FROM POINCARÉ DUALITY
TO
EVENS-LU-WEINSTEIN PAIRING

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We begin with defining the problem and with pointing out the importance of the Evens-Lu-Weinstein representation (for Lie algebroids).

1. Definition of Lie algebroids and examples

The notion of Lie algebroid comes from the paper

J. Pradines, *Théorie de Lie pour les groupoides differentiables,* Atti del Convegno Internazionale di Geometria Differenziale, Bologna, 28-30, IX, 1967.

Lie pseudoalgebra (Lie-Rinehart algebra) is its algebraic equivalent:

J. C. Herz, *Pseudo-algébres de Lie,* C. R. Acad. Sci. Paris, 263 (1953), I, 1935–1937 and II, 2289–2291.

Definition 1.1. *By a Lie algebroid we mean a system*

$$A = (A, [\![\cdot,\cdot]\!], \#),$$

in which

$A \xrightarrow{p} M$ *is a vector bundle,*

$\# : A \to TM$ *is a linear homomorphism (called the anchor),*

$(\Gamma(A), [\![\cdot,\cdot]\!])$ *is an real Lie algebra of global cross-sections of* A,

Leibniz property is satisfied:

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\#\circ\xi)(f)\cdot\eta,$$

$\xi, \eta \in \operatorname{Sec} A$, $f \in C^\infty(M)$.

From the above $\operatorname{Sec}\# : \operatorname{Sec} A \to \mathfrak{X}(M)$, $\xi \mapsto \#\circ\xi$, is a homomorphism of Lie algebras

$$\#[\![\xi,\eta]\!] = [\#(\xi), \#(\eta)]$$

see J. C. Herz (1953).

Example 1.1. *(1)* TM *tangent bundle,*

(2) $\mathfrak{g}$ *finite dimensional Lie algebra,*

(3) $TM\times\mathfrak{g}$ *trivial Lie algebroid,*

(4) $F\subset TM$ *involutive regular distribution,*

(5) Lie algebroid of a principal bundle $P(M,G)$

$$A(P) := TP/G$$

$\Gamma(A(P)) \cong \mathfrak{X}^R(P)$ *the Lie algebra of right-invariant vector fields,* *The bundle TP/G comes from*

M.Atiyah, *Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc., 85, No 1, (1957), 181-207.*

For $A(P)$ we have the so-called Atiyah sequence

$$0 \to P \times_{Ad} \mathfrak{g} \to A(P) \to TM \to 0.$$

(6) Lie algebroid of a Lie groupoid $\Phi \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$, $u : M \to \Phi$, $x \longmapsto u_x$ (the unit over x)

$$A(\Phi) = u^* (T^\alpha \Phi)$$

α-vertical vectors at the units.

(7) Lie algebroid of a vector bundle $\mathfrak{f} \to M$,

$$A(\mathrm{f}),$$
$$\Gamma(A(\mathrm{f})) = CDO(\mathrm{f}),$$

$$0 \to \mathrm{End}(\mathrm{f}) \to A(\mathfrak{f}) \to TM \to 0$$

$CDO(\mathrm{f})$ is the module of covariant differential operators. For example, if ∇ is a usual connection in $\mathfrak{f}$ then for any $X \in \mathfrak{X}(M)$ the operator $\nabla_X : \Gamma(\mathfrak{f}) \to \Gamma(\mathfrak{f})$ is a CDO with the anchor X.

(8) Lie algebroid of a TC-foliation

P.Molino, *Etude des feuilletages transversalement complets et applications, Ann.Sci. Ecole Norm. Sup., 10(3), (1977), 289-307.*

In the paper

R. Almeida, P. Molino, *Suites d'Atiyah et feuilletages transversalement complets, C. R. Acad. Sci. Paris Ser. I Math., 300 (1985),*

there is a theorem:

- *The Lie algebroid of a TC-foliation is integrable, (i.e. is isomorphic to the Lie algebroid of a principal bundle) if and only if this foliation is developable.*

Hence we obtain first examples of nonintegrable Lie algebroids: if M is simple connected and leaves of a TC-foliation are non closed then the Lie algebroid of this TC-foliation is not integrable.

(9) Lie algebroid T^*M of a Poisson manifold $(M, \{\cdot,\cdot\})$,

$$[\![df, dg]\!] = d\{f, g\},$$
$$\#(df)(g) = \{f, g\}.$$

A. Coste, P. Dazord, A. Weinstein, *Groupoides symplectiques, Publ. Dep. Math. Universite de Lyon 1, 2/A, (1987).*

etc (Jacobi manifolds, some pseudogroups)

2. Transitive Lie algebroids, Atiyah sequence, connections, representations

Definition 2.1. *Lie algebroid A is called transitive if the anchor $\# : A \to TM$ is an epimorphism. A Lie algebroid is called regular if the anchor is of a constant rank (the image $F := \operatorname{Im} \# \subset TM$ is then an involutive regular distribution). For a regular (especially for transitive) Lie algebroid we have the so-called Atiyah sequence*

$$0 \to \boldsymbol{g} \to A \to F \to 0.$$

The splitting of this sequence is called a connection in A

$$0 \to \boldsymbol{g} \to A \underset{\overleftarrow{\lambda}}{\to} F \to 0.$$

A representation of A in a vector bundle $\mathfrak{f}$ is a homomorphism of Lie algebroids

$$\nabla : A \to A(\mathfrak{f})$$

i.e. linear homomorphism compatible with the anchors and trivial curvature.

A cross-section $\nu \in \Gamma(\mathfrak{f})$ is called ∇-invariant if $\nabla_\xi(\nu) = 0$ for all $\xi \in \Gamma(A)$. The space of invariant cross-sections is denoted by $\Gamma(\mathfrak{f})_{I(\nabla)}$.

An important example is the adjoint representation

$$\operatorname{ad}_A : A \to A(\boldsymbol{g}),$$
$$(\operatorname{ad}_A)_\xi(\nu) = [\![\xi, \nu]\!].$$

The representation ad_A induces new representations in associated bundles, for example

$$\operatorname{ad}_A : A \to A\big(\bigvee^k \boldsymbol{g}^*\big).$$

3. Generalized connections, characteristic classes

The algebra of invariant sections

$$I(A) := \bigoplus^k \Gamma\big(\bigvee^k \boldsymbol{g}^*\big)_{I(\operatorname{ad}_A)}$$

is the domain of the Chern-Weil homomorphism

$$h_A : I(A) \to H_{dR}(F)$$

$H_{dR}(F)$ is the algebra of tangential cohomology

J.Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids,* Publ. Dep. Math. University de Lyon 1, 1991.

