phi<n$ and

$$\left(⨍_A\Phi\right)_x(w_1\wedge\ldots\wedge w_k)=(-1)^{nk}\Phi_x(\varepsilon_x\wedge\tilde{w}_1\wedge\ldots\wedge\tilde{w}_k)$$

if $\Phi\in\Omega_A^{n+k}(M)$, $k\geqslant 0$, $x\in M$, $w_i\in T_xM$, and $\tilde{w}_i\in A|_x$ are arbitrary vectors satisfying $\gamma(\tilde{w}_i)=w_i$. Equivalently, $⨍_A\Phi\in\Omega^k(M)$ is a uniquely determined differential form such that $\gamma^*(⨍_A\Phi)=(-1)^{nk}\iota_\varepsilon\Phi$, where $\gamma^*\colon \Omega(M)\to\Omega_A(M)$ is the pullback of differential forms, $\gamma^*(\Phi)_x(u_1\wedge\ldots\wedge u_k)=\Phi_x(\gamma(u_1)\wedge\ldots\wedge\gamma(u_k))$.

The pullback of differential forms via a (non-strong in general) homomorphism of Lie algebroids commutes with exterior derivatives [17]. The basic properties of $⨍_A$ are given in the following theorems:

Theorem 1.2.

(a) *If $(T,t)\colon (A,\varepsilon)\to(A',\varepsilon')$ is a homomorphism of vertically oriented Lie algebroids, then $t^*\circ⨍_{A'}=⨍_A\circ T^*$,*
(b) $⨍_A\circ\gamma^*=0$,
(c) $⨍_A\gamma^*\psi\wedge\Phi=\psi\wedge⨍_A\Phi$ *for arbitrary forms $\psi\in\Omega(M)$ and $\Phi\in\Omega_A(M)$,*
(d) $⨍_A\Phi\wedge\gamma^*\psi=(-1)^{nk}(⨍_A\Phi)\wedge\psi$ *for $\psi\in\Omega^k(M)$, $\Phi\in\Omega_A^{\geqslant n}(M)$,*
(e) $⨍_A$ *is an epimorphism.*

Theorem 1.3. *The operator* $\fint_A$ *commutes with the exterior derivatives* d_A *and* d_M *if and only if*

(a1) *the isotropy Lie algebras* $\boldsymbol{g}|_x$ *are unimodular, and*

(a2) *the cross-section* ε *is invariant with respect to the adjoint representation of* A *on* $\bigwedge^n \boldsymbol{g}$.

1.2.3. TUIO-Lie algebroids. The transitive Lie algebroid A possessing properties (a1) and (a2) from Theorem 1.3 is called *unimodular invariantly oriented* (*TUIO-Lie algebroid* for short). Clearly, the modular class m_A of a TUIO-Lie algebroid A is zero (m_A is equal to the characteristic class of the adjoint representation of A on the top exterior power of $\boldsymbol{g}$ [5]).

In [19] and [23] the following sources of such Lie algebroids can be found:

- the Lie algebroids of G-principal bundles for a structure Lie group G not necessarily compact or connected but satisfying $\det(\mathrm{Ad}_G\, a) = +1$, $a \in G$,
- the Lie algebroids of the TC-foliations of left cosets of nonclosed Lie subgroups in Lie groups,
- the Lie algebroids of TP-foliations on compact and simply connected manifolds.

The fundamental properties of TUIO-Lie algebroids are:

– the fibre integral $\fint_A$ yields a homomorphism in cohomology $\fint_A^{\#}\colon H_A(M) \to H(M)$,

– the cohomology algebra $H_A(M)$ satisfies the Poincaré duality: $H_A^{\bullet}(M) \cong (H_{A,c}^{\bar{n}-\bullet})^*$, $\bar{n} = \operatorname{rank} A$.

1.2.4. s-Lie algebroids and the Euler class. Cohomology theory of flat connections is developed in this paper on the subcategory of TUIO-Lie algebroids for which isotropy Lie algebras are spherical, i.e. satisfy condition (1.1). Such Lie algebroids are called briefly *s-Lie algebroids.* This class contains, for example, the following Lie algebroids:

- the Lie algebroids of G-principal bundles P such that
  - $\det(\mathrm{Ad}_G\, a) = +1$, $a \in G$ (G need not be compact or connected),
  - $H^*(\mathrm{gl}(G)) = H^*_{dR}(S^{\dim G})$.

  For example, if $n = 3$, we can take $G = \mathrm{Spin}(3)$, $SO(3)$, $O(3)$ (the last is disconnected), or $G = SL(2)$ (noncompact case). The theory of foliations of codimension 3 is a source of the principal bundles considered above;
- the Lie algebroids of some TC-foliations, among them, of the foliations of left cosets of nonclosed Lie subgroups H in Lie groups G such that $\dim \overline{H} - \dim H = 1$.

In [21], for an s-Lie algebroid (A,ε), a long exact sequence of cohomology (Gysin sequence) is constructed:

$$\ldots \longrightarrow H^p(M) \xrightarrow{D^p} H^{p+n+1}(M) \xrightarrow{\gamma^{\#}} H_A^{p+n+1}(M) \xrightarrow{f_A^{\#}} H^{p+1}(M) \xrightarrow{D^{p+1}} \ldots \tag{1.2}$$

The class $\chi_{(A,\varepsilon)} := D^0(1) \in H^{n+1}(M)$ is called the *Euler class* of (A,ε). The homomorphism D is given by $D(\alpha) = \alpha \wedge \chi_{(A,\varepsilon)}$.

