$\mathfrak{g}$-LOCALLY CONFORMAL SYMPLECTIC STRUCTURES

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1. l.c.s structures from the point of view of Lie algebroids

We recall that l.c.s. structure on a manifold M is a pair (ω, Ω) of differentiable forms on M such that
(1) ω is a real closed 1-form on M,
(2) Ω is a real non-degenerated 2-form fulfilling the property

$$d\Omega = -\omega \wedge \Omega.$$

From the non-degeneration of Ω follows that M has even dimension.

To consider l.c.s. structures and next, their generalizations, $\mathfrak{g}$-l.c.s. structures, we use Lie algebroids with trivial adjoint Lie algebra bundle $\boldsymbol{g} = M \times \mathfrak{g}$.

From the general theorem concerning the form of any transitive Lie algebroids (Mackenzie [M], Kubarski [K1]) we have:

- *Each transitive Lie algebroid on M with a trivial adjoint bundle $\boldsymbol{g} \cong M \times \mathbb{R}$ is isomorphic to*
$$A = TM \times \mathbb{R}$$
*with $\gamma = \mathrm{pr}_1 : TM \times \mathbb{R} \to TM$ as the anchor and the bracket $[\![\cdot,\cdot]\!]$ in $\operatorname{Sec} A$ is defined via some **flat** covariant derivative ∇ in $M \times \mathbb{R}$ and a 2-form $\Omega \in \Omega^2(M)$ fulfilling the Bianchi identity $\nabla\Omega = 0$ in the following way*
$$[\![(X, f), (Y, g)]\!] = ([X, Y], \nabla_X g - \nabla_Y f - \Omega(X, Y)).$$

Each flat covariant derivative in $\boldsymbol{g} = M \times \mathbb{R}$ is of the form

$$\nabla_X f = \partial_X f + \omega(X) \cdot f$$

where ω is a closed 1-differentable form on M.

The condition $\nabla\Omega = 0$ is equivalent to $d\Omega = -\omega \wedge \Omega$. Hence a transitive Lie algebroid with trivial adjoint bundle $\boldsymbol{g} = M \times \mathbb{R}$ is determined by the following data:

(*) a closed 1-form ω and a 2-form Ω such that $d\Omega = -\omega \wedge \Omega$.

The Lie algebroid obtained in this way will be denoted by $(TM \times \mathbb{R}, \omega, \Omega)$. A connection $\lambda : TM \to TM \times \mathbb{R}$ in the Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ is of the form $\lambda(X) = (X, \eta(X))$ for a 1-form $\eta \in \Omega^1(M)$. The curvature form $\Omega^\lambda(X, Y) = [\![\lambda X, \lambda Y]\!] - \lambda[X, Y]$ of a connection λ is equal to

$$\Omega^\lambda = \nabla\eta - \Omega = d\eta + \omega \wedge \eta - \Omega. \tag{1.1}$$

According to (*) the pair (ω, Ω) determinig the above Lie algebroid is precisely a locally conformal symplectic structure (l.c.s. structure, for short) on the manifold provided that the 2-form Ω is non-degenerate. Therefore our transitive Lie algebroids $TM \times \mathbb{R}$ determined by (ω, Ω) are natural generalizations of the locally conformal symplectic structures. If the 1-form ω is exact the l.c.s. structure is called **globally** conformal symplectic structure. The property that an l.c.s. structure is global can be equivalently expressed in the language of Lie algebroids: (Kadobianski, Kubarski, Kushnirevitch, Wolak [KKKW])

- *Let (ω,Ω) be an l.c.s. structure. The following conditions are equivalent:*
  - *the l.c.s. structure (ω,Ω) is globally conformal symplectic structure,*
  - *the associated Lie algebroid $A=(TM\times\mathbb{R},\omega,\Omega)$ is invariantly oriented,*
  - $H_A^{m+1}(M)=\mathbb{R}$, $m=\dim M$,
  - *the cohomology algebra $H_A(M)$ of the Lie algebroid A satisfies the Poincaré duality.*

We recall that a transitive Lie algebroid $(A,[\![\cdot,\cdot]\!],\gamma)$ is called *inavriantly oriented* [K2] if there is specified a cross section ε of the bundle $\bigwedge^n \boldsymbol{g}$, $\boldsymbol{g}:=\ker\gamma$ and $n=rank\boldsymbol{g}$, which is invariant with respect to the adjoint representation of A on $\bigwedge^n\boldsymbol{g}$. The structure Lie algebras $\boldsymbol{g}_{|x}$ are then unimodular.

