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A LOCAL PROPERTY OF THE SUBSPACES OF EUCLIDEAN SPACES

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1. Preliminares

Let $M \neq \emptyset$ be a set and C an arbitrary set of real functions defined on M. We denote by τ_C the weakest topology on M such that all functions belonging to C are continuous. For any set A contained in M we denote by $C|A$ the set of functions of the form $\alpha|A$ where $\alpha \in C$. We denote by C_A the set of all real functions on A such that for any point p of A there exists in τ_C an open neighbourhood U of p and a function $\alpha \in C$, such that $\beta|A \cap U = \alpha|A \cap U$. It is easy to verify that, for any set $A \subset M$, we have $\tau_{C_A} = \tau_{C|A} = \tau_C|A$. In particular $\tau_{C_M} = \tau_C$. We denote by scC the set of all real functions of the form $\omega(\alpha_1, ..., \alpha_n)$, where $\omega \in \mathcal{E}_n$, $\alpha_1, ..., \alpha_n \in C$ and n belongs to the set of all positive integers $\mathcal{N}$ and $\mathcal{E}_n$ is the set of all real C^∞-functions on n-dimensional Euclidean space E^n. An ordered pair (M, C) such that $C_M = C = scC$ is called to be a *differential space*. The set C is called the *differential structure* of this differential space [1], [2], [6].

For a set C of real functions defined on M, the set $(scC)_M$ is the smallest differential structure on M including the set C. $(M, (scC)_M)$ is called the *differential space generated* by C.

If (M, C) is a differential space and $A \subset M$, then (A, C_A) is also a differential space called the *differential subspace* of (M, C) [1]. It is easy to see that $C_A = (C|A)_A$.

By a vector tangent to a differential space (M, C) at a point p of M we mean any linear mapping $v : C \to E$ which fulfils Leibniz's condition at the point p:

$$v(\alpha\beta) = v(\alpha)\,\beta(p) + \alpha(p)\,v(\beta) \quad \text{for all} \quad \alpha, \beta \in C.$$

We shall denote by $(M, C)_p$ or M_p a linear space of all vectors tangent to (M, C) at the point $p \in M$.

Any real C^∞-manifold M will be identified with the differential space $(M, C^\infty(M))$, where $C^\infty(M)$ is the set of all smooth real functions

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on M. In particular, we denote $C^{\infty}(E^n)$ by $\mathcal{E}_n$ and we call the pair $(E^n, \mathcal{E}_n)$ the *n-dimensional Euclidean differential space.*

It is aesy to verify that for $\emptyset \neq M \subset E^n$, $n \in \mathcal{N}$,

$$\mathcal{E}_{nM} = \left(sc\left\{\pi^i|M; i = 1, ..., n\right\}\right)_M$$

where $\pi^i\left(\left(x^1, x^2, ..., x^n\right)\right) = x^i$ for any $\left(x^1, x^2, ..., x^n\right) \in E^n$. The topological space $(M, \tau_{\mathcal{E}_{nM}})$ is a subspace of the topological space $(E^n, \tau_{\mathcal{E}_n})$.

In the sequel the symbol τ_M will be used instead of $\tau_{\mathcal{E}_{nM}}$. Using a partition of unity it may be proved that $\mathcal{E}_{nM}$ is the set of all functions of the form $\alpha|M$, where α is a C^{∞}-function on an open set U in E^n including M. The basic result of this paper consists in the following theorem.

Theorem 1. *For any $p \in M \subset E^n$ the integer $m = \dim\left(M, \mathcal{E}_{nM}\right)_p$ is the smallest one that there exists in τ_M an open neighbourhood U of the point p which is included in an m-dimensional C^{∞}-surface of E^n.*

2. The proof of basic result

From now on we fix the integer $k > 0$, and the non empty set $M \subset E^k$. For brevity we write $\mathcal{E} := \mathcal{E}_k$, $C := \mathcal{E}_M$, $M_p := (M, C)_p$, $E_p^k := \left(E^k, \mathcal{E}_k\right)_p$.

The mappings $L_1 : M_p \to E_p^k$ and $L_2 : E_p^k \to E^k$ defined by

$$(1) \qquad \left(L_1(v)\right)(f) := v\left(f|M\right) \quad \text{for} \quad v \in K_p \quad \text{and} \quad f \in \mathcal{E},$$

$$(2) \qquad L_2(\bar{v}) := \left(\bar{v}\left(\pi^1\right), ..., \bar{v}\left(\pi^k\right)\right) \quad \text{for} \quad \bar{v} \in E_p^k,$$

are respectively a linear monomorphism and a linear isomorphism of suitable linear spaces (c.f. [1]).

