On characteristic classes in the theory of Lie algebroids

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1 Primary characteristic classes

1.1 Classical theory for principal bundles

The classical theory of primary characteristic classes for principal fibre bundles is well known. Let (P,π,M,G) be a G-pfb on M with projection $\pi : P \to M$ and structural Lie group G acting on the right on P. $P/G = M$. The domain of the Chern-Weil homomorphism for P is the space

$$I(G) := \left(\bigvee \mathfrak{g}^*\right)_I = \left(\bigvee \mathfrak{g}^*\right)_{I(Ad_G)}$$

of symmetric multilinear functions (equivalently polynomials) on $\mathfrak{g} = \mathfrak{gl}(G)$ invariant with respect to the adjoint representation $Ad_G : G \to GL(\mathfrak{g})$ of G. The Chern-Weil homomorphism for P

$$h_p : \left(\bigvee \mathfrak{g}^*\right)_I \to H_{dR}(M)$$

can be defined by

$$h_P(\Gamma) = [\chi_P(\Gamma)]$$

where for invariant k-polynomial $\Gamma \in \bigvee^k \mathfrak{g}^*$ the differential form $\chi_P(\Gamma) \in \Omega^{2k}(M)$ is such that

$$\pi^*(\chi_P(\Gamma)) = \frac{1}{k!}\Gamma\left(\Omega^k\right)$$

$\Omega \in \Omega^2(P,\mathfrak{g})$ is the curvature form of any connection ω in P, $\Gamma\left(\Omega^k\right) = \langle \Gamma, \underbrace{\Omega \vee ... \vee \Omega}_{k} \rangle$ the pairing is defined through the permanent. The image $\operatorname{Im}(h_P) \subset H_{dR}(M)$ is called the Pontryagin algebra of P. Below we can see that:

If (P,π,M,G) and (P',π',M,G') are connected principal fibre bundles on M such that their Lie algebroids $A(P)$ and $A(P')$ are isomorphic (G and G' can be nonconnected*), then $h_P = h_{P'}$.*

There are non-isomorphic pfb's having isomorphic Lie algebroids (for example P equal to the trivial $SO(3)$ pfb on $\mathbb{RP}^5$ and P' equal to the nontrivial $\mathrm{Spin}(3)$ structure on $\mathbb{RP}^5$).

The algebra $I(G)$ is well known for standard Lie groups, for example

$I(GL(n,\mathbb{R}))$ is generated by Pontryagin polynomials p_i defined for matrices $A \in \mathfrak{gl}(n,\mathbb{R})$ by

$$\det\left(\lambda \cdot \mathbf{1}_n - \frac{1}{2\pi}A\right) = \sum\nolimits_{i=0}^{n} p_i(A) \cdot \lambda^{n-i}.$$

$I(O(n))$ as above by p_{2i}.

$I(SO(2m-1))$ as above by p_{2i}.

$I(SO(2m))$ as above by p_{2i} and by the so-called Pfaffian $Pf \in \left(\bigvee^{2m} \mathfrak{so}(2m)^*\right)_I$ defined for $A = [A_{ij}] \in \mathfrak{so}(2m)$ by

$$Pf(A) = \frac{1}{2^{2m}\pi^m}\sum_{\sigma} \operatorname{sgn}\sigma \cdot A_{\sigma_1,\sigma_2} \cdot \ldots \cdot A_{\sigma_{2m-1},\sigma_{2m}}.$$

For example $Pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a.$

1.2 The Lie algebroid of a pfb (P, π, M, G).

Take the Atiyah short exact sequence ($\mathfrak{g}$ is the **right!** Lie algebra of G)

$$0 \to P \times_G \mathfrak{g} \to TP/G \to TM \to 0.$$

$P \times_G \mathfrak{g}$ is a Lie Algebra Bundle, for any $z \in P_x$

$$\hat{z} : \mathfrak{g} \cong (P \times_G \mathfrak{g})_x, v \longmapsto [(z, v)],$$

is an isomorphism of Lie algebras. Splittings of this sequence are in a 1-1 corrspondence to connections in P. TP/G is a vector bundle over M such that

$$\mathrm{Sec}(TP/G) \cong \mathfrak{X}^r(P),$$

$\mathfrak{X}^r(P)$ is the Lie algebra of G-right invariant vector fields on P. Therefore we can introduce the structure of Lie algebra $[\![\cdot,\cdot]\!]$ in the space of global sections of the vector bundle TP/G. The projection (called the *anchor*) $[\pi_*] : TP/G \to TM$ passing to the sections is a Lie algebra homomorphism $[\pi_*]([\![\xi,\eta]\!]) = [\![[\pi_*]\xi, [\pi_*]\eta]\!]$. The Leibniz condition

$$[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi,\eta]\!] + [\pi_*](\xi)(f) \cdot \eta$$

holds. Therefore

$$(TP/G, [\![\cdot,\cdot]\!], [\pi_*])$$

is a Lie algebroid on M denoted by $A(P)$ for which the linear homomorphism $[\pi_*] : TP/G \to TM$ is an epimorphism.

Definition 1 *By a Lie algebroid on a manifold M we mean the triple $(A, [\![\cdot,\cdot]\!], \#_A)$ where A is a vector bundle on M, $[\![\cdot,\cdot]\!]$ is a real Lie algebra structure in the space of global sections $\mathrm{Sec}\, A$ and $\#_A : A \to TM$ is a linear homomorphism fulfilling the Leibniz condition*

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta.$$

$\mathrm{Sec}\,\#_A : \mathrm{Sec}\,A \to \mathfrak{X}(M)$ is a homomorphism of Lie algebras. The vector field $X = \#_A \circ \xi$ is the anchor of the section $\xi \in \mathrm{Sec}\,A$. Homomorphisms of Lie algebroids preserve by definition the anchors and Lie algebra structures. In the space $\Omega(A) = \mathrm{Sec}\left(\bigwedge A^*\right)$ the differential d_A is defined by the analogous formula as for usual differential forms

$$\begin{aligned}(d_A\Phi)(\xi_0,\dots,\xi_p) &= \sum_{i=0}^{p}(-1)^i(\#_A\circ\xi_i)(\Phi(\xi_0,\dots\hat{\xi}_i\dots,\xi_p)) \\ &+\sum_{i<j}(-1)^{i+j}\Phi([\![\xi_i,\xi_j]\!],\xi_0,\dots\hat{\xi}_i\dots\hat{\xi}_j\dots,\xi_p),\end{aligned}$$

and defines the algebra of cohomology $H(A)$.

A is called **transitive** if $\#_A$ is an epimorphism. A is called *regular* if the anchor is of constant rank. The image $\mathrm{Im}\,\#_A$ is then a regular foliation. (In arbitrary case, the image $\mathrm{Im}\,\#_A$ of the anchor is a Stefan foliation).

Denote for shortness

$$A(P) := TP/G.$$

We notice that having only the Lie algebroid $A(P)$ we can not reconstruct the structural Lie group, but only its Lie algebra !

