TANGENTIAL CHERN-WEIL HOMOMORPHISM *

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Abstract

Moore and Schochet gave the Chern-Weil homomorphism of a vector bundle $\mathfrak{f}$ over a foliated space $(M, \mathcal{F})$, measuring the nonexistence of partially flat covariant derivatives. We look at this problem (restricting our interest to vector bundles over foliated manifolds) from the point of view of nontransitive Lie algebroids. We use – for our problem – the Chern-Weil homomorphism of the regular Lie algebroid coming from the Lie algebroid $A(\mathfrak{f})$ of $\mathfrak{f}$ by restricting it to the elements whose anchors are tangent to the foliation $\mathcal{F}$. Our observations lead to the conjecture that we can sometimes obtain some essentially new kinds of characteristic classes (with respect to the construction of Moore-Schochet) called *singular.*

1 Introduction

1.1 Category of regular Lie algebroids

Let F be a C^∞ constant dimensional and involutive distribution on a C^∞ Hausdorff paracompact connected manifold M. By a *regular Lie algebroid* over a foliated manifold (M, F) [5] (see also [3], [4], [7]) we mean a system

$$(A, [\![\cdot,\cdot]\!], \gamma)$$

consisting of a vector bundle A over M and mappings

$$[\![\cdot,\cdot]\!] : \mathrm{Sec}A \times \mathrm{Sec}A \to \mathrm{Sec}A, \quad \gamma : A \to TM,$$

such that:

*This paper is in final form and no version of it will be submitted for publications elsewhere

1) $(\mathrm{Sec}A, [\![\cdot,\cdot]\!])$ is a real Lie algebra;

2) γ, called by K.Mackenzie [7] an *anchor*, is a homomorphism of vector bundles, and $\mathrm{Im}\gamma = F$;

3) $\mathrm{Sec}\gamma : \mathrm{Sec}A \to \mathcal{X}(M)$, $\xi \mapsto \gamma \circ \xi$, is a homomorphism of Lie algebras;

4) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$, $\xi, \eta \in \mathrm{Sec}A$, $f \in C^\infty(M)$.

A Lie algebroid $(A, [\![\cdot,\cdot]\!], \gamma)$ is called *transitive* if $F = TM$.

Let $(A, [\![\cdot,\cdot]\!], \gamma)$ and $(A', [\![\cdot,\cdot]\!]', \gamma')$ be two Lie algebroids on the same manifold M. By a *homomorphism*

$$H : (A, [\![\cdot,\cdot]\!], \gamma) \to (A', [\![\cdot,\cdot]\!]', \gamma')$$

between them we mean a homomorphism $H : A \to A'$ of vector bundles over $id_M : M \to M$ such that:

1) $\gamma' \circ H = \gamma$;

2) $\mathrm{Sec}\,H : \mathrm{Sec}\,A' \to \mathrm{Sec}\,A$, $\xi \mapsto H \circ \xi$, is a homomorphism of Lie algebras.

A homomorphism H of Lie algebroids induces a homomorphism of the associated exact *Atiyah sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{g} & \hookrightarrow & A & \xrightarrow{\gamma} & F & \longrightarrow & 0 \\ & & \downarrow H^+ & & \downarrow H & & \downarrow & & \\ 0 & \longrightarrow & \boldsymbol{g}' & \hookrightarrow & A' & \xrightarrow{\gamma'} & F' & \longrightarrow & 0\,. \end{array}$$

A *trivial Lie algebroid* is defined to be one isomorphic to $(TM\times\mathfrak{g}, [\![\cdot,\cdot]\!], \mathfrak{pr}_1)$ where $\mathfrak{g}$ is a finite-dimensional Lie algebra and the bracket is given by the formula

$$[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - \mathcal{L}_Y\sigma - [\sigma,\eta]),$$

$X, Y \in \mathcal{X}(M)$, $\sigma, \eta \in C^\infty(M;\mathfrak{g})$. Every transitive Lie algebroid over a manifold diffeomorphic to $\mathbb{R}^\varkappa$ is trivial [6], [7].