Each s-Lie algebroid of rank 1 (i.e. with $\boldsymbol{g}|_x = \mathbb{R}$) has the trivial adjoint Lie algebra bundle $\boldsymbol{g} = M \times \mathbb{R}$ and is isomorphic to $A = A_\Omega = TM \times \mathbb{R}$ [19] with the structure $[\![\cdot,\cdot]\!]$ defined via a closed real 2-form $\Omega \in \Omega^2(M)$, $d\Omega = 0$, by the formula

$$[\![(X,f),(Y,g)]\!] = ([X,Y], -\Omega(X,Y) + \partial_X g - \partial_Y f)$$

and $\varepsilon(x) = 1$. The Euler class $\chi_{(A,\varepsilon)}$ of this s-Lie algebroid (A,ε) is equal to $[-\Omega]$ [21], and can also be expressed via the Chern-Weil homomorphism h_A of A [15]:

$$\chi_{(A,\varepsilon)} = h_A^{(2)}(-\varepsilon^*) = [\langle -\varepsilon^*, \Omega\rangle] \in \Omega^2(M)$$

where $\varepsilon^* \in \operatorname{Sec} \boldsymbol{g}^*$ is an invariant cross-section defined by $\iota_\varepsilon \varepsilon^* = 1$.

For an s-Lie algebroid (A,ε) of rank 3 (i.e. $\dim \boldsymbol{g}|_x = 3$), ε is an invariant cross-section of $\bigwedge^3 \boldsymbol{g}$ and the Euler class of A is equal to

$$\chi_{(A,\varepsilon)} = h_A^{(4)}(-2\Gamma) = \frac{1}{2}[\langle -2\Gamma, \Omega \vee \Omega\rangle] \in H^4(M)$$

[21] where Γ is an invariant cross-section of $\bigvee^2 \boldsymbol{g}^*$ such that $\Gamma_x = \varrho_x^{-1}(\varepsilon_x^*)$ for the canonical isomorphism $\varrho_x \colon \big(\bigvee^2 \boldsymbol{g}^*|_x\big)_I \xrightarrow{\cong} \bigwedge^3 \boldsymbol{g}^*|_x$, $x \in M$ ($\langle \varrho_x(\psi), x \wedge y \wedge z\rangle = \langle \psi, [x,y] \vee z\rangle$ [8, Vol. III], $\Omega \in \Omega^2(M;\boldsymbol{g})$ is the curvature form of any connection.

1.3. Statements of the main results

Our paper is organized as follows: in Section 3 we define the cohomology class $\omega_\lambda \in H_A^n(M)$ ($n = \operatorname{rank} \boldsymbol{g}$) of a flat connection $\lambda\colon TM \to A$ in an s-Lie algebroid (A,ε) and next we introduce the notion of the difference class $[\lambda,\sigma] \in H^n(M)$ for two flat connections λ and σ. Among its fundamental properties, we establish a relationship with the Euler class $\chi_{(A,\varepsilon)}$ of (A,ε): for two locally defined flat connections λ and σ on open subsets U and V respectively, given a covering of M, the Euler class $\chi_{(A,\varepsilon)}$ is equal to $\partial([\lambda,\sigma])$ where ∂ is the connecting homomorphism for the Mayer-Vietoris sequence of the triad (M,U,V) for the de Rham cohomology.

Section 4 is devoted to flat connections in an s-Lie algebroid (A, ε) over an oriented manifold M such that $\dim M = n+1$, $n = \dim \boldsymbol{g}|_x$, i.e. according to our Theorem 2.1 from [21], only in

(1) $\mathbb{R}$-Lie algebroids over M^2,

(2) so(3) and sl(2, $\mathbb{R}$)-Lie algebroids over M^4.

We define in that section the fundamental notion of the index $j_a\sigma$ of a locally defined flat connection σ with singularity at a. We prove that, for $\mathbb{R}$-Lie algebroids, each real number can be the index at a given point of some singular connection. For $\mathfrak{g} = \mathrm{so}(3)$, the set of indexes at a point is equal to the set of multiples of some real number. For $\mathfrak{g} = \mathrm{sl}(2, \mathbb{R})$, the index $j_a\sigma = 0$.

Section 4 contains the main results of the paper. We prove a version of the Euler-Poincaré-Hopf theorem for flat connections: the sum of indexes $\sum_{v=1}^{k} j_{a_v}(\sigma)$ of a flat connection σ with isolated singularities at $a_1, \dots, a_k$ is independent of the choice of the connection and $\sum_{v=1}^{k} j_{a_v}(\sigma) = \int_M^{\#} \chi_{(A,\varepsilon)}$. In Section 5 we give some important integral formulae for the index and, finally, some remarks concerning the existence of flat connections with finitely many singularities.

2. Difference class

By a *connection* in a transitive Lie algebroid A we mean a splitting $\lambda\colon TM \to A$ of the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} TM \to 0$, i.e. a homomorphism of vector bundles such that $\gamma \circ \lambda = \mathrm{id}$. If λ is a homomorphism of Lie algebroids $\lambda \circ [X, Y] = [\![\lambda \circ X, \lambda \circ Y]\!]$, $X, Y \in \mathfrak{X}(M)$, then λ is called *flat*. In this situation, the pullback of differential forms $\lambda^*\colon \Omega_A(M) \to \Omega(M)$ commutes with differentials $\lambda^* \circ d_A = d_M \circ \lambda^*$, giving—on cohomology—a homomorphism of algebras $\lambda^{\#}\colon H_A(M) \to H(M)$. Let λ be a flat connection in an s-Lie algebroid (A, ε). By virtue of the exactness of sequence (1.2), [19, Prop. 4.2.1 (b)] and [21, Cor. 3.4] we can easily show that

$$H_A(M) = \ker \int_A^{\#} \oplus \ker \lambda^{\#}, \tag{2.1}$$

$$\int_A^{\#} \Big|_{\ker \lambda^{\#}} \colon \ker \lambda^{\#} \xrightarrow{\cong} H(M). \tag{2.2}$$

By the above, there exists a uniquely determined cohomology class

$$\omega_\lambda \in \ker \lambda^{\# n} \subset H_A^n(M)$$

such that $\int_A^{\#} \omega_\lambda = 1 \in H^0(M)$. ω_λ depends on the mapping $\lambda^{\#}\colon H_A(M) \to H(M)$ only. The class ω_λ is called the *cohomology class* of a flat connection λ.

