LOCAL AND NICE STRUCTURES OF THE GROUPOID OF AN EQUIVALENCE RELATION

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ABSTRACT. A comparison between the concepts of local and nice structures of the groupoid of an equivalence relation is presented. It is shown that these concepts are closely related, and that generically they characterize the equivalence relations induced by regular foliations. The first concept was introduced by J.Pradines (1966), and studied by R.Brown and O.Mucuk (1996), while the second one was given by the first author (1987). The importance of these concepts in the nontransitive geometry is indicated.

1. LOCAL AND NICE STRUCTURES OF EQUIVALENCE RELATIONS

Let $R \subset X \times X$ be any equivalence relation on a connected paracompact C^∞-manifold X of dimension n. R becomes a topological groupoid on X with the topology induced from $X \times X$, namely

$$R = (R, \alpha, \beta, \gamma)$$

where $\alpha = (\mathrm{pr}_1)_{|R}$, $\beta = (\mathrm{pr}_2)_{|R}$, $\gamma : R_\alpha \times_\beta R \to R$, $((x,y),(z,x)) \mapsto (z,y)$. Let $\delta : R \times_\alpha R \to R$, $((x,z),(x,y)) \mapsto (y,z)$. According to the definition of Brown and Mucuk [A-B], [B-M1], [B-M2] (following the original definition of J.Pradines [P]) R is called a *locally Lie groupoid* if there exists a subset $W \subset R$ equipped with a structure of a manifold such that

1) $O_R \subset W \subset R$, $(O_R = \{(x,x)\, ;\ x \in X\})$,
2) $W = W^{-1}$,
3) a) $W_\delta = (W \times_\alpha W) \cap \delta^{-1}[W]$ is open in $W \times_\alpha W$,
   b) the restriction of δ to W_δ is smooth,
4) the restriction to W of the source and the target maps α and β are smooth and the triple (α, β, W) is smoothly locally sectionable,
5) W generates R as a groupoid.

W satisfying 1)-5) will be called a *local smooth structure* of R.

Remark 1.1. *The second part of 4) implies that $\alpha_{|W} : W \to X$ is a coregular mapping. Therefore $W \times_\alpha W = (\alpha_{|W}, \alpha_{|W})^{-1}[O_R]$ is a proper (i.e. embedded) submanifold of $W \times W$.*

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Let F be a k-dimensional foliation on X and $\Lambda = \{(U_\lambda, \phi_\lambda)\}$ be an atlas of foliated charts. The equivalence relation on X determined by the leaves of F is denoted by R_F. For a chart $(U_\lambda, \phi_\lambda)$ we write R_λ for the equivalence relation on U_λ whose equivalence classes are the plaques of U_λ

$$R_\lambda = \{(x, y) \in X \times X;\ x \in U_\lambda,\ y \in Q_\lambda^x\},$$

Q_λ^x denotes the plaque of U_λ passing through x. We write

$$W(\Lambda) = \bigcup_\lambda R_\lambda.$$

Brown and Mucuk in [B-M2] proved the following

Theorem 1.1. *If X is a paracompact foliated manifold, then a foliated atlas Λ may be chosen so that the pair $(R_F, W(\Lambda))$ is a locally Lie groupoid.*

Clearly $W(\Lambda)$ is a $n+k$-dimensional proper submanifold of $X \times X$ and $W(\Lambda)_x = \left(\alpha_{|W(\Lambda)}\right)^{-1}(x)$ is a connected k-dimensional submanifold of $W(\Lambda)$. The crucial role in the proof is played by Lemma 4.4 from [T].

On the other hand, in [K2] there is a second approach to the groupoid of an equivalence relation characterizing the fact that the family of all equivalence classes forms a k-dimensional foliation.

