New question in the flat secondary characteristic classes, a Lie algebroid approach

by

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The plane of the talk

1. The Kamber-Tondeur secondary (i.e. exotic) flat characteristic homomorphism

$$h_{P,P',\omega}: H^{\bullet}(\mathfrak{g}, H) \longrightarrow H^{\bullet}_{dR}(M).$$

The question: Is $h_{P,P',\omega}$ a nontrivial homomorphism?

2. The Koszul homomorphism

$$k^{\#}: H^{\bullet}(\mathfrak{g}/\mathfrak{h}) \rightarrow H^{\bullet}(\mathfrak{g})$$

for a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g} \subset \mathfrak{h}$. The question is: Is $k^{\#}$ a monomorphism?

3. A role of $k^{\#}$ for $h_{P,P',\omega}$ (a suitable language – a Lie algebroid language).

4. A new exotic characteristic homomorphism

$$h_{A,B} : Domain \longrightarrow H\left(A\right)$$

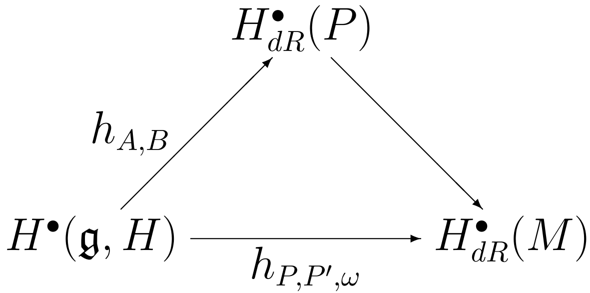
for a pair of Lie algebroids (A, B) on a manifold M where $B \subset A$.

(a) If $M = \{\bullet\}$, i.e. A, B are Lie algebras, then $h_{A,B} = \pm\, k^{\#}$.

(b) If $A = A\left(P\right)$, $B = A(P')$, then

$$h_{A,B} : H^{\bullet}\left(\mathfrak{g}, H\right) \longrightarrow H^{\bullet}\left(P\right)$$

is universal in the following sense: Let ω be a flat connection in P. Then there exists the following commutative diagram



i.e. $h_{A,B}$ factorizes $h_{P,P',\omega}$ for all flat connections.

5. We obtain some obstruction to the existence of an H-reduction in a pfb P ($P \subset H$).

Two important homomorphisms

Kamber-Tondeur characteristic homomorphism for flat principal bundles with reductions

The characteristic homomorphism (Kamber-Tondeur 1973-75)

$$\Delta_{(P,P',\omega)\#} = \Delta_{\#} : H^{\bullet}(\mathfrak{g}, H) \longrightarrow H^{\bullet}_{dR}(M) \tag{1}$$

for a G-principal fibre bundle P, a flat connection ω in P and an H-reduction $P' \subset P$ ($H \subset G$ is a closed Lie subgroup of G), is one of the most important notion in differential geometry of principal bundles. The cohomology classes from the image of the homomorphism $\Delta_{(P,P',\omega)\#}$ are called the *secondary* (also *exotic*) *flat characteristic classes* of (P, P', ω). The homomorphism $\Delta_{(P,P',\omega)\#}$ measures the independence of ω and P' (if ω is a connection in P' then $\Delta^{+}_{\#} = 0$).

F. Kamber, Ph. Tondeur, *Algèbres de Weil semisimpliciales*, C.R. Ac. Sc. Paris, t. **276** (1973), 1177–1179;

F. Kamber, Ph. Tondeur, *Homomorphisme caractéristique d'un fibré principal feuilleté*, ibid. t. **276** (1973), 1407–1410;

F. Kamber, Ph. Tondeur, *Classes caractéristiques dérivées d'un fibré principal feuilleté*, ibid. t. **276** (1973), 1449–1452.

F. Kamber, Ph. Tondeur, *Characteristic invariants of foliated bundles*, Manuscripta Mathematica, **11** (1974), 51–89.

F. Kamber, Ph. Tondeur, *Foliated Bundles and Characteristic Classes*, Lectures Notes in Math., **493**, Springer-Verlag, 1975.

Problem 1 *The fundamental question is: Is the homomorphism* $\Delta_\#$ *nontrivial,* $\Delta_\#^+ \neq 0$*, for a given triple* (P, P', ω) *?*

We recall that $H^\bullet(\mathfrak{g}, H)$, called the *relative Lie algebra cohomology of* $(\mathfrak{g}, H)$, is the cohomology space of the complex $\left(\bigwedge (\mathfrak{g}/\mathfrak{h})^{*H}, d^H\right)$ where $\bigwedge (\mathfrak{g}/\mathfrak{h})^{*H}$ is the space of invariant elements with respect to the adjoint representation of the Lie group H and the differential d^H is defined by the formula

$$\begin{aligned} &\left\langle d^H(\psi), [w_1] \wedge \ldots \wedge [w_k] \right\rangle \\ &= \sum_{i<j} (-1)^{i+j} \left\langle \psi, [[w_i, w_j]] \wedge [w_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [w_k] \right\rangle \end{aligned} \tag{2}$$

for $\psi \in \bigwedge^k (\mathfrak{g}/\mathfrak{h})^{*H}$ and $w_i \in \mathfrak{g}$. We recall, that first it was introduced by C. Chevalley and S. Eilenberg in 1948 in

C. Chevalley, S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. of Amer. Math. Soc., **63** (1948), 85–124.

The characteristic homomorphism (1) is constructed as follows. Let

$$\breve{\omega} : TP \to \mathfrak{g}$$

denote the connection form of ω. There exists a homomorphism of G-DG-algebras

$$\breve{\omega}_\wedge : \bigwedge \mathfrak{g}^* \to \Omega\left(P\right)$$

(thanks the flatness of ω) induced by the algebraic connection

$$\breve{\omega} : \mathfrak{g}^* \to \Omega\left(P\right),\ \alpha \mapsto \alpha\breve{\omega} = \langle \alpha, \breve{\omega} \rangle.$$