For two flat connections $\lambda, \sigma\colon TM \to A$, their cohomology classes $\omega_\lambda, \omega_\sigma \in H^n_A(M)$ satisfy the equality $\fint_A^{\#}(\omega_\lambda - \omega_\sigma) = 0$. The exactness of sequence (1.2) implies that there exists a cohomology class $[\lambda, \sigma] \in H^n(M)$ such that

$$\omega_\lambda - \omega_\sigma = \gamma^{\#}[\lambda, \sigma].$$

Definition 2.1. The class $[\lambda, \sigma]$ is called the *difference class* of flat connections λ and σ in the s-Lie algebroid (A, ε).

Proposition 2.2. *For flat connections λ and σ in an s-Lie algebroid (A, ε), we have $\lambda^{\#}\alpha - \sigma^{\#}\alpha = -\big(\fint_A^{\#}\alpha\big) \wedge [\lambda, \sigma]$, $\alpha \in H_A(M)$.*

P r o o f. Let us write, for $\alpha \in H_A(M)$, $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \ker \fint_A^{\#}$ and $\alpha_2 \in \ker \lambda^{\#}$. By the exactness of sequence (1.2), $\alpha_1 = \gamma^{\#}\tilde{\alpha}$ for $\tilde{\alpha} \in H(M)$. It is easy to see the equality $\alpha_2 = \gamma^{\#}(\fint_A^{\#}\alpha_2) \wedge \omega_\lambda$. Now, take $\beta := \fint_A^{\#}\alpha_2$. Then $\beta = \fint_A^{\#}\alpha_2 + \fint_A^{\#}\alpha_1 = \fint_A^{\#}\alpha$ and $\gamma^{\#}\beta \wedge \omega_\lambda = \alpha_2$, so $\alpha = \alpha_1 + \alpha_2 = \gamma^{\#}\tilde{\alpha} + \gamma^{\#}\beta \wedge \omega_\lambda$. Since $\omega_\lambda = \omega_\sigma + \gamma^{\#}[\lambda, \sigma]$, we have

$$\begin{aligned} \lambda^{\#}\alpha - \sigma^{\#}\alpha &= \lambda^{\#}(\gamma^{\#}\tilde{\alpha} + \gamma^{\#}\beta \wedge \omega_\lambda) - \sigma^{\#}(\gamma^{\#}\tilde{\alpha} + \gamma^{\#}\beta \wedge (\omega_\sigma + \gamma^{\#}[\lambda, \sigma])) \\ &= \beta \wedge \lambda^{\#}\omega_\lambda - \beta \wedge (\sigma^{\#}\omega_\sigma + \sigma^{\#}\gamma^{\#}[\lambda, \sigma]) \\ &= -\left(\fint_A^{\#}\alpha\right) \wedge [\lambda, \sigma]. \end{aligned}$$

□

Since $\gamma^{\#}$ is a monomorphism for a flat Lie algebroid A, we obtain

Corollary 2.3. *Let λ and σ be two flat connections in an s-Lie algebroid A. Then the following conditions are equivalent:*

(a) $\lambda^{\#} = \sigma^{\#}$,
(b) $\omega_\lambda = \omega_\sigma$,
(c) $[\lambda, \sigma] = 0$.

If $H^n(M) = 0$, then clearly $H^n(M) \ni [\lambda, \sigma] = 0$, and therefore $\lambda^{\#} = \sigma^{\#}$.

Lemma 2.4. *Let (A, ε) be an arbitrary s-Lie algebroid with $\operatorname{rank} \boldsymbol{g} = n$. For a representative Ψ of the Euler class $\chi_{(A,\varepsilon)}$ and an n-form $\Phi \in \Omega^n_A(M)$ such that $\fint_A \Phi = -1$ and $d_A\Phi = \gamma^*\Psi$, for any open subset $U \subset M$ and two flat connections $\lambda, \sigma\colon TU \to A|_U$, the following equalities hold:*

(1) $\omega_\sigma = [\gamma|_U^*\sigma^*(\Phi|_U) - \Phi|_U]$,
(2) $[\lambda, \sigma] = [\lambda^*\Phi|_U - \sigma^*\Phi|_U]$.

P r o o f. It is easy to see the closedness of the forms $\gamma|_U^*\sigma^*(\Phi|_U) - \Phi|_U$ and $\lambda^*\Phi|_U - \sigma^*\Phi|_U$. Since $\sigma^*(\gamma|_U^*(\sigma^*\Phi|_U) - \Phi|_U) = 0$ and $\fint_{A|_U}(\gamma|_U^*(\sigma^*\Phi|_U) - \Phi|_U) = -\fint_{A|_U}\Phi|_U = 1$, we get (1) by the definition of the cohomology class of σ.

Since $\gamma|_U^\#$ is a monomorphism the equalities

$$\omega_\lambda - \omega_\sigma = [\gamma|_U^*\lambda^*\Phi|_U - \gamma|_U^*\sigma^*\Phi|_U] = \gamma|_U^\#([\lambda^*\Phi|_U - \sigma^*\Phi|_U])$$

yield now (2). □

Theorem 2.5 (The naturality of the difference class). *Let $(T,t)\colon (A,\varepsilon) \to (A',\varepsilon')$ be a homomorphism of s-Lie algebroids such that $T_x\colon A|_x \to A|'_{tx}$, $x \in M$, are isomorphisms.*

(a) *Assume that σ, σ' are flat connections in A and A', respectively, such that $T \circ \sigma = \sigma' \circ t_*$. Then $T^\#\omega_{\sigma'} = \omega_\sigma$.*

(b) *For any two pairs of such flat connections (λ, λ'), (σ, σ'), we have $t^\#([\lambda', \sigma']) = [\lambda, \sigma]$.*

P r o o f. (a): To check (a), it is sufficient to notice that $\sigma^\#(T^\#\omega_{\sigma'}) = (T \circ \sigma)^\#\omega_{\sigma'} = (\sigma' \circ t_*)^\#\omega_{\sigma'} = t^\#\sigma'^\#\omega_{\sigma'} = 0$ and (see [19]) $\fint_A^\# T^\#\omega_{\sigma'} = t^\# \fint_{A'}^\# \omega_{\sigma'} = 1$.