The Lie algebroid $A(P)$ acts on the Lie algebra bundle $P\times_G\mathfrak{g}$ by

$$ad_{A(P)}(\xi)(\nu) = [\![\xi,\nu]\!].$$

This actions can be extended to the actions $ad^{\vee}_{A(P)}$ of $A(P)$ on the symmetric power of the dual of $P\times_G\mathfrak{g}$, $\bigvee^k(P\times_G\mathfrak{g})^*$ - i.e. on the vector bundle of polynomials. We take the space [algebra] of invariant sections

$$I(A(P)) := \bigoplus^k \mathrm{Sec}\left(\bigvee^k(P\times_G\mathfrak{g})^*\right)_I$$

of vector bundles $\bigvee^k(P\times_G\mathfrak{g})^*$. We have $\Gamma\in\mathrm{Sec}\left(\bigvee^k(P\times_G\mathfrak{g})^*\right)_I$ if and only if for every $\xi\in\mathrm{Sec}(A(P))$ and $\nu_1,\dots,\nu_k\in\mathrm{Sec}(P\times_G\mathfrak{g})$

$$[\pi_*](\xi)\langle\Gamma,\nu_1\vee\dots\vee\nu_k\rangle = \sum_{i=1}^{k}\langle\Gamma,\nu_1\vee\dots\vee[\![\xi,\nu_i]\!]\vee\dots\vee\nu_k\rangle.$$

Theorem 2 *If P is a **connected** pfb (G can be non-connected) then there exists an isomorphism of algebras*

$$\rho : I(G) \to I(A(P)).$$

(In general, we have always a monomorphism).

To understand this fact we need to explain the definition of the Lie algebroid of a vector bundle.

By a Lie algebroid of a vector bundle $\mathfrak{f}$ we mean the Lie algebroid $A(\mathfrak{f}) := A(L\mathfrak{f})$ of the $GL(n,\mathbb{R})$-pfb $L\mathfrak{f}$ of all frames of $\mathfrak{f}$ ($n = \operatorname{rank}\mathfrak{f} = \dim \mathfrak{f}_x$). The fibre $A(L\mathfrak{f})_x$ over x is canonically isomorphic to the space of $\mathbb{R}$-linear homomorphisms

$$\left\{ l : \operatorname{Sec}\mathfrak{f} \to \mathfrak{f}_x;\ \exists_{v\in T_xM}\forall_{\xi\in\operatorname{Sec}\mathfrak{f}}\forall_{f\in C^\infty(M)}\left(l(f\cdot\xi) = f\cdot l(\xi) + v(f)\cdot\xi\right)\right\}.$$

In other words, a global section $\mathfrak{L}\in A(\mathfrak{f})$ determines a **covariant derivative operator**

$$\mathfrak{L} : \operatorname{Sec}\mathfrak{f} \to \operatorname{Sec}\mathfrak{f},$$
$$\mathfrak{L}(f\cdot\xi) = f\cdot\mathfrak{L}(\xi) + X(f)\cdot\xi$$

The vector field X is here the anchor of $\mathfrak{L}$, $X = \#_{A(\mathfrak{f})}(\mathfrak{L})$. We see that the space of global sections of $A(\mathfrak{f})$ is canonically isomorphic to the space of **covariant derivative operators.** This isomorphism is an isomorphism of Lie algebras. There are many other geometric categories from which we can define a Lie Functor to the category of Lie algebroids: principal fibre bundles, vector bundles, Lie or differential groupoids, transversely complete foliations, nonclosed Lie subgroups, Poisson manifolds etc.

Proof of the Theorem.

Take the Lie algebroid $A(\mathbf{g})$ of the Lie Algebra Bundle $\mathbf{g} := P\times_G\mathfrak{g}$ and the adjoint representation $Ad_G : G\to GL(\mathfrak{g})$, $Ad_G(g) = (\tau_g)_{*e}$, where $\tau_g : G\to G$, $h\longmapsto ghg^{-1}$. Consider the Ad_G-homomorphism

$$Ad_P : P\to L\mathbf{g}$$
$$Ad_P(z) = [(z,\cdot)]$$

($[(z,\cdot)] : \mathfrak{g}\overset{\cong}{\to}\mathbf{g}_x$ is a frame of $\mathbf{g}$ at x) of the principal fibre bundles which is called the *adjoint representation of a principal fibre bundle* and consider its differential

$$ad_{A(P)} = (Ad_P)' : A(P)\to A(\mathbf{g})$$

which is equal to the adjoint representation of the Lie algebroid $A(P)$, $ad_{A(P)}(\xi)(\nu) = [\![\xi,\nu]\!]$. The representation Ad_P can be lifted standardly to the homomorphism $Ad_P^\vee : P\to L\left(\bigvee^k\mathbf{g}^*\right)$ of pfb's with respect to the induced homomorphism $Ad_G^\vee : G\to GL\left(\bigvee^k\mathfrak{g}^*\right)$, as well as $ad_{A(P)}$ can be lifted to the homomorphism of Lie algebroids $ad^{\vee}_{A(P)} : A(P) \to A\left(\bigvee^k \mathbf{g}^*\right)$. The analogous property

$$\left(Ad^{\vee}_P\right)' = ad^{\vee}_{A(P)}$$

holds. To prove our theorem we need the following definitions and Lemmas:

Definition 3 *Let $\mu : G \to GL(V)$ be a representation of a Lie group G in a finite dimensional vector space V and $\mathfrak{f}$ a vector bundle with the typical fibre V. Let $F : P \to L\mathfrak{f}$ be a μ-homomorphism of principal fibre bundles. A section $\xi \in \operatorname{Sec}\mathfrak{f}$ is called F-invariant if there exists a vector $v \in V$ such that*

$$F(z)(v) = \xi_{\pi z} \quad \textit{for all } z \in P.$$

Denote by $(\operatorname{Sec}\mathfrak{f})_{I(F)}$ the space of all F-invariant sections of $\mathfrak{f}$.

Lemma 4 *Denote by $V_{I(\mu)}$ the subspace of V of μ-invariant vectors. Then, for $v \in V_{I(\mu)}$, the function*

$$\xi_v : M \to \mathfrak{f}, \quad x \longmapsto F(z)(v)$$

where $z \in P_x$, is a correctly defined smooth section of $\mathfrak{f}$ and

$$V_{I(\mu)} \overset{\cong}{\to} (\operatorname{Sec}\mathfrak{f})_{I(F)}, \quad v \longmapsto \xi_v,$$

is an isomorphism.

Therefore applying Lemma 4 to the representation $Ad^{\vee}_P : P \to L\left(\bigvee^k \mathbf{g}^*\right)$ we have

$$I^k(G) = \left(\bigvee^k \mathfrak{g}^*\right)_{I(Ad_G)} \cong \left(\operatorname{Sec}\bigvee^k \mathbf{g}^*\right)_{I\left(Ad^{\vee}_P\right)}.$$

Definition 5 *Let $T : A \to A(\mathfrak{f})$ be a homomorphism of Lie algebroids (T is called a representation of A in a vector bundle $\mathfrak{f}$). A section $\xi \in \operatorname{Sec}\mathfrak{f}$ is called T-invariant (or T-parallel) if $T(v)(\xi) = 0$ for all $v \in A$. Denote by $(\operatorname{Sec}\mathfrak{f})_{I^{\circ}(T)}$ the space of all T-invariant sections of $\mathfrak{f}$.*

Lemma 6 *Let $F : P \to L\mathfrak{f}$ be a μ-homomorphism of principal fibre bundles and $F' : A(P) \to A(\mathfrak{f})$ its differential. The spaces of invariant sections $(\operatorname{Sec}\mathfrak{f})_{I(F)}$ and $(\operatorname{Sec}\mathfrak{f})_{I^{\circ}(F')}$ under F and its differential F' are related by*

(a) $(\operatorname{Sec}\mathfrak{f})_{I(F)} \subset (\operatorname{Sec}\mathfrak{f})_{I^{\circ}(F')}$,

*(b) if P is **connected** (nothing is assumed about the connectedness of G), then*

$$(\operatorname{Sec}\mathfrak{f})_{I(F)} = (\operatorname{Sec}\mathfrak{f})_{I^{\circ}(F')}.$$

In consequence, applying to the representation $ad^{\vee}_{A(P)} : A(P) \rightarrow A\left(\bigvee^k \mathbf{g}^*\right)$ we have

$$I^k(G) \cong \left(\operatorname{Sec}\bigvee^k \mathbf{g}^*\right)_{I\left(Ad^{\vee}_P\right)} \subset \left(\operatorname{Sec}\bigvee^k \mathbf{g}^*\right)_{I^o\left(ad^{\vee}_{A(P)}\right)} = I^k(A(P)).$$

If P is connected we have $I^k(G) \cong (I^o)^k(A(P))$.