The homomorphism $\breve{\omega}_\wedge$ can be restricted to H-basic elements

$$\breve{\omega}_H : \left(\bigwedge \mathfrak{g}^*\right)_H \to \Omega\left(P\right)_H,$$

and according to the isomorphisms

$$\left(\bigwedge \mathfrak{g}^*\right)_H \cong \bigwedge \left(\mathfrak{g}/\mathfrak{h}\right)^{*H} \quad \text{and} \quad \Omega\left(P\right)_H \cong \Omega\left(P/H\right)$$

gives a DG-homomorphism of algebras

$$\breve{\omega}_H : \bigwedge \left(\mathfrak{g}/\mathfrak{h}\right)^{*H} \to \Omega\left(P/H\right).$$

Composing it with $s^* : \Omega(P/H) \to \Omega(M)$ where $s : M \to P/H$ is the cross-section determined by the H-reduction P', we obtain a homomorphism of DG-algebras

$$\Delta_{P,P',\omega} : \bigwedge (\mathfrak{g}/\mathfrak{h})^{*H} \xrightarrow{\breve{\omega}_H} \Omega(P/H) \xrightarrow{s^*} \Omega(M).$$

Passing to cohomology we obtain characteristic homomorphism (1).

Theorem 2 *If H-reductions P'_1 and P'_2 are H-homotopic, then*

$$\Delta_{(P,P'_1,\omega)\#} = \Delta_{(P,P'_2,\omega)\#}.$$

Therefore, the nontriviality of $\Delta_{(P,P',\omega)\#}$ implies that there is no homotopic changing of P' that TP' contains ω.

Theorem 3 *If $K \subset H \subset G$ where K is a maximal compact subgroup and H is closed, then two H-reductions are homotopic, so (1) is independent on the choose of the H-reduction P'.*

Theorem 4 *The homomorphism $\Delta_{(P,P',\omega)\#}$ on the level of forms is given by the following formula*

$$\left(\Delta_{P,P',\omega}(\psi)\right)_x (w_1 \wedge \ldots \wedge w_k) = \langle \psi, [\breve{\omega}_z(\tilde{w}_1)] \wedge \ldots \wedge [\breve{\omega}_z(\tilde{w}_k)] \rangle \qquad (3)$$

where $z \in P'_{|x}$, $w_i \in T_xM$, $\tilde{w}_i \in T_zP'$, $\pi'_* \tilde{w}_i = w_i$ ($\breve{\omega} : TP \to \mathfrak{g}$ denotes the connection form of ω).

Back

Close

The key observation is that we can eleminate the connection form $\breve{\omega}$ from this formula and use only the horizontal lifting ω^h of vectors via ω and an arbitrary connection λ in P'. Namely, we have

Theorem 5 *Consider an auxiliary connection λ in P' and its extension to P. Let $\breve{\lambda} : TP \to \mathfrak{g}$ be its connection form. Then we have*

$$(\Delta\psi)_x (w_1 \wedge \ldots \wedge w_k) = \langle \psi, [-\breve{\lambda}_z \omega^h_z (w_1)] \wedge \ldots \wedge [-\breve{\lambda}_z \omega^h_z (w_k)] \rangle. \quad (4)$$

The Koszul homomorphism

Consider a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h} \subset \mathfrak{g}$ and the inclusion

$$k : \left(\bigwedge \mathfrak{g}^*\right)_{i_h=0,\theta_h=0} \hookrightarrow \bigwedge \mathfrak{g}^*$$

($(\bigwedge \mathfrak{g}^*)_{i_h=0,\theta_h=0}$ denotes the space of basic elements, i.e. $\mathfrak{h}$-horizontal and $\mathfrak{h}$-invariant). We notice that $(\bigwedge \mathfrak{g}^*)_{i_h=0,\theta_h=0} = \bigwedge (\mathfrak{g}/\mathfrak{h})^{*H}$ if H is connected). The homomorphism k commutes with differentials giving a homomorphism on cohomology

$$k^{\#} : H^{\bullet}(\mathfrak{g}/\mathfrak{h}) \rightarrow H^{\bullet}(\mathfrak{g}) .$$

This homomorphism was considered in the following work by Koszul:

J-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bulletin de la Société Mathématique de France, **78** (1950), 65–127.

We recall that $\mathfrak{g}$ is called reductive if $\mathfrak{g} = Z_{\mathfrak{g}} \oplus \mathfrak{g}'$ for some semisimple Lie algebra $\mathfrak{g}'$ ($Z_{\mathfrak{g}}$ is the center of $\mathfrak{g}$). Let $ad_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathrm{End}\,(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$, $ad_{\mathfrak{g}}\,(x)\,(y) = [x, y]$ and let $ad_{\mathfrak{g},\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g} \longrightarrow \mathrm{End}\,(\mathfrak{g})$ be the restriction of $ad_{\mathfrak{g}}$, i.e. $ad_{\mathfrak{g},\mathfrak{h}}$ is the adjoint representation of $\mathfrak{h}$ in $\mathfrak{g}$.

We recall that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that the representation $ad_{\mathfrak{g},\mathfrak{h}}$ is semisimple is called **reductive** in $\mathfrak{g}$ (clearly, if $\mathfrak{h}$ is reductive in $\mathfrak{g}$ then $\mathfrak{h}$ is also reductive).

The key question for $k^{\#}$ is: When it is a monomorphism?

See

[GHV] Greub & Halperin & Vanstone, *Connections, Curvature, and Cohomology,* Academic Press, New York, San Francisco, London, 1976.

Theorem 6 (GHV based on ideas given by Koszul) *If* $\mathfrak{h}$ *is reductive in* $\mathfrak{g}$, *then* $k^{\#}$ *is a monomorphism if and only if the homomorphism* $H^{\bullet}(\mathfrak{g}) \to H^{\bullet}(\mathfrak{h})$ *induced by the inclusion* $\mathfrak{h} \hookrightarrow \mathfrak{g}$ *is surjective (in this case* $\mathfrak{h}$ *is called noncohomologous* 0 *in* $\mathfrak{g}$*).*

The tables I-III in the 3rd volume of this book contain many examples of such pairs.

