

Nonunital Lie-Rinehart algebras

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1 Nonunital Lie-Rinehart algebras

Let R be a **commutative ring with a unit** and A an arbitrary **commutative** R -algebra (**not necessarily with unit**!). The set of derivations of the algebra A is denoted by $\text{Der}(A)$. It is an A -submodule of the A -module $\text{End}_A(M)$ and a Lie R -algebra.

Example 1 For manifolds. If $A = C^\infty(P)$ is the algebra of smooth functions on a manifold P , then $\text{Der}(A)$ and the Lie algebra of vector fields on P are isomorphic, $\text{Der}(A) \cong \mathfrak{X}(P)$.

Definition 2 A Lie (R, A) -algebra (Lie-Rinehart algebra) is the pair (L, ω) , where L is a Lie R -algebra and A -module and $\omega : L \rightarrow \text{Der}(A)$ is an A -linear homomorphism of Lie algebras such that the following Leibniz condition holds

$$[\alpha, a \cdot \beta] = a \cdot [\alpha, \beta] + \omega(\alpha)(a) \cdot \beta, \quad \alpha, \beta \in L, \quad a \in A.$$

Example 3 For manifolds: Lie-Rinehart algebra induced by a Lie algebroid. Let $(E, [\cdot, \cdot], \gamma_E)$ be a Lie algebroid on M , i.e. the system for which E is a vector bundle on M , the module $(\Gamma(E), [\cdot, \cdot])$ of global cross-sections of E is an \mathbb{R} -Lie algebra, $\gamma_E : E \rightarrow TM$ is a linear homomorphism of vector bundles (called the anchor) fulfilling the Leibniz axiom

$$[[\xi, f \cdot \eta]] = f \cdot [[\xi, \eta]] + \gamma_L(\xi)(f) \cdot \eta, \quad f \in C^\infty(M), \quad \xi, \eta \in \Gamma(E)$$

Then the system

$$(\Gamma(E), [\cdot, \cdot], \omega),$$

where $\omega : \Gamma(E) \rightarrow \text{Der}(C^\infty(M))$ is defined by $\omega(\alpha)(a) = (\gamma_E \circ \alpha)(a)$ for $\alpha \in \Gamma(E)$, $a \in C^\infty(M)$, is a $(\mathbb{R}, C^\infty(M))$ -Lie-Rinehart algebra.

Lie-Rinehart algebras are therefore algebraically suitable to Lie algebroids, however they appeared considerably earlier,

- in 1953 (Herz) under the name of *pseudo-algèbre de Lie*.

Next they appeared independently more than ten times under different names, for example:

- *regular restricted Lie algebra extension* (Hochschild, 1955),
- *Lie d-ring* (Palais, 1961),
- *(R, C)-Lie algebra* (Rinehart, 1963),
- *Lie algebra with an associated module structure* (Hermann, 1967),
- *Lie module* (Nelson, 1967),
- *(A, C)-system* (Ne'eman, 1971; Kostant and Sternberg, 1990),
- *sheaf of twisted Lie algebras* (Kamber and Tondeur, 1971),
- *algèbre de Lie sur C/R* (Illusie, 1972),
- *Lie algebra extension* (Teleman, 1972),
- *Lie-Cartan pairs* (Kastler and Stora, 1985),
- *Atiyah algebras* (Beilinson and Schechtmann, 1988; Manin, 1988) and
- *differential Lie algebra* (Kosmann-Schwarzbach and Magri, 1990).

We prefer *Lie-Rinehart algebra* according to:

- J. Huebschmann, *Poisson cohomology and quantization*, J. für die Reine und Angew. Math. 408 (1990), 57-113. 9.

Example 4 (1) (Algebra Poissona) Let R be commutative ring and A – a commutative R -algebra together with a R -Lie algebra structure $\{\cdot, \cdot\} : A \times A \rightarrow A$, such that for $a \in A$ the homomorphism $\{a, \cdot\} : A \rightarrow A$ is

a differentiation of A , $\{a, b \cdot c\} = b \cdot \{a, c\} + \{a, b\} \cdot c$, $a, b, c \in A$. The pair $(A, \{\cdot, \cdot\})$ is called a Poisson algebra. The mapping

$$\mu : A \rightarrow \text{Der}_R(A), \quad a \mapsto \{a, \cdot\} \quad (1)$$

is a homomorphism of A -modules fulfilling the Leibniz condition

$$\{a, b \cdot c\} = b \cdot \{a, c\} + \mu(a)(b) \cdot c, \quad a, b, c \in A.$$

Therefore $(A, \{\cdot, \cdot\}, \mu)$ is a Lie-Rinehart algebra over (R, A) with the anchor μ .

Particularly, Poisson algebras coming from Poisson manifolds.