1.2 Sources of Lie algebroids

Differential Geometry has discovered many objects which determine Lie algebroids, such as, for example, differential groupoids, principal bundles, vector bundles, transversally complete foliations, nonclosed Lie subgroups, Poisson manifolds, some complete closed pseudogroups, and actions of Lie groups on manifolds.

On the other hand, if $(A, [\![\cdot,\cdot]\!], \gamma)$ is a transitive Lie algebroid on M and (M,F) is a foliated manifold, then $A^F := \gamma^{-1}[F] \subset A$ forms, in an evident manner, a regular Lie algebroid over (M,F). For example, a principal bundle or a vector bundle over a foliated manifold determines an object which is the fundamental tool serving our problem. For the other examples, see "*Characteristic homomorphisms of regular Lie algebroids*", Poster, European Congress of Mathematics, Paris, 1992, contained in [6].

1.3 Representations and invariant cross-sections

Let $\mathfrak{f}$ be a vector bundle over M. We recall [5] (see also [7]) that the fibre $A(\mathfrak{f})_{|x}$ of the Lie algebroid $A(\mathfrak{f})$ of $\mathfrak{f}$ over x consists of all linear homomorphisms l :Sec$\mathfrak{f} \to \mathfrak{f}_{|x}$, called $\mathfrak{f}$-*vectors*, for which there exists a vector $u \in T_xM$ such that

$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu(x), \quad f \in C^\infty(M),\, \nu \in \mathrm{Sec}\mathfrak{f}.$$

u is the anchor of l and is denoted by $q(l)$. A cross-section $\xi \in \mathrm{Sec}\, A(\mathfrak{f})$ defines a covariant differential operator $\mathcal{L}_\xi : \mathrm{Sec}\,\mathfrak{f} \to \mathrm{Sec}\,\mathfrak{f}$, $\mathcal{L}_\xi(\nu)(x) = \xi_x(\nu)$, $x \in M$; $\mathcal{L}_{[\![\xi,\eta]\!]} = [\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_\xi \circ \mathcal{L}_\eta - \mathcal{L}_\eta \circ \mathcal{L}_\xi$ [7].

The Lie algebroid $A(\mathfrak{f})$ is (locally) described in the following:

Theorem 1.3.1 [5; 5.4.4] *Let $\psi : U \times V \to p^{-1}[U]$ be a local trivialization of a vector bundle $\mathfrak{f}$ (with V as a typical fibre). For $\nu \in \mathrm{Sec}\,\mathfrak{f}$, denote by ν_ψ the function $U \ni x \mapsto \psi_{|x}^{-1}(\nu_x) \in V$. Then the mapping*

$$\bar{\psi} : TU \times \mathrm{End}(V) \longrightarrow A(\mathfrak{f})_{|U}$$

such that

$$\bar{\psi}(v,a)(\nu) = \psi_{|x}(\partial_v(\nu_\psi) + a(\nu_\psi(x)))$$

when $v \in T_xU$ and $a \in \mathrm{End}(V)$, is an isomorphism of Lie algebroids. □

By a *representation* of a Lie algebroid A on $\mathfrak{f}$ [5], [7] we mean a homomorphism $T : A \to A(\mathfrak{f})$ of Lie algebroids. A cross-section $\nu \in \mathrm{Sec}\,\mathfrak{f}$ is called T*–invariant* if

$$T(v)(\nu) = 0 \quad \text{for all} \quad v \in A.$$

Denote by $(\mathrm{Sec}\,\mathfrak{f})_{I^\circ(T)}$ (or $(\mathrm{Sec}\,\mathfrak{f})_{I^\circ}$ when there is no misunderstanding) the space of all T-invariant cross-sections of $\mathfrak{f}$.

