Universal secondary characteristic homomorphism of pairs of regular Lie algebroids $B \subset A$

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1 Abstract

There are three theories of secondary characteristic classes on Lie algebroids, first was given by J.Kubarski [1991], [K_2], [K_5], [K_6], the next was given by R.L.Fernandes [2002] [F] and the last by M.Crainic [2003] [Cr]. These theories generalize the classical secondary characteristic classes for principal bundles and foliations.

The purpose of this lecture is to present a universal secondary characteristic homomorphism

$$h_{(B,A)} : H(\boldsymbol{g}, B) \to H(A)$$

for a pair (B, A), $B \subset A$, of regular Lie algebroids (nonflat in general) over the same foliated manifold (M, F) [especially for transitive ones] where $\boldsymbol{g} = \ker \#_A$. This homomorphism $h_{(B,A)}$ has the following property: for an arbitrary (nonregular in general) Lie algebroid L on M and for a flat L-connection in A (i.e. a homomorphism of Lie algebroids) $\nabla : L \to A$, the superposition

$$\nabla^{\#} \circ h_{(B,A)} : H(\boldsymbol{g}, B) \to H(L)$$

describes — for the suitable (B, A, ∇) —

a) the classical secondary "flat" characteristic homomorphism for a principal bundle with a reduction and flat connection, see for example [K+T$_1$]-[K+T$_4$];

b) the Crainic characteristic classes for a representation of L on a vector bundle [Cr].

2 Application to principal bundles

We start with an application of the universal homomorphism to principal bundles, probably not mentioned earlier in the literature.

Theorem 1 *If G is a compact connected group and P' is a connected H-reduction in a G-principal bundle $P = P(M, G)$ (nonflat in general), then there exists a "universal" homomorphism of algebras*

$$h_{(P',P)} : H(\mathfrak{g}, H) \longrightarrow H_{dR}(P).$$

given on the level of forms by the following direct formula:

$$\left(\Delta_{(P',P)}\psi\right)(z; w_1 \wedge ... \wedge w_k) = \langle \psi, [-\omega(z; w_1)] \wedge ... \wedge [-\omega(z; w_k)] \rangle,$$

where ω is the form of a connection on P extending an arbitrary connection in P'.

For every flat connection λ in the bundle P the classical secondary characteristic homomorphism $h_{(P',P,\lambda)} : H(\mathfrak{g}, H) \longrightarrow H_{dR}(M)$ for (P', P, λ) is factorized by $h_{(P',P)}$, i.e. the diagram below commutes

$$\begin{array}{ccc} & H_{dR}(P) & \\ {}^{h_{(P',P)}}\nearrow & & \searrow^{\lambda^{\#}} \\ H(\mathfrak{g}, H) & \xrightarrow[h_{(P',P,\lambda)}]{} & H_{dR}(M) \end{array}$$

where $\lambda^{\#}$ on the level of right-invariant forms is given as the pullback of forms,

$$\lambda^* : \Omega^r(P) \longrightarrow \Omega(M),$$
$$\lambda^*(\phi)(x; u_1 \wedge ... \wedge u_k) = \phi(z; \tilde{u}_1 \wedge ... \wedge \tilde{u}_k)$$

and $z \in P_{|x}$, $\tilde{u}_i$ *is the horizontal lift of* u_i. *[Recall that* $H^r_{dR}(P) := H(\Omega^r(P)) \simeq H_{dR}(P)$.*]*

In the general case (noncompact or nonconnected Lie group G*) we must change the algebra* $H_{dR}(P)$ *for the algebra cohomology of right-invariant vector fields* $H^r(P)$.

It seems to be interesting the following question:

— Is the homomorphism $h_{(P',P)} : H(\mathfrak{g}, H) \longrightarrow H^r(P)$ a monomorphism?

3 Definition of the homomorphism for a pair $B \subset A$ of regular Lie algebroids and homotopy property

Definition 2 *Consider a regular Lie algebroid*$(A, [\![\cdot,\cdot]\!], \#_A)$ *and its Lie subalgebroid* $(B, [\![\cdot,\cdot]\!], \#_B)$, $B \subset A$, *both over the same regular foliated manifold* (M, F),

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{g} & \underset{\omega^{j\circ\lambda}}{\overset{i}{\rightleftarrows}} & A & \overset{\#_A}{\longrightarrow} & F & \longrightarrow & 0 \\
 & & \uparrow\cup & & \uparrow\cup\, j & & \| & & \\
0 & \longrightarrow & \boldsymbol{h} & \longrightarrow & B & \underset{\lambda}{\overset{\#_B}{\rightleftarrows}} & F & \longrightarrow & 0
\end{array}$$