As notice Huebschmann a Poisson structure on an arbitrary commutative algebra A over a commutative unital ring R gives rise to a structure of an (R, A) -Lie algebra in the sense of Rinehart on the A -module of Kähler differentials for A .

(2) **For manifolds:** We have important example of Lie algebroid: transformation Lie algebroid $M \times \mathfrak{g}$ for a differential manifold M , Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a homomorphism of Lie algebras $\lambda : \mathfrak{g} \rightarrow \mathfrak{X}(M)$. The anchor is defined by $\rho : M \times \mathfrak{g} \rightarrow TM$, $\rho(x, v) = \lambda(v)(x)$ after the identification $\Gamma(M \times \mathfrak{g}) \cong C^\infty(M; \mathfrak{g})$ we have the Lie bracket $\llbracket \cdot, \cdot \rrbracket : \Gamma(M \times \mathfrak{g}) \times \Gamma(M \times \mathfrak{g}) \rightarrow \Gamma(M \times \mathfrak{g})$ by

$$\llbracket s, t \rrbracket(x) = [s_x, t_x] + ((\lambda(s_x)) \cdot t)(x) - ((\lambda(t_x)) \cdot s)(x)$$

$s, t \in C^\infty(M; \mathfrak{g})$, $x \in M$. Clearly, the Lie algebroid $(M \times \mathfrak{g}, \llbracket \cdot, \cdot \rrbracket, \rho)$ induce a Lie-Rinehart algebra $C^\infty(M; \mathfrak{g}) \cong C^\infty(M) \otimes_{\mathbb{R}} \mathfrak{g}$ over $(\mathbb{R}, C^\infty(M))$.

There is an important generalization of this transformation Lie-Rinehart algebra: Let R be a commutative unital ring and A – a commutative R -algebra (not necessarily with unit!) and L – finitely generated Lie algebra over R , it means there exists a set $\{\alpha_1, \dots, \alpha_n\} \subset L$, that arbitrary element $\alpha \in L$ can be represent in the form

$$\alpha = \sum_{k=1}^n m_k \alpha_k + \sum_{k=1}^n r_k \cdot \alpha_k$$

$m_k \in Z$, $r_k \in R$. Then in the R -module $A \otimes_R L$ can be introduced a structure of a A -module with the multiplication

$$A \times (A \otimes_R L) \longrightarrow A \otimes_R L,$$

$$\left(b, a \otimes \left(\sum_{k=1}^n m_k \cdot \alpha_k + \sum_{k=1}^n r_k \cdot \alpha_k \right) \right) \mapsto \sum_{k=1}^n (m_k ab + r_k ab) \otimes \alpha_k.$$

Let

$$\gamma : L \longrightarrow \text{Der}_R(A)$$

be an arbitrary Lie algebra homomorphism. Then in $A \otimes_R L$ we can introduce a bracket

$$[a_1 \otimes l_1, a_2 \otimes l_2] = a_1 \cdot a_2 \otimes [l_1, l_2] + a_1 \cdot \gamma(l_1)(a_2) \otimes l_2 - a_2 \cdot \gamma(l_2)(a_1) \otimes l_1$$

$a_1 \otimes l_1, a_2 \otimes l_2 \in A \otimes_R L$. Take the anchor

$$\omega : A \otimes_R L \longrightarrow \text{Der}_R(A),$$

$$\omega(a \otimes l)(b) = a \cdot \gamma(l)(b).$$

In this way we obtain a structure of a (R, A) -Lie algebra in $A \otimes_R L$ generalizing the transformation Lie algebroid.

In 1990-99 J.Huebschmann wrote a series of papers

- (1) J. Huebschmann, *Poisson cohomology and quantization*, J. für die Reine und Angew. Math. 408 (1990), 57-113. 9.
- (2) —, *Lie-Rinehart algebras, Gerstenhaber algebras, and Batalin-Vilkovisky algebras*, Annales de l'Institut Fourier 48 (1998), 425-440.
- (3) —, *Duality for Lie-Rinehart algebras and the modular class*, 1998, preprint.
- (4) —, *Extensions of Lie-Rinehart algebras and the Chern-Weil construction*, Contemp. Math. 227 147-176, 1999.

relating to Lie-Rinehart algebras. In (1) - the base of the series - the author wrote [p.70₇] "**Let A be an algebra over R , not necessarily with 1**" and repeated this sentence in the context of commutative algebras on next pages. However, all the technical tools which were used, are appropriate in the case of unital algebras only. There are some simple **anomalies** in the theory of A -modules over nonunital R -algebra A , one of them concern tensor products.