1.3 Chern-Weil homomorphism for Lie algebroids.

Now we pass to the construction of **the Chern-Weil homomorphism for Lie algebroids.**

First, take a single transitive Lie algebroid A with the Atiyah sequence $0 \rightarrow \mathbf{g} \rightarrow A \overset{\#_A}{\rightarrow} TM \rightarrow 0$. The Lie algebroid A acts on the Lie Algebra Bundle $\mathbf{g}$ by $ad_A : A \rightarrow A(\mathbf{g})$, $ad_A(\xi)(\nu) = [\![\xi,\nu]\!]$, $\xi \in \operatorname{Sec} A$, $\nu \in \operatorname{Sec}\mathbf{g}$. This action can be extended on the bundle $\bigvee^k \mathbf{g}^*$ to the action $ad_A : A \rightarrow A\left(\bigvee^k \mathbf{g}^*\right)$. Let $(I^o)^k(A)$ denotes the space of invariant sections of $\bigvee^k \mathbf{g}^*$. $I^o(A) := \bigoplus (I^o)^k(A)$ is an algebra. The value $\Gamma_x \in \bigvee^k \mathbf{g}^*_x$ at x of any invariant section is a vector invariant with respect to the adjoint representation $ad_{\mathbf{g}_x}$ of the isotropy Lie algebra $\mathbf{g}_x$, Each invariant vector $v \in \bigvee^k \mathbf{g}^*_x$ can be extended uniquely to some invariant section of the bundle $\bigvee^k \mathbf{g}^*$ over some open subset containing x (for example if this neighbourhood is contractible). Sometimes v can be extended on the whole M. Namely, to see this consider an arbitrary general representation $T : A \rightarrow A(\mathfrak{f})$ of A on a vector bundle $\mathfrak{f}$.

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathbf{g} & \overset{T^+}{\longrightarrow} & \operatorname{End}\mathfrak{f} \\ \downarrow & & \downarrow \\ A & \overset{T}{\longrightarrow} & A(\mathfrak{f}) \\ \downarrow & & \downarrow \\ TM & = & TM \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

$T^+_x : \mathbf{g}_x \rightarrow \operatorname{End}(\mathfrak{f}_x)$ is a representation of the Lie algebra $\mathbf{g}_x$ on the vector bundle $\mathfrak{f}_x$. Denote by $(\mathfrak{f}_x)_{I^o}$ the subspace of T^+_x-invariant vector and take the subbundle $\mathfrak{f}_I := \bigcup_{x\in M} (\mathfrak{f}_x)_{I^o} \subset \mathfrak{f}$. The representation T gives a usual flat covariant derivative ∇ in $\mathfrak{f}_{I^o}$ and section ξ of this bundle is parallel under ∇ iff ξ is T-invariant. ∇ is defined by: for $v \in T_xM$ and arbitrary lifted vector $\tilde{v} \in A_x$, $\#_A(\tilde{v}) = v$ we put $\nabla_v\xi = T(\tilde{v})(\xi)$. Taking the holonomy homomorphism

$\pi_1(M) \to (\mathfrak{f}_x)_{I^o}$ for ∇ we see that ξ is T-invariant iff $\xi_x \in (\mathfrak{f}_x)_{I^o}$ for any point $x \in M$ and at any arbitrary fixed point x_o the vector ξ_{x_o} is invariant under the holonomy homomorphism, $\xi_{x_o} \in (\mathfrak{f}_{x_o})_{I^o}^{\pi_1(M)}$.

Take a connection $\omega : TM \to A$ (i.e. $\#_A \circ \omega = \mathrm{id}_{TM}$) and its curvature form $\Omega \in \Omega^2(M; \mathbf{g})$

$$\Omega(X,Y) = [\![\omega(X), \omega(Y)]\!] - \omega([X,Y]).$$

The Chern-Weil homomorphism of A is defined by

$$\begin{aligned} h_A &: I^o(A) \to H(M) \\ (I^o)^k(A) \ni \Gamma &\longmapsto \frac{1}{k!}[\langle \Gamma, \Omega \vee ... \vee \Omega \rangle]. \end{aligned} \tag{1}$$

If there exists a flat connection then $h_A^+ = 0$. The analogous construction can be made analogously for regular Lie algebroid over a foliated manifold (M, F), we must use the algebra of tangential differential forms $\Omega(F)$ and its cohomology $H(F)$ instead of $\Omega(M)$ and $H(M)$.

Theorem 7 *If $A(P)$ is a Lie algebroid of a principal fibre bundle (P, π, M, G), then the diagram commutes*

$$\begin{array}{ccc} I^o(A(P)) & & \\ & \searrow h_{A(P)} & \\ \rho \uparrow & & H(M) \\ & \nearrow h_P & \\ I(G) & & \end{array}$$

If P is connected, then under the identification $I(G) = I^o(A(P))$ we have $h_P = h_{A(P)}$.

1.4 The Chern-Weil homomorphism for pairs of Lie algebroids

Take a pair of Lie algebroids (A, L) on a manifold M and assume that A is transitive (we may assume only that A is regular). let $0 \to \mathbf{g} \to A \overset{\#_A}{\to} TM \to 0$ be the Atiyah sequence of A, $\mathbf{g} = \ker \#_A$.

Definition 8 *By a L-connection in A we mean a linear homomorphism*

$$\nabla : L \to A$$

compatible with the anchors $\#_A \circ \nabla = \#_L$. By a curvature form of ∇ we shall mean the 2-form $\Omega_\nabla \in \Omega^2(L; \mathbf{g})$ defined by

$$\Omega_\nabla(\xi, \eta) = [\![\nabla \circ \xi, \nabla \circ \eta]\!].$$

If $L = TM$ then ∇ is a splitting of the Atiyah sequence $0 \to \mathbf{g} \to A \to TM \to 0$, i.e. a usual connection in A. If $L = T^*M$ is the Lie algebroid of a Poisson manifold $(M, \{\cdot,\cdot\})$ and $A = A(P)$ we have the so-called **contravariant connection** in a principal fibre bundle P, if $A = A(\mathfrak{f})$ we have the so-called contravariant connection in a vector bundle $\mathfrak{f}$ (see Fernandes). If $0 \to L' \to L \to L'' \to 0$ is a extension of Lie algebroids, then any splitting $\nabla : L'' \to L$ is a L''-connection in L (Huebschmann). We add that a flat L-connection in $A(\mathfrak{f})$ is the same as L-covariant derivative $\nabla_\xi \nu$ in a vector bundle $\mathfrak{f}$, $\xi \in \operatorname{Sec} L$, $\nu \in \operatorname{Sec} \mathfrak{f}$, i.e. an operator $\nabla_\xi \nu$ fulfilling the usual Koszul axioms with the following difference:

$$\nabla_\xi (f\nu) = f\nabla_\xi \nu + \#_L(\xi)(f)\,\nu.$$

The following superposition

$$L \overset{\#_L}{\to} TM \overset{\omega}{\to} A$$

(where $\omega : TM \to A$ is a connection in A) is an example of a L-connection in A.