Theorem 7 *If* $(\mathfrak{g}, \mathfrak{h})$ *is a reductive pair (i.e.* $\mathfrak{g}$ *is reductive and* $\mathfrak{h}$ *is reductive in* $\mathfrak{g}$*) then* $k^{\#}$ *is a monomorphism if and only if* $H^{\bullet}(\mathfrak{g}/\mathfrak{h})$ *is generated (as an algebra) by* 1 *and elements of odd degree.*

Example 8 (see the book by Greub&Halperin&Vanstone) *The following are examples of such pairs:*

$(End\mathbb{R}^n, \mathfrak{so}(n))$ *for* n *odd,*

$(\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(k, \mathbb{C}))$ *for* $k < n$,

$(\mathfrak{so}(2n+1), \mathfrak{so}(2k+1))$, $k < n$,

$(\mathfrak{so}(2n), \mathfrak{so}(2k+1))$, $k < n$,

etc.

The Koszul homomorphism $k^{\#}$ has a great meaning in the first classes, i.e. in the Chern-Weil homomorphism h, of the principal bundles $\pi : G \to G/H$ and in the calculation of $H(G/H)$ (G and H are Lie groups having $\mathfrak{g}$ and $\mathfrak{h}$ as Lie algebras).

For example, we have

Theorem 9 (GHV, Volume III, p . 466) *Let G be a compact connected Lie group, $H \subset G$ a compact connected Lie subgroup of G. Then the conditions are equivalent:*

(1) $\mathfrak{h}$ *is noncohomologous* 0 *in* $\mathfrak{g}$,

(2) $\pi^{\#} : H(G/H) \to H(G)$ *is a monomorphism,*

(3) $h^{+} = 0$,

(4) $H(G/H)$ *is generated by* 1 *and elements of odd degree.*

The main aim of my talk is to show that the Koszul homomorphism possesses also a great meaning for the secondary (i.e. exotic) flat characteristic homomorphisms.

We will also construct a new exotic characteristic homomorphism having close connection with the standard exotic flat homomorphism and with the Koszul homomorphism, and for which the key question is on the monomorphicity, not on the nontriviality.

Lie algebroids are suitable objects for which we obtain the comparision of $\Delta_{(P,P',\omega)\#}$ and $k^{\#}$.

Definition 10 *A **Lie algebroid** on a smooth manifold M is a triple $(L, [\![\cdot,\cdot]\!], \#_L)$ where L is a vector bundle on M, $(\operatorname{Sec} L, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, $\#_L : L \to TM$ is a linear homomorphism of vector bundles and the following Leibniz condition is satisfied*

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_L(\xi)(f)\cdot\eta, \quad f\in C^\infty(M),\ \xi,\eta\in\operatorname{Sec} L.$$

- The anchor $\#_L$ is bracket-preserving, $\#_L\circ[\![\xi,\eta]\!] = [\#_L\circ\xi, \#_L\circ\eta]$ for all $\xi,\eta\in\operatorname{Sec} L$.
- The kernel $\mathbf{g}_x$ of $(\#_L)_x : L_x \to T_xM$ is a Lie algebra, called an *isotropy Lie algebra*.

If the anchor $\#_L$ is epimorphism, then L is called *transitive*.

- For transitive case the exact sequence

$$0 \to \boldsymbol{g} \to L \to TM \to 0$$

  is called the *Atiyah sequence* of L.

- **Simply examples:** TM (tangent bundle) and $\mathfrak{g}$ (a finitely dimensional Lie algebra).

- **Nontrivial examples:**

— $A(P)$ — the Lie algebroid of a principal bundle P,

$$A(P) = TP/G, \qquad \Gamma(A(P)) \cong \mathfrak{X}^r(P)$$

- the Lie algebra of right invariant vector fields. Here $\boldsymbol{g}_x \cong \mathfrak{gl}^o(G)$ – the right Lie algebra of G.

— $A(\mathfrak{f})$ — Lie algebroid of a vector bundle $\mathfrak{f}$,

$l \in A(\mathfrak{f})_x \iff l : \Gamma(\mathfrak{f}) \to \mathfrak{f}_x$ and
$\exists_{u \in T_x M} \forall_{\nu \in \Gamma(\mathfrak{f})} \forall_{f \in C^\infty(M)} \; (l(f \cdot \nu) = f(x) \cdot l(\nu) + (\#l)(f) \cdot \nu_x)$.

$$\Gamma(A(\mathfrak{f})) \cong CDO(\mathfrak{f})$$

is the module of covariant derivative operators, i.e. the module of linear operators $\alpha : \Gamma(\mathfrak{f}) \to \Gamma(\mathfrak{f})$ with anchors, $\alpha(f\nu) = f\alpha(\nu) + X(f)\nu$, $X = \#\alpha$. We have

$$A(\mathfrak{f}) \cong A(L\mathfrak{f}).$$

- **The sources of nontransitive Lie algebroids:** differential groupoids, Poisson manifolds, actions of Lie algebras on manifolds.

◊ By a **connection** in transitive Lie algebroid L we mean a splitting $\nabla : TM \to L$ of the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \longrightarrow L \underset{\underset{\nabla}{\longleftarrow}}{\longrightarrow} TM \longrightarrow 0,$$

i.e. a linear homomorphism $\nabla : TM \to L$ commuting with anchors $\#_L \circ \nabla = id_{TM}$. If $L = A(P)$ for a principal bundle P then there is a bijection between connections in $A(P)$ and usual connections in P. If $L = A(\mathfrak{f})$ for a vector bundle $\mathfrak{f}$ then there exists a bijection between connections in $A(\mathfrak{f})$ and linear affine connections in $\mathfrak{f}$.

We will use more general connections for Lie algebroids.

Definition 11 *Let L and A be two Lie algebroids on M. By L-**connection** in A we mean a linear homomorphism $\nabla : L \longrightarrow A$ commuting with anchors*

$$\begin{array}{ccc} L & \xrightarrow{\quad\nabla\quad} & A \\ & {\scriptstyle \#_L}\searrow \quad \swarrow{\scriptstyle \#_A} & \\ & TM. & \end{array}$$

- If $A = T^*M$ is a Lie algebroid of a Poisson manifold and $L = A(P)$ or $L = A(\mathfrak{f})$ then an L-connection in A fulfils great role in the Poisson geometry (Vaisman, Fernandes).
- For a transitive Lie algebroid A with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \longrightarrow A \longrightarrow TM \longrightarrow 0$ and an L-connection ∇ in A we associate the curvature tensor

$$\Omega^{\nabla} \in \Omega^2(L; \boldsymbol{g}), \quad \Omega^{\nabla}(\xi_1, \xi_2) = [\![\nabla_{\xi_1}, \nabla_{\xi_2}]\!] - \nabla_{[\![\xi_1,\xi_2]\!]}.$$