1.1 Nonunital modules

Let R be an arbitrary ring (in general it need not be commutative or unital). Recall that a *MODULE* over R is an abelian group $(M, +)$ with an action $R \times M \rightarrow M$ satisfying the following three conditions:

$$(M1) \quad r \cdot (s \cdot m) = (r \cdot s) \cdot m,$$

$$(M2) \quad (r + s) \cdot m = r \cdot m + s \cdot m,$$

$$(M3) \quad r \cdot (m + n) = r \cdot m + r \cdot n, \quad r, s \in R \text{ and } m, n \in M.$$

If R has a unit 1_R , then M is called *R-UNITAL* if

$$(M4) \quad 1_R \cdot m = m, \quad \text{where } m \in M.$$

When M is unital we do not assume axiom (M4) a priori.

The category of left [right] R -modules fulfilling (M1)-(M3) is denoted by ${}_R\mathcal{M}^0$ [\mathcal{M}_R^0] and the category left [right] **unital** R -modules over **unital** ring R by ${}_R\mathcal{M}$ [\mathcal{M}_R].

In some nonunital cases the role of the condition (M4) can be played by one of the following:

$$(M4') \quad \{m \in M; \forall r \in R, r \cdot m = 0\} = \bigcap_{r \in R} \ker \rho_r = \{0\}, \text{ where } \rho_r : M \rightarrow M, m \mapsto r \cdot m.$$

$$(M4'') \quad \text{any element } m \in M \text{ has a representation } m = \sum_{i=1}^n r^i \cdot m_i \text{ for some } r^i \in R, m_i \in M,$$

however there is no reason to create new categories with Axiom (M4') or (M4''), because these properties are not preserved by the functor Hom or a tensor product.

A simple example of an R -module is an abelian group M with the action $r \cdot m = 0$; it is called the *TRIVIAL MODULE*.

The general definition of an algebra comes from

- N.Jacobson, *Structure of rings*, AMS, 190 Hope Street, Providence, R.I. 1956.
- W.Greub & S.Halperin & R.Vanstone, *Connections, Curvature and Cohomology*, Vol. I, Academic Press 1972).

Definition 5 Let R be a commutative ring. An ALGEBRA over R is an R -module A with a ring structure given by an R -bilinear map $\cdot : A \times A \rightarrow A$, i.e. in particular

$$(r \cdot a) \cdot b = a \cdot (r \cdot b) = r \cdot (a \cdot b), \quad a, b \in A, r \in R.$$

A HOMOMORPHISM of R -algebras $\varphi : A \rightarrow B$ is a homomorphism of R -modules with $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for $a, b \in A$.

The "pure" ring corresponding to the R -algebra A is denoted by \underline{A} ($\underline{\cdot}$ is a functor from the category of R -algebr to the category of rings).

Lemma 6 If R has a unit and A is R -unital module, than R -algebra A has a unit if and only if there is a homomorphism of rings $\eta : R \rightarrow \underline{A}$ such that

$$\eta(r) \cdot a = a \cdot \eta(r) = r \cdot a, \quad r \in R, a \in A, \quad (2)$$

(under (2) we have $\eta(1_R) = 1_A$). If R is not unital, then this condition does not imply unitality of the algebra A (an example is a tensor algebra $T(V)$ for an arbitrary R -module $V \in {}_R\mathcal{M}^0$ over a nonunital ring R).

The source of studying of nonunital algebras are, for example, **ideals in unital algebras**. A more concrete examples of nonunital algebras are algebras of differential operators of rank $\geq k \geq 1$ on \mathbb{R}^n (being commutative in the case of constant coefficients).

UNITALIZATION

— Define

$$R_+ = \mathbb{Z} \bigoplus R \quad (3)$$

(direct sum of abelian groups) and the module action by

$$r \cdot (k \oplus s) = 0 \oplus (k \cdot r + r \cdot s), \quad k \in \mathbb{Z}, r, s \in R.$$

Hence we obtain a module $R_+ \in {}_R\mathcal{M}^0$ such that for $p = 1_{\mathbb{Z}} \oplus 0$ we have $r \cdot p = 0 \oplus r \neq 0$ (if $r \neq 0$).

— The module R_+ **is not** R -unital (even if R is unital).

— It appears that R_+ is a free module in the category ${}_R\mathcal{M}^0$, generated by only one element.

— Let R be a ring. In the R -module R_+ there is a structure of a unital ring given by

$$(k \oplus r) \cdot (l \oplus s) = kl \oplus (ks + lr + rs). \quad (4)$$

An element $1_{\mathbb{Z}} \oplus 0$ is a unit of R_+ . R is isomorphic to the ideal $\{0_{\mathbb{Z}}\} \oplus R$ of R_+ . If R is commutative, then R_+ is a commutative R -algebra. The ring R_+ is called a *UNITALIZATION* of R .