By the Chern-Weil homomorphism of the pair (A, L) we mean

$$h_{L,A} : I^o(A) \to H(L)$$

defined by the formula identical to (1). The image of $h_{L,A}$ is the Pontryagin algebra of the pair (L, A),

$$\operatorname{Pont}(L, A) := \operatorname{Im} h_{L,A}.$$

Theorem 9

$$\begin{array}{ccc} I^o(A) & \xrightarrow{h_A} & H(M) \\ & {\scriptstyle h_{L,A}}\searrow & \downarrow (\#_L)^* \\ & & H(L) \end{array}$$

In particular $(\#_L)^*[\operatorname{Pont} A] = \operatorname{Pont}(L, A)$.

Consider $L = A$, $\nabla = \mathrm{id}_A : A \to A$ is a flat A-connection in A, so $h^+_{A,A} = 0$. Therefore

$$\begin{array}{ccc} I(A) & \xrightarrow{h_A} & H(M) \\ & {\scriptstyle h^+_{A,A}=0}\searrow & \downarrow (\#_L)^* \\ & & H(A) \end{array}$$

$\operatorname{Pont}^* A \subset \ker(\#_L)^*$. In this way we have simply proof of the well known fact concerning principal fibre bundle $\pi : P \to M$, $\operatorname{Pont}(P) \subset \ker \pi^*$.

1.5 Problem

Let $(A, [\![\cdot,\cdot]\!], \#_A)$ and $\mathfrak{f}$ be a transitive Lie algebroid and a vector bundle on a manifold M, respectively. Assume that $F \subset TM$ is a C^∞ constant dimensional involutive distribution and $\mathcal{F}$ – the foliation determined by F. We recall that

A and F give rise to the regular Lie algebroid over (M,F) where we put $A^F := (\#_A)^{-1}[F] \subset A$; see. Its Atiyah sequence is

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A^F \xrightarrow{\#_A^F} F \longrightarrow 0\,,$$

where $\boldsymbol{g}$ is the Lie algebra bundle adjoint of A, and $\#_A^F := \#_A | A^F$. Any representation $T: A \to A(\mathfrak{f})$ of A on $\mathfrak{f}$ restricts to the representation

$$T^F = T|A^F : A^F \longrightarrow A(\mathfrak{f})$$

of A^F on $\mathfrak{f}$.

Lemma 10 For $\mathcal{F}$-basic functions $f^i \in \Omega_b^\circ(M,\mathcal{F})$ and T-invariant cross-sections $\nu_i \in \operatorname{Sec}\mathfrak{f}$, $\sum_i f^i \cdot \nu_i$ is a T^F-invariant cross-section, in other words,

$$\Omega_b^\circ(M,\mathcal{F}) \cdot (\operatorname{Sec}\mathfrak{f})_{I^\circ(T)} \subset (\operatorname{Sec}\mathfrak{f})_{I^\circ(T^F)}\,.$$

In general, the above inclusion can not be replaced by the equality, which means that not every T^F-invariant cross-section is of the form $\sum_i f^i \cdot \nu_i$ for $\mathcal{F}$-basic functions f^i and T-invariant cross-sections ν_i. As an example we can consider the Möbius band with the foliation F by meridians. Equip M with a flat Riemannian structure for which the fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are orthonormal base. Let P be $O(2,\mathbb{R})$-principal bundle of orthonormal frames of TM and $A(P)$ let be the Lie algebroid of P with the Atiyah sequence $0 \to Sk(TM,TM) \to A(P) \to TM \to 0$.

The Pfaffian $Pf \in Sk(2,\mathbb{R})$ is not $O(2,\mathbb{R})$ invariant but there exists an invariant section of $Sk(TM,TM)^*$ of the form $Pf \cdot g$ where g is some basic function equaling zero at one leaf of F.

Definition 11 *Each T^F-invariant cross-section $\nu \in (\operatorname{Sec}\mathfrak{f})_{I^\circ(T^F)}$ not belonging to $\Omega_b^\circ(M,\mathcal{F}) \cdot (\operatorname{Sec}\mathfrak{f})_{I^\circ(T)}$ will be called* singular. *The characteristic class corresponding to any singular cross-section will be also called* singular.

Problem 12 *Find an example of nontrivial singular characteristic class.*

Let P be a G-principal bundle on M, whereas $F \subset TM$ and $\mathcal{F}$ are as above.

Definition 13 By the *tangential Chern-Weil homomorphism of P over (M,F)* we mean the Chern-Weil homomorphism $h_{A(P)^F}$ of the regular Lie algebroid $A(P)^F$.

Let $\mathfrak{g}$ be the right Lie algebra of G and $(\bigvee \mathfrak{g}^\star)_{I(G)}$ — the algebra of G-invariant elements.

Theorem 14 *Let $G_\circ$ be the connected component containing the unit of G. If each $G_\circ$-invariant element of $\bigvee \mathfrak{g}^\star$ is G-invariant, then the domain of the homomorphism $h_{A(P)^F}$ is equal to $\Omega_b^\circ(M,\mathcal{F}) \cdot I^\circ(A(P))$ ($\cong \Omega_b^\circ(M,\mathcal{F}) \cdot (\bigvee \mathfrak{g}^\star)_{I(G)}$ when P is connected).*

Consider a nonorientable Riemannian vector bundle $\mathfrak{f}$ of rank $2m$ and a connected $O(2m;\mathbb{R})$-principal bundle P of orthonormal frames of $\mathfrak{f}$, and the transitive Lie algebroid $A = A(P)$. We have $\mathrm{Pont}^{2m}(P) = \mathrm{Pont}^{2m}(A) = 0$ (and, of course, $\mathrm{Pont}^{k}(P) = 0$ for $k > 2m$).

Problem 15 *Using singular characteristic classes find an example of a nonorientable Riemannian vector bundle f and an involutive distribution F with orientable leaves for which*

$$\mathrm{Pont}^{2m}(A^F) \neq 0.$$

2 Secondary flat characteristic classes for Lie algebroids

2.1 Classical theory for principal bundles

Consider the triple (P, P', ω) where $P = (P, \pi, M, G)$ is a G-principal fibre bundle, P' is his H-reduction ($H \subset G$ is a closed Lie subgroup of G), and ω is a flat connection in P. Equivalently (according to Lehmann approach) we can consider two ideals J_1 and J_2 in the algebra of invariant polynomials $I(G)$

$$\begin{aligned} J_1 &= I^+(G), \\ J_2 &= \ker(I(G) \to I(H)). \end{aligned}$$

The characteristic homomorphism

$$\Delta_{\#\, P,P',\omega} = \Delta_{\#} : H^*(\mathfrak{g}, H) \longrightarrow H_{dR}(M)$$

is one of the most important notion in differential geometry of principal bundles. The cohomology classes from the image of the homomorphism $\Delta_{\#\, P,P',\omega}$ are called the *secondary flat characteristic classes* of (P, P', ω).

Since this homomorphism is an invariant of the class of homotopic H-reductions and measures the incompatibility of the flat structure ω with a given H-reduction, the nontriviality of $\Delta_{\#\, P,P',\omega}$ implies that there is no homotopic changing of P' containing the connection ω.