A cross-section ε of the bundle $\bigwedge^n\boldsymbol{g}$ is invariant if and only if, in any open subset $U\subset M$ on which ε is of the form $\varepsilon_{|U}=(h_1\wedge\ldots\wedge h_n)_{|U}$, $h_i\in\operatorname{Sec}\boldsymbol{g}$, we have, for all $\xi\in\operatorname{Sec}A$,

$$\sum_{i=1}^{n}(h_1\wedge\ldots\wedge[\![\xi,h_i]\!]\wedge\ldots\wedge h_n)_{|U}=0.$$

In the case $A=(TM\times\mathbb{R},\omega,\Omega)$ we have $n=1$ and $\boldsymbol{g}=M\times\mathbb{R}$ and a positive function $\varepsilon\in C^\infty(M)=\operatorname{Sec}(M\times\mathbb{R})$ is invariant if and only if ε is ∇-constant, $\nabla\varepsilon=0$. The condition $\nabla\varepsilon=0$ is equivalent to $\omega=d(-\ln(\varepsilon))$.

Two l.c.s. structures (ω,Ω) and (ω',Ω') on a manifold M are called *conformally equivalent* if

$$\Omega'=\frac{1}{a}\Omega,\ \ \omega'=\omega+\frac{da}{a},$$

for a nonwhere vanishing function a on M (non-singular for short).

If two l.c.s. structures (ω',Ω') and (ω,Ω) on a manifold M are conformally equivalent then the associated Lie algebroids $A'=(TM\times\mathbb{R},\omega',\Omega')$ and $(TM\times\mathbb{R},\omega,\Omega)$ are isomorphic via the isomorphism

$$\begin{gathered}H:(TM\times\mathbb{R},\omega',\Omega')\rightarrow(TM\times\mathbb{R},\omega,\Omega)\\ H(X,f)=(X,a\cdot f)\end{gathered}$$

where $a\in C^\infty(M)$ is a non-singular smooth function (i.e. $a(x)\neq0$ for all $x\in M$). The isomorphism $H:A'\rightarrow A$ of the above form will be called a *conformal isomorphism.*

We add that the general form of a homomorphism $H:TM\times\mathbb{R}\rightarrow TM\times\mathbb{R}$ of vector bundles commuting with anchors $\gamma=pr_1$ is as follows

(**) $$H(X,f)=H_{\eta,a}(X,f):=(X,\eta(X)+a\cdot f),$$

for $\eta\in\Omega^1(M)$ and $a\in C^\infty(M)$.

Proposition 1. *(A) The following conditions are equivalent:*

(1) *H is a homomorphism of Lie algebroids,*
(2) (a) $\nabla\eta=\Omega-a\cdot\Omega'$,
    (b) $\nabla_X(a\cdot f)=a\cdot\nabla'_X f$,
(3) (a) $d\eta+\omega\wedge\eta=\Omega-a\cdot\Omega'$,

(b) $a\cdot(\omega'-\omega)=da.$

The homomorphism H is an isomorphism of Lie algebroids if and only if a is non-singular. Conditions (1), (2), (3) are then equivalent to

(4) (a) $\Omega'=\frac{1}{a}\cdot(\Omega-d\eta-\omega\wedge\eta)$,

(b) $\omega'=\omega+\frac{da}{a}$.

(B) For arbitrary Lie algebroid $A'=(TM\times\mathbb{R},\omega',\Omega')$ and a data η, a, a – non-singular, the differential forms $\omega=\omega'-\frac{da}{a}$, $\Omega=a\cdot\Omega'+d\eta+\omega\wedge\eta$ fulfil the condition $d\Omega=-\omega\wedge\Omega$, i.e. the data (ω,Ω) determines a Lie algebroid $A=(TM\times\mathbb{R},\omega,\Omega)$ and $H_{\eta,a}:A'\to A$ given by (**) is an isomorphism of Lie algebroids

Clearly

$$H_{\eta,a}=H_{\eta,1}\circ H_{0,a},$$

see the diagram

$$\begin{array}{ccc} A'=(TM\times\mathbb{R},\omega',\Omega') & \xrightarrow{H_{\eta,a}} & (TM\times\mathbb{R},\omega,\Omega)=A \\ \searrow H_{0,a} & & H_{\eta,1}\nearrow \\ & (TM\times\mathbb{R},\omega,a\cdot\Omega') & \end{array}$$

it means that if A' is isomorphic to A then there exists a Lie algebroid $A''=(TM\times\mathbb{R},\omega,\Omega'')$, $\Omega''=a\cdot\Omega'$ conformally isomorphic to A, i.e. such that $[A]$, $[A'']\in \mathit{Opext}\,(TM,\nabla,M\times\mathbb{R})$ =the set of isomorphic classes of Lie algebroids having the same representation ∇ (a flat covariant derivative ∇).