There exist a homomorphism of algebras (see [B+K])

$$h_{(B,A)} : H(\boldsymbol{g}, B) \to H(A)$$

in which

$$H(\boldsymbol{g}, B) = H\left(\left(\operatorname{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\right)_I, \delta\right)$$

is the relative cohomology algebra of the complex $((\operatorname{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*)_I, \delta)$ *of the* B*-invariant cross-sections of the vector bundle* $\bigwedge (\boldsymbol{g}/\boldsymbol{h})^*$ *with respect to the adjoint representation of* B *in* $\bigwedge (\boldsymbol{g}/\boldsymbol{h})^*$ *induced by* $ad_{B,\boldsymbol{g}}$ *of* B *in the vector bundle* $\boldsymbol{g}/\boldsymbol{h}$ *defined by*

$$ad_{B,\boldsymbol{g}}(\xi)([\nu]) = [[\![\xi, \nu]\!]], \ \xi \in \operatorname{Sec} B, \ \nu \in \operatorname{Sec} \boldsymbol{g}.$$

The differential δ *for invariant cross-sections is defined by*

$$\langle \delta\Psi, [\nu_1] \wedge ... \wedge [\nu_{k+1}] \rangle = \sum_{i<j} (-1)^{i+j+1} \langle \Psi, [[\nu_i, \nu_j]] \wedge [\nu_1] \wedge ...\hat{\imath}...\hat{\jmath}... \wedge [\nu_{k+1}] \rangle$$

$\Psi \in \operatorname{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*_{I^\circ(B)}$, $\nu_i \in \operatorname{Sec} \boldsymbol{g}$.

The characteristic secondary characteristic homomorphism $h_{(B,A)}$ on the level of differential forms is defined with the help of an auxiliary taken connection λ in B by the formula

$$h_{(B,A)}[\Psi] = \Delta_{(B,A)}(\Psi)$$

$$\Delta_{(B,A)} : \left(\operatorname{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*\right)_I \to \Omega(A)$$

$$\Delta_{(B,A)}(\Psi)(x; v_1 \wedge \ldots \wedge v_k) = \langle \Psi_x, \left[-\omega^{j\circ\lambda}(x; v_1)\right] \wedge \ldots \wedge \left[-\omega^{j\circ\lambda}(x; v_k)\right].$$

One can prove the commutativity of $\Delta_{(B,A)}$ with differential δ and d_A which gives a homomorphism on cohomology.

The following gives the fundamental homotopic property of the homomorphism $h_{(B,A)}$.

Theorem 3 *If $B, B' \subset A$ are homotopic Lie subalgebroids of A (both over (M, F)) then there exist an isomorphism of algebras*

$$H(\boldsymbol{g}, B) \cong H(\boldsymbol{g}', B) \tag{1}$$

under which

$$h_{(B,A)} = h_{(B',A)}.$$

Recal that two Lie subalgebroids B_0, $B_1 \subset A$ (both over (M, F)) are said to be *homotopic* if there exists a Lie subalgebroid $B \subset T\mathbb{R} \times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times F)$ such that for $t \in \{0, 1\}$

$$\upsilon_x \in B_{t|x} \Longleftrightarrow (\theta_t, \upsilon_x) \in B_{|(t,x)}.$$

B is called a *subalgebroid joining B_0 with B_1*.

See [K_5] to compare the relation of homotopic subbundles of a principal bundle with the relation of homotopic subalgebroids.

4 Homomorphism for (B, A, ∇)

Now consider additionally a homomorphism of Lie algebroids

$$\nabla : L \to A$$

where L is any (nonregular in general) Lie algebroid on the manifold M.

Remark 4 *The characteristic classes from the image of the Chern–Weil homomorphism* $h_{L,A}:I(A)\rightarrow H(L)$ *of the pair* (L,A) *[B+K+W] are obstructions for the existence of a flat* L*-connection in* A*.* $h_{L,A}=0$ *if there exists a flat* L*-connection in* A*.* $I(A)$ *is the space of invariant cross–sections of* $\bigvee^{k}\boldsymbol{g}^{*}$ *with respect to the representation induced by the adjoint one* $\mathrm{ad}_{A}:A\rightarrow\mathrm{CDO}(\boldsymbol{g})$*.*