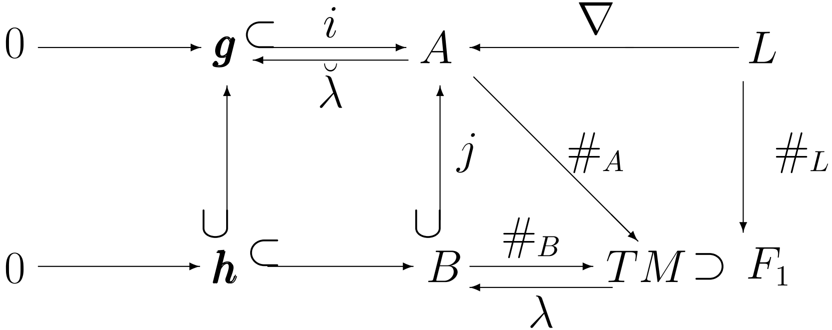
The flatness of ∇ (i.e. the vanishing of Ω^{∇}) is equivalent to a fact that $\nabla : L \longrightarrow A$ is a homomorphism of Lie algebroids.

◊ Let $\Omega(L) = \Gamma(\bigwedge L^*)$ denote the algebra of differential forms on L and $d_L : \Omega(L) \longrightarrow \Omega(L)$ the standard operator of differentiation and $H(L) = H(\Omega(L), d_L)$ the cohomology algebra of $(\Omega(L), d_L)$.

Exotic flat characteristic homomorphism in the category of Lie algebroids

Consider the triple (A, B, ∇) where $B \subset A$ are transitive Lie algebroids on M and $\nabla : L \longrightarrow A$ a flat L-connection in A (L is an arbitrary Lie algebroid, IRREGULAR also). We add that A and B can be regular over the same foliated manifold, but for our lecture we assume the transitivity of them. The triple (A, B, ∇) we will called an $\boxed{FS\text{-Lie algebroid.}}$

In the diagram below $\lambda : TM \to B$ is an arbitrary auxiliary connection in B. Then $j \circ \lambda : TM \to A$ is a connection in A. Let $\breve{\lambda} : A \to \boldsymbol{g}$ be its connection form.



(5)

Define a homomorphism of algebras

$$\Delta : \mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \longrightarrow \Omega(L), \tag{6}$$

$$(\Delta\Psi)_x (w_1 \wedge \ldots \wedge w_k) = \left\langle \Psi_x, \left[-\breve{\lambda}(\nabla w_1)\right] \wedge \ldots \wedge \left[-\breve{\lambda}(\nabla w_k)\right] \right\rangle,$$

$w_i \in L_{|x}$; compare with formula (4):

$$(\Delta\psi)_x (w_1 \wedge \ldots \wedge w_k) = \langle \psi, [-\breve{\lambda}_z \omega_z^h (w_1)] \wedge \ldots \wedge [-\breve{\lambda}_z \omega_z^h (w_k)] \rangle. \tag{4}$$

Consider the subalgebra

$$\left(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^{*}\right)_{I_B}$$

of invariant cross-sections with respect to the adjoint representation of B on $\bigwedge (\boldsymbol{g}/\boldsymbol{h})^{*}$ induced by

$$ad_{B,\mathbf{h}} : B \longrightarrow A(\boldsymbol{g}/\boldsymbol{h}),\ ad_{B,\mathbf{h}}(\xi)([\nu]) = [\![\xi, \nu]\!].$$

Clearly, $\Psi \in (\mathrm{Sec} \bigwedge^{k} (\boldsymbol{g}/\boldsymbol{h})^{*})_{I_B}$ if and only if

$$(\#_B \circ \xi)\langle \Psi, [\nu_1] \wedge ... \wedge [\nu_k]\rangle$$
$$= \sum_{j=1}^{k} (-1)^{j-1} \langle \Psi, [\![j \circ \xi, \nu_j]\!] \wedge [\nu_1] \wedge ...\hat{j}... \wedge [\nu_k]\rangle$$

for all $\xi \in \mathrm{Sec}\, B$ and $\nu_j \in \mathrm{Sec}\, \boldsymbol{g}$. In particular, for $X \in \mathfrak{X}(M)$ and $\xi = \lambda \circ X$ we have

$$X \langle \Psi, [\nu_1] \wedge ... \wedge [\nu_k]\rangle \tag{7}$$
$$= \sum_{j=1}^{k} (-1)^{j-1} \langle \Psi, [\![j \circ \lambda \circ X, \nu_j]\!] \wedge [\nu_1] \wedge ...\hat{j}... \wedge [\nu_k]\rangle$$

We have a differential $\bar{\delta}$ of degree $+1$ which acts in the algebra $(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^{*})_{I_B}$ of invariant elements in such a way that

$$\begin{aligned} &\left\langle \bar{\delta}\Psi, [\nu_1] \wedge \ldots \wedge [\nu_{k+1}] \right\rangle \\ &= -\sum_{i<j} (-1)^{i+j} \left\langle \Psi, [[\nu_i, \nu_j]] \wedge [\nu_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [\nu_{k+1}] \right\rangle \end{aligned} \tag{8}$$

The cohomology algebra

$$H^{\bullet}(\boldsymbol{g}, B) := H^{\bullet}\left((\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^{*})_{I_B}, \bar{\delta} \right)$$

will be called the *relative cohomology algebra of the pair* (A, B) *of Lie algebroids,* $B \subset A$.

Theorem 12 *The homomorphism* $\Delta : \mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \longrightarrow \Omega(L)$ *commutes with the differentials* $\bar{\delta}$ *and* d_L *giving a homomorphism on cohomology*

$$\Delta_{(A,B,\nabla)\#} : H^{\bullet}(\boldsymbol{g}, B) \to H^{\bullet}(L).$$

$\Delta_{(A,B,\nabla)\#}$ is called the *exotic flat characteristic homomorphism* for (A, B, ∇).

Fundamental properties:

- If $\Delta^{+} \neq 0$ then $\nabla : L \to A$ is not a connection in B (i.e. $\mathrm{Im}\,\nabla$ is not contained in the subalgebroid B).