Let (ω',Ω') and (ω,Ω) be l.c.s. structures. We observe that the isomorphism $H_{\eta,a}:A'\to A$ given by (**) is equivalent to conformal equivalence of the associated l.c.s. structures if and only if $\eta=0$.

How we can formulate the problem of existence of l.c.s. structures? We have the simple

Proposition 2. *Any Lie algebroid $A'=(TM\times\mathbb{R},\omega',\Omega')$ is isomorphic to $A=(TM\times\mathbb{R},\omega,\Omega)$ with Ω non-degenerate (i.e. (ω,Ω) is an l.c.s. structure) if and only if there exists in A' a connection for which the curvature tensor is non-degenerate.*

Proof. Let $H_{\eta,a}:A'\to A$ be an isomorphism of Lie algebroids

$$\begin{array}{ccccccccc} 0 & \to & M\times\mathbb{R} & \to & (TM\times\mathbb{R},\omega',\Omega') & \underset{\overleftarrow{\lambda'}}{\to} & TM & \to & 0 \\ & & \downarrow H^{+}_{\eta,a} & & \downarrow H_{\eta,a} & & \| & & \\ 0 & \to & M\times\mathbb{R} & \to & (TM\times\mathbb{R},\omega,\Omega) & \underset{\overleftarrow{\lambda}}{\to} & TM & \to & 0 \end{array}$$

$H^{+}_{\eta,a}(f)=a\cdot f.$. For arbitrary connections λ' and λ in A' and A, respectivelly such that $H_{\eta,a}\circ\lambda'=\lambda$ we have the following equality for curvature tensors

$$\Omega^{\lambda}=H^{+}_{\eta,a}\circ\Omega^{\lambda'}.$$

Therefore, if Ω is nondegenerate and λ' is a connection such that $H_{\eta,a}\circ\lambda'=\lambda$ where $\lambda(v)=(v,0)$, then $\Omega^{\lambda}=-\Omega$ and, clearly, $\Omega^{\lambda'}$ is non-degenerate.

Conversely, if $\lambda'(X)=(X,\eta(X))$ is any connection in A' such that $\Omega^{\lambda'}$ is non-degenerate, then $H_{-\eta,1}$ is an isomorphism of A' on $A:=\left(TM\times\mathbb{R},\omega',-\Omega^{\lambda'}\right)$ (see (1.1)) and $\left(\omega',-\Omega^{\lambda'}\right)$ is an l.c.s. structure. □

So, the problem of existing of l.c.s. structures can be precisely formulated as follows:

Existing problem: We introduce into the class of pairs (ω,Ω) fulfilling (*), i.e. $d\Omega=-\omega\wedge\Omega$, the equivalence relation

r) $(\omega',\Omega')\approx(\omega,\Omega)\equiv$ the Lie algebroids $A'=(TM\times\mathbb{R},\omega',\Omega')$ and $A=(TM\times\mathbb{R},\omega,\Omega)$ are isomorphic, i.e. there exists $\eta\in\Omega^1(M)$ and $a\in C^\infty(M)$, $a(x)\neq 0$ for all $x\in M$, such that (4a), (4b) hold: (4a) $\Omega'=\frac{1}{a}(\Omega-d\eta-\omega\wedge\eta)$, (4b) $\omega'=\omega+\frac{da}{a}$.

We can ask: does there in every (in given) equivalence class $[(\omega',\Omega')]$ exist (ω,Ω) being a l.c.s. structure; equivalently, does there in the Lie algebroid

$$A'=(TM\times\mathbb{R},\omega',\Omega')$$

exist a connection with non-degenerate curvature tensor, i.e. equivalently, does exist a 1-form $\eta\in\Omega^1(M)$ such that $d\eta+\omega\wedge\eta-\Omega$ is a non-degenerate.