Let $\partial_i f(p)$ denote i-th partial derivative of the function $f \in \mathcal{E}$ at the point $p \in E^k$, $i = 1, ..., k$. If we denote $f_{|h}(p) := h^i \partial_i f(p)$ where $h = \left(h^1, ..., h^k\right) \in E^k$, we have

$$(3) \qquad \bar{v}(f) = \partial_i f(p)\, \bar{v}\left(\pi^i\right) = f_{|L_2(\bar{v})}(p) \quad \text{for} \quad \bar{v} \in E_p^k,$$

(the sumation convention is used here). Let $L := L_2 \circ L_1 : M_p \to E^k$ and

$$(4) \qquad \bar{M}_p = \left\{L(\bar{v}) \in E^k; \bar{v} \in M_p\right\}.$$

It is aesy to see that the mapping $L : M_p \to \bar{M}_p$ makes these linear spaces isomorphic to each other. We have

$$(5) \qquad \begin{cases} L(v) = \left(v\left(\pi^1|M\right), ..., v\left(\pi^k|M\right)\right) & \text{for} \quad v \in M_p, \\ v\left(f|M\right) = f_{L(v)} \quad \text{for} \quad v \in M & \text{and} \quad f \in \mathcal{E}_k. \end{cases}$$

Lemma 1. *For $p \in M$, $h \in E^k$, $k \in \mathcal{N}$ the following properties are equivalent:*

(a) $h \in \bar{M}_p$,

(b) *there exists a mapping $\bar{v} : \mathcal{E}|M \to E$ such that*

$$\bar{v}\,(f|M) = f_{|h}\,(p) \quad \textit{for} \quad f \in \mathcal{E}.$$

Proof. The implication (a)$\Longrightarrow$(b) follows immediately from (4) and (5) by putting $\bar{v} := v|\,(\mathcal{E}|M)$, $v \in M_p$ and $h = L\,(v)$.

In order to prove the implication (b)$\Longrightarrow$(a) let us suppose that h fulfils (b) and consider the set of functions $\mathcal{E}|M$. From (b) it follows that $\bar{v}$ is the linear mapping of $\mathcal{E}|M$ into E fulfilling the Leibniz's condition at the point p:

$$\bar{v}\,(\alpha\beta) = \bar{v}\,(\alpha)\,\beta\,(p) + \alpha\,(p)\,\bar{v}\,(\beta) \quad \text{for} \quad \alpha, \beta \in \mathcal{E}|M.$$

By using this conditions and linearity of $\bar{v}$ one can easy verify that $\bar{v}\,(\alpha) = 0$ for each function $\alpha \in \mathcal{E}|M$ equal to 0 at an open neighbourhood of the point p. As a consequence of this the mapping $v : C \to E$ defined by

$$v\,(\alpha) := \bar{v}\,(f|M) \quad \text{for} \quad \alpha \in C,$$

where $f \in \mathcal{E}$ is a function such that $f|U = \alpha|U$ for some set $U \in \tau_M$ including the point p, is well defined. The function v is linear and fulfils Leibniz's condition so it belongs to M_p. For $i = 1, ..., k$ we have $v\,(\pi^i|M) = \pi^i_{|h}\,(p) = \bar{v}\,(\pi^i|M) = h^i$, where $h = \left(h^1, ..., h^k\right)$, so from (5) we have: $L\,(v) = \left(h^1, h^2, ..., h^k\right) = h$. The Lemma is proved. □

Lemma 2. *For $h \in E^k$ and $p \in M$ the following conditions are equivalent:*

(a) $h \in \bar{M}_p$,

(b) $f_{|h}\,(p) = 0$ *for any $f \in \mathcal{E}$ equal to 0 on M.*

Proof. It is easy to see that the conditions (b) in Lemmas 1 and 2 are equivalent to each other. □

For any $f \in \mathcal{E}$ and $p \in E^k$ we donote $\operatorname{grad} f\,(p) := (\partial_1 f\,(p)\,, ..., \partial_k f\,(p))$.