- **Functoriality.** Let (A, B, ∇) and (A', B', ∇') be two pairs of FS-Lie algebroids on M and M', respectively, and let $H : A' \to A$ be a homomorphism of Lie algebroids over a mapping $f : M \to M'$ such that $H[B'] \subset B$. Let $h : L' \to L$ be also a homomorphism of Lie algebroids over f such that $\nabla \circ H = h \circ \nabla'$. We write then

$$(H, h, f) : (A', B', \nabla') \to (A, B, \nabla) .$$

Let $H^{+\#} : H^{\bullet}(\boldsymbol{g}, B) \to H^{\bullet}(\boldsymbol{g}', B')$ be the homomorphism of cohomology algebras induced by the pullback $H^{+*} : \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^* \to \mathrm{Sec} \bigwedge^k (\boldsymbol{g}'/\boldsymbol{h}')^*$. Then the following diagram

$$\begin{array}{ccc} H^{\bullet}(\boldsymbol{g}, B) & \xrightarrow{\Delta_{(A,B,\nabla)\#}} & H^{\bullet}(L) \\ {\scriptstyle H^{+\#}} \downarrow & & \downarrow {\scriptstyle h^{\#}} \\ H^{\bullet}(\boldsymbol{g}', B') & \xrightarrow[\Delta_{(A',B',\nabla')\#}]{} & H^{\bullet}(L') \end{array}$$

commutes.

- **Homotopy invariance and rigidity**. Two Lie subalgebroids B_0, $B_1 \subset A$ (both over the same foliated manifold M are said to be homotopic if there exists a Lie subalgebroid $B \subset T\mathbb{R} \times A$ over $\mathbb{R} \times M$ such that for $t \in \{0,1\}$, $x \in M$

$$v_x \in B_{t|x} \text{ if and only if } (\theta_t, v_x) \in B_{|(t,x)}. \tag{9}$$

B is called a *subalgebroid joining* B_0 *with* B_1.

- REMARK. If $A = A(P)$ and $B_i = A(P_i)$, P_i being H-reductions of P then $A(B_1)$ is homotopic to $A(B_2)$ if and only if there exists $a \in G$ such that the principal subbundles $R_a[P_1]$ and P_2 are homotopic, i.e. cross-sections $s_i : P/H \longrightarrow M$ determining of reductions are homotopic. [If B_i and G are connnected, then B_1 is homotopic to B_2 if and only the principal subbundles P_1 and P_2 are homotopic].

Theorem. If B_0, $B_1 \subset A$ are homotopic subalgebroids of A and $\nabla : L \longrightarrow A$ is a flat L-connection in A, then characteristic homomorphisms $\Delta_{(A,B_0,\nabla)\#} : H^\bullet(\boldsymbol{g}, B_0) \longrightarrow H^\bullet(L)$ and $\Delta_{(A,B_1,\nabla)\#} : H^\bullet(\boldsymbol{g}, B_1) \longrightarrow H_L(M)$ are equivalent in this sense that there exists an isomorphism of algebras $\alpha : H^\bullet(\boldsymbol{g}, B_0) \xrightarrow{\simeq} H^\bullet(\boldsymbol{g}, B_1)$ such that

$$\begin{array}{ccc} H^\bullet(\boldsymbol{g}, B_0) & \xrightarrow{\quad\alpha\quad} & H^\bullet(\boldsymbol{g}, B_1) \\ & \Delta_0 \searrow \qquad \swarrow \Delta_1 & \\ & H^\bullet(L) & \end{array}$$

- **Corollary.** If $\Delta^+_{(A,B,\nabla)\#} \neq 0$ then ∇ is not a connection in any subalgebroid homotopic to B.

Particular cases of the universal exotic characteristic homomorphism

The exotic universal characteristic homomorphism of principal fibre subbundles

Consider a G-principal bundle P and its H-reduction $P' \subset P$ where $H \subset G$ is a closed Lie subgroup and a flat connection ω in P. ω determines the connection $\nabla : TM \to A(P)$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{g} & \hookrightarrow & A(P) & \underset{\overleftarrow{\ \nabla\ }}{\longrightarrow} & TM & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \to & \boldsymbol{h} & \hookrightarrow & A(P') & \longrightarrow & TM & \to & 0 \end{array}$$

As well as we have two triples, algebroids triple $(A(P), A(P'), \nabla)$ and bundles triple (P, P', ω).

Theorem 13 *If P' is connected then there exists an isomorphism of algebras $H^\bullet(\mathfrak{g}, H) \cong H^\bullet(\boldsymbol{g}, A(P'))$ such that the diagram*

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}, H) & & \\ & \searrow^{\Delta_{(P,P',\omega)\#}} & \\ \cong \downarrow & & H^\bullet(M) \\ & \nearrow_{\Delta_{(A(P),A(P'),\nabla)\#}} & \\ H^\bullet(\boldsymbol{g}, A(P')) & & \end{array}$$

commutes.

It means, that $\Delta_{(A(P),A(P'),\nabla)\#}$ is equivalent to the classical case considered by Kamber and Tondeur.

The case with usual connection in a Lie algebroid

Let $L = TM$ and $\nabla : TM \to A$ be a flat connection in A. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{g} & \hookrightarrow & A & \underset{\overleftarrow{\nabla}}{\longrightarrow} & TM & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \to & \boldsymbol{h} & \hookrightarrow & B & \longrightarrow & TM & \to & 0 \end{array}$$

and characteristic homomorphism $\Delta_{(A,B,\nabla)\#} : H^{\bullet}(\boldsymbol{g}, B) \to H^{\bullet}(M)$ given by Kubarski and published in 1993 (as a sketch) and in 2001 in detail.

The case (nearly) equivalent to Crainic theory

Consider a vector bundle $\mathfrak{f}$ with a Riemannian metric h. Then we have the Lie algebroid $A(\mathfrak{f})$ of $\mathfrak{f}$ and the Lie subalgebroid $A(\mathfrak{f}, h) \subset A(\mathfrak{f})$ of the Riemannian reduction. More precisely, for $\alpha \in \Gamma(A(\mathfrak{f}))$

$$\alpha \in \Gamma(A(\mathfrak{f}, h)) \Longleftrightarrow$$
$$\Longleftrightarrow h(\alpha(\nu), \mu) = (\#\alpha)(h(\nu, \mu)) - h(\nu, \alpha(\mu)), \quad \nu, \mu \in \Gamma(\mathfrak{f}).$$

Let $\nabla : L \to A(\mathfrak{f})$ be a flat L-covariant derivative in $\mathfrak{f}$ (i.e. a representation of L in $\mathfrak{f}$). The characteristic homomorphism of the triple $(A(\mathfrak{f}), A(\mathfrak{f}, h), \nabla)$ is closely connected to the Crainic characteristic classes (2003).