Let $M \in {}_R\mathcal{M}^0$, where R is any ring. Then in the abelian group M we can introduce a structure of an R_+ -module given by

$$(k \oplus r) \cdot m = km + rm, \quad k \in \mathbb{Z}, r \in R, m \in M.$$

The R_+ -module obtained this way will be denoted by $M_{(+)}$ and called the *unitalisation* of the R -module M . If M and N are R -modules, then a mapping $\varphi : M \rightarrow N$ is a homomorphism of R -modules if and only if φ is a homomorphism of the induced R_+ -modules. Thus we have

$$\text{Hom}_R(M, N) = \text{Hom}_{R_+}(M_{(+)}, N_{(+)}) . \quad (5)$$

1.2 Modules over nonunital algebras

Definition 7 Let R be a commutative ring and A any R -algebra. A -module is an R -module M which has an \underline{A} -module structure in the category $\underline{A}\mathcal{M}^0$ such that the multiplication $A \times M \rightarrow M$ is R -bilinear.

Definition 8 A mapping $\varphi : M \rightarrow N$ is, by definition, a homomorphism of A -modules M and N if φ is both homomorphism of R -modules and \underline{A} -modules. The category of left [right] A -modules is denoted by ${}_A\mathcal{M}^0$ [\mathcal{M}_A^0]. If the algebra A is unital, then the category of left [right] \underline{A} -unital A -modules is denoted by ${}_A\mathcal{M}$ [\mathcal{M}_A].

In the category of nonunital modules over R -algebras take places some "anomalies" concerning some disconcordance between structures of R -modules and structures of \underline{A} -modules. One is given below, the second - more below - on the ground of tensor product.

Example 9 (First anomaly) If M and N are A -modules, then in general \underline{A} -linearity of a mapping $\varphi : M \rightarrow N$ does not imply R -linearity. For example if $M \neq 0$ is an R -module and A is an R -algebra, then M is an \underline{A} -module under the trivial multiplication $A \times M \rightarrow M$, $(a, m) \mapsto 0$. Observe that each additive mapping $\varphi : M \rightarrow M$ is then \underline{A} -linear but it need not be R -linear. For example by injectivity of the group of rational number \mathbb{Q} there is an additive mapping $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ such that $\text{Im } \alpha = \mathbb{Q}$. Clearly, α is not \mathbb{R} -linear however is A -linear for every R -algebra A and trivial structure of A -module.

Here we give examples of classes of A -modules N such that for the mappings $\varphi : M \rightarrow N$ do not take place the above anomaly (i.e. such that the equality holds $\text{Hom}_A(M, N) = \text{Hom}_{\underline{A}}(M, N)$):

Example 10 *The equality holds $\text{Hom}_A(M, N) = \text{Hom}_{\underline{A}}(M, N)$ if*

- A is unital and N is \underline{A} -unital.
- N fulfils axiom M_4' with respect to the ring A , i.e. $\{n \in N; \forall a \in A, an = 0\} = 0$.
- there exists a ring homomorphism $\eta : R \rightarrow A$ fulfilling $\eta(r) \cdot a = a \cdot \eta(r) = r \cdot a$.

Definition 11 *An A -module M which possess a structure of a ring $M \times M \rightarrow M$ being an A -bilinear mapping (i.e. R -bilinear and \underline{A} -bilinear) is called an A -algebra.*

The set of endomorphisms $\text{End}_A(M)$ is an R -module and R -algebra with respect to composition. If A is commutative, then $\text{End}_A(M)$ is an A -module and A -algebra.

UNITALIZATION

Let

$$A_+ = R \bigoplus A$$

(direct sum of R -modules) and define the structure of A -module as follows

$$a \cdot (r \oplus b) = 0 \oplus (ra + ab), \quad a, b \in A, r \in R.$$

For $e = 1_R \oplus 0$ we have $a \cdot e = 0 \oplus a \neq 0$, if $a \neq 0$.

— In next sections we show that A_+ is a free A -module (in the category ${}_A\mathcal{M}^0$), with one generator.

Lemma 12 *If R is a commutative ring with a unit 1_R and A is an R -algebra and an R -unital module, then there is an R -algebra structure in A -module A_+ . given by*

$$(r \oplus a) \cdot (s \oplus b) = rs \oplus (rb + sa + ab). \quad (6)$$

$a, b \in A, r, s \in R$. The element $1_R \oplus 0$ is a unit of A_+ . If A is commutative, then A_+ is a commutative A -algebra.

If R is a commutative ring with a unit and M is an A -module, then we can introduce a unital A_+ -module structure in the R -module M given by

$$(r \oplus a) \cdot m = rm + am.$$

The A_+ -module obtained this way will be denoted by $M_{(+)}$.

- A mapping $\varphi : M \rightarrow N$ is a homomorphism of A -modules if and only if it is a homomorphism of the appropriate induced A_+ -modules.