Theorem 1.2 (J.Kubarski [K2, Th. 3])**.** *If X is a paracompact manifold, then the following conditions are equivalent:*

   (1) *the family of all equivalence classes of R is a k-dimensional foliation,*
   (2) *there exists a subset $W \subset R$ such that*
      (i) *$O_R \subset W \subset R$,*
      (ii) *W is a proper $n + k$-dimensional C^∞-submanifold of $X \times X$,*
      (iii) *$\alpha_{|W} : W \to X$ is a submersion,*
      (iv) *For $(x, y) \in R$, the set*

$$D_{(x,y)}[W_y] \cap W_x$$

      is open in the manifold $W_x = \left(\alpha_{|W}\right)^{-1}(x)$, where $D_{(x,y)} : R_y \to R_x$, $(y, z) \mapsto (x, z)$,
      (v) *W generates R as a groupoid,*
      (vi) *the manifolds W_x are connected.*

We can choose W to be symmetric, i.e. $W = W^{-1}$. The crucial role in the proof of the theorem is played by the *nice covering* [H-H, 1981] and the Frobenius theorem in the version [D, p.86]. The manifold W satisfying the above conditions (i)-(vi) is said to be a *nice structure of R* (thanks to the nice covering used in the proof, see also [K3]). In the above theorem it is proved that $W(\Lambda)$ is a nice structure if Λ is any nice covering.

The aim of this note is to explain the relations between the concepts of local and nice structures of an equivalence relation. It occurs that under mild assumptions the concepts are not only equivalent but also identical. Moreover each of them can characterize the (regular) foliation relations among equivalence relations, and the axioms of a nice structure are considerably simpler. In the final section it is indicated how these concepts are related with leaf preserving diffeomorphisms on a foliated manifold and automorphisms of a regular Poisson structure.

Finally let us mention that in [K2] there is also given a characterization (by suitable axioms of a subset $W \subset R$) of some wider class of equivalence relations R on X for which the family of all arcwise connected components of all equivalence classes of R is a regular foliation and every equivalence class of R has a countable number of such components. As a consequence a new short proof of classical Godement's theorem on division is presented.

2. Local structure $\implies$ nice structure

First we show that under mild assumptions any local structure is a nice one.

Proposition 2.1. *If $W \subset R$ is a local smooth structure such that W is a proper $n+k$-dimensional C^∞-submanifold of $X \times X$ and $W_x = \left(\alpha_{|W}\right)^{-1}(x)$ are connected then W carries a nice structure. Consequently, the family of equivalence classes of R forms a k-dimensional foliation.*

Proof. We have only to prove the condition (iv). For this purpose fix arbitrarily $(x_0, y_0) \in R$ and consider the smooth mapping

$$i_{x_0,y_0} : W_{x_0} \longrightarrow W \times_\alpha W, \quad (x_0, z) \longmapsto ((x_0, z), (x_0, y_0)).$$

It is easy to see that

$$(2.1) \qquad (i_{x_0,y_0})^{-1}[W_\delta] = D_{(x_0,y_0)}[W_{y_0}] \cap W_{x_0}.$$

" $\subset$ " Let $(x_0, z) \in (i_{x_0,y_0})^{-1}[W_\delta]$, then $(x_0, z) \in W_{x_0}$ and $((x_0, z), (x_0, y_0)) \in W_\delta$ which means that $\delta((x_0, z), (x_0, y_0)) = (y_0, z) \in W \cap R_{y_0} = W_{y_0}$ and $(x_0, z) = D_{(x_0,y_0)}(y_0, z) \in D_{(x_0,y_0)}[W_{y_0}] \cap W_{x_0}$.
" $\supset$ " Let $(y_0, z) \in W_{y_0}$ and $(x_0, z) \in W_{x_0}$, then

$$\delta(i_{x_0,y_0}(x_0, z)) = \delta((x_0, z), (x_0, y_0)) = (y_0, z) \in W_{y_0} \subset W.$$

Finally (2.1) implies (iv). $\qquad \square$

3. Nice structure $\implies$ local structure

In view of Theorems 1.2 and 1.1 each nice structure admits a local structure (as a suitable subset). The problem is whether every nice structure *is* a local one.

This problem amounts to reducing the axioms of a local smooth structure. The first step consists in the following

Proposition 3.1. *If $W \subset R$ fulfils (i)-(iii) and $W = W^{-1}$ then the triple (α, β, W) is smoothly locally sectionable.*