Lemma 3. *Let $p = (0, ..., 0) \in M \subset E^k$ and $e_i = (0, 0, ..., 0, 1, 0, ..., 0)$ (1 in the i-th position), $1 \le i \le k$. If $m := \dim M_p$, $1 \le m \le k-1$ and $e_1, ..., e_m \in \bar{M}_p$ then there exists functions $f^{m+1}, ..., f^k \in \mathcal{E}$ equal to 0 on M and such that $\partial_i f^j\,(p) = \delta^j_i$, where $\delta^j_i = 1$ for $i = j$ and $\delta^j_i = 0$ for $i \ne j$.*

Proof. Let the assumptions of the Lemma be satisfied. Then

$$\bar{M}_p = \operatorname{Lin}\,(e_1, ..., e_m)\,, \tag{6}$$

where $\mathrm{Lin}\,(e_1, ..., e_m)$ is the linear subspace of E^k spanned by $e_1, ..., e_m$. We put $K := \{h \in E^k; h = \mathrm{grad}\, f\,(p)\,,\ f \in \mathcal{E} \text{ and } f = 0 \text{ on } M\}$. K is a linear subspace of E^k and $e_i \perp K$ with respect to the canonical scalar product in E^k so $K \subset \mathrm{Lin}\,(e_{m+1}, ..., e_k)$ (see Lemma 2). We shall prove more, namely that $K = \mathrm{Lin}\,(e_{m+1}, ..., e_k)$. If the above equality is not satisfied, then there exists a non-zero vector $h \in \mathrm{Lin}\,(e_{m+1}, ..., e_k)$, such that $K \perp h$. Hence $f_{|h}\,(p) = \mathrm{grad}\, f\,(p) \cdot h = 0$ for $f = 0$ on M, and $h \in \bar{M}_p$ (Lemma 2), but this contradicts (6). From above equality we obtain the existence of functions $f_{m+1}, ..., f_k \in \mathcal{E}_k$ equal to zero on M, such that $\mathrm{grad}\, f_j\,(p) = e_j$ or equivalently $\partial_i f^j\,(p) = \delta_i^j$. The Lemma is proved. □

Proposition 1. *Let $p \in M \subset E^k$. If $0 < m := \dim M_p \leq k$ then there exist non empty sets: U open in τ_M and V open in $\tau_{\mathcal{E}_m}$, and regular 1-1 C^∞-mapping $\phi : V \to E^k$ such that*

$$p \in U \subset \{\phi\,(u) \in E^k; u \in V\}\,.$$

Proof. If $m = k$, the proposition evidently holds. We suppose that $1 \leq m < k$. We can assume, without loss of generality, that $p = (0, ..., 0) \in E^k$ and $\bar{M}_p = \mathrm{Lin}\,(e_1, ..., e_m)$. We denote $q = (x^1, ..., x^k) = (u, w)$ where $u = (x^1, ..., x^m)$ and $w = (x^{m+1}, ..., x^k)$. Let f^j, $j = m+1, ..., k$, are functions as in Lemma 3. We define a mapping $F : E^k \to E^{k-m}$ by

$$F\,(q) := \left(f^{m+1}\,(q)\,, ..., f^k\,(q)\right) \quad \text{for} \quad q \in E^k.$$

This mapping has the following properties:

(a) $F\,(q) = F\,(u, w) = 0$ for $q = (u, w) \in M$,
(b) F is C^∞-mapping,
(c) F is regular at the point $p = (\bar{u}, \bar{w})\,.$
From the inverse mapping theorem it follows that there exists:
(d) a set $U' \in \tau_{E^k}$ such that $p \in U'$,
(e) a set $V \in \tau_{R^m}$ such that $\bar{u} \in V$,
(f) a C^∞-mapping $\psi : V \to E^{k-m}$ such that for any $u \in V$ we have $F\,(u, \psi\,(u)) = 0$,
(g) if $F\,(q) = 0$ and $q = (u, w) \in U'$ then $u \in V$ and $w = \psi\,(u)$.
It is evident that $U := U' \cap M$, V and $\phi\,(u) := (u, \psi\,(u))$ for $u \in V$ fulfil conditions of Proposition 1.

□

Now, we examine the case of $\dim M_p = 0$, which was not considered above.

Proposition 2. *Let $p \in M \subset E^k$. If $\dim M_p = 0$ then the point p is isolated in M.*

Proof. Let us set $|q| := \sqrt{(x^1)^2 + ... + (x^k)^2}$ for any $q = \left(x^1, ..., x^k\right) \in E^k$.

Let us assume the point p is not isolated. Then there exists a sequence (p_i) of points of M different from p and convergent to p. For the sequence $h_n := \frac{p_n - p}{|p_n - p|}$, $n \in \mathcal{N}$ of points of S^{k-1} we can find a subsequences h_{n_i} convergent to a point $h \in S^{k-1}$. One can easy see that for any $f \in \mathcal{E}$

$$\lim_{i\to\infty} \frac{f(p_{n_i}) - f(p)}{|p_{n_i} - p|} = f_{|h}(p).$$

It easy to see that left side of this sequence defines mapping $\bar{v} : \mathcal{E}|M \to E$ such that $\bar{v}(f|M) = f_{|h}(p)$. From Lemma 1 $h \in \bar{M}_p$ so $\dim M_p \neq 0$, which ends of the proof. □

Theorem 1 results easily from Propositions 1 and 2. In that Theorem an non-empty discrete subset of E^n is called a 0-dimensional C^∞-surface in E^n.