M. Crainic, *Differentiable and algebroid cohomology, Van Est isomorphisms, and characteristic classes,* Commentarii Mathematici Helvetici, **78** (2003), 681–721.

Back
Close

Theorem 14 *Let $\mathfrak{f}$ be nonorientable or orientable odd rank. Then ($n = rank\mathfrak{f}$)*

$$H^{\bullet}\left(End\mathfrak{f}, A\left(\mathfrak{f}, h\right)\right) \cong H^{\bullet}\left(\mathfrak{gl}\left(n, \mathbb{R}\right), O\left(n\right)\right) \cong \bigwedge\left(y_1, y_3, ..., y_{2n'-1}\right)$$

where n' is the largest odd integer $\leq n$, and y_{2k-1} are represented by the multilinear trace forms. The characteristic homomorphism $\Delta_{(A(\mathfrak{f}),A(\mathfrak{f},h),\nabla)\#}$ is given by

$$\Delta_{(A(\mathfrak{f}),A(\mathfrak{f},h),\nabla)\#}\left(y_{2k-1}\right) = \frac{\left(-1\right)^k \cdot \left(4k-3\right)!}{2^{4k-3} \cdot \left(2k-1\right)! \cdot \left(2k-2\right)!} u_{4k-3}\left(\mathfrak{f}, \nabla\right),$$

where $u_{4k-3}\left(\mathfrak{f}, \nabla\right)$ are the Crainic characteristic classes for the representation ∇ of L in $\mathfrak{f}$ given by Chern-Simons forms.

Crainic classes: $u_{2k-1}(\mathfrak{f}) = [u_{2k-1}(\mathfrak{f},\nabla)] \in H^{2k-1}(L)$ are defined in such a way that

$$u_{2k-1}(\mathfrak{f},\nabla) = i^{k+1}\,\mathrm{cs}_k(\nabla,\nabla^h) = (-1)^{\frac{k(k+1)}{2}}\,\mathrm{cs}_k(\nabla,\nabla^h), \qquad (10)$$

k is odd natural (only odd k gives nontrivial classes for real $\mathfrak{f}$) and

$$\mathrm{cs}_k\left(\nabla,\nabla^h\right) = \int_0^1 \mathrm{ch}_k\left(\nabla^{\mathrm{aff}}\right) \in \Omega^{2k-1}(L)$$

for the affine combination $\nabla^{\mathrm{aff}} = (1-t)\cdot\tilde{\nabla} + t\cdot\tilde{\nabla}^h : T\mathbb{R}\times A \longrightarrow A\left(\mathrm{pr}_2^*\,\mathfrak{f}\right)$ is defined by the formula

$$\left(\int_0^1 \mathrm{ch}_k(\nabla^{\mathrm{aff}})\right)_{\xi_1,\dots,\xi_{2k-1}} = \int_0^1 \mathrm{ch}_k(\nabla^{\mathrm{aff}})_{\frac{\partial}{\partial t},\xi_1,\dots,\xi_{2k-1}}|_{(t,\bullet)}dt$$

for $\xi_1,\dots,\xi_{2k-1} \in \mathrm{Sec}\,L$, where $\tilde{\nabla}$ we denote here the lifting of an arbitrary L-connection $\nabla : L \to A(\mathfrak{f})$ through the projection $\mathrm{pr}_2 : \mathbb{R}\times M \to M$:

$$\tilde{\nabla} : T\mathbb{R}\times L \to A\left(\mathrm{pr}_2^*\,\mathfrak{f}\right), \qquad \tilde{\nabla}_{(v_t,\xi_x)}\left(\nu\circ\mathrm{pr}_2\right) = \nabla_{\xi_x}(\nu)$$

where $T\mathbb{R} \times L$ is the Cartesian product of Lie algebroids; $T\mathbb{R} \times L$ is isomorphic to the pull-back $\mathrm{pr}_2^{\wedge} L$ of L via pr_2 .

Theorem 15 *Let $\mathfrak{f}$ be orientable even rank $2m$. Then*

$$\begin{aligned}&H\left(\operatorname{End}\mathfrak{f}, A\left(\mathfrak{f}, \{h, \mathrm{v}\}\right)\right)\\ &\cong H\left(gl\left(2m, \mathbb{R}\right), SO\left(2m\right)\right) \cong \bigwedge\left(y_1, y_3, \ldots, y_{2m-1}, y_{2m}\right)\end{aligned}$$

where additionally y_{2m} is determined by a complicated manner using some Pfaffian form.

Crainic does not consider the class y_{2m}.

The universal homomorphism (the case $\nabla = id_A : A \to A$)

Consider the case $L = A$ and a trivial flat A-connection in A, $\nabla = id_A : A \to A$. Then the fundamental diagram is as follows

$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{g} & \hookrightarrow & A & \underset{\nabla = id}{\overset{=\!=}{\longleftarrow}} & A & & \\ & & \uparrow & & \uparrow & & \downarrow & & \\ 0 & \to & \boldsymbol{h} & \hookrightarrow & B & \longrightarrow & TM & \to & 0. \end{array}$$

Really, we obtain here a pair of Lie algebroids (B, A) where $B \subset A$ and the characteristic homomorphism

$$\Delta_{(A,B)\#} : H^{\bullet}(\boldsymbol{g}, B) \to H^{\bullet}(A)$$

for (A, B) . The obtained characteristic homomorphism is a new homomorphism for the theory of exotic classes and it is universal in the following sense:

Theorem 16 (Factorizing theorem) *The universal characteristic homomorphism $\Delta_{(A,B)\#}$ factorizes all flat characteristic exotic homomorphisms: for every flat connection $\nabla : L \to A$ we have*

$$\Delta_{(A,B,\nabla)\#} = \nabla^{\#} \circ \Delta_{(A,B)\#},$$

$$\Delta_{(A,B,\nabla)\#} : H^{\bullet}(\boldsymbol{g}, B) \xrightarrow{\Delta_{(A,B)\#}} H^{\bullet}(A) \xrightarrow{\nabla^{\#}} H^{\bullet}(L)$$

(the equality holds for the level of differential forms, $\nabla^{\#}$ is induced on cohomology by ∇, which is possible thanks the flatness of ∇ – then ∇ commutes with differentials).