This approach generalizes known constructions given by Teleman 1972 (present in this conference),

N. Teleman, *Cohomology of Lie algebras*, Global Analysis and its applications, Vol III, Intern. Course, Trieste 1972, Vienna 1974,

—, *A characteristic ring of a Lie algebra extension*, Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. (8), 498-506 and 708-711, 1972.

He proved that for any principal bundle $P(M,G)$ with connected Lie group G we have $I(A) \cong I(\mathfrak{g})$, the space of G-invariant polynomials, and

$$h_{A(P)} = h_P.$$

It turns out that this holds for all principal bundles with connected total space P

J.Kubarski (1991), *The Chern-Weil ...*

I.W.Belko, *Characteristic classes of transitive Lie algebroids* (in Russian, Minsk, Belarus, 1994, pp.192 (Preprint).

— *Characteristic homomorphism of a Lie algebroid* (in Russian, News of Belarus Academy of Sciences, Serie of physical-mathematical sciences, number 1, 1997.

To consider nonregular Lie algebroids we must generalize the notion of a connection.

Definition 3.1. *Let A and B be two Lie algebroids on M. Usually, B is transitive (or may be regular) but A is arbitrary!*

By an A-connection in a Lie algebroid B we mean a linear homomorphism

$$\nabla : A \to B$$

compatible with the anchors

$$\begin{array}{ccc} A & \stackrel{\nabla}{\longrightarrow} & B \\ \#_A \searrow & & \swarrow \#_B \\ & TM & \end{array}$$

Seeing the Atiyah sequence of the transitive Lie algebroid B

$$0 \to \boldsymbol{g} \to B \underset{\overleftarrow{\nabla=\lambda}}{\to} TM \to 0$$

we observe that the above definition of a A-connection in B generalizes the standard TM-connection in the Lie algebroid B.

By an A-connection in a vector bundle $\mathfrak{f}$ we mean an A-connection in the Lie algebroid $A(\mathfrak{f})$ of f

$$\nabla : A \to A(\mathfrak{f}).$$

Equivalently, $\nabla : A \to A(\mathfrak{f})$ is an operator $\nabla_\xi \nu$, $\xi \in \Gamma(A)$, $\nu \in \Gamma(f)$, which satisfies the known Koszul axioms with the following modification: $\nabla_\xi : \Gamma(\mathfrak{f}) \to \Gamma(\mathfrak{f})$ is a covariant derivative operator with the ancjor $\#(\xi) \in \mathfrak{X}(M)$

$$\nabla_{\xi}(f \cdot \nu) = f \cdot \nabla_{\xi}\nu + \#(\xi)(f) \cdot \nu.$$

By an A-connection in a principal bundle P we mean an A-connection in the Lie algebroid $A(P)$

$$\nabla : A \to A(P).$$

The first examples of generalized connections $\nabla : A \to B$ appear in Poisson geometry in the papers by Vaisman (called a contravariant derivative) for $A = T^*M$, $B = A(P)$, P is any principal bundle.

I. Vaisman, On the geometric quantization of Poisson manifolds, J. of Math. Physics 32 (1991), 3339-3345.

Next we have contravariant connections on vector bundles in the papers by R.L.Fernades, for $A = T^*M$ and $B = A(\mathfrak{f})$ $\mathfrak{f}$, is any vector bundle,

R.L.Fernandes, *Contravariant Connections on Poisson Manifolds,* in Proceedings of the Summer School on Differential Geometry, Eds. A. M. Breda, F. Carvalho, J. Costa, B. Wegner, Univ. Coimbra, 1999.

For A-connections in principal bundles and in vector bundles see the papers by Fernandes and Crainic

R.L.Fernandes, *Lie Algebroids, Holonomy and Characteristic Classes*, Advances in Mathematics 170, (2002) 119-179.

M.Crainic, *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes,* Commentarii Mathematici Helvetici, to appear.

M.Crainic, R.L.Fernandes, *Secondary Characteristic Classes of Lie Algebroids*, (preprint Sept. 2003).

In the last papers there are characteristic "flat" classes for A-representations in a vector bundle $\mathfrak{f}$.

There are characteristic classes for

$$(A, B, \lambda)$$

where $B \subset A$ are regular Lie algebroids over the same foliated manifold and λ is a usual flat [or partially flat] connection in A.

J.Kubarski, *Algebroid nature of the characteristic classes of flat bundles*, Homotopy and geometry, Banach Center Publications, Vol. 45, Inst. of Math. Polish Academy of Sciences, Warszawa 1998, 199-224.

J.Kubarski, *The Weil algebra and the secondary characteristic homomorphism of regular Lie algebroids.* Lie Algebroids and Related Topics in Differential Geometry, Banach Center Publicationes, Volume 54, 135-173, IMPAN Warszawa 2001.

This theory generalizes the Kamber-Tondeur theory for flat or partially flat characteristic classes of principal bundles (by putting $A = A(P)$ the Lie algebroid of a principal bundle $P(M, G)$, and B - the Lie subalgebroid of its H-reduction, $H \subset G$).

There are the so-called intrinsic characteristic classes of arbitrary Lie algebroids which generalize secondary characteristic classes for foliations.

M.Crainic, R.L.Fernandes, *Secondary Characteristic Classes of Lie Algebroids*, (preprint Sept. 2003).

DIFFICULTIES: for arbitrary Lie algebroid the underlying characteristic foliation ($\operatorname{Im}\# \subset TM$) is singular (in the sense of P.Stefan).

How can one use the idea of secondary characteristic classes for regular foliations? The last can be described as the characteristic classes of the Bott connection in the normal bundle to the foliation. The Bott connection plays the role of the "adjoint representation".

The key to construct these classes for arbitrary Lie algebroid is some generalization of the adjoint representation.

4. Evens-Lu-Weinstein representation

For arbitrary Lie algebroid A the image $F = \operatorname{Im}\#$ is not a vector bundle, and we must consider the $\mathbb{Z}_2$-super vector bundle

$$\mathfrak{f} = A \oplus TM$$

together with two $\mathbb{R}$-linear connections (not $C^\infty(M)$-linear !)

$$\begin{aligned} &\nabla^0 : \Gamma(A) \to CDO(A), \\ &\nabla^0_\xi \eta = [\![\xi, \eta]\!], \end{aligned}$$

$$\begin{aligned} &\nabla^1 : \Gamma(A) \to CDO(TM), \\ &\nabla^1_\xi (X) = [\#(\xi), X]. \end{aligned}$$

The role of the adjoint representation for arbitrary Lie algebroid A is played by the pair (∇^0, ∇^1) or equivalently by the $\mathbb{R}$-linear representation [flat connection] of A in the super-bundle $A \oplus TM$.

$$\begin{aligned} &\mathrm{Ad} : \Gamma(A) \to \Gamma(A \oplus TM), \\ &\mathrm{Ad}_\xi (\eta \oplus X) = [\![\xi, \eta]\!] \oplus [\#(\xi), X]. \end{aligned}$$

- The fundamental property of Ad is: the induced representation in the top-power vector bundle

  $$Q_A = \Lambda^{top} A \otimes \Lambda^{top} T^* M$$

  (denoted rather by D)

  $$D : \Gamma(A) \to CDO(Q_A)$$

  is a usual i.e. $C^\infty(M)$-connection! The case is: Ad is a connection "up to homotopy".