Proof. Since $\beta = \alpha \circ \iota$, and the inverse map $\iota : W \to W$, $(x, y) \mapsto (y, x)$ is a diffeomorphism, β is a submersion, too. For $(x, y) \in W$, $V_1 := \ker \alpha_{*(x,y)}$ and $V_2 := \ker \beta_{*(x,y)}$ are k-dimensional and $V_1 \cap V_2 = 0$. Choose an $(n-k)$-dimensional vector subspace $V \subset T_{(x,y)}W$ such that

$$(3.1) \qquad T_{(x,y)}W = V \bigoplus V_1 \bigoplus V_2$$

and find a k-dimensional vector subspace $V_0 \subset V_1 \bigoplus V_2$ transversal to both V_1 and V_2. For example if $(u_1, ..., u_k)$ and $(v_1, ..., v_k)$ are basis of V_1 and V_2, respectively, we can take $V_0 = \mathrm{Lin}\{u_1 + v_1, ..., u_k + v_k\}$. Putting

$$U = V \bigoplus V_0$$

we have that $U \cap V_2 = 0$, and this implies that $\alpha_* : U \to T_x X$ is an isomorphism. Since $\alpha : W \to X$ is a submersion there exists a local cross-section s of α near x such that $s_* T_x X = U$. It is clear that

$$(\beta \circ s)_* : T_x X \to T_y X$$

is an isomorphism: if $(\beta \circ s)_* (v) = \theta_y$ then $s_* (v) \in V_2 \cap U = 0$ which yields $v = 0$. From this $\beta \circ s$ is a diffeomorphism near the point x. $\qquad\square$

To finish the problem of reduction of axioms of a local smooth structure we have to prove that the conditions (i)-(iv) together with the symmetry 2) $(W = W^{-1})$ imply condition 3).

Let F be any foliation on a manifold X and denote by L_x the leaf of F through x. Fix a point $x_0 \in X$ and $y_0 \in L_{x_0}$. Assume that U and V are two foliated coordinate neighbourhoods of x_0 and y_0 respectively, and $\varphi : U \to V$ is a foliated diffeomorphism (i.e. $\varphi(x) \in L_x$) onto an open subsets of V. Set

$$R_\varphi = \left\{ (x,y) \in U \times V ; x \in U, \ y \in Q_V^{\varphi(x)} \right\} \subset U \times V.$$

Clearly, R_φ is an $n+k$-dimensional proper submanifold of $U \times V$. Of course, if $U' \subset U$ is any coordinate subneighbourhood of x_0 and $\varphi_0 : U' \to V$ and $\varphi_1 : U' \to \varphi[U']$ are induced mappings then R_{φ_0} and R_{φ_1} are open submanifolds of R_φ.

Lemma 3.1. *Let $\varphi, \psi : U \to V$ be any foliated diffeomorphisms such that $\varphi(x_0), \psi(x_0)$ belong to the same plaque of V, i.e. $Q_V^{\varphi(x_0)} = Q_V^{\psi(x_0)}$. If $R_\varphi \cap R_\psi$ is an open subset of R_φ (therefore also of R_ψ) then there exists a coordinate neighbourhood U_1 such that $x_0 \in U_1 \subset U$ and $\varphi(x), \psi(x)$ lie on the same plaque of V for all $x \in U_1$, i.e. $Q_V^{\varphi(x)} = Q_V^{\psi(x)}$, $x \in U_1$.*

Proof. The topology on R_φ and R_ψ is induced from $U \times V$, so there exist open neighbourhoods $\Omega, \Omega' \subset U \times V$ such that $R_\varphi \cap R_\psi = \Omega \cap R_\varphi = \Omega' \cap R_\psi$. Then $\tilde{\Omega} = \Omega \cap \Omega'$ is an open subset of $U \times V$ and

$$\tilde{\Omega} \cap R_\varphi = \Omega \cap \Omega' \cap R_\varphi = \Omega' \cap R_\varphi \cap R_\psi = R_\varphi \cap R_\psi \cap R_\varphi = R_\varphi \cap R_\psi.$$

Analogously $\tilde{\Omega} \cap R_\psi = R_\varphi \cap R_\psi$. Therefore

$$\{x_0\} \times Q_V^{\varphi(x_0)} = \{x_0\} \times Q_V^{\psi(x_0)} \subset R_\varphi \cap R_\psi = \tilde{\Omega} \cap R_\varphi = \tilde{\Omega} \cap R_\psi \subset U \times V.$$