1.4 Connections, curvature and the Chern-Weil homomorphism of regular Lie algebroids

Let A be any regular Lie algebroid over a foliated manifold (M, F). Any splitting $\lambda : F \to A$ of the exact Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \underset{\xleftarrow{\lambda}}{\xrightarrow{\gamma}} F \longrightarrow 0$$

is called a *connection* in A. The tangential differential form

$$\Omega \in \Omega^2_F(M, \boldsymbol{g}) = \mathrm{Sec}(\bigwedge^2 F^\star \bigotimes \boldsymbol{g})$$

defined by

$$\Omega(X,Y)=\lambda\circ[X,Y]-[\![\lambda\circ X,\lambda\circ Y]\!],\quad X,Y\in \operatorname{Sec}F,$$

is called the *curvature tensor* of λ. If $A=A(P)^F$, P being a principal bundle and $A(P)=TP/G$ – its Lie algebroid [4], [7], then there is a bijection between connections in A and partial connections in P over F.

By the *adjoint representation* $ad_A : A\to A(\boldsymbol{g})$ of a regular Lie algebroid A we mean the one defined by $ad_A(v)(\sigma)=[\![\xi,\sigma]\!](x)$ where ξ is an arbitrarily taken cross-section of A such that $\xi(x)=v\in A_{|x}$ and $\sigma\in\operatorname{Sec}\boldsymbol{g}$. The mapping ad_A induces canonically a representation $\bigvee^k ad_A^\natural$ of A on the symmetric power $\bigvee^k\boldsymbol{g}^\star$ denoted, for short, also by ad_A. Let $I^{\circ k}(A):=(\operatorname{Sec}(\bigvee^k\boldsymbol{g}^\star))_{I^\circ}$ be the space of invariant elements with respect to this representation. $I^\circ(A):=\bigoplus^{k\geq 0}I^{\circ k}(A)$ forms an algebra. We recall that $\Gamma\in I^{\circ k}(A)$ if and only $\Gamma\in\operatorname{Sec}(\bigvee^k\boldsymbol{g}^*)$ and

$$\begin{aligned}\forall_{\xi\in\operatorname{Sec}A}\,\forall_{\sigma_1,\dots,\sigma_k\in\operatorname{Sec}\boldsymbol{g}}\quad (\quad &(\gamma\circ\xi)\langle\Gamma,\sigma_1\vee\ldots\vee\sigma_k\rangle\\ =\ &\sum_{i=1}^{k}\langle\Gamma,\sigma_1\vee\ldots\vee[\![\xi,\sigma_i]\!]\vee\ldots\vee\sigma_k\rangle\ )\end{aligned}$$

Theorem 1.4.1 [5] *For $\Gamma\in I^{\circ k}(A)$ and the curvature tensor Ω of a connection in A, the real tangential form*

$$\beta(\Gamma)=\tfrac{1}{k!}\langle\Gamma,\underbrace{\Omega\vee\cdots\vee\Omega}_{k\ times}\rangle\in\Omega_F^{2k}(M)$$

is closed. The mapping

$$h_A : I^\circ(A)\to H_F(M),\ \ \Gamma\mapsto[\beta(\Gamma)],\tag{1.4.1}$$

called the Chern-Weil homomorphism *of A, is a homomorphism of algebras independent of the choice of a connection.* □

Remark 1.4.2 $H_F(M)$ in the above theorem denotes the *tangential cohomology space* of (M,F) being the cohomology space of the complex $(\Omega_F(M),d^F)$ of real tangential differential forms, where d^F is the standard differentiation defined in an elementary way in terms of local coordinates [8] or, equivalently, by the global formula, the same as for usual real differential forms.