We recall that $H^*(\mathfrak{g}, H)$, called the *relative Lie algebra cohomology,* is the cohomology space of the complex

$$\left(\left(\bigwedge (\mathfrak{g}/\mathfrak{h})^*\right)_I, d^H\right)$$

where $\left(\bigwedge (\mathfrak{g}/\mathfrak{h})^*\right)_I$ is the space of invariant elements with respect to the adjoint representation of the Lie group H and the differential d^H is defined via the formula

$$\left\langle d^H(\psi), [w_1] \wedge \ldots \wedge [w_k] \right\rangle = \sum_{i<j} (-1)^{i+j} \left\langle \psi, [[w_i, w_j]] \wedge [w_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [w_k] \right\rangle$$

for $\psi \in \bigwedge^k (\mathfrak{g}/\mathfrak{h})^*_I$ and $w_i \in \mathfrak{g}$. The homomorphism $\Delta_{\#\, P,P',\omega}$ on the level of forms is given by the following direct formula

$$(\Delta\psi)(x; w_1 \wedge ... \wedge w_k) = \langle \psi, [\omega(z; \tilde{w}_1)] \wedge ... \wedge [\omega(z; \tilde{w}_k)] \rangle$$

where $x \in M$, $z \in P'$, $\pi z = x$, $w_i \in T_x M$, $\tilde{w}_i \in T_z P'$, $\pi'_* \tilde{w}_i = w_i$ ($\omega : P \to \mathfrak{g}$ is the connection form of a given flat connection).

The relative Lie algebra cohomology $H(\mathfrak{g}, H)$ is well known (Kamber-Tondeur, Godbillon). For example

$$H(gl(n,\mathbb{R}), O(n)) \cong \bigwedge (y_1, y_3, ..., y_{2n'-1})$$

where n' is the largest odd integer $\leq n$, and we have by definition $y_{2k-1} \in H^{4k-3}(gl(n,\mathbb{R}), O(n))$ are represented by the multilinear trace form $\tilde{y}_k \in \bigwedge (gl(n,\mathbb{R})/Sk(n,\mathbb{R}))^*$

$$\tilde{y}_{2k-1}([A_1], ..., [A_{4k-3}]) = \sum_{\sigma \in S_{4k-3}} \operatorname{sgn}\sigma \ \operatorname{tr}\left(\tilde{A}_{\sigma(1)} \circ ... \circ \tilde{A}_{\sigma(4k-3)}\right) \qquad (2)$$

where $\tilde{A}_i = \frac{1}{2}\left(A_i + A_i^T\right)$ - the symmetrization of A_i.

$$\begin{aligned} H(gl(2m-1,\mathbb{R}), SO(2m-1)) &= H(gl(2m-1,\mathbb{R}), O(2m-1)) \\ &= \bigwedge (y_1, y_3, ..., y_{2n'-1}) \end{aligned}$$

$$H(gl(2m,\mathbb{R}), SO(2m)) = \bigwedge (y_1, y_3, ..., y_{2n'-1}, (Sf)_{2m})$$

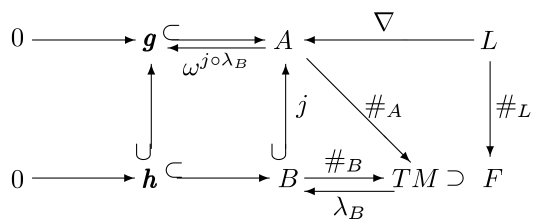
where $(Sf)_{2m}$ is the called skew symmetric Pfaffian.

2.2 Algebroid's generalization

Consider a triple

$$(A, B, \nabla)$$

consisting of transitive Lie algebroid A and its transitive Lie subalgebroid B and (nonregular, in general) Lie algebroid L on M and a flat L-connection $\nabla : L \to A$. In the diagram below $\lambda_B : TM \to B$ means an arbitrary auxiliary connection in B. Then $j \circ \lambda_B : TM \to A$ is a connection in A. Let $\omega^{j\circ\lambda_B} : A \to \boldsymbol{g}$ be its connection form.



(D)

Clearly, A and B can be regular Lie algebroids over the same foliated manifold (M, F). The constructed characteristic homomorphism for the triple (A, B, ∇) is measuring the incompatibility of the flat structure with a given subalgebroid and has homotopic properties in analogy to to the classical case of principal bundles.

Example 16 *1.* For $L = TM$ we obtain the case in which the connection ∇ is a connection in A,

2. For $L = TM$ and $A = TP/G$ and $B = TP'/H$ *(*P' is an H-reduction of P*)* we obtain the classical case equivalent to the standard case of principal bundles.

3. For $L = A$ and $\nabla = \mathrm{id}_A$ we consider only the Lie algebroid A and its Lie subalgebroid B. This case produces a characteristic homomorfism for the inclusion $\mathrm{B} \subset \mathrm{A}$, in particular for the inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, and in particular for inclusion of principal bundles $P' \subset P$ which finally produces a new theorem for principal bundles.

It is the main goal of my talk.

4. Let $\mathfrak{f}$ be a vector bundle equipped with a Riemannian metric g. If $A = A(\mathfrak{f})$ (i.e. A is the Lie algebroid of the principal bundle of frames $L\mathfrak{f}$) and $B \subset A$ is a Riemannian reduction $B = A(\mathfrak{f}, \{g\})$ (more precisely B is the Lie algebroid of the principal bundle of orthogonal frames). We obtain the case equivalent to the one considered by M. Crainic of the characteristic exotic characteristic classes for a representation of any Lie algebroid L in a vector bundle.

To construct the characteristic homomorphism for (A, B, ∇) we notice that for a general connection $\nabla : L \to A$ does not exist a suitable notion of a connection form. The connection form was used in the direct formula for the classical case. We must do a characteristic homomorphism without any "connection form". In the classical case (P, P', ω) take auxiliarily a connection λ in P' and extend it to a connection in P. Let $\omega^\lambda : P \to \mathfrak{g}$ be the connection form. Then it appears that the characteristic homomorphism for (P, P', ω) can be defined on the lewel of differential forms via

$$(\Delta_* \Psi)_x (w_1 \wedge ... \wedge w_k) = \left\langle \Psi_x, \left[-\omega^\lambda \circ \hat{w}_1\right] \wedge ... \wedge \left[-\omega^\lambda \circ \hat{w}_k\right] \right\rangle$$

for $\Psi \in \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$, $x \in M$, $w_i \in T_x M$ and $\hat{w}_i$ being the horizontal lifting of w_i with respect to the flat connection ω taken at the beginning.

In the general case (A, B, ∇) we define the homomorphism

$$\omega_{B,\nabla} : L \longrightarrow \boldsymbol{g}/\boldsymbol{h}$$

given by

$$\omega_{B,\nabla}(w) = \left[- \left(\omega^{j \circ \lambda_B} \circ \nabla\right)(w)\right].$$

Remark 17 *(1) It is important observation that* $\omega_{B,\nabla}$ *does not depend on the choice of an auxiliary connection* λ_B $(\lambda_B - \lambda'_B$ *takes values in* $\boldsymbol{h})$

(2) $\omega_{B,\nabla} = 0$ *if* ∇ *takes values in* B.

Definition 18 *Define the homomorphism of algebras*

$$\Delta : \mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \longrightarrow \Omega(L) \tag{3}$$

by

$$(\Delta\Psi)(x; w_1 \wedge ... \wedge w_k) = \langle \Psi_x, \omega_{B,\nabla}(w_1) \wedge ... \wedge \omega_{B,\nabla}(w_k) \rangle,$$

$\Psi \in \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$, $x \in M$, $w_i \in L_{|x}$.