We add that for a fixed closed form ω, i.e. a flat covariant derivative $\nabla_X f=\partial_X f+\omega(X)\cdot f$ in the trivial bundle $M\times\mathbb{R}$, the classification of Lie algebroids of the form $(TM\times\mathbb{R},\omega,\cdot)$ up to isomorphism is as follows: for the class of isomorphic Lie algebroids $Opext(TM,\nabla,M\times\mathbb{R})$ we have [Mackenzie give the full answer for the classification]

$$Opext(TM,\nabla,M\times\mathbb{R})=H^2_\nabla(M;\mathbb{R}).$$

To sum up we see that important l.c.s's notions can be translated into the Lie algebroid's language. We have the following table:

l.c.s.	**Lie algebroid**
$(M,\omega,\Omega)\equiv$ ω is closed, $d\Omega=-\omega\wedge\Omega$.	$A=TM\times\mathbb{R}$ with anchor $\gamma=pr_1:TM\times\mathbb{R}\to TM$, bracket $[\![(X,f),(Y,g)]\!]=$ $([X,Y],\nabla_X g-\nabla_Y f-\Omega(X,Y))$ where $\nabla_X g=\partial_X g+\omega(X)\cdot g$ ∇ is flat and $\nabla\Omega=0$.
globally c.s. $\equiv$ ω is exact	A is invariantly oriented
two l.c.s. structures (ω',Ω') and (ω,Ω) on M are conformally equivalent $\equiv$ $\omega'=\omega+\frac{da}{a}$, $\Omega'=\frac{1}{a}\Omega$	the corresponding Lie algebroids are isomorphic via $H_{0,a}:TM\times\mathbb{R}\to TM\times\mathbb{R}$, $H(X,f)=(X,a\cdot f)$ $a\in C^\infty(M)$, $a(x)\neq 0$ for all x.

2. GENERALIZATIONS: $\mathfrak{g}$-l.c.s. structures and Lie algebroids

We generalize l.c.s. structures to $\mathfrak{g}$-l.c.s. structures in which we can consider an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ instead of the commutative Lie algebra $\mathbb{R}$ [KK]. From the general theorem on the form of Lie algebroids, mentioned above, we have [M], [K1];

Theorem 1. *Each transitive Lie algebroid with a trivial adjoint bundle of Lie algebras $M\times\mathfrak{g}$ is isomorphic to $TM\times\mathfrak{g}$ with $\gamma=\mathrm{pr}_1 : TM\times\mathfrak{g}\to TM$ as the anchor and the bracket*

$$[\![(X,\sigma),(Y,\eta)]\!] = \left([X,Y],\nabla_X\eta-\nabla_Y\sigma+[\sigma,\eta]-\Omega(X,Y)\right)$$

in $\mathrm{Sec}\,A$ *is defined via the following data (∇,Ω): a covariant derivative ∇ in the trivial vector bundle $M\times\mathfrak{g}$ and a 2-form $\Omega\in\Omega^2(M;\mathfrak{g})$ fulfilling the conditions:*

(1) $R^{\nabla}_{X,Y}\sigma=-[\Omega(X,Y),\sigma]$, *$R^{\nabla}$ being the curvature tensor of ∇,*
(2) $\nabla_X[\sigma,\eta]=[\nabla_X\sigma,\eta]+[\sigma,\nabla_X\eta]$, $\sigma,\eta\in C^\infty(M;\mathfrak{g})$,
(3) $\nabla\Omega=0$.

The Lie algebroid obtained in the above way via the data (∇,Ω) fulfilling (1)÷(3) above will be denoted here by

$$(TM\times\mathfrak{g},\nabla,\Omega).$$

The form $-\Omega$ is the curvature form of the connection $\lambda : TM\to TM\times\mathfrak{g}$, $\lambda(v)=(v,0)$, in this Lie algebroid $(TM\times\mathfrak{g},\nabla,\Omega)$.

$$0\to M\times\mathfrak{g}\to TM\times\mathfrak{g}\underset{\overleftarrow{\lambda}}{\to} TM\to 0.$$