(b): By virtue of the fact that $\gamma^\#$ is a monomorphism, we only need to notice

$$\begin{aligned} \gamma^\# t^\#[\lambda', \sigma'] &= (t_* \circ \gamma)^\#[\lambda', \sigma'] = (\gamma' \circ T)^\#([\lambda', \sigma']) \\ &= T^\#(\omega_{\lambda'} - \omega_{\sigma'}) = \omega_\lambda - \omega_\sigma = \gamma^\#[\lambda, \sigma]. \end{aligned}$$

□

The folowing theorem gives a relationship between the Euler class and the difference class (compare with the classical theorem for sphere bundles, for example [8]).

Theorem 2.6. *Let $\{U, V\}$ be an open cover of M and let $\lambda_U\colon TU \to A|_U$ and $\sigma_V\colon TV \to A|_V$ be flat connections in (A,ε) over U and V, respectively (U, V need not be connected). Consider the Mayer-Vietoris sequence of the triad $\{M, U, V\}$ for the usual real de Rham cohomology and let $\partial\colon H(U \cap V) \to H(M)$ be the connecting homomorphism. Then*

$$\chi_{(A,\varepsilon)} = \partial[\lambda, \sigma]$$

where $\lambda = \lambda_U|_{U\cap V}$ and $\sigma = \sigma_V|_{U\cap V}$.

P r o o f. For the inclusions $j_1\colon U \cap V \hookrightarrow U$ and $j_2\colon U \cap V \hookrightarrow V$ according to Lemma 2.4, $[\lambda, \sigma] = [\lambda^*\Phi|_{U\cap V} - \sigma^*\Phi|_{U\cap V}] = [j_1^*(\lambda_U^*\Phi|_U) - j_2^*(\sigma_V^*\Phi|_V)]$. Since $d(\lambda_U^*\Phi|_U) = \lambda_U^* d_{A|_U}\Phi|_U = \lambda_U^*\gamma|_U^*\Psi|_U = \Psi|_U$, analogously $d(\sigma_V^*\Phi|_V) = \Psi|_V$, we get—via the construction of ∂—$\partial[\lambda, \sigma] = [\Psi] = \chi_{(A,\varepsilon)}$. □

3. The index of a flat connection at an isolated singular point and the Euler number

By a *local connection with singularity at a point* $a \in M$ in a Lie algebroid A we mean the connection

$$\sigma\colon\ T\dot{U} \to A|_{\dot{U}},\ a \in U \subset M\ \ (U \text{ is open}),\ \ \dot{U} = U \setminus \{a\}.$$

Let (A, ε) be an arbitrary s-Lie algebroid over an $n+1$-dimensional oriented manifold M ($n \geqslant 1$) with $n = \operatorname{rank} \boldsymbol{g}$ and let $\sigma\colon\ T\dot{U} \to A|_{\dot{U}}$ be a local connection with singularity at $a \in U \subset M$. Take additionally a neighbourhood $V \ni a$ such that $V \subset U$ and $V \cong \mathbb{R}^{n+1}$. $A|_V$ possesses ([29], [22]) a global flat connection $\lambda\colon\ TV \to A|_V$. Denote $\lambda|_{\dot{V}}$ ($\dot{V} = V \setminus \{a\}$) by $\dot{\lambda}$ and consider the difference class $[\dot{\lambda}, \sigma|_{\dot{V}}] \in H^n(\dot{V})$. Let $\alpha_V\colon\ H^n(\dot{V}) \xrightarrow{\cong} \mathbb{R}$ be the canonical mapping [8, Vol. I] ($\dot{V}$ has the orientation induced from M). Using the same arguments as in the theory of sphere bundles [8, Vol. I] and taking into account Corollary 2.3 we check that the number $\alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$ is independent of the auxiliary flat connection λ and of the neighbourhood V. It means that $\alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$ depends only on the choice of σ.

Definition 3.1. The number $\alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$ is called the *index of* σ at a and is denoted by

$$j_a(\sigma).$$

Proposition 3.2 (Naturality of the index). *Let $(\hat{A}, \hat{\varepsilon})$ be another s-Lie algebroid over an oriented $n+1$-dimensional manifold $\hat{M}$ and let $(T, t)\colon\ (\hat{A}, \hat{\varepsilon}) \to (A, \varepsilon)$ be a homomorphism of s-Lie algebroids fulfilling the conditions*

$$T_x\colon\ \hat{A}|_x \to A|_{tx},\ \ x \in M,\ \ \textit{is an isomorphism},$$
$$t\colon\ \hat{M} \to M\ \ \textit{is a diffeomorphism onto an open subset.}$$

Let $a \in M$, $\hat{a} \in \hat{M}$, $t(\hat{a}) = a$. Take a local flat connection $\sigma\colon\ T\dot{U} \to A|_{\dot{U}}$ with singularity at a. Then the mapping $T^{\#}\sigma\colon\ T\dot{W} \to \hat{A}|_{\dot{W}}$, $W = t^{-1}[U]$, $\dot{W} = W \setminus \{\hat{a}\}$ defined by $(T^{\#}\sigma)(v) = T|_{pv}^{-1}(\sigma(t_*v))$ is a flat connection in $\hat{A}$ with singularity at $\hat{a}$, and $j_{\hat{a}}(T^{\#}\sigma) = j_a(\sigma)$.