Observe that Δ can be written as superposition $\Delta = \nabla^* \circ \Delta_o$,

$$\Delta : \mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \xrightarrow{\Delta_o} \Omega(A) \xrightarrow{\nabla^*} \Omega(L)$$

where ∇^* is the pullback of forms and Δ_o is the homomorphism given for particular case of flat connection $\nabla = \mathrm{id}_A$, so that

$$\begin{aligned}(\Delta_o\Psi)(x; \upsilon_1 \wedge ... \wedge \upsilon_k) &= \langle \Psi_x, \omega_{B,id_A}(\upsilon_1) \wedge ... \wedge \omega_{B,id_A}(\upsilon_k) \rangle \\ &= \left\langle \Psi_x, \left[-\omega^{j\circ\lambda_B}(\upsilon_1)\right] \wedge ... \wedge \left[-\omega^{j\circ\lambda_B}(\upsilon_1)\right] \right\rangle\end{aligned}$$

for $\Psi \in \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$, $x \in M$, $\upsilon_i \in A_{|x}$.

In the algebra $\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*$ we distinguish the subalgebra $\left(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\right)_{I^o(B)}$ of invariant sections with respect to an adjoint representation of B in $\bigwedge (\boldsymbol{g}/\boldsymbol{h})^*$. $\Psi \in \left(\mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*\right)_{I^o(B)}$ if and only if

$$(\gamma_B \circ \xi) \langle \Psi, [\nu_1] \wedge ... \wedge [\nu_k] \rangle = \sum_{j=1}^{k} (-1)^{j-1} \langle \Psi, [[j \circ \xi, \nu_j]] \wedge [\nu_1] \wedge ...\hat{\jmath}... \wedge [\nu_k] \rangle$$

for all $\xi \in \mathrm{Sec}\, B$ and $\nu_j \in \mathrm{Sec}\, \boldsymbol{g}$. In particular, if $\Psi \in \left(\mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*\right)_{I^o(B)}$ then for $X \in \mathfrak{X}(F)$ and $\xi = \lambda_B \circ X$ we have

$$X \langle \Psi, [\nu_1] \wedge ... \wedge [\nu_k] \rangle = \sum_{j=1}^{k} (-1)^{j-1} \langle \Psi, [[j \circ \lambda_B \circ X, \nu_j]] \wedge [\nu_1] \wedge ...\hat{\jmath}... \wedge [\nu_k] \rangle . \tag{4}$$

In the space $\left(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\right)_{I^o(B)}$ of invariant cross–sections we have a differential $\bar{\delta}$ defined by

$$\left\langle \bar{\delta}\Psi, [\nu_1] \wedge ... \wedge [\nu_k] \right\rangle = -\sum_{i<j} (-1)^{i+j} \langle \Psi, [[\nu_i, \nu_j]] \wedge [\nu_1] \wedge ...\hat{\imath}...\hat{\jmath}... \wedge [\nu_k] \rangle ,$$

$\Psi \in \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*_{I^o(B)}$, $\nu_i \in \mathrm{Sec}\, \boldsymbol{g}$.

$H(\boldsymbol{g}, B)$ is the cohomology algebra of $\left(\left(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\right)_{I^o(B)}, \bar{\delta}\right)$.

Theorem 19 *The homomorphism Δ commutes with the differentials $\bar{\bar{\delta}}$ and d_L.*

Corollary 20 *Δ and Δ_o induce the homomorphisms in cohomology*

$$\Delta_{\#(A,B,\nabla)} : H\left(\boldsymbol{g}, B\right) \overset{\Delta_{o\,\#}}{\longrightarrow} H\left(A\right) \overset{\nabla^{\#}}{\longrightarrow} H\left(L\right). \tag{5}$$

The map $\Delta_{\#}$ is called the *characteristic homomorphism* of the triple (A, B, ∇). Of course, Δ_o is the characteristic homomorphism of the pair $\left(A, B\right)$, $B \subset A$.

Remark 21 We see that for a pair of transitive Lie algebroids (A, B), $B \subset A$, [they can be both regular over the same foliation] and for an arbitrary element $\kappa \in H\left(\boldsymbol{g}, B\right)$ there exists a "universal" cohomology class $\Delta_{o\,\#}\left(\kappa\right) \in H\left(A\right)$ such that for any (nonregular in general) Lie algebroid L on M and a flat L-connection $\nabla : L \to A$ the equality holds

$$\Delta_{\#}\left(\kappa\right) = \nabla^{\#}\left(\Delta_{o\,\#}\left(\kappa\right)\right).$$

Problem 22 *Is the characteristic homomorphism $\Delta_{o\,\#} : H\left(\boldsymbol{g}, B\right) \longrightarrow H\left(A\right)$ a monomorphism for a given $B \subset A$?*

The characteristic homomorphism $\Delta_{\#(A,B,\nabla)} : H\left(\boldsymbol{g}, B\right) \longrightarrow H\left(L\right)$ has functoriality property and is invariant under homotopic subalgebroids.

Definition 23 *Two Lie subalgebroids B_0, $B_1 \subset A$ (both over (M, F)) are said to be* homotopic *if there exists a Lie subalgebroid $B \subset T\mathbb{R}\times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times F)$ such that for $t \in \{0, 1\}$*

$$\upsilon_x \in B_{t|x} \Longleftrightarrow (\theta_t, \upsilon_x) \in B_{|(t,x)}. \tag{6}$$

B is called a subalgebroid joining B_0 with B_1.

This relation is closely related to the relation of homotopic subbundles of a principal bundle.

Theorem 24 (The first homotopy independence) *If B_0, $B_1 \subset A$ are homotopic subalgebroids of A and $\nabla : L \to A$ is a flat L-connection in A, characteristic homomorphisms $\Delta_{\#} : H\left(\boldsymbol{g}, B_0\right) \to H\left(L\right)$ and $\Delta_{\#} : H\left(\boldsymbol{g}, B_1\right) \to H\left(L\right)$ are equivalent in this sense that there exists an isomorphism $\alpha : H\left(\boldsymbol{g}, B_0\right) \overset{\simeq}{\longrightarrow} H\left(\boldsymbol{g}, B_1\right)$ of algebras such that the diagram*

$$\begin{array}{ccc} H(\boldsymbol{g}, B_0) & & \\ \downarrow \alpha\,\simeq & \searrow^{\Delta_{\#}} & \\ & & H_L(M) \\ & \nearrow_{\Delta_{\#}} & \\ H(\boldsymbol{g}', B_1) & & \end{array}$$

commutes.

Definition 25 *Let H_0, $H_1 : L' \to L$ be homomorphisms of Lie algebroids. By a* homotopy joining H_0 *to* H_1 *we mean a homomorphism of Lie algebroids*

$$H : T\mathbb{R} \times L' \longrightarrow L$$

such that $H(\theta_0, \cdot) = H_0$ and $H(\theta_1, \cdot) = H_1$ where θ_0 and θ_1 are null vector tangent bundle of R at 0 and 1, respectively. We say that H_0 and H_1 are homotopic *and write $H_0 \sim H_1$.*

The homotopy $H : T\mathbb{R} \times L' \longrightarrow L$ determines a chain homotopy operator [**?**] which implies that $H_0^{\#} = H_1^{\#} : H_L(M) \to H_{L'}(M')$.