We see, that no class from the kernel $\ker \Delta_{(A,B)\#}$ of the universal homomorphism has a meaning to investigating of the problem: Does the connection $\nabla : L \to A$ belong to the geometry of B? (i.e. *Does the image of ∇ is contained in B?*).

Therefore, the following question is important:

NEW QUESTION IN THE THEORY OF EXOTIC FLAT CLASSES:

Does the universal homomorphism

$$\Delta_{(A,B)\#} : H^{\bullet}(\boldsymbol{g}, B) \longrightarrow H^{\bullet}(A)$$

is a monomorphism?

If not, then the more important role is played by the algebra

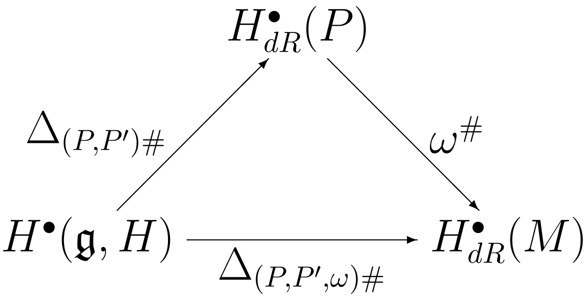
$$H^{\bullet}(\boldsymbol{g}, B) / \ker \Delta_{(A,B)\#}.$$

Universal homomorphism for reductions of principal bundles

Theorem 17 (a) *If G is a compact connected Lie group and P' is a* connected *H-reduction in a G-principal bundle P, $H \subset G$ (H may be nonconnected), then there exists a "universal" characteristic homomorphism*

$$\Delta_{(P,P')\#} : H^{\bullet}(\mathfrak{g}, H) \longrightarrow H^{\bullet}_{dR}(P)$$

acting from the algebra $H^{\bullet}(\mathfrak{g}, H)$ to the total cohomology $H^{\bullet}_{dR}(P)$.If P is flat, and ω is a flat connection in the principal bundle P, the characteristic homomorphism $\Delta_{(P,P',\omega)\#} : H^{\bullet}(\mathfrak{g}, H) \longrightarrow H^{\bullet}_{dR}(M)$ is factorized by $\Delta_{(P,P')\#}$, i.e. the diagram



commutes where $\omega^{\#}$ *on the level of right-invariant forms* $\Omega^r(P)$ *is given as the pullback of forms,*

$$\omega^* : \Omega^r(P) \longrightarrow \Omega(M),$$

$$\omega^*(\phi)_x (u_1 \wedge \ldots \wedge u_k) = \phi_z (\tilde{u}_1 \wedge \ldots \wedge \tilde{u}_k)$$

where $z \in P_{|x}$, $\tilde{u}_i$ *is the* ω*-horizontal lift of* u_i. [*Recall that* $H^{r\bullet}_{dR}(P) := H^\bullet(\Omega^r(P)) \simeq H^\bullet_{dR}(P)$.]

(b) *In the general case (for a noncompact or a nonconnected Lie group* G*) there exists a characteristic homomorphism*

$$\Delta_{(P,P')\#} : H^\bullet(\mathfrak{g}, H) \longrightarrow H^{r\bullet}_{dR}(P)$$

of algebras ($H^{r\bullet}_{dR}(P)$ *denotes the cohomology algebra of right invariant differential forms) which factorizes the characteristic homomorphisms for every flat connection (it means, the analogous diagram commutes).*

(c) *The homomorphism* $\Delta_{(P,P')\#}$ *on the level of forms is given by the following formula*

$$\left(\Delta_{P,P'}\psi\right)_z (w_1 \wedge \ldots \wedge w_k) = \left\langle \psi, [-\lambda_z (w_1)] \wedge \ldots \wedge [-\lambda_z (w_k)] \right\rangle ,$$

where λ *is the form of a connection on* P *extending an arbitrary connection on* P'*.*

The case of Lie algebras

Consider a pair of finitely dimensional Lie algebras $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$, and the universal characteristic homomorphism $\Delta_{(\mathfrak{g},\mathfrak{h})\#} : H^\bullet(\mathfrak{g}, \mathfrak{h}) \to H^\bullet(\mathfrak{g})$ for a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$ and give a class of such pairs for which $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is a monomorphism. This homomorphism is strictly closed to the Koszul homomorphism.

Theorem 18 *There exists an isomorphism of algebras (induced by the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$) such that*

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}, \mathfrak{h}) & & \\ \downarrow \cong & \searrow^{\Delta_{(\mathfrak{g},\mathfrak{h})\#}} & \\ & & H^\bullet(\mathfrak{g}) \\ & \nearrow_{k^\#} & \\ H^\bullet(\mathfrak{g}/\mathfrak{h}) & & \end{array}$$

i.e. $h_{(\mathfrak{g},\mathfrak{h})\#}$ is equivalent to the Koszul homomorphism.

If $k^\#$ is a monomorphism, then $h_{(\mathfrak{g},\mathfrak{h})\#}$ is a monomorphism as well.

Example 19 Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair of Lie algebras $(\mathfrak{h} \subset \mathfrak{g})$, $s : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ the canonical projection. Theorems IX and X from the book by Greub-Halperin-Vanstone yield that $k^{\#}$ is injective if and only if $H^{\bullet}(\mathfrak{g}/\mathfrak{h})$ is generated by 1 and elements of odd degree. Therefore, it follows that $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is injective if and only if $H^{\bullet}(\mathfrak{g}, \mathfrak{h})$ is generated by 1 and elements of odd degree.

We must add that in a wide class of pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{h}$ is reductive in $\mathfrak{g}$ the homomorphism $k^{\#}$ is injective if and only if $\mathfrak{h}$ is noncohomologous to zero (n.c.z. in short) in $\mathfrak{g}$ (i.e. if the homomorphism of algebras $H^{\bullet}(\mathfrak{g}) \to H^{\bullet}(\mathfrak{h})$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is surjective).