Remark 1.4.3 Homomorphism (1.4.1) for the Lie algebroid $A(P)$ of a connected principal bundle $P=P(M,G)$ agrees with the classical Chern-Weil homomorphism $h_P : I(G)\to H_{dR}(M)$ of P [2] in the sense that there exists a natural isomorphism $\alpha : I(G)\to I^\circ(A)$ of algebras such that

$$h_{A(P)}\circ\alpha=h_P$$

We pay our attention to the fact that this holds although in the Lie algebroid $A(P)$ there is no direct information about the structure Lie group G of P which may be disconnected !

Remark 1.4.4 There are transitive Lie algebroids which do not come from principal bundles but have nontrivial Chern-Weil homomorphisms (Lie algebroids of some transversally complete foliations have this property [5]).

1.5 The inverse image of a regular Lie algebroid and of a representation

In [5] there is a construction of some important (from the technical point of view) notion of the *inverse image* $f^{\wedge}A$ of a regular Lie algebroid A over (M, F) by a morphism $f : (M', F') \to (M, F)$ of the category of foliated manifolds (i.e. a smooth mapping $f : M' \to M$ such that $f_\star[F'] \subset F$). In this paper we shall only take the above notion for the inclusion $i_L : (L, TL) \hookrightarrow (M, F)$ where L is a leaf of the foliation F. Then $i_L^{\wedge}(A)$ is the transitive Lie algebroid for which

1) the total space is equal to $TL \times_{i_{L\star},\gamma} A = \{(v, w) \in TL \times A;\ v = \gamma w\}$;
2) the projection $pr_1 : TL \times_{i_{L\star},\gamma} A \to TL$ is the anchor;
3) the Lie bracket in $\mathrm{Sec}(i_L^{\wedge} A)$ is defined uniquely by demanding that, for $\xi, \eta \in \mathrm{Sec}\, A$,

$$[\![(\gamma \circ \xi \mid L, \xi \mid L), (\gamma \circ \eta \mid L, \eta \mid L)]\!] = ([\gamma \circ \xi, \gamma \circ \eta] \mid L, [\![\xi, \eta]\!] \mid L).$$

Clearly, the total space of the Lie algebroid $i_L^{\wedge}(A)$ is isomorphic to the restriction $A_{|L}$ via the linear isomorphism $A_{|L} \ni w \mapsto (\gamma w, w) \in i_L^{\wedge}(A)$. Therefore we can identify $i_L^{\wedge}(A)$ with $A_{|L}$ to obtain a transitive Lie algebroid on L. The bracket in $\mathrm{Sec}(A_{|L})$ satisfies the condition

$$[\![\xi|L, \eta|L]\!] = [\![\xi, \eta]\!]|L, \quad \xi, \eta \in \mathrm{Sec}\, A.$$

We shall call $A_{|L}$ the *restriction* of A to the leaf L.

The second important "technical" notion is the inverse image $f^\star T$ of a representation $T : A \to A(\mathfrak{f})$ over $f : (M', F') \to (M, F)$; see [5]. For the inclusion i_L, we obtain in this way the *restriction* $T_{|L} : A_{|L} \to A(\mathfrak{f}_{|L})$ of T to L as a representation such that

$$T_{|L}(v)(\nu|L) = T(v)(\nu) \ \text{ for } \ \nu \in \mathrm{Sec}\, \mathfrak{f}.$$

1.6 Representation of principal bundles on vector bundles

Denote by $L\mathfrak{f}$ the $GL(V)$-principal bundle of all frames $z : V \xrightarrow{\cong} \mathfrak{f}_{|x},\ \ x \in M$ (V being the typical fibre of $\mathfrak{f}$). Let $\mu : G \to GL(V)$ be a homomorphism of Lie groups.