3. Corollaries

We say, that differential space (N, D) can be diffeomorphically embeded into the differential space (L, H) if there exists a subset $L' \subset L$ such that $(L', H_{L'})$ and (N, D) are diffeomorphic to each other. In the sequel we shall consider only differential spaces (N, D) such that any point $p \in N$ has a neighbourhood V such that (V, D_V) can be embedded into $\left(E^{n(p)}, \mathcal{E}_{n(p)}\right)$ for some $n(p) \in \mathcal{N}$. From Theorem 1 we obtain:

Corollary 1. *For a point p of the differential space (N, D) there exist a set $V \in \tau_D$ and an n-dimensional C^∞-manifold $\left(\tilde{N}, C^\infty\left(\tilde{N}\right)\right)$, $n := \dim N_p$, such that $p \in V \subset \tilde{N}$ and $D_V = C^\infty\left(\tilde{N}\right)_V$. The inequality*

$$\dim M_q \leq \dim M_p$$

is fulfiled for any point $q \in V$.

Corollary 2. *If (N, D) is a differential space such that (N, τ_D) is separable and if there exists $n \in \mathcal{N}$ such that for any $p \in N$ $\dim (N, D)_p \leq n$, then topological dimension of (N, τ_D) does not exceed n.*

Proof. This results easily from Corollary 1. □

Differential spaces which have tangent spaces of constant dimension are the most interesting. For a differential space (N, D) and $i = 0, 1, ...$ we shall denote by N^i union of all sets $V \in \tau_D$ such that $\dim (N, D) = i$

for any $q \in V$. If N^i is not empty then (N^i, D_{N^i}) is a differential subspace of (N, D) and for any $q \in N^i$ $\dim(N^i, D_{N^i}) = i$. From Corollary 1 we obtain

Corollary 3. *For any differential space* (N, D) *the set* $\bigcup_{i=0}^{\infty} N^i$ *is open and dense in the topological space* (N, τ_D).

Proof. For any subset $A \subset N$ we denote its closure in (N, τ_D) by $\bar{A}$. We shall use mathematical induction. Let $p \in N$. It is easy to see, that $\dim(N, D)_p \geq 0$. If $\dim(N, D)_p = 0$ then the point p is isolated in (N, τ_D) and $p \in N^0$, see Corollary 1, $p \in \overline{\bigcup_{i=0}^{\infty} N^i}$. Suppose. that for any $q \in N$ such that $0 \leq \dim(N, D)_q \leq m - 1$ we have $q \in \overline{\bigcup_{i=0}^{\infty} N^i}$. For any point p$\in N$ such that $\dim(N, D)_p = m$ there exists an open neighbourhood V of p such that $\dim(N, D)_q \leq m$ for any point $q \in V$ (Corollary 1). Let $U \in \tau_D$ be a set containing the point p. If for any $q \in U \cap V$ $\dim(N, D)_q = m$, then $p \in N^m$ and $p \in \overline{\bigcup_{i=0}^{\infty} N^i}$. If it is not true then there exists a point $q \in U \cap V$, such that $\dim(N, D) \leq m - 1$. From the induction hypothesis. the point $q \in \overline{\bigcup_{i=0}^{\infty} N^i}$, so $U \cap \bigcup_{i=0}^{\infty} N^i \neq \emptyset$. This is true for any set $U \in \tau_D$ containing the point p, so we have $p \in \overline{\bigcup_{i=0}^{\infty} N^i}$. The corollary is proved. □

By virtue of Corollary 1 any point p of differential space (N, D) such that $\dim(N, D)_p = k$ has a neighbourhood V such that (V, D_V) can be diffeomorphically embedded in $\left(E^k, \mathcal{E}_k\right)$. Hence it is interesting to consider the differential subspace $(M, \mathcal{E}_{kM})$ of $\left(E^k, \mathcal{E}_k\right)$ for which there exists a point $p \in M$ such that $\dim(M, C)_p = k$.

Corollary 4. *Let* $p \in M \subset E^k$. $\dim(M, \mathcal{E}_{kM})_p = k$ *if and only if for any* $f \in \mathcal{E}_k$ *equal to* 0 *on* M $\partial_i f(p) = 0$ *for* $i = 1, 2, ..., m$.