S.Evens, J-H.Lu, and A.Weinstein, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*, Quart. J. Math. Oxford 50 (1999), 417-436,

This generalizes the constructions for Poisson geometry from

A.Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. 23 (1997), 379-394.

The property "up to homotopy" of the connection Ad means that there exists a linear homomorphism $\partial : A \oplus TM \to A \oplus TM$ such that $(A \oplus TM, \partial)$ is a super-complex, here $\partial = \partial^0 \oplus \partial^1$, $\partial^0 = \# : A \to TM$, $\partial^1 = 0 : TM \to A$, and there exists a "homotopy" $H_{(\xi,f)} \in \mathrm{End}\,(A \oplus TM)\,,\ \xi \in \Gamma(A)\,,\ f \in C^\infty(M)\,,$ such that the linearity of Ad with respect to the bottom index $\xi \in \Gamma(A)$ is "up to homotopy"

$$\mathrm{Ad}_{f\xi}(s) = f \cdot \mathrm{Ad}_\xi(s) + \left[H_{(f\xi)}, \partial\right](s)\,, \quad s \in \Gamma(A \oplus TM)\,.$$

Here we have $H^0_{(f\xi)} = 0 : A \to TM$, $H^1_{(f\xi)} : TM \to A$, $H^1_{(f\xi)}(X) = -X(f) \cdot \xi$.

Remark 4.1. *Since publishing the paper by S.Evens, J-H.Lu, and A.Weinstein it has been observed the important role of $\mathbb{R}$-linear objects: connections, differential forms e.t.c. in the characteristic classes and next in algebraic topology.*

We notice that for a transitive Lie algebroid A with the Atiyah sequence $0 \to \boldsymbol{g} \to A \overset{\#}{\to} TM \to 0$ we have

$$A \cong \boldsymbol{g} \oplus TM$$

(isomorphism is given by a connection $\lambda : TM \to A$) so

$$A \oplus T^*M = \boldsymbol{g} \oplus TM \oplus T^*M \cong \boldsymbol{g}$$

and

$$\begin{aligned} Q_A &= \Lambda^{top} A \otimes \Lambda^{top} T^*M \\ &= \Lambda^{top}(\boldsymbol{g} \oplus TM) \otimes \Lambda^{top} T^*M \\ &= \Lambda^{top}\boldsymbol{g} \otimes \Lambda^{top} TM \otimes \Lambda^{top} T^*M \\ &= \Lambda^{top}\boldsymbol{g} \end{aligned}$$

For regular Lie algebroid $0 \to \boldsymbol{g} \to A \overset{\#}{\to} F \to 0$ we have $A \cong \boldsymbol{g} \oplus TM$, and $TM \cong F \oplus \left(TM_{/F}\right)$, therefore

$$\begin{aligned} Q_A &= \Lambda^{top} A \otimes \Lambda^{top} T^*M \\ &= \Lambda^{top}\boldsymbol{g} \otimes \Lambda^{top} F \otimes \Lambda^{top} F^* \otimes \Lambda^{top}\left(TM_{/F}\right)^* \\ &= \Lambda^{top}\boldsymbol{g} \otimes \Lambda^{top}\left(TM_{/F}\right)^*. \end{aligned}$$

The first factor $\Lambda^{top}\boldsymbol{g}$ is the object of acting of the usual adjoint representation, the second factor $\Lambda^{top}\left(TM_{/F}\right)^{*}$ is the object of acting of the Bott connection.

The representation $D : A \to A\left(Q_A\right)$ of A in the vector bundle Q_A is defined by the formula

$$D : A \to A\left(Q_A\right),$$
$$D_\xi\left(Y \otimes \varphi\right) = L_\xi Y \otimes \varphi + Y \otimes L_{\#(\xi)}\varphi$$

where $Y \in \Gamma\left(\Lambda^{top}A\right)$, $\varphi \in \Gamma\left(\Lambda^{top}T^{*}M\right) = \Omega^{top}\left(M\right)$, and $L_\xi Y$ is the Schouten bracket

$$L_\xi Y = [\xi, Y]$$

defined for simple tensor $Y = \xi_1 \wedge ... \wedge \xi_r$ ($r = \operatorname{rank} A$), $\xi_i \in \Gamma\left(A\right)$, by the formula

$$[\xi, Y] = [\xi, \xi_1 \wedge ... \wedge \xi_r] = \sum_i \xi_1 \wedge ... \wedge [\![\xi, \xi_i]\!] \wedge ... \wedge \xi_r.$$

$L_{\#(\xi)}\varphi$ is the usual Lie derivation of the top-differential form φ on M.

For transitive Lie algebroid A we fix a connection $\lambda : TM \to A$. Then we have an isomorphism

$$Q_A = \Lambda^{top}\boldsymbol{g}$$

for which

$$D = \operatorname{ad}{}_A^{top}.$$

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5. Main problem

To consider simultaneously nonorientable manifolds we multiply the bundle Q_A by the orientation bundle $or\left(M\right)$

$$Q_A^{or} := Q_A \otimes or\left(M\right)$$

and the E-L-W representation D by the canonical A-representation ∂_A^{or} on the orientation bundle

$$\left(\partial_A^{or}\right)_\xi\left(\nu\right) = \left(\partial^{or}\right)_{\#\xi}\left(\nu\right)$$

where ∂^{or} is the canonical flat structure in $or\left(M\right)$

$$D^{or} = D \otimes id + id \otimes \partial_A^{or}.$$

Let

$$\Omega\left(A, Q_A^{or}\right) = \Gamma\left(\Lambda A^{*} \otimes Q_A^{or}\right)$$

denotes the space of A-differential forms with values in Q_A^{or}. Since D^{or} is a representation, we have the diferential operator

$$d_{D^{or}} : \Omega\left(A, Q_A^{or}\right) \to \Omega\left(A, Q_A^{or}\right)$$

$$d_{D^{or}}\left(f\right)\left(\xi_0,...,\xi_t\right)=\sum_i\left(-1\right)^i D^{or}_{\xi_i}\left(f\left(\xi_0,...\hat{\imath}...,\xi_t\right)\right)+\sum_{i<j}\left(-1\right)^{i+j}f\left([\![\xi_i,\xi_j]\!],...\hat{\imath}...\hat{\jmath}...\right)$$