Theorem 26 (The second homotopy independence) *If ∇_0, $\nabla_1 : L \to A$ are homotopic flat L-connections of A then characteristic homomorphisms are equal $\Delta_{\#(A,B,\nabla_0)} = \Delta_{\#(A,B,\nabla_1)}$.*

2.3 Application to principal bundles

Taking a **connected** principal bundle $P = P(M, G)$ with a structure Lie group G and a **connected** H-reduction $P' \subset P$ and using the isomorphism of algebras ρ we have the commutative diagram

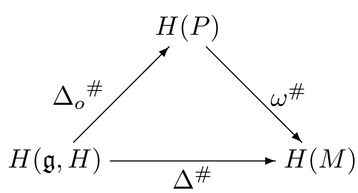
$$\begin{array}{ccccc} H(\mathfrak{g}, H) & \stackrel{\Delta_0}{\longrightarrow} & H^r_{dR}(P) & \longrightarrow & H_{dR}(P) \\ \cong\downarrow \rho & & \| & & \\ H(\boldsymbol{g}, A(P)) & \stackrel{\Delta_0}{\longrightarrow} & H_{A(P)}(M) & & . \end{array}$$

as well as we obtain

Theorem 27 *If G is a compact connected group and P' is a connected H-reduction in an G-principal bundle P, then there exists a "universal" homomorphism $\Delta_o^{\#}$ acting from the algebra $H(\mathfrak{g}, H)$ to the total cohomology $H_{dR}(P)$,*

$$\Delta_o^{\#} : H(\mathfrak{g}, H) \longrightarrow H_{dR}(P).$$

In the case of flat principal bundle P for every flat connection ω in the bundle P the characteristic homomorphism $\Delta^{\#} : H(\mathfrak{g}, H) \longrightarrow H_{dR}(M)$ is factorized by $\Delta_o^{\#}$, i.e. the diagram below commutes



where $\omega^{\#}$ on the level of right-invariant forms Ω^r is given as the pullback of forms,

$$\omega^* : \Omega^r(P) \longrightarrow \Omega(M),$$

$$\omega^*(\phi)(x; u_1 \wedge ... \wedge u_k) = \phi(z; \tilde{u}_1 \wedge ... \wedge \tilde{u}_k)$$

where $z \in P_{|x}$, $\tilde{u}_i$ is the horizontal lift of u_i. [Recall that $H^r_{dR}(P) := H(\Omega^r(P)) \simeq H_{dR}(P)$.] In the general case (noncompact or nonconnected Lie group G) there exists a homomorphism $\Delta^{\#}_o : H(\mathfrak{g}, H) \longrightarrow H^r_{dR}(P)$ of algebras which factorizes the characteristic homomorphism for every flat connection. The homomorphism $\Delta^{\#}_o$ on the level of forms is given by the following direct formula:

$$(\Delta_o \psi)(z; w_1 \wedge ... \wedge w_k) = \langle \psi, [-\omega(z; w_1)] \wedge ... \wedge [-\omega(z; w_k)] \rangle,$$

where ω is the form of a connection on P extending an arbitrary connection on P'.

It seems to be interesting the following question:
— Is the homomorphism $\Delta^{\#}_o : H(\mathfrak{g}, H) \longrightarrow H(P)$ a monomorphism?

2.4 Application to finitely dimensional Lie algebras

A pair $(\mathfrak{h}, \mathfrak{g})$, $\mathfrak{h} \subset \mathfrak{g}$, of finite dimensional Lie algebras, we have a characteristic homomorphism

$$\Delta_o : H(\mathfrak{g}, \mathfrak{h}) = \left(\bigwedge (\mathfrak{g}/\mathfrak{h})^* \right)_{I^o} \rightarrow H(\mathfrak{g})$$

$$(\Delta_o \psi)(w_1 \wedge ... \wedge w_k) = (-1)^k \langle \psi, [w_1] \wedge ... \wedge [w_k] \rangle$$

($H(\mathfrak{g}, \mathfrak{h}) = H(\mathfrak{g}, H)$ for aritrary connected Lie group having $\mathfrak{h}$ as its Lie algebra). The homomorphism Δ_o can be nontrivial. For example for $\mathfrak{g} = gl(n, \mathbb{R})$ and $\mathfrak{h} = \mathfrak{so}(n)$ $(= (Sk(n, \mathbb{R})\ )$ the trace formula tr : $\mathfrak{g}/\mathfrak{h} \rightarrow \mathbb{R}$ is invariant and gives nontrivial element in the cohomology.

If Δ_o is not trivial than the identical homomorphism id : $\mathfrak{g} \rightarrow \mathfrak{g}$ is not homotopic to any homomorphism from $\mathfrak{g}$ to $\mathfrak{g}$.

Let $\mathfrak{h}_1$ be the next Lie algebra and $\phi : \mathfrak{h}_1 \rightarrow \mathfrak{g}$ be any homomorphism of Lie algebras. If $\Delta_{(\mathfrak{g}, \mathfrak{h}, \phi)} = \phi^{\#} \circ \Delta_o$ is not trivial then ϕ can not be homotopic to any homomorphism of Lie algebras $\mathfrak{h}_1 \rightarrow \mathfrak{h}$.

2.5 Crainic characteristic classes

Take a vector bundle $\mathfrak{f}$ and its Lie algebroid $A(\mathfrak{f})$ as well as a Riemannian metric h in $\mathfrak{f}$. The metric h yields the Lie subalgebroid $B = A(\mathfrak{f}, \{h\})$. We recall that $\mathcal{L} \in \mathrm{Sec}(A(\mathfrak{f}, \{h\})) \iff \mathcal{L} \in \mathrm{Sec}(A(\mathfrak{f}))$ and for each sections $\xi, \eta \in \mathrm{Sec}(\mathfrak{f})$ the formula holds

$$h(\mathcal{L}(\xi), \eta) = h(\xi, \eta) - h(\xi, \mathcal{L}(\eta)).$$

Two Lie subalgebroids $B_i = A(\mathfrak{f}, \{h_i\})$, $i = 1, 2$, corresponding to Riemannian metrics h_i are homotopic Lie subalgebroids.The Atiyah sequences for $A(\mathfrak{f})$ and $A(\mathfrak{f}, \{h\})$ are

$$0 \longrightarrow End(\mathfrak{f}) \longrightarrow A(\mathfrak{f}) \longrightarrow TM \longrightarrow 0,$$
$$0 \longrightarrow Sk(\mathfrak{f}) \longrightarrow A(\mathfrak{f}, \{h\}) \longrightarrow TM \longrightarrow 0.$$

If the vector bundle $\mathfrak{f}$ is nonorientable (nonoriented), then the characteristic homomorphism $\Delta_{\#} : H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\})) \to H(L)$ corresponding to $(A(\mathfrak{f}), A(\mathfrak{f}, \{h\}), \nabla)$ produces the Crainic characteristic classes. Indeed, using the isomorphism κ from Theorem **??** and the classical relation (Kamber-Tondeur, Godbillon) we have

$$H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\})) \cong H(gl(n, \mathbb{R}), O(n)) \cong \bigwedge(y_1, y_3, ..., y_{2n'-1})$$

where n' is the largest odd integer $\leq n$, and we have by definition $y_{2k-1} \in H^{4k-3}(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\}))$ are represented by the multilinear trace form $\tilde{y}_k \in \Gamma\left(\bigwedge(End(\mathfrak{f})/Sk(\mathfrak{f}))^*\right)$

$$\tilde{y}_{2k-1}([A_1], ..., [A_{4k-3}]) = \sum_{\sigma \in S_{4k-3}} \operatorname{sgn}\sigma \ \operatorname{tr}\left(\tilde{A}_{\sigma(1)} \circ ... \circ \tilde{A}_{\sigma(4k-3)}\right) \qquad (7)$$

where $\tilde{A}_i = \frac{1}{2}(A_i + A_i^*)$ is the symmetrization of A_i with respect to the inner scalar product induced by the metric h.