The equality $\#_{L}^{\#}\circ h_{A}=h_{L,A}$ *(compare with [Cr] and [F] for* $A=\mathrm{CDO}(\mathfrak{f})$ *and* $A=A(P)$*) connects the Chern-Weil homomorphism* $h_{L,A}$ *with* h_{A} *(the last is the Chern-Weil homomorphism of a single regular Lie algebroid* A*, obtained earlier by J.Kubarski in [*K_1*]).*

For arbitrary vector bundle $\mathfrak{f}$ *since the Lie algebroid* $\mathrm{CDO}(\mathfrak{f})$ *is integrable,* $\mathrm{CDO}(\mathfrak{f})=A(L(\mathfrak{f}))$*, the algebra of invariants* $I(\mathrm{CDO}(\mathfrak{f}))$ *is canonically isomorphic to the algebra of invariant polynomials on the Lie algebra* $\mathrm{gl}(n,\mathbb{R})$ *[*K_1*]. The case of* L*-connections in a vector bundle* $\mathfrak{f}$ *was considered by Crainic [Cr]. The case of* L*-connections in* $A=A(P)$ *corresponds to the case considered by Fernandes [F].*

The superposition

$$h_{(B,A,\nabla)}=\nabla^{\#}\circ h_{(B,A)}:H(\boldsymbol{g},B)\rightarrow H(L)$$

has the following properties:

1) on the level of forms is defined by the formula

$$h_{(B,A,\nabla)}([\Psi])=\Delta_{(B,A,\nabla)}(\Psi)$$

$$\Delta_{(B,A,\nabla)}(\Psi)(x;v_{1}\wedge...\wedge v_{k})=\left\langle\Psi_{x},\left[-\omega^{j\circ\lambda}(\nabla v_{1})\right]\wedge...\wedge\left[-\omega^{j\circ\lambda}(\nabla v_{k})\right]\right\rangle,$$

2) If B,B' are homotopic Lie subalgebroids then $h_{(B,A,\nabla)}=h_{(B',A,\nabla)}$ under isomorphism (1),

3) $h_{(B,A,\nabla)}=0$ if ∇ takes values in a Lie subalgebroid B' homotopic to B,

4) If $\nabla,\ \nabla':L\rightarrow A$ are homotopic homomorphisms of Lie algebroids then $h_{(B,A,\nabla)}=h_{(B,A,\nabla')}$,

5) $h_{(B,A)}=h_{(B,A,id_{A})}$.

Recall the definition of homotopy between homomorphisms of Lie algebroids.

Definition 5 *[K_4] Let H_0, $H_1 : L \to A$ be homomorphisms of Lie algebroids. By a* homotopy joining H_0 *to* H_1 *we mean a homomorphism of Lie algebroids*

$$H : T\mathbb{R} \times L \longrightarrow A$$

such that $H(\theta_0, \cdot) = H_0$ and $H(\theta_1, \cdot) = H_1$ where θ_0 and θ_1 are null vector tangent bundle of R at 0 and 1, respectively.

5 EXAMPLES

5.1 On the ground of principal bundles

Let $P' \subset P$ be a H-reduction of a G-principal bundle $P = P(M, G)$, $H \subset G$, and let λ be a flat connection in P.

The fundamental question is:

- is λ a connection in P' ? or less, is λ a connection in another H-reduction P'' homotopic to P' ?

The secondary characteristic homomorphism $h_{(P',P,\lambda)} : H(\mathfrak{g}, H) \to H_{dR}(M)$, investigated intensively in the seventies for the triple (P', P, λ), describes the obstructions to this fact. We look at this homomorphism from the point of view of Lie algebroids and especially from the point of view of the secondary characteristic homomorphism of the pair of Lie algebroids $(A(P'), A(P))$ of principal bundles P' and P.

A flat connection λ in P is equivalent to a connection $\lambda : TM \to A(P) = TP/G$ in the transitive Lie algebroid $A(P) = TP/G$ of P. Let $0 \to \boldsymbol{g} \to A(P) \to TM \to)$ be the Atiyah sequence of P ($\boldsymbol{g} = P \times_{Ad} \mathfrak{g}$, $\mathfrak{g}$ denotes the Lie algebra of G).

Theorem 6 *If P' is connected H-reduction (H may not be connected), then there exist an isomorphism of algebras*

$$H(\mathfrak{g}, H) \overset{\kappa}{\cong} H(\boldsymbol{g}, A(P'))$$

under which

$$h_{(P'P,\lambda)} = h_{(A(P'),A(P),\lambda)}$$

where $h_{(P',P,\lambda)} : H(\mathfrak{g}, H) \to H_{dR}(M)$ is the classical "flat" secondary characteristic homomorphism.