In view of the above, examples below yield that the secondary characteristic homomorphism for the reductive pair of Lie algebras $(\mathrm{End}(V), \mathrm{Sk}(V))$ is a monomorphism for any odd dimensional vector space V and not a monomorphism for even dimensional.

Example 20 (The pair of Lie algebras $(\mathrm{End}(V), \mathrm{Sk}(V))$) (a) Let V be a vector space of odd dimension, $\dim V = 2m-1$. Then

$$\begin{aligned} H^\bullet(\mathrm{End}(V), \mathrm{Sk}(V)) &\cong H^\bullet(\mathfrak{gl}(2m-1, \mathbb{R}), O(2m-1)) \\ &\cong \bigwedge(y_1, y_3, \ldots, y_{2m-1}) \end{aligned}$$

where $y_{2k-1} \in H^{4k-3}(\mathrm{End}(V), \mathrm{Sk}(V))$ are represented by the multilinear trace forms. We conclude from the previous example that $\Delta_{(\mathrm{End}(V), \mathrm{Sk}(V))\#}$ is injective.

(b) In the case of even dimensional vector space V $(\dim V = 2m)$ we have

$$\begin{aligned} H^\bullet(\mathrm{End}(V), \mathrm{Sk}(V)) &\cong H^\bullet(\mathfrak{gl}(2m, \mathbb{R}), SO(2m)) \\ &\cong \bigwedge(y_1, y_3, \ldots, y_{2m-1}, y_{2m}) \end{aligned}$$

where y_{2k-1} are the same as above and $y_{2m} \in H^{2m}(\mathfrak{gl}(2m, \mathbb{R}), SO(2m)) \cong H^{2m}(\mathrm{End}(V), \mathrm{Sk}(V))$ is some nonzero element. Therefore, the homomorphism $\Delta_{(\mathrm{End}(V), \mathrm{Sk}(V))\#}$ is not a monomorphism.

The aim of the talk:
the relation of the Koszul homomorphism with the exotic flat characteristic homomorphism

Consider a pair (A, B) of transitive Lie algebroids, $B \subset A$, $x \in M$, and a pair of adjoint Lie algebras $(\boldsymbol{g}_x, \boldsymbol{h}_x)$. Clearly, the inclusion $\iota_x : (\boldsymbol{g}_x, \boldsymbol{h}_x) \to (A, B)$ is a homomorphism of pairs of Lie algebroids over $\{*\} \hookrightarrow M$. The functoriality property and the factorizing theorem give rise to the commutative diagram

$$\begin{array}{ccccc} \Delta_{(A,B,\nabla)\#} : \; H^{\bullet}(\boldsymbol{g}, B) & \xrightarrow{\Delta_{(A,B)\#}} & H^{\bullet}(A) & \xrightarrow{\nabla^{\#}} & H^{\bullet}(L) \\ \downarrow \iota_x^{+\#} & & \downarrow \iota_x^{\#} & & \\ H^{\bullet}(\boldsymbol{g}_x, \boldsymbol{h}_x) & \xrightarrow{\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}} & H^{\bullet}(\boldsymbol{g}_x) & & \end{array}$$

joining the Koszul homomorphism for Lie algebras to the exotic flat characteristic homomorphism (via the universal flat exotic characteristic homomorphism).

We see that if $\iota_x^{+\#}$ and the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}$ are monomorpisms, then $\Delta_{(A,B)\#}$ is also a monomorphism.

About a monomorphicity of the universal exotic characteristic homomorphism for a pair of transitive Lie algebroids

Obviously, if the left and down homomorphisms in the last diagram are monomorphisms, then $\Delta_{(A,B)\#}$ is a monomorphism as well.

- The homomorphism $\iota_x^{+\#}$ is a monomorphism if each invariant element $v \in \left(\bigwedge \left(\boldsymbol{g}_x/\boldsymbol{h}_x\right)^*\right)_{I^o(\mathfrak{h}_x)}$ can be extended to a global invariant cross-section of the vector bundle $\bigwedge \left(\boldsymbol{g}/\boldsymbol{h}\right)^*$.

Remark 21 ι_x^{+*} *on the cross-sections is always a monomorphism, therefore, if it is also an epimorphism then it is an isomorphism.*

Such a situation holds, for example, for integrable Lie algebroids A and B ($B \subset A$), i.e. if $A = A(P)$ for some principal G-bundle P and $B = A(P')$, where P' is a reduction of P with **connected** structural Lie group $H \subset G$. In consequence, we obtain the following theorem joining the Koszul homomorphism with exotic characteristic classes.

Theorem 22 *Let (A, B) be a pair of Lie algebroids, $B \subset A$, and let $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ be a pair of adjoint Lie algebras at $x \in M$. If*

(1) A is an integrable Lie algebroid via a principal fibre bundle P, $A = A(P)$,

(2) the structure Lie group of the connected reduction P' such that $A(P') = B$ is a ***connected*** *Lie group,*

(3) the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x, \boldsymbol{h}_x)\#}$ for the pair $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ is a monomorphism (the many examples was given previously),

then $\Delta_{(A,B)\#} : H(\boldsymbol{g}, H) \longrightarrow H(A(P))$ *is a monomorphism as well.*

A conlusion about the existence of an H-reduction

From the above we have the conclusion.

Conclusion 23 *If* $H \subset G$ *is a connected Lie subgroup of* G, $(\mathfrak{g}, \mathfrak{h})$ *is a n.c.z. pair and an* H*-reduction exists, then there exists a monomorphism*

$$H(\mathfrak{g}, H) \longrightarrow H(A(P)) = H(\Omega^r(P)).$$

Therefore, for every s

$$\dim H^s(\mathfrak{g}, H) \leq \dim H^s(\Omega^r(P)).$$

Theorem 24 *If* H *is a conected Lie subgroup of* G, $(\mathfrak{g}, \mathfrak{h})$ *is a n.c.z. pair and there exists* $s \in \mathbb{N}$ *such that*

$$\dim H^s(\mathfrak{g}, H) > \dim H^s(\Omega^r(P)),$$

then there exists no H*-reduction in* P.