More generally, the curvature form of an arbitrary connection $\lambda(X)=(X,\eta(X))$, $\eta\in\Omega^1(M;\mathfrak{g})$, is given by

$$\Omega^\lambda(X,Y)=(\nabla\eta)(X,Y)+[\eta X,\eta Y]-\Omega(X,Y).$$

We write the covariant derivative ∇ in the trivial bundle $M\times\mathfrak{g}$ in the form

$$\nabla_X\sigma=\partial_X\sigma+\omega(X)(\sigma)$$

for a 1-form $\omega\in\Omega^1(M;\mathrm{End}\,\mathfrak{g})$. The curvature tensor R^{∇} of ∇ is equal to

$$R^{\nabla}_{X,Y}\sigma=d\omega(X,Y)(\sigma)+[\omega(X),\omega(Y)](\sigma).$$

Proposition 3. *The conditions (1)-(3) characterizing the data (∇,Ω) determining the Lie algebroid $(TM\times\mathfrak{g},\nabla,\Omega)$ can be express as follows*

- *the condition (1) is equivalent to*

$$d\omega(X,Y)(\sigma)+[\omega(X),\omega(Y)](\sigma)=-[\Omega(X,Y),\sigma],$$

- *the condition (2) is equivalent to $\omega_x\in\mathrm{Der}(\mathfrak{g})$, i.e. ω_x is a differentiation of the Lie algebra $\mathfrak{g}$,*
- *the condition (3) is equivalent to*

$$d\Omega=-\omega\wedge\Omega$$

(the values of forms ω and Ω are multiplied with respect to the 2-linear homomorphism $\mathrm{End}\,\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, $(a,\sigma)\mapsto a\circ\sigma$, *where* $(a\circ\sigma)_x=a_x(\sigma_x)$.

Definition 1. *The pair* (∇, Ω) *determinig the above Lie algebroid*

$$(TM \times \mathfrak{g}, \nabla, \Omega)$$

will be called $\mathfrak{g}$***-locally conformal symplectic structure*** ($\mathfrak{g}$***-l.c.s. structure****, for short) on our manifold provided that the 2-form* Ω *is non-degenerate in the following sense: for each point* $x \in M$ *the mapping* $T_xM \to L(T_xM, \mathfrak{g})$, $v \mapsto \Omega_x(v, \cdot)$, *is a monomorphism.*

We notice that if $\dim \mathfrak{g} \geq 2$ there is no dimensional obstructions to the existing of an non-degenerate tensors:

Lemma 1. *For arbitrary vector spaces* V *and* $\mathfrak{g}$ *such that* $\dim \mathfrak{g} \geq 2$ *there exists a 2-linear skew-symmetric non-degenerate tensor* $\Omega \in \Omega^2(V; \mathfrak{g})$.

Proof: Let $(e_1, ..., e_n)$ be a basis of $\mathfrak{g}$. If $\dim V$ is even, then there exists a real 2-linear skew-symmetric non-degenerate tensor, say Ω_0. The form $\Omega := \Omega_0 \cdot e_1 \in \Omega^2(V; \mathfrak{g})$ is non-degenerate. If $\dim V = 2k + 1$ and $(v_1, ..., v_{2k+1})$ is a basis of V and $u^1, ..., u^{2k+1}$ is a dual basis, then put

$$\Omega_0 = u^1 \wedge u^2 + ... + u^{2k-1} \wedge u^{2k},$$
$$\Omega_1 = u^{2k} \wedge u^{2k+1}.$$

The form $\Omega := \Omega_0 \cdot e_1 + \Omega_1 \cdot e_2$ is non-degenerate. □

Definition 2. *A* $\mathfrak{g}$*-l.c.s. structure is called* ***globally*** *conformal symplectic structure if the associated Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *is invariantly oriented.*

Theorem 2. *If the Lie algebra* $\mathfrak{g}$ *is unimodular with no centre, then each* $\mathfrak{g}$*-l.c.s. structure is globally c.s. structure.*

Proof. According to the classifying theorem [Mackenzie] if $\mathfrak{g}$ is with no centre then for the trivial LAB $\boldsymbol{g} = M \times \mathfrak{g}$ there exists exactly one, up to isomorphism, a transitive Lie algebroid A with the adjoint LAB $\boldsymbol{g} = M \times \mathfrak{g}$. Therefore, A must be isomorphic to the trivial Lie algebroid $A = TM \times \mathfrak{g}$ with the data $(\partial, 0)$. If additionally $\mathfrak{g}$ is unimodular then this Lie algebroid is invariantly oriented: $\varepsilon(x) \equiv \varepsilon_o \in \bigwedge^n \mathfrak{g}$ is an invariant cros-section. □