Now we can choose neighbourhoods U_1 and V_1 such that $x_0 \in U_1 \subset U$, $\varphi(x_0) \in V_1 \subset V$ and $U_1 \times V_1 \subset \tilde{\Omega}$. We can also assume that $\varphi[U_1] \subset V_1$. Then

$$U_1 \times V_1 \cap R_\varphi \subset \tilde{\Omega} \cap R_\varphi = \tilde{\Omega} \cap R_\psi \subset R_\psi = \bigcup_{\tilde{x} \in U} \{\tilde{x}\} \times Q_V^{\psi(\tilde{x})}.$$

Consequently, for all $x \in U_1$ we obtain

$$Q_V^{\varphi(x)} \cap V_1 \subset Q_V^{\psi(x)},$$

i.e. $\varphi(x) \in Q_V^{\psi(x)}$, or equivalently $Q_V^{\varphi(x)} = Q_V^{\psi(x)}$, $x \in U_1$. $\qquad\square$

The following two propositions describe further properties of nice structures.

Proposition 3.2. *If $W \subset R$ fulfils (i)-(vi) and $W = W^{-1}$ then for any $(x_0, y_0) \in W$ there exist coordinate neighbourhoods U_{x_0} and U_{y_0} of x_0 and y_0, respectively, and a foliated diffeomorphism $\phi : U_{x_0} \to U_{y_0}$ such that $R_\phi \subset W$.*

Proof. Suppose $W \subset R$ fulfils (i)-(vi) and $W = W^{-1}$. Then, by Theorem 1.2, the family of all equivalence classes of R is a k-dimensional foliation, say F. Observe that $F_W := \alpha^* F$ is a $2k$-dimensional foliation on W, and $F_W = \beta^* F$.

Let $(x_0, y_0) \in W$. There exist an open neighborhood Ω of (x_0, y_0) in W and vector fields $X_1, \ldots, X_k, Y_1, \ldots, Y_k, Z_1, \ldots, Z_{n-k}$ defined on Ω and spanning $T_{(x_0,y_0)}W$ at (x_0, y_0) with the properties that $X_1, \ldots, X_k$ are sections of the bundle $T F_W \cap \ker \beta_*$ and $Y_1, \ldots, Y_k$ are sections of the bundle $T F_W \cap \ker \alpha_*$. Denote by φ_t^X the local flow of a vector field X. There is $\epsilon > 0$ and Ω', an open subset in Ω, such that

$$U_\epsilon^{n+k} \ni t \mapsto \varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_k}^{X_k} \circ \varphi_{t_{k+1}}^{Y_1} \circ \cdots \circ \varphi_{t_{2k}}^{Y_k} \circ \varphi_{t_{2k+1}}^{Z_1} \circ \cdots \circ \varphi_{t_{n+k}}^{Z_{n-k}}((x_0, y_0)) \in \Omega'$$

is a diffeomorphism and, consequently, an F_W-foliated chart. Here $U_\epsilon^m = \{t \in \mathbb{R}^m : |t_i| < \epsilon\}$ is the ϵ-cube centered at the origin. It follows that

$$\psi_1 : U_\epsilon^n \ni t \mapsto \alpha \left(\varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_k}^{X_k} \circ \varphi_{t_{k+1}}^{Z_1} \circ \cdots \circ \varphi_{t_n}^{Z_{n-k}}((x_0, y_0)) \right) \in U \subset X$$

and

$$\psi_2 : U_\epsilon^n \ni t \mapsto \beta \left(\varphi_{t_1}^{Y_1} \circ \cdots \circ \varphi_{t_k}^{Y_k} \circ \varphi_{t_{k+1}}^{Z_1} \circ \cdots \circ \varphi_{t_n}^{Z_{n-k}}((x_0, y_0)) \right) \in V \subset X$$

are F-foliated charts at x_0 and y_0, respectively. Moreover, $\alpha|_{\Omega'} : \Omega' \to U$ and $\beta|_{\Omega'} : \Omega' \to V$ are foliation preserving submersions. Therefore the opens $U_{x_0} = U$, $V_{y_0} = V$ and the mapping

$$\phi := \psi_2 \circ \psi_1^{-1} : U \to V$$

satisfy the required condition. $\qquad\square$

Proposition 3.3. *Let $W \subset R$ be a nice structure of an equivalence relation R, such that $W = W^{-1}$. Take $x_0 \in X$ and $y_0, z_0 \in W_{x_0}$. We assume that all the points are pairwise distinct and $z_0 \in W_{y_0}$. Then there exist coordinate neighbourhoods U_{x_0}, U_{y_0}, U_{z_0} of x_0, y_0, z_0, respectively, and foliated diffeomorphisms $\varphi : U_{x_0} \to U_{y_0}$, $\psi : U_{y_0} \to U_{z_0}$, such that*