Since A_+ has a unit we obtain the following equalities

$$\text{Hom}_A(M, N) = \text{Hom}_{A_+}(M_{(+)}, N_{(+)}) = \text{Hom}_{\underline{A}_+}(M_{(+)}, N_{(+)}) .$$

2 Tensor product of A -modules

2.1 Standard definitions

Let R be a commutative ring and A any R -algebra. For A -modules $M \in \mathcal{M}_A^0$ and $N \in {}_A\mathcal{M}^0$ we have two tensor products

- $M \otimes_R N$ in the category of R -modules (it is R -module since R is commutative),
- $M \otimes_{\underline{A}} N$ in the category of \underline{A} -modules (it is an abelian group and \underline{A} -module if A is commutative).

Lemma 13 *In the abelian group $M \otimes_{\underline{A}} N$ there exist two structures of a left R -module given by*

$$(1) \quad r \cdot (m \otimes_{\underline{A}} n) = (r \cdot m) \otimes_{\underline{A}} n,$$

$$(2) \quad r \cdot (m \otimes_{\underline{A}} n) = m \otimes_{\underline{A}} (r \cdot n).$$

These structures are equal if and only if the canonical mapping $M \times N \rightarrow M \otimes_{\underline{A}} N$, $(m, n) \mapsto m \otimes_{\underline{A}} n$ is R -bilinear in the structure (1) (equivalently in (2)); equivalently, if there exists a mapping

$$M \otimes_R N \longrightarrow M \otimes_{\underline{A}} N, \quad m \otimes_R n \longmapsto m \otimes_{\underline{A}} n, \quad (7)$$

equivalently, if we can change-over scalars from R : $(r \cdot m) \otimes_{\underline{A}} n = m \otimes_{\underline{A}} (r \cdot n)$.

Example 14 (Second anomaly) *These two structures in the last lemma need not to be equal. It means that the mapping γ need not exist. In fact, let A be an arbitrary commutative \mathbb{R} -algebra and give \mathbb{R} the trivial structures of a right and left A -module ($\mathbb{R} \times A \rightarrow \mathbb{R}$, $(r, a) \mapsto 0$, $A \times \mathbb{R} \rightarrow \mathbb{R}$, $(a, r) = 0$). Then $\mathbb{R} \otimes_{\underline{A}} \mathbb{R} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$. For $r_0 \notin \mathbb{Q}$ we have $r_0 \otimes_{\mathbb{Z}} 1_{\mathbb{R}} \neq 1_{\mathbb{R}} \otimes_{\mathbb{Z}} r_0$, which means that the two structures of \mathbb{R} -modules are different. Indeed, if $\varphi : \mathbb{R} \rightarrow \mathbb{Q}$ is an additive mapping, then there exist a homomorphism of abelian groups $f : \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}$ such that $f(r \otimes_{\mathbb{Z}} t) = \varphi(r) \cdot t$ (such a mapping φ exists, because \mathbb{Q} is an injective module). Hence $f(r_0 \otimes_{\mathbb{Z}} 1_{\mathbb{R}}) \in \mathbb{Q}$ and $f(1_{\mathbb{R}} \otimes_{\mathbb{Z}} r_0) \notin \mathbb{Q}$.*

Example 15 *The above two structures of R -modules are equal to each other if*

- *A has a unit and M or N is \underline{A} -unital,*
- *A is commutative and $P = M \otimes_{\underline{A}} N$ satisfies the axiom (4') for the ring \underline{A} .*

2.2 Tensor product of A -modules

We introduce new notion of the tensor product in the category of A -modules for arbitrary algebra A over commutative unital ring R void from the above anomaly (i.e. enable us to change-over scalars from R).

— Let R be a commutative ring with a unit and A any R -algebra. If $M \in \mathcal{M}_A^0$, $N \in {}_A\mathcal{M}^0$ and $P \in {}_R\mathcal{M}^0$, then a mapping $f : M \times N \rightarrow P$ is called *R -bilinear*, if $f(m, \cdot)$ and $f(\cdot, n)$ are R -linear and $f(m \cdot a, n) = f(m, a \cdot n)$ for any $m \in M$, $n \in N$, $a \in A$.

Definition 16 *A tensor product of A -modules M and N is the pair $(M \otimes_A N, \otimes_A)$, where $M \otimes_A N$ is an R -module in the category ${}_R\mathcal{M}^0$ and $\otimes_A : M \times N \rightarrow M \otimes_A N$ is an R -bilinear mapping which satisfies the following condition: if $f : M \times N \rightarrow P$ is an R -bilinear mapping into an arbitrary R -module P , then there exists a unique homomorphism of R -modules φ such that $\varphi(m \otimes_A n) = f(m, n)$.*