Take the pairing on A-differential forms ($r=\operatorname{rank}A$), the subscript c stands for compactly supported differential forms,

$$\Omega^j\left(A\right)\times\Omega^{r-j}_{D^{or},c}\left(A;Q^{or}_A\right)\to\Omega^r_{D^{or},c}\left(A;Q^{or}_A\right),\quad\left(f,g\right)\longmapsto f\wedge g,$$

we have

$$d_{D^{or}}\left(f\wedge g\right)=d_Af\wedge g+\left(-1\right)^{|f|}f\wedge d_{D^{or}}g$$

which produces the E-L-W pairing in cohomology

$$F:H^j\left(A\right)\times H^{r-j}_{D^{or},c}\left(A;Q^{or}_A\right)\to H^r_{D^{or},c}\left(A;Q^{or}_A\right).$$

REMARK on the top group of cohomology $H^r_{D^{or},c}\left(A;Q^{or}_A\right)$. Consider an isomorphism of vector bundles

$$\tilde{\rho}:\Lambda^rA^*\otimes Q_A\xrightarrow{\cong}\Lambda^mT^*M,$$

$$\Psi\otimes\left(Y\otimes\mu\right)\longmapsto\left\langle\Psi,Y\right\rangle\cdot\mu,$$

$[Q_A=\Lambda^{top}A\otimes\Lambda^{top}T^*M]$. $\tilde{\rho}$ induces an isomorphism of modules of sections

$$\tilde{\rho}:\Omega^r\left(A;Q_A\right)\xrightarrow{\cong}\Omega^m\left(M\right).$$

Assume orientability of M. In the mentioned paper by S.Evens, J-H.Lu, and A.Weinstein, there is the following version of Stokes Theorem:

Theorem 5.1 (E-L-W Stokes Theorem)**.** *For $\Psi'\in\Omega^{r-1}\left(A\right)$ and $Y\otimes\mu\in\Gamma\left(Q_A\right)$ the following eqality holds*

$$\tilde{\rho}\left(d_D\left(\Psi'\otimes Y\otimes\mu\right)\right)=\left(-1\right)^{r-1}\left(\iota_{\#_A\left(\Psi'\lrcorner Y\right)}\mu\right).$$

Consequently, if the form $\Psi'\otimes Y\otimes\mu$ has compact support then

$$\int_M\tilde{\rho}\left(d_D\left(\Psi'\otimes Y\otimes\mu\right)\right)=0.$$

Take the following linear homomorphism

$$\tilde{\rho}^{r-1}:\Lambda^{r-1}A^*\otimes Q_A\to\Lambda^{m-1}T^*M,\quad\left(\Psi'\otimes Y\otimes\mu\right)\mapsto\left(-1\right)^{r-1}\iota_{\#_A\left(\Psi'\lrcorner Y\right)}\mu,$$

and notice the commutativity of the diagram

$$\begin{array}{ccc}\Omega^{r-1}_c\left(A;Q_A\right) & \xrightarrow{\tilde{\rho}^{r-1}_c} & \Omega^{m-1}_c\left(M\right)\\ \downarrow{\scriptstyle d_D} & & \downarrow{\scriptstyle d_{dR}}\\ \Omega^r_c\left(A;Q_A\right) & \xrightarrow{\tilde{\rho}_c} & \Omega^m_c\left(M\right).\end{array}$$

From this we deduce that $\tilde{\rho}_c : \Omega_c^r(A; Q_A) \xrightarrow{\cong} \Omega_c^m(M)$ induces an $\mathbb{R}$-linear epimorphism in cohomology

$$\tilde{\rho}_{c,\#} : H_{D,c}^r(A; Q_A) \twoheadrightarrow H_c^r(M) \cong \mathbb{R}.$$

For arbitrary (oriented or nonoriented manifold) we prove this analogously: we multiply vector bundles by the orientation bundle $or(M)$ and use the Stokes theorem for densities:

$$\tilde{\rho}_{c,\#}^{or} : H_{D^{or},c}^r(A; Q_A^{or}) \twoheadrightarrow H_c^r(M, or(M)) \cong \mathbb{R}.$$

For transitive Lie algebroid we have important

Proposition 5.1. *If A is a transitive Lie algebroid, then $\tilde{\rho}_{c,\#}^{or}$ is an isomorphism.*

Composing the cohomology pairing with $\tilde{\rho}_{c,\#}^{or}$ we obtain the E-L-W cohomology pairing

$$H^j(A) \times H_{D^{or},c}^{r-j}(A; Q_A^{or}) \to \mathbb{R},$$

$$\left(\left[\Psi^j\right], \left[\Psi^{r-j} \otimes X \otimes \mu \otimes e\right]\right) \longmapsto \int_M^{\#,or} \langle \Psi^j \wedge \Psi^{r-j}, X \rangle \cdot \mu \otimes e.$$

Problem 1. *The authors S.Evens, J-H.Lu, and A.Weinstein say:*

- *— The problem of when it is non-degenerate is very interesting.*

In the cited paper there is an example of nonregular Lie algebroid for which this pairing is not necessary nondegenerate. This is the Lie transformation algebroid associated with the infinitesimal action $\gamma : \mathfrak{g} \to \mathfrak{X}(M)$ of a finitely dimensional Lie algebra $\mathfrak{g}$ on a manifold M. We recall the construction of this Lie algebroid: the trivial bundle $\mathfrak{g} \times M$ over M has the structure of a Lie algebroid, with the anchor given by $\rho(v, x) = \gamma(v)_x$, and Lie bracket

$$[\![\xi, \eta]\!](x) = [\xi(x), \eta(x)] + (\gamma(\xi(x))(\eta))(x) - (\gamma(\eta(x))(\eta))(x),$$

$\xi, \eta \in C^\infty(M, \mathfrak{g}) \cong \Gamma(\mathfrak{g} \times M)$ and $x \in M$. The vector field $X = x^N \frac{d}{dx}$ on $\mathbb{R}$ $(N \geq 2)$ defines an action of the 1-dimensional Lie algebra $\mathfrak{g} = \mathbb{R}^1$ on $M = \mathbb{R}$. Let $A = \mathbb{R}^1 \times \mathbb{R}$ be the tansformation Lie algebroid. Then

$$\dim H^1(a) = 1, \quad \dim H_{D,c}^1(A, Q_A) \geq N,$$

therefore the pairing

$$H^1(A) \times H_{D^{or},c}^1(A; Q_A) \to \mathbb{R}$$

is not nondegenerate, i.e. $H^1(A) \ncong \left(H_{D^{or},c}^1(A; Q_A)\right)^*$.