P r o o f. The fact that $T^{\#}\sigma$ is a flat connection follows trivially from the assumption that T gives an isomorphism of $\hat{A}$ onto $A|_{t[\hat{M}]}$. Restricting the domain of σ, if necessary, we may assume that $U \subset t[\hat{M}]$. Choose a neighbourhood $V \ni a$ small enough to be $V \subset U$, and both V and $\hat{V} = t^{-1}[V]$ are diffeomorphic to $\mathbb{R}^{n+1}$. Then $\hat{A}|_{\hat{V}}$ and $A|_V$ are trivial. Take an arbitrary flat connection $\lambda\colon\ TV \to A|_V$. Then $T^{\#}\lambda\colon\ T\hat{V} \to A|_{\hat{V}}$ is flat and, plainly, $j_a\sigma = \alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$

and $j_{\hat{a}}(T^{\#}\sigma) = \alpha_{\hat{V}}([(T^{\#}\lambda)^{\cdot}, (T^{\#}\sigma)|_{\hat{V}\setminus\{\hat{a}\}}])$. Since $(T^{\#}\lambda)^{\cdot} = T^{\#}(\lambda|_{\dot{V}}) = T^{\#}(\dot{\lambda})$ and $(T^{\#}\sigma)|_{\hat{V}\setminus\{\hat{a}\}} = T^{\#}(\sigma|_{\dot{V}})$, the naturality of the difference class yields

$$j_{\hat{a}}(T^{\#}\sigma) = \alpha_{\dot{V}}([T^{\#}\dot{\lambda}, T^{\#}(\sigma|_{\dot{V}})]) = \alpha_{\dot{V}} t^{\#}[\dot{\lambda}, \sigma|_{\dot{V}}] = \alpha_{V}[\dot{\lambda}, \sigma|_{\dot{V}}] = j_a(\sigma).$$

□

The main goal of this article is a theorem joining the index sum $\sum j_{a_v}(\sigma)$ of any flat connection with a finite number of singularities $\{a_1, \ldots, a_k\}$ to the Euler class of a Lie algebroid.

Theorem 3.3 (The Euler-Poincaré-Hopf theorem for flat connections). *Let (A, ε) be an s-Lie algebroid of rank n over an oriented compact manifold M of dimension $n+1$ and let $\sigma\colon T(M \setminus \{a_1, \ldots, a_k\}) \to A$ be a flat connection with singularities at points $a_1, \ldots, a_k$. Then the Euler class $\chi_{(A,\varepsilon)} \in H^{n+1}(M)$ is given by the formula*

$$\chi_{(A,\varepsilon)} = \left(\sum_{v=1}^{k} j_{a_v}(\sigma)\right) \cdot \omega_M,$$

where ω_M is the orientation class of M; equivalently, $\int_M^{\#} \chi_{(A,\varepsilon)} = \sum_{v=1}^{k} j_{a_v}(\sigma)$. In particular, the index sum $\sum_{v=1}^{k} j_{a_v}(\sigma)$ is independent of the choice of the connection.

P r o o f. For each $v = 1, \ldots, k$, choose a neighbourhood $U_v \ni a_v$ diffeomorphic to $\mathbb{R}^{n+1}$ and such that the sets U_v are pairwise disjoint. Put $U = \bigcup U_v$, $V = M \setminus \{a, \ldots, a_k\}$. Then $M = U \cup V$ and $U \cap V = \bigcup \dot{U}_v$ where $\dot{U}_v = U_v \setminus \{a_v\}$. Take arbitrary flat connections $\tilde{\lambda}_v\colon TU_v \to A|_{U_v}$, $v = 1, \ldots, k$. The family $\{\tilde{\lambda}_v\}$ determines one flat connection $\tilde{\lambda}\colon TU \to A|_U$ such that $\tilde{\lambda}|_{U_v} = \tilde{\lambda}_v$. Define $\check{\lambda} = \tilde{\lambda}|_{U\cap V}$ and $\check{\sigma} = \sigma|_{U\cap V}$. According to Theorem 2.6, $\chi_{(A,\varepsilon)} = \partial[\check{\lambda}, \check{\sigma}]$. Next, let $\lambda_v = \tilde{\lambda}_v|_{\dot{U}_v}$ and $\sigma_v = \sigma|_{\dot{U}_v}$. Then $[\check{\lambda}, \check{\sigma}] = \bigoplus_v [\lambda_v, \sigma_v]$. By [8, Prop. VII Chap. VI, Vol. I] $\int_M^{\#} \circ \partial = \alpha$, where $\alpha\colon \bigoplus_v H^n(\dot{U}_v) \to \mathbb{R}$ is given by $\bigoplus \beta_v \mapsto \sum \alpha_{U_v}(\beta_v)$. Therefore we get

$$\int_M^{\#} \chi_{(A,\varepsilon)A} = \int_M^{\#} \partial[\check{\lambda}, \check{\sigma}] = \int_M^{\#} \partial\Big(\bigoplus_{\nu}[\lambda_v, \sigma_v]\Big) = \alpha\Big(\bigoplus_{\nu}[\lambda_v, \sigma_v]\Big)$$
$$= \sum_{v=1}^{k} \alpha_{U_v}[\lambda_v, \sigma_v] = \sum_{v=1}^{k} j_{a_v}(\sigma).$$

□

The sum

$$j_{(A,\varepsilon)} = \sum_{v=1}^{k} j_{a_v}(\sigma)$$

is called the *Euler number of the s-Lie algebroid* (A,ε). According to Theorem 5.4 from [21], the Euler number $j_{(A,\varepsilon)}$ is not—in general—an invariant of the cohomology algebra of A and has nothing in common with the Euler-Poincaré characteristic of A. The last one, when considered for TUIO-Lie algebroids ($\dim M + \operatorname{rank} \boldsymbol{g}$ is odd), is always 0 [23].