Below we give the construction of a tensor product of A -modules as a quotient. Let

$$M \otimes_A N = R(M \times N) / R_A(M \times N)$$

where $R_A(M \times N)$ be an R -submodule of a free R -module $R(M \times N)$ generated by the elements

$$\begin{aligned} & (r^1 \cdot m_1 + r^2 \cdot m_2, n) - r^1(m_1, n) - r^2(m_2, n), \\ & (m, s^1 \cdot n_1 + s^2 \cdot n_2) - s^1(m, n_1) - s^2(m, n_2), \\ & (m \cdot a, n) - (m, a \cdot n), \end{aligned}$$

where $a \in A$, $r^i, s^i \in R$ (the pair (m, n) is identified with $1_R \cdot (m, n)$) and define

$$\otimes_A : M \times N \rightarrow M \otimes_A N, \quad (m, n) \mapsto [(m, n)].$$

Lemma 17 *If one of modules M or N is R -unital, then the above construction $(M \otimes_A N, \otimes_A)$ is a tensor product of A -modules, i.e. in particular, $M \otimes_A N$ is a R -module such that $r \cdot (m \otimes_A n) = (r \cdot m) \otimes_A n = m \otimes_A (r \cdot n)$.*

Examples (15) give the equalities $M \otimes_{\underline{A}} N = M \otimes_A N$.

3 Free and projective modules in nonunital categories

3.1 Nonunital free⁰ modules

Let R be a (nonunital in general) ring and $M \in {}_R\mathcal{M}^0$. If $B \subset M$ is a nonempty subset, then an R -submodule $M(B)$ of M generated by B is the smallest R -submodule of M which contains B . It is given by

$$M(B) = \left\{ \sum_i k^i m_i + \sum_j r^j m'_j : k^i \in \mathbb{Z}, r^j \in R, m_i, m'_j \in B \right\}.$$

The module M is generated by B if $M(B) = M$.

Definition 18 (1) *A nonempty subset $B \subset M$ is called a basis⁰ (more precisely an R -basis⁰) of M if for any module $N \in {}_R\mathcal{M}^0$ and map $f : B \rightarrow N$ there is a unique homomorphism of R -modules $\varphi : M \rightarrow N$ with $\varphi|_B = f$.*

(2) *An R -module M is called free⁰ if has a basis⁰.*

For the category of unital modules over unital ring we have standard definitions of a basis and a free module.

Lemma 19 B is a basis⁰ of a module M if and only if B is a basis of a unital R_+ -module $M_{(+)}$.

Let $R(B)$ denotes the R -module consisting of the set of functions $\rho : B \rightarrow R$ such that $\rho(b) = 0$ for all but a finite number of $b \in B$.

Lemma 20 If $M \in {}_R\mathcal{M}^0$ is a free⁰ R -module and $B \subset M$ is his basis⁰, then

$$M \cong \mathbb{Z}(B) \bigoplus R(B)$$

with the structure of R -module defined by $r \cdot (\sigma \oplus \rho) = 0 \oplus (\sigma r + r\rho)$.

Corollary 21 Any free⁰ R -module with one generator (the ring R can be unital or not) is isomorphic to R_+ .

Lemma 22 For commutative ring R we have $\text{End}_R(R_+) \cong R_+$.

3.2 Nonunital free⁰ A -modules

Assume R is an unital commutative ring and A is an R -algebra (as an R -module, A is R -unital, $A \in {}_R\mathcal{M}$). Analogously we define A -base⁰ and A -free⁰ A -module.

Definition 23 For A -module $L \in {}_A\mathcal{M}^0$ a nonempty subset $B \subset L$ is called A -base⁰ of L if for each A -module $N \in {}_A\mathcal{M}^0$ and function $f : B \rightarrow N$ there exists exactly one homomorphism of A -modules $\varphi : L \rightarrow N$ extending f . An A -module l is called A -free⁰ if has an A -basis⁰.

If $L \in {}_A\mathcal{M}^0$ is an A -free⁰ A -module and $B \subset L$ is his A -basis⁰, then

Lemma 24 If $L \in {}_A\mathcal{M}^0$ is an A -free⁰ A -module and $B \subset L$ is his A -basis⁰, then L is isomorphic to A -module

$$L \cong R(B) \bigoplus \underline{A}(B)$$

with the structure given by $a \cdot (\sigma \oplus \eta) = 0 \oplus (\sigma \cdot a + a \cdot \eta)$. In particular, A -free⁰ module with one generator is isomorphic $z_{A_+} = R \bigoplus A$. A -base⁰ is $1_R \oplus 0$.

3.3 Nonunital projective modules

R -module $M \in {}_R\mathcal{M}^0$ is called *projective*⁰, if for any epimorphism $\varphi : N \rightarrow P$ of modules in the category ${}_R\mathcal{M}^0$ the sequence

$$\mathrm{Hom}_R(M, N) \xrightarrow{\mathrm{Hom}(id, \varphi)} \mathrm{Hom}_R(M, P) \longrightarrow 0$$

is exact. It can be easily checked that M is projective⁰ \iff the induced R_+ -module $M_{(+)} \in {}_{R_+}\mathcal{M}$ is projective $\iff M_{(+)}$ is a direct term of a free R_+ -module in the category ${}_{R_+}\mathcal{M}$.