$$R_\varphi, \ R_\psi, \ R_{\psi \circ \varphi} \subset W.$$

Proof. Denote by L_x the leaf of the foliation determined by R. Take continuous and without selfintersections arc $\alpha : [0, 1] \to \beta [W_{x_0}] \subset L_{x_0}$ such that $\alpha(0) = x_0$, $\alpha(1/2) = y_0$ and $\alpha(1) = z_0$. By Proposition 3.2 there exist coordinate neighbourhoods U_0, U_1, U_2 of the points x_0, y_0, z_0, respectively, and foliated diffeomorphisms $\varphi : U_1 \to U_2$, $\psi : U_2 \to U_3$, such that $R_\varphi, R_\psi \subset W$. We will prove that there exists a coordinate subneighbourhood U_{x_0} of x_0 contained in U_1 such that $R_{\psi \circ \varphi|_{U_{x_0}}} \subset W$ from which our proposition follows immediately.

Put
$$I = \{t \in [0, 1] ; \exists_{\varphi_t, \psi_t}, \ R_{\varphi_t}, R_{\psi_t}, R_{\psi_t \circ \varphi_t} \subset W\}$$

where $\varphi_t : U_t \to U_t'$, $\psi_t : U_t' \to U_3$, are foliated diffeomorphisms (into) such that $x_0 \in U_t \subset U_1$, $\varphi_t(x_0) = \alpha(t)$, $\psi_t(\alpha(t)) = z_0$. We prove that I is nonempty open-closed subset of $[0, 1]$.

• $0 \in I$. Indeed, by Proposition 3.2 we can find coordinate neighbourhoods U_{y_0} and U_{z_0} of y_0 and z_0 and a foliated diffeomorphism $\psi : U_{y_0} \to U_{z_0}$ such that $R_\psi \subset W$. Taking $U_{x_0} = U_{y_0}$ and $\varphi_0 = id_{U_{x_0}}$ we obtain the conclusion.

• *I is open.* Indeed, let $t_0 \in I$ and $\varphi_{t_0} : U_{t_0} \to U_{t_0}'$, $\psi_{t_0} : U_{t_0}' \to U_3$ satisfy the properties from definition of I. There exists $\varepsilon > 0$ such that if $|t - t_0| < \varepsilon$ then

$\alpha\left(t\right) \in Q_{U'_{t_0}}^{\alpha(t_0)}$. Consider an arbitrary foliated diffeomorphism $c_t : U'_{t_0} \to U'_{t_0}$ such that $c_t\left(\alpha\left(t\right)\right) = \alpha\left(t_0\right).$ Then we put

$$\varphi_t = c_t^{-1} \circ \varphi_{t_0}, \quad \psi_t = \psi_{t_0} \circ c_t.$$

• *I is closed.* Let $t_0 \in \bar{I}.$ Choose arbitrarily coordinate neighbourhoods U_{t_0}, U'_{t_0} of $x_0\,\alpha\left(t_0\right)$ respectively, and foliated diffeomorphisms $\varphi_{t_0} : U_{t_0} \to U'_{t_0}, \psi_{t_0} : U'_{t_0} \to U_3,$ such that $\varphi_{t_0}\left(x_0\right) = \alpha\left(t_0\right), \psi_{t_0}\left(\alpha\left(t_0\right)\right) = z_0,$ and $R_{\varphi_{t_0}}, R_{\psi_{t_0}} \subset W.$ We can find $t \in I$ such that $\alpha\left(t\right) \in Q_{U'_{t_0}}^{\alpha(t_0)}$ and φ_t, ψ_t from definition of $I.$ We can assume that $U'_t \subset U'_{t_0},$ and that $\psi_{t_0}(\alpha(t)) \in Q_{U_3}^2.$ Consider on U'_t two foliated diffeomorphisms ψ_t and $\psi'_{t_0} = \psi_{t_0}|U'_t.$ By assumption, R_{ψ_t} and $R_{\psi_{t_0}}$ (so also $R_{\psi'_{t_0}}$) are open subsets of $W.$ Since $\alpha\left(t\right)$ and $\alpha\left(t_0\right)$ lie on the same plaque of U'_{t_0} and $\psi_{t_0}\left(\alpha\left(t_0\right)\right) = z_0 = \psi_t\left(\alpha\left(t\right)\right)$ lie on the same plaque of U_3 we obtain that

$$\emptyset \neq \{\alpha(t)\} \times Q_{U_3}^{\psi_t(\alpha(t))} \subset R_{\psi_t} \cap R_{\psi'_{t_0}}.$$