6. Results

The following are the main theorems:

Theorem 6.1 (On non-degeneracy)**.** *If A is transitive, then the pairing $H^j(A)\times H^{r-j}_{D^{or},c}(A;Q^{or}_A)\to\mathbb{R}$ is non-degenerated, i.e. $H^j(A)\cong\left(H^{r-j}_{D^{or},c}(A;Q^{or}_A)\right)^*$.*

Theorem 6.2 (On the Exceptionality)**.** *Let A be a transitive Lie algebroid. For a line bundle ξ and a representation $\nabla : A\to A(\xi)$ the following conditions are equaivalent:*

(a) $H^r_{\nabla,c}(A;\xi)\neq 0$,

(b) $H^r_{\nabla,c}(A;\xi)=\mathbb{R}$ *and the pairing* $H^j(A)\times H^{r-j}_{\nabla,c}(A;\xi)\to H^r_{\nabla,c}(A;\xi)\cong\mathbb{R}$ *is nondegenerate, i.e.* $H^j(A)\cong\left(H^{m+n-j}_{\nabla,c}(A;\xi)\right)^*$,

(c) $(\xi,\nabla)\sim(Q^{or}_A,D^{or})$.

The first to prove the "nondegeneracy" theorem is the following general Poincaré duality:

Theorem 6.3 (THEOREM I)**.** *Assume that M is a connected m-dimensional manifold (oriented or not) and ξ_1, ξ_2 are two flat vector bundles with flat covariant derivatives ∇_1 and ∇_2 respectively. Denote by $or(M)$ the orientation bundle with canonical flat structure ∂^{or}. If $F:\xi_1\times\xi_2\to or(M)$ is a pairing (i.e. 2-linear homomorphism) of vector bundles compatible with the flat structures $(\nabla_1,\nabla_2,\partial^{or})$ and non-degenerate at least at one point (therefore, at every) then the induced pairing in cohomology*

$$H^j_{\nabla_1}(M,\xi_1)\times H^{m-j}_{\nabla_2,c}(M,\xi_2)\xrightarrow{F_\#}H^m_{\partial^{or},c}(M,or(M))\xrightarrow{\int_M^{or,\#}}\mathbb{R}$$

is nondegenerate in the sense that

$$H^j_{\nabla_1}(M,\xi_1)\overset{\cong}{\to}\left(H^{m-j}_{\nabla_2,c}(M,\xi_2)\right)^*.$$

This theorem generalizes the classical Poincaré duality for oriented as well as for non-oriented manifolds, and as well as the Poincaré duality for d_ω-cohomologies. The last from

F. Guedira, A. Lichnerowicz, *Géometrie des algébres de Lie locales de Kirillov,*J. Math. Pures Appl. 63 (1984), 407-484.

The second step concerns finitely-dimensional Lie algebras.

Theorem 6.4 (THEOREM II)**.** *Assume $\mathfrak{g}$ is an arbitrary n-dimensional Lie algebra and $\nabla_1,\nabla_2:\mathfrak{g}\to L_{\mathbb{R}}$ are two representations of $\mathfrak{g}$ in $\mathbb{R}$. Denote by $\nabla_{\text{trad}}:\mathfrak{g}\to L_{\mathbb{R}}$ the trace-representation $(\nabla_{\text{trad}})_a=\operatorname{tr}(\operatorname{ad}_a)\cdot\operatorname{id}$. Then the top group of cohomology $H^n_{\text{trad}}(\mathfrak{g})$ of $\mathfrak{g}$ with respect to ∇_{trad} is nonzero, $H^n_{\text{trad}}(\mathfrak{g})\overset{\cong}{\to}\mathbb{R}$, and if the multiplication of reals is compatible with respect to $(\nabla_1,\nabla_2,\nabla_{\text{trad}})$ then the induced pairing in cohomology*

$$H^i_{\nabla_1}(\mathfrak{g})\times H^{n-i}_{\nabla_2}(\mathfrak{g})\to H^n_{\text{trad}}(\mathfrak{g})\cong\mathbb{R}$$

is nondegenerate, i.e.

$$H^i_{\nabla_1}(\mathfrak{g}) \xrightarrow{\cong} \left(H^{n-i}_{\nabla_2}(\mathfrak{g})\right)^* .$$

In particular, for $(\nabla_1, \nabla_2, \nabla_{\mathrm{trad}}) = (0, \nabla_{\mathrm{trad}}, \nabla_{\mathrm{trad}})$ *we obtain the nondegenerate pairing*

$$H^i(\mathfrak{g}) \times H^{n-i}_{\mathrm{trad}}(\mathfrak{g}) \to H^n_{\mathrm{trad}}(\mathfrak{g}) \cong \mathbb{R},$$

i.e.

$$H^i(\mathfrak{g}) \xrightarrow{\cong} \left(H^{n-i}_{\mathrm{trad}}(\mathfrak{g})\right)^* .$$

For unimodular Lie algebra $\mathfrak{g}$ *the usual Poincaré duality for* $\mathfrak{g}$ *is obtained in this way.*

This theorem follows from some modification of the beautiful Chern-Hirzebruch-Serre Lemma from

S.S.Chern, F.Hirzebruch, J-P. Serre, *On the index of a fibered manifold*, Proc. AMS, 8 (1957), 587-596,

concerning Poincaré differentiations. The modification replaces algebras with pairings. The asumptions on finite dimensionality is superfluous.

Lemma 6.1. *Let* $A_s = \bigoplus_{i=0}^n A_s^i$, $d_s : A_s \to A_s$, $s = 1, 2, 3$, *be three graded differential* $\mathbb{R}$*-vector spaces such that*

1. $d_s\left[A_s^i\right] \subset A_s^{i+1}$,
2. $d_s^2 = 0$,
3. $d_3\left[A_3^{n-1}\right] = 0$,
4. $A_3^n \cong \mathbb{R}$, $A_3^i = 0$ *for* $i > n$.