Let R be a commutative ring with a unit and A any R -algebra. A -module $M \in {}_A\mathcal{M}^0$ is called projective⁰, if for any epimorphism $\varphi : N \rightarrow P$ of modules in the category ${}_A\mathcal{M}^0$ the sequence

$$\mathrm{Hom}_A(M, N) \xrightarrow{\mathrm{Hom}(id, \varphi)} \mathrm{Hom}_A(M, P) \longrightarrow 0$$

is exact. The A -module M is projective⁰ $\iff M$ is a direct term of an free⁰ A -module.

4 Picard group

4.1 Picard group of a nonunital commutative ring

Definition 25 Let R be a commutative ring. A module $M \in {}_R\mathcal{M}^0$ is called *invertible*⁰ if it is invertible as a unital R_+ -module i.e. there exists a module $N \in {}_R\mathcal{M}^0$ such that $M \otimes_R N \cong R_+$ (we recall that R_+ is a free⁰ module with one generator).

Lemma 26 If R is a commutative ring and R -modules M, N are invertible⁰, then $M \otimes_R N$ is an invertible⁰ R -module.

Now we can define the Picard group of any commutative ring nonunital R . Let $[M]^0$ denotes the class of R -modules which are isomorphic to $M \in {}_R\mathcal{M}^0$. In the set

$$\mathrm{Pic}^0(R)$$

of isomorphism classes of invertible⁰ R -modules we define multiplication $[M]^0 \cdot [N]^0 = [M \otimes_R N]^0$. Hence we obtain an abelian group with the unit $[R_+]^0$ and the inverse $([M]^0)^{-1} = [\mathrm{Hom}_R(M, R_+)]^0$.

Evidently

$$\mathrm{Pic}^0(R) = \mathrm{Pic}(R_+).$$

4.2 Picard group of an R -algebra A

Let R be a commutative ring with a unit. Now we generalize our consideration to the case of commutative R -unital R -algebras A .

Definition 27 An A -module $M \in {}_A\mathcal{M}^0$ is called *invertible⁰*, if M is invertible as an A_+ -module, i.e. $M \otimes_A N \cong A_+$ for some $N \in {}_A\mathcal{M}^0$.

Remark 28 If an algebra A has a unit, then: (a) an invertible A -module is invertible as an \underline{A} -module as well; (b) $\text{Pic}(A) = \text{Pic}(\underline{A})$.

Let A be a commutative nonunital R -algebra. Similarly as in the case of R -modules, we can prove that an invertible⁰ A -module is projective⁰ and finitely generated. Moreover, if $M \otimes_A N \cong A_+$, then $N \cong \text{Hom}_A(M, A_+)$.

Notice that a tensor product of invertible⁰ A -modules is an invertible⁰ A -module. Hence we can introduce a structure of an abelian group in the set

$$\text{Pic}^0(A)$$

of isomorphism classes $[M]^0$ of invertible⁰ A -modules. $[A_+]^0$ is the unit and the inverse of $[M]^0$ is $[\text{Hom}_A(M, A_+)]^0$.

—If A -module is invertible⁰ then $\text{End}_A(M) \cong A_+ = R \oplus A$ (as A -modules and commutative A -algebras). We notice that $\text{End}_{\underline{A}}(M) \cong (\underline{A})_+ = \mathbb{Z} \oplus \underline{A}$ but this is different object than $A_+ = R \oplus A$.

We have the following isomorphisms

$$\text{Pic}^0(A) \cong \text{Pic}(A_+) \cong \text{Pic}(\underline{A}_+).$$

5 Lie-Rinehart algebras, continuation

Studies in the nonunital case were initiated by J.Huebschmann, however some of his proofs and techniques use unitality, e.g. (1) properties of a tensor product of A -modules, (2) existence of a covariant L -derivative in a projective module (3) construction of a Picard group. Below we consider Lie (R, A) -algebras for any nonunital R -algebra A (R is commutative unital ring). We notice that for any A -module M the R -module $\text{End}_R(M)$ is also A -module.

Definition 29 Let (L, ω) be a Lie (R, A) -algebra and M any A -module. A covariant L -derivative in M is an R -linear map $\nabla : L \rightarrow \text{End}_R(M)$ which satisfy (a) ∇ is A -linear, (b) $\nabla_\alpha : M \rightarrow M$ is a covariant operator with the anchor $\omega(\alpha)$, i.e.

$$\nabla_\alpha(a \cdot m) = a \cdot \nabla_\alpha m + \omega(\alpha)(a) \cdot m.$$

The theorem below in the unital case is given by Huebschmann.