4. Integral formulae

Proposition 4.1. *For a trivial s-Lie algebroid $A = TM \times \mathfrak{g}$ vertically oriented by a tensor $0 \neq \varepsilon_0 \in \bigwedge^n \mathfrak{g}$ ($n = \dim \mathfrak{g}$) and equipped with a "standard" flat connection $\tau_0 \colon TM \to TM \times \mathfrak{g}$, $v \mapsto (v, 0)$, we have:*

(a) *If $\sigma \colon TM \to A$ is a flat connection, then its cohomology class ω_σ is given by $\omega_\sigma = -\hat{\sigma}^\#[\varphi_0] \times 1 + 1 \times [\varphi_0]$ where $\hat{\sigma} = \mathrm{pr}_2 \circ \sigma \colon TM \to \mathfrak{g}$ and $\varphi_0 \in \bigwedge^n \mathfrak{g}^*$ is a tensor such that $\iota_{\varepsilon_0}\varphi_0 = 1$. In particular $\omega_{\tau_0} = 1 \times [\varphi_0]$.*

(b) *The difference class $[\tau_0, \sigma]$ is equal to $[\tau_0, \sigma] = \hat{\sigma}^\#[\varphi_0]$.*

Proof. Using the fact that the projection $\mathrm{pr}_2 \colon TM \to \mathfrak{g}$ is a nonstrong homomorphism of Lie algebroids [17], we can easily see that

$$\sigma^\#(-\hat{\sigma}^\#[\varphi_0] \times 1 + 1 \times [\varphi_0]) = 0,$$
$$\int_A^\# (-\hat{\sigma}^\#[\varphi_0] \times 1 + 1 \times [\varphi_0]) = 1.$$

Now (a) follows from the definition of the cohomology class ω_σ whereas (b) may be obtained from the definition of the difference class and the equality

$$\omega_{\tau_0} - \omega_\sigma = 1 \times [\varphi_0] - (-\hat{\sigma}^\#[\varphi_0] \times 1 + 1 \times [\varphi]) = \gamma^\# \hat{\sigma}^\#[\varphi_0].$$

□

Corollary 4.2. *For arbitrary flat connections $\lambda, \sigma \colon TM \to TM \times \mathfrak{g}$ we have*

$$[\lambda, \sigma] = (\hat{\sigma}^\# - \hat{\lambda}^\#)[\varphi_0].$$

Put $\dot{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$ and let $\mathfrak{g}$ be any n-dimensional unimodular Lie algebra $\mathfrak{g}$. Take tensors $0 \neq \varepsilon_0 \in \bigwedge^n \mathfrak{g}$, $\varphi_0 \in \bigwedge^n \mathfrak{g}^*$ joined by the relation $\iota_{\varepsilon_0}(\varphi_0) = 1$. Fix a flat connection $\sigma \colon T\dot{\mathbb{R}}^{n+1} \to T\dot{\mathbb{R}}^{n+1} \times \mathfrak{g}$ in the trivial Lie algebroid $A = T\dot{\mathbb{R}}^{n+1} \times \mathfrak{g}$ (oriented by the tensor ε_0). Let $i \colon S^n \hookrightarrow \dot{\mathbb{R}}^{n+1}$ be the inclusion.

Proposition 4.3. *The index $j_0(\sigma)$ of σ is given by the formula*

$$j_0(\sigma) = \int_{S^n} \sigma_S^*(\varphi_0) \tag{4.1}$$

where σ_S is a nonstrong homomorphism of Lie algebroids defined as the composition $\sigma_S\colon TS^n \xhookrightarrow{i_*} T\dot{\mathbb{R}}^{n+1} \xrightarrow{\hat{\sigma}} \mathfrak{g}$.

Proof. Let $\tau\colon T\mathbb{R}^{n+1} \to T\mathbb{R}^{n+1} \times \mathfrak{g}$, $v \mapsto (v, 0)$, be the "standard" flat connection. Then, according to [8, Vol. I] and Prop. 4.1 (b),

$$\begin{aligned} j_0\sigma = \alpha_{\mathbb{R}^{n+1}}([\dot{\tau}, \sigma]) = \alpha_{\mathbb{R}^{n+1}}\hat{\sigma}^{\#}[\varphi_0] = \int_{S^n} i^{\#}\hat{\sigma}^{\#}[\varphi_0] \\ = \int_{S^n} (\hat{\sigma} \circ i_*)^{\#}[\varphi_0] = \int_{S^n} (\sigma_S)^{\#}[\varphi_0]. \end{aligned}$$

□

Treat now $\hat{\sigma}\colon T\dot{\mathbb{R}}^{n+1} \to \mathfrak{g}$ as a 1-form on $\dot{\mathbb{R}}^{n+1}$ with values in $\mathfrak{g}$ and take the exterior n-product $\hat{\sigma}\wedge\ldots\wedge\hat{\sigma} \in \Omega^n(\dot{\mathbb{R}}^{n+1}; \bigwedge^n \mathfrak{g})$. We have $\hat{\sigma}^*(\varphi_0) = n!^{-1}(\varphi_0)_*(\hat{\sigma}\wedge\ldots\wedge\hat{\sigma})$. Therefore

$$j_0(\sigma) = \frac{1}{n!}\varphi_0\Big(\int_{S^n} i^*(\hat{\sigma}\wedge\ldots\wedge\hat{\sigma})\Big). \tag{4.2}$$