Let

$$\cdot : A_1 \times A_2 \to A_3$$

be a pairing such that

5. $d_3(x \cdot y) = d_1 x \cdot y + (-1)^{\deg x} x \cdot d_2 y$,
6. $\cdot : A_1^r \times A_2^{n-r} \to A_3^n \cong \mathbb{R}$, $r = 0, 1, ..., n$ *are nondegenerate in the sense that the induced mappings*

$$i_r : A_1^r \xrightarrow{\cong} \left(A_2^{n-r}\right)^*$$

are linear isomorphisms.

Then the induced homomorphisms in cohomology

$$\cdot : H^r(A_1, d_1) \times H^{n-r}(A_2, d_2) \to H^n(A_3, d_3) \cong \mathbb{R}$$

are nondegenerate as well, i.e. the induced linear homomorphism

$$i'_r : H^r(A_1, d_1) \to \left(H^{n-r}(A_2, d_2)\right)^*$$

are linear isomorphisms.

To prove the main "nondegeneracy" theorem for Lie algebroids, we use the two above theorems and some theorem concerning a pairing $\cdot : {}^1A \times {}^2A \to {}^3A$ between graded filtered differential $\mathbb{R}$-vector spaces and theirs spectral sequnces.

Given three graded filtered differential $\mathbb{R}$-vector spaces

$$\left({}^rA = \bigoplus_{i\geq 0} {}^rA^i,\ {}^rd,\ {}^rA_j\right), \quad r = 1, 2, 3, \tag{6.1}$$

denote for simplicity

$${}^rH := H\left({}^rA,\ {}^rd\right).$$

Assume

$$\cdot : {}^1A \times {}^2A \to {}^3A$$

preserves gradations and filtrations

$${}^1A^s \cdot {}^2A^t \subset {}^3A^{s+t}, \tag{6.2}$$

$${}^1A_j \cdot {}^2A_k \subset {}^3A_{j+k}, \tag{6.3}$$

and that the differentials rd satisfy the compatibility condition

$${}^3d\,(x \cdot y) = {}^1dx \cdot y + (-1)^{\deg x}\, x \cdot {}^2dy. \tag{6.4}$$

Clearly, there exists a multiplication of cohomology classes

$$\cdot : {}^1H^j \times {}^2H^k \to {}^3H^{j+k}, \quad ([x], [y]) \mapsto [x \cdot y]\,.$$

Let

$$\left({}^rE_s^{j,i},\ {}^rd_s\right)$$

be the spectral sequences of the graded filtered differential $\mathbb{R}$-vector spaces (6.1). The following main result of this chapter generalizes Corollary 12 from

J.Kubarski, A.S.Mishchenko, *Lie algebroids: spectral sequences and signature,* Sbornik: Mathematics, Volume 194(2003), Number 7, Pages 1079-1103.

Theorem 6.5 (THEOREM III)**.** *Given three graded filtered differential $\mathbb{R}$-vector spaces (6.1) and a pairing $\cdot : {}^1A \times {}^2A \to {}^3A$ satisfying (6.2), (6.3), (6.4), assume that the filtrations are regular in the sense ${}^rA_0 = {}^rA$ and that the second terms ${}^rE_2^{j,i}$ lie in the rectangular $0 \leq j \leq m$, $0 \leq i \leq n$ and that ${}^3E_2^{(m+n)} = {}^3E_2^{m,n} \cong \mathbb{R}$.*

If the multiplication in the second terms

$$\langle\cdot,\cdot\rangle_2 : {}^1E_2^{(j)} \times {}^2E_2^{(m+n-j)} \to {}^3E_2^{m,n} \cong \mathbb{R}$$

is nondegenerate in the sense that

$$ {}^{1}E_{2}^{(j)} \xrightarrow{\cong} \left({}^{2}E_{2}^{(m+n-j)}\right)^{*}, \quad x \mapsto \langle x, \cdot \rangle_{2}, $$

is a linear isomorphism, then

(a) ${}^{3}H^{m+n} \cong \mathbb{R}$,
(b) ${}^{r}H^{t} = 0$ *for* $t > m+n$,
(c) *the multiplication in cohomology classes*

$$ \langle \cdot, \cdot \rangle_{H} : {}^{1}H^{j} \times {}^{2}H^{m+n-j} \longrightarrow {}^{3}H^{m+n} \cong \mathbb{R} \tag{6.5} $$

is nondegenerate as well, i.e.

$$ {}^{1}H^{j} \xrightarrow{\cong} \left({}^{2}H^{m+n-j}\right)^{*}, \quad [x] \mapsto \langle [x], \cdot \rangle_{H}, $$

is a linear isomorphism.

The above theorem is applied to Hochschild-Serre spectral sequence for transitive Lie algebroids. These spectral sequences for transitive Lie algebroids were considered in the book by Mackenzie

K. C. H. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, 124, 1987,

and in the paper

V.Itskov, M.Karashev, and Y.Vorobjev, *Infinitesimal Poisson Cohomology,* Amer. Math. Soc. Transl. (2) Vol. 187, 1998.

We fix a transitive Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \#_{A})$ with the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \overset{\#_A}{\to} TM \to 0$ and a representation $\nabla : A \to A(\mathfrak{f})$ of a Lie algebroid A on a vector bundle $\mathfrak{f}$. ∇ is a homomorphism of Lie algebroids, then ∇ induces a homomorphism of vector bundles $\nabla^{+} : \boldsymbol{g} \to \operatorname{End}(\mathfrak{f})$

$$ \begin{array}{ccc} \boldsymbol{g} & \xrightarrow{\nabla^{+}} & \operatorname{End}(\mathfrak{f}) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\nabla} & A(\mathfrak{f}) \end{array} $$

and $\nabla_{x}^{+} : \boldsymbol{g}_{x} \to \operatorname{End}(\mathfrak{f}_{x})$ is a representation of the isotropy Lie algebra $\boldsymbol{g}_{x}$ in the vector space $\mathfrak{f}_{x}$.

For the Evens-Lu-Weeinstein representation $D : A \to A(Q_{A})$ we have:

Lemma 6.2. (a) $D^{+} : \boldsymbol{g} \to \operatorname{End}(Q_{A})$ *is defined by*

$$ D_{\sigma}^{+} = (\nabla_{\mathrm{trad}})_{\sigma} = \operatorname{tr}(\operatorname{ad}_{\sigma}) \cdot \mathrm{id}, \quad \sigma \in \Gamma(\boldsymbol{g}). $$

(b) $H_{D^{+}}^{n}(\boldsymbol{g}, Q_{A}) = \Lambda^{n}\boldsymbol{g}^{*} \otimes Q_{A}$, $H_{D^{or+}}^{n}(\boldsymbol{g}, Q_{A}^{or}) = \Lambda^{n}\boldsymbol{g}^{*} \otimes Q_{A}^{or}$.