Since R_{ψ_t} and $R_{\psi'_{t_0}}$ are $n+k$-dimensional submanifolds of W we see that $R_{\psi_t} \cap R_{\psi'_{t_0}}$ is nonempty open subset of the manifold R_{ψ_t} as well as of $R_{\psi'_{t_0}}.$ By Lemma (3.1) there exists a coordinate neighbourhood $\tilde{U}'_t \subset U'_t$ such that ψ_t and ψ'_{t_0} maps each plaque in $\tilde{U}'_t$ into the same plaque in $U_3.$ By the well known Lemma 4.4. from [T] we can diminish the set $\tilde{U}'_t$ in such a way that each plaque of U'_{t_0} cuts the set $\tilde{U}'_t$ along at most one plaque of $\tilde{U}'_t.$ Consider the saturation $\check{U}'_{t_0}$ of $\tilde{U}'_t$ in U'_{t_0} by plaques of the last. Clearly, $\alpha\left(t_0\right) \in \check{U}'_{t_0},$ and let $\check{U}_{t_0} \subset U_{t_0}$ be a coordinate neighbourhood of x_0 such that $\varphi_{t_0}\left[\check{U}_{t_0}\right] \subset \check{U}'_{t_0}.$ The restrictions of φ_{t_0} to $\check{U}_{t_0}$ and ψ_{t_0} to $\check{U}'_{t_0}$ we denote by $\check{\varphi}_{t_0}$ and $\check{\psi}_{t_0},$ respectively. Notice that $R_{\check{\psi}_{t_0} \circ \check{\varphi}_{t_0}} \subset W.$ In fact, take $(x, z) \in R_{\check{\varphi}_{t_0} \circ \check{\varphi}_{t_0}},$ i.e. $x \in \check{U}_{t_0}$ and $z \in Q_{U_3}^{\check{\varphi}_{t_0} \circ \check{\varphi}_{t_0}(x)}.$ There exists $y \in \tilde{U}'_t$ such that y and $\check{\varphi}_{t_0}\left(x\right)$ lie on the same plaque of $\check{U}'_{t_0}.$ Therefore, the plaques passing through $\psi_{t_0}\left(y\right)$ and $\psi_{t_0}\left(\check{\varphi}_{t_0}\left(x\right)\right)$ are the same, and

$$\{y\} \times Q_{U_3}^{\psi_{t_0}(y)} = \{y\} \times Q_{U_3}^{\psi_t(y)} \subset W,$$

which implies that $(x, z) \in W.$

From the above $I = [0, 1]$ and our proposition is proved. □

Now we can prove our main result.

Theorem 3.1. *If $W \subset R$ is a nice structure of an equivalence relation on a manifold X then W is a local smooth structure of R.*

Proof. It remains to prove the property that $W_\delta = (W \times_\alpha W) \cap \delta^{-1}[W]$ is open in $W \times_\alpha W$ (it is clear that $\delta : W_\delta \to W$ is smooth). Take $(y_0, x_0), (y_0, z_0) \in W$ such that $(x_0, z_0) \in W.$ Then $((y_0, x_0), (y_0, z_0)) \in (W \times_\alpha W) \cap \delta^{-1}[W]$ and $\delta\left((y_0, x_0), (y_0, z_0)\right) = (x_0, z_0).$ By Proposition 4 we can find coordinate neighbourhoods $U_{x_0}, U_{y_0}, U_{z_0}$ of the points $x_0, y_0, z_0,$ respectively, and foliated diffeomorphisms $\varphi : U_{x_0} \to U_{y_0}, \psi : U_{y_0} \to U_{z_0}$ such that $R_\varphi, R_\psi, R_{\psi \circ \varphi} \subset W.$ By the symmetry $W = W^{-1}$ we obtain that $R_\varphi^{-1} \subset W.$ Clearly