We recall the indirect definition of $h_{(P',P,\lambda)}$ on the level of differential forms:

$$h_{(P',P,\lambda)}([\psi]) = \Delta_{(P',P,\lambda)}(\psi)$$

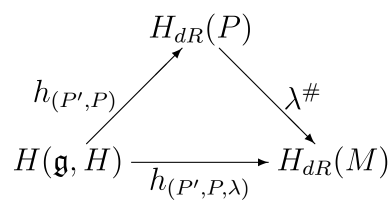
$\Delta_{(P',P,\lambda)}(\psi)(x; w_1 \wedge ... \wedge w_k) = \langle \psi_x, [+\omega(z; \tilde{w}_1)] \wedge ... \wedge [+\omega(z; \tilde{w}_k)] \rangle$, where $x \in M$, $z \in P_{|x}$, $\tilde{w}_i \in T_z P' \subset T_z P$ is a P'-horizontal lifting of w_i.

According to isomorphism $H(\mathfrak{g}, H) \overset{\kappa}{\cong} H(\boldsymbol{g}, A(P'))$ from theorem (6) and the superposition

$$h_{(A(P'),A(P),\lambda)} = \lambda^{\#} \circ h_{(A(P'),A(P))}$$

we obtain Theorem 1, i.e. that $h_{(P',P,\lambda)}$ is factorized by the universal homomorphism

$$h_{(P',P)} = h_{(A(P'),A(P))} : H(\mathfrak{g}, H) = H(\boldsymbol{g}, A(P')) \to H(A(P)) = H^r(P)$$



5.2 Crainic characteristic classes

Consider a vector bundle $\mathfrak{f}$ and its Lie algebroid $\mathrm{CDO}(\mathfrak{f})$. The cross-sections of $\mathrm{CDO}(\mathfrak{f})$ are covariant derivative operators of $\mathfrak{f}$. Equivalently $\mathrm{CDO}(\mathfrak{f})$ can be described as the Lie algebroid of the $\mathrm{GL}(n;\mathbb{R})$-principal bundle $L\mathfrak{f}$ of frames of $\mathfrak{f}$.

Let L be an arbitrary nonregular (in general) Lie algebroid L on M and ∇ a representation of L on $\mathfrak{f}$, equivalently given by the homomorphism of Lie algebroids

$$\nabla : L \to \mathrm{CDO}(\mathfrak{f}).$$

Crainic classes [Cr] $u_{2k-1}(\mathfrak{f})$ of ∇ live in $H(L)$ and one of the direct formula for these classes is as follows: let h be any Riemannian metric in $\mathfrak{f}$ and ∇^h the adjoint L-connection in *f*, then

$$u_{2k-1}(\mathfrak{f}) = [u_{2k-1}(\mathfrak{f}, \nabla)] \in H(L)$$

where $u_{2k-1}(\mathfrak{f},\nabla)\in\Omega^{2k-1}(L)$ are defined by

$$u_{2k-1}(\mathfrak{f},\nabla)=(-1)^{\frac{k(k+1)}{2}}\operatorname{cs}_k\left(\nabla,\nabla^h\right),\quad k \text{ is odd},$$
$$\operatorname{cs}_k\left(\nabla,\nabla^h\right)=\int_\Delta \operatorname{ch}_k\left(\nabla^{\mathrm{aff}}\right)$$

for $\nabla^{\mathrm{aff}}=t\nabla+(t-1)\nabla^h$.

We look at this classes from the point of view of universal characteristic homomorphism of a pair of Lie algebroids. Consider in this purpose the reduction of $\mathrm{CDO}(\mathfrak{f})$ coming from a Riemannian metric h, i.e. a Lie subalgebroid

$$\mathrm{CDO}(\mathfrak{f},\{h\})\subset\mathrm{CDO}(\mathfrak{f}).$$

For example, $\mathrm{CDO}(\mathfrak{f},\{h\})$ can be realized as the Lie algebroid of a suitable connected reduction of the $\mathrm{GL}(n)$-principal bundle of frames $L\mathfrak{f}$ to orthonormal frames $L(\mathfrak{f},\{h\})$. Taking the canonical isomorphism of Lie algebroids $\Phi_{\mathfrak{f}}:A(L\mathfrak{f})\to\mathrm{CDO}(\mathfrak{f})$ [K$_1$] we put

$$\mathrm{CDO}(\mathfrak{f},\{h\})=\Phi_{\mathfrak{f}}\left[A\left(L\left(\mathfrak{f},\{h\}\right)\right)\right].$$