Consider the $C^{\infty}(M)$-module of A-differential $\mathfrak{f}$-valued forms $\Omega(A;\mathfrak{f})$ with natural gradation, differental and Hochschild-Serre filtration

$$ \Omega_{j} = \Omega_{j}(A;\mathfrak{f}) $$

of $C^{\infty}(M)$-modules where Ω_j^t, $t \geq 0$, consists of all those t-differential forms f for which $f(\gamma_1, ..., \gamma_t) = 0$ whenever $t - j + 1$ of the arguments γ_i belongs to $\mathfrak{b}(\boldsymbol{g})$. Analogously, we consider a submodule of $C^{\infty}(M)$-differential forms with compact support $\Omega_c^t(A; \xi)$. We obtain in this way a graded filtered differential space

$$(6.6) \qquad \left(\Omega(A; \mathfrak{f}) = \bigoplus_t \Omega^t(A; \mathfrak{f}), d_{A,\nabla}, \Omega_j\right)$$

and its spectral sequence

$$(6.7) \qquad \left(E_{A,s}^{j,i}, d_{A,\nabla,s}\right);$$

and we get a graded filtered differential space with compact support

$$(6.8) \qquad \left(\Omega_c(A; \mathfrak{f}) = \bigoplus_t \Omega_c^t(A; \mathfrak{f}), d_{A_c,\nabla}, \Omega_{c,j}\right)$$

and its spectral sequence

$$(6.9) \qquad \left(E_{A_c,s}^{j,i}, d_{A_c,\nabla,s}\right).$$

Using the results from
G.Hochschild, and J.-P.Serre, *Cohomology of Lie algebras,* Ann. Math. 57, 1953, 591-603.
modyfing it to $C^{\infty}(M)$-cocycles we can give the formulas for the second terms of the spectral sequences:

Lemma 6.3. *The homomorphisms*

$$\Psi_{A,2} : E_{A,2}^{j,i} \to H_{\nabla^i}^j\left(M; H_{\nabla^+}^i(\boldsymbol{g}, \mathfrak{f})\right), \quad [f] \mapsto \left[\,(-1)^{ji}\left[\bar{f}_j\right]\,\right],$$
$$\Psi_{A_c,2} : E_{A_c,2}^{j,i} \to H_{\nabla^i,c}^j\left(M; H_{\nabla^+}^i(\boldsymbol{g}, \mathfrak{f})\right), \quad [f] \mapsto \left[\,(-1)^{ji}\left[\bar{f}_j\right]\,\right],$$

are isomorphisms of $C^{\infty}(M)$-modules, where for an auxiliary connection $\lambda : TM \to A$ and for $f \in Z_{A,1}^{j,i} \subset \Omega^{j+i}(A; \mathfrak{f})$ we define

$$\bar{f}_j \in \Omega^j\left(M; \Lambda^i \boldsymbol{g}^* \otimes \mathfrak{f}\right)$$

by the formula

$$\bar{f}_j(X_1, ..., X_j)(\sigma_1, ..., \sigma_i) = f(\lambda X_1, ..., \lambda X_j, \sigma_1, ..., \sigma_i),$$

and ∇^i is a usual flat covariant derivative in the vector bundle $H_{\nabla^+}^i(\boldsymbol{g}, \mathfrak{f})$ defined by the formula: for a d_{∇^+}-cocycle $f \in \Lambda^i \boldsymbol{g}^* \otimes \mathfrak{f}$

$$\nabla_X^i([f]) = \left[\left(\mathcal{L}_A^i\right)_{\lambda X}(f)\right]$$

where

$$\left(\mathcal{L}_A^i\right)_{\lambda X}(f)(\sigma_1, ..., \sigma_i) = \nabla_{\lambda X}(f(\sigma_1, ..., \sigma_i)) - \sum_t f(\sigma_1, ..., [\![\lambda X, \sigma_t]\!], ..., \sigma_i).$$

For $\mathfrak{f}=Q_A$ and for the E-L-W representation $\nabla = D : A \to A(Q_A)$ we have:

Proposition 6.1. *The bundles $H^n_{D^{or}+}(\boldsymbol{g}, Q^{or}_A)$ and $or(M)$ are isomorphic in the category of flat bundles*

$$\left(H^n_{D^{or}+}(\boldsymbol{g}, Q^{or}_A), \nabla^n\right) \sim (or(M), \partial^{or}).$$

Assume that A is a transitive Lie algebroid with three representations

$$\nabla_r : A \to A(\mathfrak{f}_r), \quad r=1,2,3,$$
$$\mathfrak{f}_3 = Q^{or}_A, \quad \nabla_3 = D^{or} \text{ (E-L-W repr.)}$$

and a pairing

$$F : \mathfrak{f}_1 \times \mathfrak{f}_2 \to Q^{or}_A$$

compatible with the representations $(\nabla_1, \nabla_2, \nabla_3)$. Roughly speaking, the multiplication of cochains

$$\wedge : \Lambda^j \boldsymbol{g}^* \otimes \mathfrak{f}_1 \times \Lambda^i \boldsymbol{g}^* \otimes \mathfrak{f}_2 \to \Lambda^{j+i} \boldsymbol{g}^* \otimes Q^{or}_A$$

gives the pairing of cohomology vector bundles

$$\wedge : H^j_{\nabla_1^+}(\boldsymbol{g}, \mathfrak{f}_1) \times H^i_{\nabla_2^+}(\boldsymbol{g}, \mathfrak{f}_2) \to H^{j+i}_{D^{or}+}(\boldsymbol{g}, Q^{or}_A). \tag{6.10}$$

Immediately from THEOREM I we obtain

Theorem 6.6. *If the pairing (6.10)*

$$\wedge : H^i_{\nabla_1^+}(\boldsymbol{g}, \mathfrak{f}_1) \times H^{n-i}_{\nabla_2^+}(\boldsymbol{g}, \mathfrak{f}_2) \to H^n_{D^{or}+}(\boldsymbol{g}, Q^{or}_A) \cong or(M) \tag{6.11}$$

is non-degenerate, then the same holds for the pairing
(6.12)

$$H^j_{\nabla^i}\big(M; H^i_{\nabla_1^+}(\boldsymbol{g}, \mathfrak{f}_1)\big) \times H^{m-j}_{\nabla^{n-i},c}\big(M; H^{n-i}_{\nabla_2^+}(\boldsymbol{g}, \mathfrak{f}_2)\big) \to H^m_{\nabla^n,c}\big(M; H^n_{D^{or}+}(\boldsymbol{g}, Q^{or}_A)\big) \xrightarrow{\int_M^\#} \mathbb{R}.$$