$$R_\varphi^{-1} \times_\alpha R_\psi \subset W \times_\alpha W$$

and $R_\varphi^{-1} \times_\alpha R_\psi$ is open in $W \times_\alpha W.$ From $R_{\psi \circ \varphi} \subset W$ we have

$$\delta\left[R_\varphi^{-1} \times_\alpha R_\psi\right] = R_\psi \circ R_\varphi \subset R_{\psi \circ \varphi} \subset W$$

which implies that $R_\varphi^{-1} \times_\alpha R_\psi \subset W \times_\alpha W \cap \delta^{-1}[W]$. $\square$

4. FINAL REMARKS

A) Let us observe that the submanifold W that appears in the both definitions plays a crucial role in constructing a special chart of the leaf preserving diffeomorphism group of a regular foliation F on X, cf.[R1]. Namely, in the construction of a chart on this group the notion of a foliated local addition is needed. By a *foliated local addition* we mean a smooth mapping $\Phi : TF \supset U \to X$ such that

(1) $\Phi(0_x) = x$, and

(2) the mapping $(\pi, \Phi) : TF \supset U \to U' \subset W$ is a diffeomorphism, where U is some neighborhood of the zero section, U' is open in W, and $\pi : TF \to X$ is the canonical projection.

Consequently, it is shown that this group carries the structure of an infinite-dimensional regular Lie group. Next the existence of W enables as well to show that the group of automorphisms of a regular Poisson manifold is a regular Lie group. It follows that the flux homomorphism can be considered for regular Poisson manifolds and a characterization of Hamiltonian diffeomorphisms can be given. It occurs that under natural assumption on the group of periods the Hamiltonian diffeomorphisms form a splitted Lie subgroup (see [R1] for all this). One can say that the existence of W usually ensures that nontransitive geometries with the set of orbits forming a regular foliations do not differ essentially from their transitives counterparts.

Analogous problem for foliations with singularities is open as W cannot be then defined. Similar difficulties arise when the group of foliation preserving diffeomorphisms (i.e. sending each leaf to a leaf) is considered.

Introducing Lie group structures on diffeomorphism groups can be viewed in a more general context of groupoids, cf.[R2]. Namely in the "non-foliated" case one can consider the group of global bisections of a Lie groupoid rather than diffeomorphism groups, but this method fails in the nontransitive case. The reason is the "holonomic imperative" (see, e.g., [B-M1]) according to which any groupoid structure over the equivalence relation R of a foliation F contains the information of the holonomy of F.

B) What does it look like the "local-nice" problem for groupoids? A forthcoming paper will be devoted to this problem. Here we mention only that in [K3] there is a generalization of the notion of a nice structure for groupoids. Hovewer in [K3] there are considered groupoids on manifolds for which the total space possesses a structure stronger than topology but weaker than differential manifold, namely the structure of a differential space in the Sikorski sense ([S1], [S2]). Hausdorff manifolds carry a natural structure of a differential space consistent with the topology. Since any subset of a differential space has the induced structure of a differential space, any equivalence relation $R \subset X \times X$ on a manifold X or the groupoid $\Phi^R = (\alpha, \beta)^{-1}[R] \subset \Phi$, where Φ is any transitive Lie groupoid on X and R is any equivalence relation on X, are examples of groupoids with the structure of a differential space. Moreover, R and Φ^R are then differential spaces of the class $\mathcal{D}_0$ ([W1],[W2], [K-K]) [i.e. locally at a neighbourhood of every point there are extended to a manifold of dimension equal to the dimension of the tangent space to this differential space at this point]. To any nice structure W of Φ one can attach a Lie algebroid $A(\Phi)$ on X.

By a *nice groupoid* [K3] we mean a groupoid Φ on a manifold X together with nice structures W of Φ and W_0 of the induced equivalence relation R_Φ on X such that the mapping $(\alpha, \beta) : W \to W_0$ is a submersion. In [K3, Th.4.29] there are given some natural conditions for a groupoid Φ of the class $\mathcal{D}_0$ which imply that it is a nice groupoid. Many ideas from [K3] can be realized (in our preliminary opinion) on abstract or topological groupoids on manifolds as well.

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