Proof. We get this immediately from Lemma 2, as $e_i \in \bar{M}_p$, $i = 1, ..., k$. □

Corollary 5. *Let* $\emptyset \neq M \subset E^k$. *Then* $\dim(M, \mathcal{E}_{kM})_p = k$ *for any* $p \in M$ *if and only if for any* $f \in \mathcal{E}_k$ *equal to* 0 *on* M *all partial derivatives of any order are equal to* 0 *on* M.

Proof. This corollary follows easily, by induction, from Corollary 4. □

By virtue of above Corollary, any subset $M \subset E^k$ such that $(M, \mathcal{E}_{kM})$ has the constant dimension k, has the same property, as any open set of E^k: the value of the partial derivatives of a function $f \in \mathcal{E}_k$ at a point $p \in m$ are uniquely determined by the values of the function on M. For a differential space (N, D) a linear mapping $X : D \to D$ such that $X(\alpha\beta) = X(\alpha)\beta + \alpha X(\beta)$ is called a *vector field* on (N, D) [1]. It easy to see that for any point $p \in N$ the function $X_p : D \to E$ defined by $X_p(\alpha) := (X\alpha)(p)$ for $\alpha \in D$ is a vector belonging to $(N, D)_p$.

Corollary 6. *Let (N, D) be a differential space. A point p belongs to $\bigcup_{i=0}^{\infty} N^i$ if and only if there exists vector fields $X_1, ..., X_s$ on (N, D) such that $\left\{X_{1p,...,X_{sp}}\right\}$ is the basis of $(N, D)_p$.*

Proof. If $X_1, ..., X_k$ are such vector fields on (N, D) that $X_{1p}, ..., X_{kp}$ is a basis of $(N, D)_p$ then there exists a set $V \in \tau_D$ such that $p \in V'$ and $X_{1q}, ..., X_{kq}$ are lineary independent for any $q \in V'$ (cf. [1]). As there exists an open neighbourhood V'' of p such that for any $q \in V''$ $\dim(N, D)_q \leq k$ (Corollary 1), for any $q \in V' \cap V''$ $\dim(N, D)_q = k$ and $p \in N^k \subset \bigcup_{i=0}^{\infty} N^i$.

Now we shall prove the other implication. For the point $p \in N^0$ the proof is trivial. Let $p \in N^k$, $k > 0$ and U be such an open neighbourhood of the point p that (U, D_U) is diffeomorphic to $(V, \mathcal{E}_{kV})$ for certain $V \subset E^k$ and $\dim(V, \mathcal{E}_{kV})_q = k$ for any $q \in V$. It is sufficient to prove Corollary for $(V, \mathcal{E}_{kV})$.

For $q \in V$ and $\alpha \in \mathcal{E}_{kV}$ there exists an open neighbourhood V_q of q and a function $f_{\alpha,q} \in \mathcal{E}$ such that $\alpha|V_q = f_{\alpha,q}|V_q$. By virtue of Corollary 5 the functions $X_i : \mathcal{E}_{kV} \to \mathcal{E}_{kV}$, $i = 1, 2, .., k$, defined for $\alpha \in \mathcal{E}_{kV}$, by

$$(X_i\alpha)(q) = \partial_i(f_{\alpha,q})(q) \quad \text{for} \quad q \in V$$

are well defined. It can be easily verified that they are vector fields on $(V, \mathcal{E}_{kV})$ and $X_{1q}, ..., X_{kq}$ is the basis of $(V, \mathcal{E}_{kV})_q$ for any $q \in V$. □

4. Examples

Example 1. *Let $M \subset E^k$ be dense in E^k. Then by Corollary 1 the dimension of $(M, \mathcal{E}_{kM})_p$ is k for any $p \in M$.*

Example 2. *The graph of the function $f : E \to E$ which is x^2 for $x > 0$ and 0 for $x \leq 0$ has the tangent space of dimension 1 at all points except for the point $(0, 0)$, where it has tangent space of dimension 2. It result easily from Corollary 1.*

Example 3. *The graph of the function $g : E \to E$ of class C^1 which is not of class C^∞ at any point is a differential subspace of $(E^2, \mathcal{E}_2)$ of constant dimension 2. It results easily from Corollary 1.*

Example 4. *Let $M \subset E^k$. If topological dimension of any non empty open subset of M is k then $\dim(M, \mathcal{E}_{kM})_p = k$ for any $p \in M$. This follows easily from Corollary 2.*

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