This last pairing is $\pm$ equal to the multiplication of the second term of the spectral sequences

$${}^1E^{j,i}_{A,2} \times {}^2E^{m-j,n-i}_{A_c,2} \xrightarrow{\wedge} {}^3E^{m,n}_{A_c,2} \xrightarrow{\cong} \mathbb{R}$$

so the last one is nondegenerate as well,

$${}^1E^{j,i}_{A,2} \cong \left({}^2E^{m-j,n-i}_{A_c,2}\right)^*,$$

and the theorem on spectral sequences THEOREM III implies that the multiplication of cohomology classes

$$\langle \cdot, \cdot \rangle_H : H^j_{\nabla_1}(A; \mathfrak{f}_1) \times H^{m+n-j}_{\nabla_2,c}(A; \mathfrak{f}_2) \to H^{m+n}_{D^{or},c}(A; Q^{or}_A) \xrightarrow{\cong} \mathbb{R}$$

is nondegenerate as well, i.e.

$$H^j_{\nabla_1}(A; \mathfrak{f}_1) \cong \left(H^{m+n-j}_{\nabla_2,c}(A; \mathfrak{f}_2)\right)^*.$$

We can apply the above theorem for the pairing

$$F:(M\times\mathbb{R})\times Q_A^{or}\to Q_A^{or}$$

and representations

$$\left(\partial_A^{or},D^{or},D^{or}\right).$$

By the Lemma (6.2) saying that $D^{+}:\boldsymbol{g}\to\operatorname{End}\left(Q_A\right)$ is defined by

$$D_\sigma^{+}=\left(\nabla_{\mathrm{trad}}\right)_\sigma=\operatorname{tr}\left(\operatorname{ad}_\sigma\right)\cdot\mathrm{id},\quad \sigma\in\Gamma\left(\boldsymbol{g}\right),$$

and by the THEOREM II (i.e. some consequence of the version of the Chern-Hirzebruch-Serre Lemma) which says that the pairing

$$H^i\left(\mathfrak{g}\right)\times H_{\mathrm{trad}}^{n-i}\left(\mathfrak{g}\right)\to H_{\mathrm{trad}}^{n}\left(\mathfrak{g}\right)\cong\mathbb{R}$$

is nondegenerate we obtain that in each point $x\in M$ the nondenerate pairing (6.11)

$$\wedge:H^i\left(\boldsymbol{g}_x\right)\times H_{D^{or+}}^{n-i}\left(\boldsymbol{g}_x,\left(Q_A^{or}\right)_x\right)\to H_{D^{or+}}^{n}\left(\boldsymbol{g}_x,\left(Q_A^{or}\right)_x\right)\cong or\left(M\right)_x\cong\mathbb{R}.$$

The last theorem gives then:

Theorem 6.7. *The E-L-W cohomology pairing*

$$H^j\left(A\right)\times H_{D^{or},c}^{m+n-j}\left(A;Q_A^{or}\right)\to\mathbb{R},$$

is nondegenerate.

From the Exceptionality of the E-L-W representation we can obtain the full answer for the question: for what Lie algebroids the real cohomology algebra is an algebra with the "Poincaré duality"

Theorem 6.8. *The following conditions are equivalent*

(a) $H_{\partial_A,c}^{m+n}\left(A\right)\neq0,$
(b) $H_{\partial_A,c}^{m+n}\left(A\right)\cong\mathbb{R}$ *and* $H\left(A\right)$ *is a Poincaré algebra, i.e. the pairing* $H^j\left(A\right)\times H_c^{m+n-j}\left(A\right)\to H_c^{m+n}\left(A\right)\cong\mathbb{R}$ *is non-degenerate,* $H^j\left(A\right)\cong\left(H_c^{m+n-j}\left(A\right)\right)^{*},$
(c) $\left(Q_A^{or},D^{or}\right)\sim\left(M\times\mathbb{R},\partial_A\right),$
(e) $\left(\Lambda^n\boldsymbol{g},ad_A^n\right)\sim\left(or\left(M\right),\partial_A^{or}\right),$ *that is the holonomy homomorphism of* $\left(\Lambda^n\boldsymbol{g},ad_A^n\right)$ *is the same as for the orientation bundle* $\left(or\left(M\right),\partial_A^{or}\right).$
(f) *A is orientable vector bundle and the modular class of A is zero* $\theta_A=0$. *(in particular, the isotropy Lie algebras $\boldsymbol{g}_x$ are unimodular).*

For oriented manifold M we obtain only the so-called transitive, unimodular invariantly oriented Lie algebroids, i.e. such that there exists a global nonsingular section $\varepsilon\in\Gamma\left(\Lambda^n\boldsymbol{g}\right)$ which is ad_A^n-constant. These Lie algebroids and its Poincaré property were discovered in the paper

J.Kubarski, Poincaré duality for transitive unimodular invariantly oriented Lie algebroids, Topology and Its Applications, Vol 121, 3, June 2002, 333-355.

In the end I would like to formulate two question.

- QUESTION 1. Is for arbitrary (nonregular) Lie algebroid A the cohomology E-L-W pairing
$$H^{j}(A)\times H^{r-j}_{D^{or},c}(A;Q^{or}_{A})\to H^{r}_{D^{or},c}(A;Q^{or}_{A})\to\mathbb{R}$$
weakly nondegenerate in the following sense:
— the 2-linear homomorphism $F:V\times W\to\mathbb{R}$ will be called "weakly" non-degenerate if both null spaces are zero, i.e. if
$$N_1=\{v\in V;\ F(v,\cdot)\}=0,$$
$$N_2=\{w\in W;\ F(\cdot,w)\}=0.$$
Equivalently, if the induced linear homomorphisms
$$V\to L(W;\mathbb{R}),\quad v\longmapsto F(v,\cdot),$$
$$W\to L(V,\mathbb{R}),\quad w\longmapsto F(\cdot,w),$$
are monomorphisms.
- QUESTION 2. Is for arbitrary (nonregular) Lie algebroid A the E-L-W representaion
$$D^{or}:A\to A(Q^{or}_{A})$$
exceptional from the point of view of the top group of cohomology?
More precisely, let A be a transitive Lie algebroid. For a line bundle $\mathfrak{f}$ and a representation $\nabla:A\to A(\mathfrak{f})$: are the following conditions equivalent?
$H^{r}_{\nabla,c}(A;\mathfrak{f})\neq 0,$
$(\mathfrak{f},\nabla)\cong(Q^{or}_{A},D^{or})\,.$