Proposition 30 If $M \in {}_A\mathcal{M}^0$ is projective⁰, then for any Lie-Rinehart algebra (L, ω) there is a covariant L -derivative in M .

We recall, that if ∇ is a covariant L -derivative in an A -module M , then ∇ induces the covariant exterior derivative (denoted by ∇ as well) $d^\nabla : \text{Alt}_A(L, M) \rightarrow \text{Alt}_A(L, M)$ given by the standard formula

$$\begin{aligned} & (d^\nabla f)(\alpha_1, \dots, \alpha_n) \\ &= \sum (-1)^{i-1} \nabla_{\alpha_i}(f(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n)) + \sum_{i < j} (-1)^{i+j} f([\alpha_i, \alpha_j], \alpha_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_n). \end{aligned}$$

Lemma 31 (Bianchi) (The Bianchi identity)

$$d^{\tilde{\nabla}}(R^\nabla) = 0.$$

where $R^\nabla : L \times L \rightarrow \text{End}_A(M)$ is the curvature tensor $R_{\alpha, \beta}^\nabla = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha, \beta]}$ and $\tilde{\nabla} : L \rightarrow \text{End}_R(\text{End}_A(M))$ is the induced covariant L -derivative in the A -module $\text{End}_A(M)$ given by

$$\tilde{\nabla}_\alpha(\varphi) = [\nabla_\alpha, \varphi], \quad \varphi \in \text{End}_A(M).$$

For invertible⁰ modules M we have important fact (following by: $\text{End}_A(M) \cong A_+$ is a commutative algebra)

Lemma 32 Let (L, ω) be a Lie-Rinehart (R, A) -algebra and M any unital invertible⁰ A -module. If $\nabla : L \rightarrow \text{End}_R(M)$ a covariant L -derivative in M , then the induced covariant L -derivative $\tilde{\nabla}$ in the adjoint A -module $\text{End}_A(M)$ is flat. Moreover, $\tilde{\nabla}$ does not depend on the choice of ∇ .

Flatness imply that $d^{\tilde{\nabla}} \circ d^{\tilde{\nabla}} = 0$ i.e. $d^{\tilde{\nabla}}$ is a differentiation inducing cohomology space. Moreover, by the Bianchi identity $d^{\tilde{\nabla}}(R^{\nabla}) = 0$, we have the cohomology class

$$[R^{\nabla}] \in H^2(\text{Alt}_A^2(L; \text{End}_A(M))).$$

Proposition 33 *The cohomology class $[R^{\nabla}]$ does not depend on the choice of a covariant L -derivative ∇ .*

Using once again $\text{End}_A(M) \cong A_+$ we can modify the tensor $R^{\nabla} : L \times L \rightarrow \text{End}_A(M)$ and we obtain a tensor $\bar{R}^{\nabla} : L \times L \rightarrow A_+$.

Theorem 34 *If (L, ω) is a Lie-Rinehart (R, A) -algebra, then for any covariant L -derivative in an invertible⁰ A -module M a map*

$$\text{Pic}^0(A) \longrightarrow H^2(\text{Alt}_A(L; A_+)), \quad [M]^0 \longmapsto [\bar{R}^{\nabla}]$$

is a homomorphism of abelian groups.

Remark 35 *Let (L, ω) be any (R, A) -algebra where R is a commutative ring with a unit and A is an arbitrary commutative nonunital R -algebra. Using unitalisation of rings, modules and algebras we can define "unital" (R_+, A_+) -algebra $(L_{(+)}, \omega_+)$ as follows: firstly we define a homomorphism of A -modules and Lie R -algebras*

$$\begin{aligned} \text{Der}(A) &\rightarrow \text{Der}(A_+), \quad \delta \mapsto \delta_+, \\ \delta_+(r \oplus a) &= (0 \oplus \delta a). \end{aligned}$$

Secondly we define

$$\omega_+ : L_{(+)} \rightarrow \text{Der}(A_+), \quad \alpha \mapsto \omega(\alpha)_+.$$

We can easily to check that $(L_{(+)}, \omega_+)$ is (R_+, A_+) -algebra, i.e. ω_+ is A_+ -linear homomorphism of Lie algebras fulfilling Leibniz condition. Immediately, we can of course obtain a homomorphism of abelian groups (J.Huebschmann)

$$\text{Pic}(A_+) \longrightarrow H^2(\text{Alt}_{A_+}(L_{(+)}; A_+)), \quad [M] \longmapsto [\bar{R}^{\nabla}],$$

which is concordant with the above one obtained by using only nonunital categories of rings, modules and algebras.

The above consideration of nonunital objects can be used further to some generalization of nonunital Lie-Rinehart algebras, as Leibniz algebras with anchor, and next to consideration of some secondary characteristic classes.