Theorem 3. *Write* $\nabla_X \sigma = \partial_X \sigma + \omega(X)(\sigma)$ *for* $\omega \in \Omega^1(M; \operatorname{End} \mathfrak{g})$. *The Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *is invariantly oriented (i.e.* (∇, Ω) *is a globally conformal symplectic structure) if and only if* $\mathfrak{g}$ *is unimodular and* $\operatorname{tr} \omega$ *is an exact form. Let* $e_1, ..., e_n$ *be a basis of* $\mathfrak{g}$. *For a non-singular function* $f \in C^\infty(M)$ *the element* $\varepsilon = f \cdot e_1 \wedge ... \wedge e_n$ *is an invariant cross-section if and only if*

$$\operatorname{tr} \omega = d(-\ln |f|).$$

Let $(e_1, ..., e_n)$ be a basis of $\mathfrak{g}$ with the structure constants c_{ij}^k. The covariant derivative ∇ determines a matrix of 1-forms $\omega_i^j \in \Omega^1(M)$ by

$$\nabla_X e_i = \sum_j \omega_i^j(X) e_j.$$

Analogously we have a collection of 2-forms Ω^j by

$$\Omega_{X,Y} = \Omega_{X,Y}^j e_j.$$

We interpret the data (1)÷(3) concerning (∇, Ω) in the tems of the matrix ω_i^j and the collection Ω^j and the structure constants c_{ij}^k.

Proposition 4. *The conditions (1)-(3) characterizing the data* (∇, Ω) *determining the Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *can be expressed as follows.*

- *The condition (1) is equivalent to*

$$-\sum_j \Omega_{X,Y}^j \cdot c_{j,i}^r = d\omega_i^r (X,Y) - \left(\sum_j \omega_i^j (X) \omega_j^r (Y) - \omega_i^j (Y) \omega_j^r (X) \right),$$

- *the condition (2) is equivalent to*

$$\sum_k c_{ij}^k \cdot \omega_k^r (X) = \sum_k \left(\omega_i^k (X) c_{kj}^r - \omega_j^k (X) c_{ki}^r \right),$$

- *the condition (3) is equivalent to*

$$d\Omega^j = -\sum_i \Omega^i \wedge \omega_i^j.$$

Two $\mathfrak{g}$-l.c.s. structures $(\nabla', \Omega'), (\nabla, \Omega)$ on a manifold M will be called $\mathfrak{g}$-*conformally equivalent* if the associated Lie algebroids are isomorphic via an isomorphism of the special form (called $\mathfrak{g}$-*conformal*) $H(X,\sigma) = (X, a(\sigma))$ for some mapping $a : M \to Aut(\mathfrak{g})$. Then the equivalent relations between the data (∇, Ω) and (∇', Ω') are as follows:

- $\Omega' = a^{-1} \circ \Omega$,
- $a \circ \nabla'_X (\sigma) = \nabla_X (a \circ \sigma)$.

We use the notation $a \circ \sigma$ for the cross-section defined by $(a \circ \sigma)_x = a_x (\sigma_x)$.

Writing ∇' and ∇ with using 1-forms $\omega', \omega \in \Omega^1 (M; \operatorname{End} \mathfrak{g})$ (as above) the last condition can be equivalently written in the form

$$\omega(X) \circ a = -\partial_X a + a \circ \omega'(X).$$

In the terms of the matrices $\omega_i^{j\prime}$ and ω_i^j this condition is equivalent to

$$\omega_i^{\prime j} (X) \cdot a_j^k - a_i^j \cdot \omega_j^k (X) = \partial_X \left(a_i^k \right).$$

The general form of a homomorphism $H : TM \times \mathfrak{g} \to TM \times \mathfrak{g}$ commuting with anchors pr_1 is as follows

$$H(X,\sigma) = H_{\eta,a}(X,\sigma) = (X, \eta(X) + a \circ \sigma) \tag{2.1}$$

for $\eta \in \Omega^1 (M; \mathfrak{g})$, $a \in C^\infty (M, \operatorname{End} \mathfrak{g})$. Consider two Lie algebroids

$$A' = (TM \times \mathfrak{g}, \nabla', \Omega') \quad \text{and } A = (TM \times \mathfrak{g}, \nabla, \Omega)$$

Proposition 5. *The following conditions are equivalent.*

(1) H *is a homomorphism of of Lie algebroids* $H : A' \to A$,
(2) (a) a_x *is a homomorphism of Lie algebras,*
  (b) $(\nabla \eta)(X,Y) + [\eta(X), \eta(Y)] = (\Omega - a\Omega')(X,Y)$,
  (c) $a \circ \nabla'_X \sigma = \nabla_X (a \circ \sigma) + [\eta(X), a \circ \sigma]$,