Example 4.4. Each trivial Lie algebroid $A = T\dot{\mathbb{R}}^{n+1} \times \mathfrak{g}$ is integrable: $A = A(P)$ for $P = \dot{\mathbb{R}}^{n+1} \times G$ where G is an arbitrary Lie group with the Lie algebra $\mathfrak{g}$. A connection $\sigma\colon T\dot{\mathbb{R}}^{n+1} \to A$ induces a connection $H \subset T(\dot{\mathbb{R}}^{n+1} \times G)$ in the principal bundle $\dot{\mathbb{R}}^{n+1} \times G$, and the flatness of σ means the integrability of H. Assume that a leaf L of the foliation H is the graph of some function $f\colon \dot{\mathbb{R}}^{n+1} \to G$. (If $n \geqslant 2$, then such a function always exists which follows from the simple connectedness of $\dot{\mathbb{R}}^{n+1}$ and the reduction theorem [12].) Therefore $\hat{\sigma}(v) = R^{-1}_{f(x)*}(f_*v)$, $R_{f(x)}$ which is the right translation on G by $f(x)$, and $f^*(\Delta_R) = \big\langle \varphi_0, n!^{-1}(\hat{\sigma}\wedge\ldots\wedge\hat{\sigma})\big\rangle$ for the right-invariant n-form $\Delta_R \in \Omega^n_R(G)$ giving φ_0 at the unity e of G.

(A) If G is compact, n-dimensional, oriented by Δ_R and the Lie algebra of G is spherical, then as a consequence of (4.1) and (4.2) we have

$$j_0(\sigma) = \int_{S^n} (f|_{S^n})^*\Delta_R = \deg(f|_{S^n}) \cdot \int_G \Delta_R. \tag{4.3}$$

As a corollary (taking any mapping $f\colon \dot{\mathbb{R}}^{n+1} \to S^n$ such that $f|_{S^n} = \mathrm{id}_{S^n}$), we obtain the existence of a local, flat singular connection having a nonzero index at the singularity.

Formula (4.3) yields that the set of real numbers which are the indexes at a given point of singular *local*, flat connections coming from functions is discrete (more exactly, is equal to the set of multiplicities of $\int_G \Delta_R$). Such a situation takes place, for example, for all flat connections in any so(3)-Lie algebroid over M^4 (since we can take $G = SO(3)$).

(B) If G is not compact, then $\Delta_R = d(\Theta)$ for some Θ and

$$j_0(\sigma) = \int_{S^n} (f|_{S^n})^* \Delta_R = \int_{S^n} d(f^*|_{S^n}\Theta) = 0.$$

Such a situation takes place, for example, in any sl$(2,\mathbb{R})$-Lie algebroid over M^4 (since we can take $G = SL(2,\mathbb{R})$). Clearly, this fact can be noticed immediately by using a base e, f, g of sl$(2,\mathbb{R})$ such that $[e,f] = g$, $[f,g] = 2f$, $[g,e] = 2e$,

For an $\mathbb{R}$-Lie algebroid there are singular flat connections which do not come from functions even locally, see the example below.

Example 4.5. In any $\mathbb{R}$-Lie algebroid over M^2 we can construct a local, flat and singular connection whose index is a preassigned real number. Indeed, since $\mathbb{R}$ is abelian, therefore the flatness of σ is equivalent to the closedness of the 1-form $\hat{\sigma}$ on M^2. In this case, the product $k \cdot \hat{\sigma}$, $k \in \mathbb{R}$, also gives a flat connection. Therefore, if $\sigma\colon T\dot{\mathbb{R}}^2 \to T\dot{\mathbb{R}}^2 \times \mathbb{R}$, $v \mapsto (v, \hat{\sigma}(v))$, is a flat connection with a nonzero index at 0, $j_0(\sigma) \neq 0$, then, for an arbitrary real number $k \in \mathbb{R}$, the mapping

$$\tau\colon T\dot{\mathbb{R}}^2 \longrightarrow T\dot{\mathbb{R}}^2 \times \mathbb{R}, \quad v \longmapsto \left(v, \frac{k}{j_0(\sigma)} \cdot \hat{\sigma}(v)\right),$$

is a flat connection with $j_0(\tau) = k$. Except for a discrete set of real numbers, this connection does not come from a function. More explicitly, considering $\varepsilon_0 = 1 \in \mathbb{R}$ and taking $\hat{\tau} = \frac{k}{2\pi}(x/(x^2+y^2)\,\mathrm{d}y - y/(x^2+y^2)\,\mathrm{d}x)$, $k \in \mathbb{R}$, we have $j_0(\tau) = \int_{S^1} \hat{\tau} = k$.

Example 4.6. In the Hopf S^1-bundle $P = (S^3 \to S^2)$, for two different points $p_1, p_2 \in S^2$ and for any real number $k \in \mathbb{R}$ there exists a global flat connection σ_k with two singularities at $\{p_1, p_2\}$, such that the index $j_{p_1}(\sigma_k)$ is equal to k. Indeed, since the Euler class of P is equal to the orientation class of S^2, any flat connection λ with a singularity at $\{p_1\}$ has the index at p_1 equal to 1 (assuming $\int_G \Delta_R = 1$). Take $p_2 \neq p_1$ and $M = S^2 \setminus \{p_2\}$. Since M is contractible, $P|_M$ is trivial, $P|_M \cong M \times S^1$. The connection λ determines a connection $\overline{\lambda}\colon T(M\backslash\{p_1\}) \to A((M\backslash\{p_1\}) \times S^1) = T(M\backslash\{p_1\}) \times \mathbb{R}$. Take $\hat{\sigma} = \mathrm{pr}_2 \circ \overline{\lambda}\colon T(M\backslash\{p_1\}) \to \mathbb{R}$. For an arbitrary real number $k \in \mathbb{R}$,

$$\bar{\sigma}_k\colon T(M\backslash\{p_1\}) \longrightarrow T(M\backslash\{p_1\}) \times \mathbb{R}, \quad v \longmapsto (v, k \cdot \hat{\sigma}(v)),$$

is a flat connection. $\bar{\sigma}_k$ determines a flat connection σ_k in P with a singularity at $\{p_1, p_2\}$, such that $j_{p_1}(\sigma_k) = k$.