Theorem 3.3.2 [K$_3$] says that $l\in\mathrm{CDO}(\mathfrak{f},\{h\})_{|x}\Longleftrightarrow l\in\mathrm{CDO}(\mathfrak{f})_{|x}$ and for an arbitrary local trivialization $\psi:U\times\mathbb{R}^n\to\mathfrak{f}_{|U}$ of Riemannian bundle $\mathfrak{f}$ (i.e. $\psi_x:\mathbb{R}^n\to\mathfrak{f}_{|x}$ is an isometry), the endomorphism

$$\mathbb{R}^n\ni u\longmapsto\psi_x^{-1}\left(l\left(\psi\left(\cdot,u\right)\right)\right)\in\mathbb{R}^n$$

belongs to the Lie algebra $\mathfrak{so}(n)$. On the other hand we have: $\mathcal{L}\in\mathrm{Sec}(\mathrm{CDO}(\mathfrak{f},\{h\}))\Longleftrightarrow\mathcal{L}\in\mathrm{Sec}(\mathrm{CDO}(\mathfrak{f}))$ and for each sections $\xi,\eta\in\mathrm{Sec}(\mathfrak{f})$ the formula holds

$$h(\mathcal{L}(\xi),\eta)=h(\xi,\eta)-h(\xi,\mathcal{L}(\eta)).$$

The Atiyah sequences for $\mathrm{CDO}(\mathfrak{f})$ and $\mathrm{CDO}(\mathfrak{f},\{h\})$ are

$$0\longrightarrow\mathrm{End}(\mathfrak{f})\longrightarrow\mathrm{CDO}(\mathfrak{f})\longrightarrow TM\longrightarrow0,$$
$$0\longrightarrow\mathrm{Sk}(\mathfrak{f})\longrightarrow\mathrm{CDO}(\mathfrak{f},\{h\})\longrightarrow TM\longrightarrow0.$$

Two Lie subalgebroids $B_i = \mathrm{CDO}(\mathfrak{f}, \{h_i\})$, $i = 1, 2$, corresponding to Riemannian metrics h_i are homotopic Lie subalgebroids [K$_5$].

For the pair of Lie algebroids $(\mathrm{CDO}(\mathfrak{f}, \{h\}), \mathrm{CDO}(\mathfrak{f}))$ we have the universal secondary characteristic homomorphism

$$h_{(\mathrm{CDO}(\mathfrak{f},\{h\}),\mathrm{CDO}(\mathfrak{f}))} : H(\mathrm{End}(\mathfrak{f}), \mathrm{CDO}(\mathfrak{f}, \{h\})) \to H(\mathrm{CDO}(\mathfrak{f})).$$

For arbitrary nonregular (in general) Lie algebroid L on M we obtain the secondary characteristic homomorphism

$$h_{(\mathrm{CDO}(\mathfrak{f},\{h\}),\mathrm{CDO}(\mathfrak{f}),\nabla)} : H(\mathrm{End}(\mathfrak{f}), \mathrm{CDO}(\mathfrak{f}, \{h\})) \to H(L)$$

According to classical isomorphisms (see the papers and the book by Kamber-Tondeur) and Theorem 6 we obtain the isomorphisms

$$\bigwedge\left(y^1, ..., y^{n'}\right) \overset{\text{K-T}}{\cong} H(\mathrm{gl}(n, \mathbb{R}), O(n)) \underset{\kappa}{\cong} H(\mathrm{End}(\mathfrak{f}), \mathrm{CDO}(\mathfrak{f}, \{h\}))$$

where n' is the largest odd integer $\leq n$.

The characteristic class $\tilde{y}^k$ in $H(\mathrm{gl}(n, \mathbb{R}), O(n))$ corresponding to y^k is defined by

$$\tilde{y}^k([A_1], ..., [A_{2k-1}]) = \sum \mathrm{sgn}\,\sigma \cdot \mathrm{tr}\left(\tilde{A}_{\sigma(1)} \circ ... \circ \tilde{A}_{\sigma(2k-1)}\right)$$

where

$$\tilde{A}_i = \frac{A_i + A_i^T}{2}$$

is the symmetrization of A_i.

Theorem 7 $h_{(\mathrm{CDO}(\mathfrak{f},\{h\}),\mathrm{CDO}(\mathfrak{f}))}\left(\kappa\left(\tilde{y}^k\right)\right) = c_k \cdot u_{2k-1}(\mathrm{f})$ *for some real* c_k.

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