(3) *In the basis* $e_1, ..., e_n$ *and for the matrix* a_i^j *defined by* $a(e_i) = \sum_j a_i^j (e_j)$
  (a) a_x *is a homomorphism of Lie algebras,*
  (b) $d\eta^k(X,Y) - (\eta^i \wedge \omega_i^k)(X,Y) + \eta^i(X) \cdot \eta^j(Y) \cdot c_{ij}^k =$
  $= (\Omega^k - \Omega'^i \cdot a_i^k)(X,Y),$
  (c) $\sum_j \omega_i'^j(X) \cdot a_j^k = \sum_j a_i^j \cdot \omega_j^k(X) + \partial_X a_i^k + \eta^j(X) \cdot a_i^s \cdot c_{js}^k.$

The homomorphism $H_{\eta,a}$ *is an isomorphism of Lie algebroids if and only if* a_x *is an isomorphism of Lie algebras.*

If (∇', Ω') and (∇, Ω) are $\mathfrak{g}$-l.c.s structures and A' and A are corresponding Lie algebroids, then the isomorphism $H_{\eta,a}$ given by (2.1) is equivalent to conformal equivalence of the associated $\mathfrak{g}$-l.c.s structures (∇', Ω') and (∇, Ω) if and only if $\eta = 0$.

Analogously, we we can put the problem of existence of l.c.s. structures. We have firstly the simple

Proposition 6. *Any Lie algebroid* $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ *is isomorphic to* $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ *with* Ω *non-degenerate (i.e.* (∇, Ω) *is a* $\mathfrak{g}$*-l.c.s. structure) if and only if there exists in* A' *a connection for which the curvature tensor is non-degenerate.*

Existing problem: We introduce into the class of pairs (∇, Ω) fulfilling (1)-(3) from Theorem 1, the equivalence relation

r$\mathfrak{g}$) $(\nabla', \Omega') \approx (\nabla, \Omega) \equiv$
$\equiv$ the Lie algebroids

$$A' = (TM \times \mathfrak{g}, \nabla', \Omega') \quad \text{and} \quad A = (TM \times \mathfrak{g}, \nabla, \Omega)$$

are isomorphic,

i.e. there exist $\eta \in \Omega^1(M; \mathfrak{g})$, $a \in C^\infty(M, \operatorname{Aut} \mathfrak{g})$ such that (2b-c) from Prop. 5 holds:

$$(\nabla \eta)(X,Y) + [\eta(X), \eta(Y)] = (\Omega - a\Omega')(X,Y)$$

and

$$a \circ \nabla'_X \sigma = \nabla_X (a \circ \sigma) + [\eta(X), a \circ \sigma].$$

We can ask does there in every (in given) equivalence class $[(\nabla', \Omega')]$ exist (∇, Ω) being a $\mathfrak{g}$-l.c.s. structure; equivalently, does there in the Lie algebroid

$$A' = (TM \times \mathfrak{g}, \nabla', \Omega')$$

exist a connection with non-degenerate curvature tensor, i.e. equivalently, does there exists a 1-form $\eta \in \Omega^1(M; \mathfrak{g})$ such that the 2-form $(\nabla \eta)(X,Y) + [\eta X, \eta Y] - \Omega(X,Y)$ is a non-degenerate.

It would be interesting to investigate the group of all compactly supported diffeomorphisms of M that preserve the $\mathfrak{g}$-l.c.s. structure up to $\mathfrak{g}$-conformal equivalence (analogously as was given for usual l.c.s. structures by Haller and Rybicki [HR]).

We add that two extreme cases: (1) $\mathfrak{g}$ commutative (for example $\mathfrak{g} = \mathbb{R}$) and (2) $\mathfrak{g}$ semisimple, are quite different. In the second case all Lie algebroids of the form $(TM \times \mathfrak{g}, \nabla, \Omega)$ (i.e. with the trivial adjoint Lie algebra $M \times \mathfrak{g}$) are isomorphic, clearly to the trivial one $TM \times \mathfrak{g}$ with the structure given by the data $(\partial, 0)$. We add that not each isomorphism is $\mathfrak{g}$-conformal. This Lie algebroid is invariantly oriented.

References

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