Finally, we give some remarks concerning the existence of a connection with a finite number of singularities. We start with the case $\mathfrak{g} = \mathbb{R}$.

Proposition 4.7. *In each invariantly oriented* $\mathbb{R}$*-Lie algebroid over an arbitrary manifold* M *for which* $H^2(M) = 0$ *there exists a flat connection, in particular, when* M *is 2-dimensional non-compact.*

P r o o f. Consider any invariantly oriented $\mathbb{R}$-Lie algebroid $A = A_\Omega = TM \times \mathbb{R}$ over M, see Section 1.2.4. Each connection $\lambda\colon TM \to TM \times \mathbb{R}$ has the form $\lambda(v) = (\overline{\lambda}(v), v)$ for a 1-differential form $\overline{\lambda} \in \Omega^1(M)$. A simple calculation shows that λ is flat if and only if $d(\overline{\lambda}) = \Omega$. If $H^2(M) = 0$, such a 1-form exists. □

As a corollary we get

Corollary 4.8. *In each s-Lie algebroid of rank* 1 *over a compact* 2*-manifold* M *there exists a flat connection with a given finite non-empty set of isolated singularities.*

If a so(3)-Lie algebroid over a compact 4-manifold comes from a Spin(3)-principal bundle, then, of course, it possesses a flat connection with one singularity (since such a cross-section of the sphere bundle exists [8, Vol. I]). In the general case, the problem is open.

The problem for $\mathrm{sl}(2, \mathbb{R})$-Lie algebroids looks differently. Namely, by the main Theorem 3.3 and Example 4.4 (B) we have that $\chi_{(A,\varepsilon)} = 0$ for any invariantly oriented $\mathrm{sl}(2, \mathbb{R})$-Lie algebroid (A, ε) over a compact connected oriented manifold M^4 admitting a flat connection with a finite number of isolated singularities.

More precisely, we have the following proposition.

Proposition 4.9. *Each invariantly oriented* $\mathrm{sl}(2, \mathbb{R})$*-Lie algebroid* (A, ε) *over a compact connected oriented* 4*-manifold admitting a flat connection with a finite number of isolated singularities is flat, i.e. admits a global flat connection.*

P r o o f. Let σ be a flat connection with finite number of singularities. We can remove the singularities. It suffices to check this locally. If σ is a flat connection in $T\dot{\mathbb{R}}^4 \times \mathrm{sl}(2, \mathbb{R})$ then σ is given by a function $f\colon \dot{\mathbb{R}}^4 \to SL(2, \mathbb{R})$. Using the fact that the third group of homotopy of $SL(2, \mathbb{R})$ is zero, $\pi_3(SL(2, \mathbb{R})) = 0$, we can find $\overline{f}\colon \mathbb{R}^4 \to SL(2, \mathbb{R})$ such that $f(x) = \overline{f}(x)$ for $\|x\| \geqslant \varepsilon$ for a given small ε. This implies that we may remove the singularity at 0. □

Remark 4.10. We explain reasons for which the index theory of singular flat connections is limited to the spherical case. For this purpose, take a flat connection λ in a transitive Lie algebroid A, with a singularity at a point $a \in M$, i.e. a flat connection in $A|_{M\backslash\{a\}}$. Let $a \in U \subset M$ be a coordinate neighbourhood of a, $U \cong \mathbb{R}^{n+1}$. Clearly [29], the restriction $A|_U$ is a trivial Lie algebroid, therefore $A|_U$ is the Lie algebroid of a trivial principal bundle $A|_U = A(U \times G)$ for some Lie group G. Assume here that it is the case when we can choose a compact connected Lie group G, and that $\dim G = n$ (the "dimensional" restriction is analogous to that in Theorem 1.1). The connection λ restricted to $U\backslash\{a\}$ determines a flat connection H in the principal bundle $U \times G$, with singularity at the point a. If $n \geqslant 2$, $U\backslash\{a\}$ is simply connected and the holonomy of H is trivial, therefore the reduction theorem [12] yields that H is determined by a cross-section $f\colon U\backslash\{a\} \to U \times G$ (i.e. $H_{(x,e)} = \operatorname{Im} f_{*x}$). By (4.3), the index $j_a\lambda$ of a flat connection λ is defined in such a way that $j_a\lambda = \deg(\overline{f}|_{S^n}) \cdot k$ (for some fixed—for a given point a—real number k) where $\overline{f} =: \mathrm{pr}_2 \circ f$. Then, by degree theory [8, Vol. I, 6.5, Cor. I], we know that the relation $\deg(\overline{f}|_{S^n}) \neq 0$ implies the injectivity of $(\overline{f}|_{S^n})^{\#}\colon H(G) \to H(S^n)$, which gives the relations $H^p(G) = 0$ for $1 \leqslant p \leqslant n-1$ meaning the same for the Lie algebra $\mathrm{gl}(G)$ of G. This explains the limitation to the spherical case.

It seems that it is possible to consider arbitrary compact Lie groups keeping in mind what follows: the real cohomology of a compact connected Lie group is isomorphic to the cohomology algebra of the product of r spheres ($r = \operatorname{rank} G = \dim T$, T—the maximal torus of G) of odd dimension and the sum of this dimensions, $\sum_{\nu=1}^{r} g_\nu$, is equal to $n = \dim G$. Therefore we may consider the case of the base manifold M of dimension $n+1$ and a flat connection with singularities lying inside the set diffeomorphic to the product $N = S^{g_1} \times \ldots \times S^{g_r}$ of $g_1, \ldots, g_r$-spheres (contained in some coordinate neighbourhood).

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