In the case of oriented vector bundle with a metric volume v, the matric h and v induce an $SO(n, \mathbb{R})$-reduction $L(\mathfrak{f}, \{h, \mathrm{v}\})$ of the frames bundle $L\mathfrak{f}$ of $\mathfrak{f}$. The Atiyah sequences for $A(\mathfrak{f}, \{h, \mathrm{v}\})$ is

$$0 \longrightarrow Sk(\mathfrak{f}) \longrightarrow A(\mathfrak{f}, \{h, \mathrm{v}\}) \longrightarrow TM \longrightarrow 0.$$

Consider the characteristic homomorphism $\Delta_{\#} : H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\})) \to H(L)$ corresponding to $(A(\mathfrak{f}), A(\mathfrak{f}, \{h, \mathrm{v}\}), \nabla)$. Therefore

- if $n = 2m - 1$ is odd, then

  $$H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\})) \cong H(gl(2m-1, \mathbb{R}), SO(2m-1)) \cong H(gl(2m-1, \mathbb{R}), O(2m-1)),$$

- if $n = 2m$ is even, then

  $$H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\})) \cong H(gl(2m, \mathbb{R}), SO(2m)) \cong \bigwedge\left(y_1, y_3, ..., y_{2n'-1}, \left[\overline{\mathrm{Sf}}\right]_{2m}\right)$$

  where where n' is the largest odd integer $< 2m$, $y_{2k-1} \in H^{4k-3}(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\}))$ are represented by the multilinear trace form $\tilde{y}_k \in \Gamma\left(\bigwedge(End(\mathfrak{f})/Sk(\mathfrak{f}))^*\right)$ defined by (7), and $\left[\overline{\mathrm{Sf}}\right]_{2m} \in H^{2m}(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\}))$ is represented by skew Pffafian $\overline{\mathrm{Sf}}_{2m} \in \Gamma\left(\bigwedge^{2m}(End(\mathfrak{f})/Sk(\mathfrak{f}))^*\right)$,

  $$\begin{aligned}\overline{\mathrm{Sf}}_{2m}([f_1], ..., [f_{2m}]) &= d\left(\overline{\mathrm{Sf}}\right)\left(\widetilde{f}_1, ..., \widetilde{f}_{2m}\right) \\ &= \sum_{\substack{\sigma \in S_{2m} \\ \sigma(1)<\sigma(2) \\ \sigma(3)<...<\sigma(2m)}} \operatorname{sgn}\sigma \ \mathrm{Sf}\left(\left[\widetilde{f}_{\sigma(1)}, \widetilde{f}_{\sigma(2)}\right], \widetilde{f}_{\sigma(3)}, ..., \widetilde{f}_{\sigma(2m)}\right)\end{aligned}$$

where $f_1, ..., f_{2m} \in \mathrm{End}(\Gamma(\mathfrak{f}))$, d is the differential on the algebra $\bigwedge (End(\mathfrak{f}))^*$,

$$d(\phi)(f_1, ..., f_n) = \sum\nolimits_{p<q} (-1)^{p+q} \phi([f_p, f_q], f_1, ...\hat{p}...\hat{q}..., f_n)$$
$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1)<\sigma(2) \\ \sigma(3)<...<\sigma(n)}} \mathrm{sgn}(\sigma)\ \phi\left(\left[f_{\sigma(1)}, f_{\sigma(2)}\right], f_{\sigma(3)}, ..., f_{\sigma(n)}\right)\ ,$$

and $\overline{\mathrm{Sf}} \in \Gamma\left(\bigwedge^{2m-1} (End(\mathfrak{f})/Sk(\mathfrak{f}))^*\right)$ is described by the formula

$$\overline{\mathrm{Sf}}(f_1, ..., f_{2m-1}) = \sum_{\sigma \in S_{2m-1}} \mathrm{sgn}\,\sigma\, \overline{\mathrm{Pf}}\left(f_{\sigma(1)}, \left[f_{\sigma(2)}, f_{\sigma(3)}\right], ..., \left[f_{\sigma(2m-2)}, f_{\sigma(2m-1)}\right]\right)$$
$$= \sum_{\sigma \in S_{2m-1}} \mathrm{sgn}\,\sigma\ \left(e, \alpha\left(f_{\sigma(1)}\right) \wedge \alpha\left[f_{\sigma(2)}, f_{\sigma(3)}\right] \wedge ... \wedge \alpha\left[f_{\sigma(2m-2)}, f_{\sigma(2m-1)}\right]\right),$$

$\overline{\mathrm{Pf}} \in \mathrm{Sym}^m(\mathrm{End}(\mathfrak{f}); C^\infty(M))$, $\overline{\mathrm{Pf}}(f_1, ..., f_m) = (e, \alpha(f_1) \wedge ... \wedge \alpha(f_m))$, e is a non-zerro cross-section of $\Gamma\left(\bigwedge^{\mathrm{top}} \mathfrak{f}\right)$, $\alpha : \Gamma(\mathrm{End}\,\mathfrak{f}) \to \Gamma\left(\bigwedge^2 \mathfrak{f}\right)$ is given by $(\alpha(\varphi), X \wedge Y) = (\varphi X, Y)$, $\varphi \in \Gamma(\mathrm{End}\,\mathfrak{f})$, $X, Y \in \Gamma(\mathfrak{f})$.

Theorem 28

$$\Delta_\#(\tilde{y}_{2k-1}) = (-1)^{\frac{(k+1)(k+2)}{2}} \cdot \frac{(2k-1)!}{2^{2k-1} \cdot k! \cdot (k-1)!}\ [u_{2k-1}(\mathfrak{f}, \nabla)]$$

where $u_{2k-1}(\mathfrak{f}, \nabla)$ represent the Crainic characteristic classes.

Explicit formula use any metric h in $\mathfrak{f}$ and the symmetric-values form $\theta = \nabla - \nabla^h$ where ∇ is any flat L-connection in $\mathfrak{f}$ and ∇^h is the adjoint L-connection induced by the metric h.

∇^h is defined by

$$\#_L(a)(h(\xi, \eta)) = h(\nabla_a \xi, \eta) + h\left(\xi, \nabla_a^h \eta\right)$$

dla dowolnych $a \in \mathrm{Sec}(A)$, $\xi, \eta \in \Gamma(\mathfrak{f})$.

$$u_{2k-1}(\mathfrak{f}, \nabla) = (-1)^{\frac{k(k+1)}{2}} cs_k\left(\nabla, \nabla^h\right),$$

k - odd (we add that only odd k gives nontrivial classes for real $\mathfrak{f}$) and

$$cs_k\left(\nabla, \nabla^h\right) = \int_0^1 ch_k\left(\nabla^{aff}\right) = (-1)^{k-1} \frac{k! \cdot (k-1)!}{(2k-1)!} \cdot \mathrm{tr}\left(\underbrace{\theta \wedge ... \wedge \theta}_{2k-1}\right) \in \Omega^{2k-1}(L)$$

for the affine combination $\nabla^{aff} = t \cdot \nabla + (1-t) \cdot \nabla^h$ and $ch_k\left(\nabla^{aff}\right) = \mathrm{tr}\left(R^{\nabla^{aff}}\right)^k$.

We add that Crainic have lost the skew Pfaffian $\left[\overline{\mathrm{Sf}}\right]_{2m}$ in oriented vector bundle of even rank.