Definition 1.6.1 By a *μ-representation* of a principal bundle $P = P(M, G)$ on $\mathfrak{f}$ we mean [5] a μ-homomorphism $H : P \to L\mathfrak{f}$ between principal bundles. The homomorphism H after its differentiation gives a homomorphism $dH : A(P) \to A(L\mathfrak{f})$ of Lie algebroids [4], [7]; on the other hand, $A(L\mathfrak{f})$ is canonically isomorphic to the Lie algebroid $A(\mathfrak{f})$ via some isomorphism $\Phi_{\mathfrak{f}} : A(L\mathfrak{f}) \to A(\mathfrak{f})$ defined by

$$\Phi_{\mathfrak{f}}([v])(\nu) = u(\partial_v(\tilde{\nu})),$$

where $v \in T_u(L\mathfrak{f}),\ u \in L\mathfrak{f},\ \nu \in \operatorname{Sec}\mathfrak{f}$, whereas

$$\tilde{\nu} : L\mathfrak{f} \to V, \quad w \mapsto w^{-1}(\nu(\pi w)),$$

see [5]. The superposition $H' = \Phi_{\mathfrak{f}} \circ dH : A(P) \to A(\mathfrak{f})$ is called the *differential* of H.

Let $H : P \to L\mathfrak{f}$ be a μ-representation of P on $\mathfrak{f}$. A cross-section $\nu \in \operatorname{Sec}\mathfrak{f}$ is called *H-invariant* if there exists a vector $v \in V$ such that $H(z)(v) = \nu_{\pi(z)}$ for all $z \in P$. Let $(\operatorname{Sec}\mathfrak{f})_{I(H)}$ denote the space of all H-invariant cross-sections of $\mathfrak{f}$. The space $(\operatorname{Sec}\mathfrak{f})_{I(H)}$ is isomorphic to the space of μ-invariant elements $V_{I(\mu)}$ via the isomorphism $V_{I(\mu)} \to (\operatorname{Sec}\mathfrak{f})_{I(H)}$ given by $v \mapsto \nu_v$ where $\nu_v(x) = F(z)(v)$ for all $z \in P_{|x},\ x \in M$ [5; 5.5.2].

The crucial role in the sequel of the theory is played by the following

Theorem 1.6.2 [5; 5.5.3] *The spaces of invariant cross-sections* $(\operatorname{Sec}\mathfrak{f})_{I(H)}$ *and* $(\operatorname{Sec}\mathfrak{f})_{I^\circ(H')}$ *under a representation $H : P \to L\mathfrak{f}$ and its differential $H' : A(P) \to A(\mathfrak{f})$ are related by* $(\operatorname{Sec}\mathfrak{f})_{I(H)} \subset (\operatorname{Sec}\mathfrak{f})_{I^\circ(H')}$. *If P is connected (nothing is assumed about the connectedness of G !), then*

$$(\operatorname{Sec}\mathfrak{f})_{I(H)} = (\operatorname{Sec}\mathfrak{f})_{I^\circ(H')}. \ \square$$

Let $\boldsymbol{g}$ be the LAB adjoint of $A(P)$. By the *adjoint representation* of P we mean the $(Ad_G : G \to GL(V))$-representation $Ad_P : P \to L\boldsymbol{g}$ defined by $Ad_P(z) = \hat{z}$ where $\hat{z} : \mathfrak{g} \to \boldsymbol{g}_{|\pi(\mathfrak{z})}$ is an isomorphism of Lie algebras defined by $\hat{z}(v) = [(A_z)_{\star e}(v)]$, where $A_z : G \to P,\ a \mapsto za$, ($\mathfrak{g}$ denotes here the right! Lie algebra of G). According to [5], we have

$$(Ad_P)' = ad_{A(P)}.$$

2 Partially invariant cross-sections

Let $(A, [\![\cdot,\cdot]\!], \gamma)$ and $\mathfrak{f}$ be a transitive Lie algebroid and a vector bundle on a manifold M, respectively. Assume that $F \subset TM$ is a C^∞ constant dimensional involutive distribution and $\mathcal{F}$ – the foliation determined by F. We recall that A and F give rise to the regular Lie algebroid over (M, F) where we put $A^F := \gamma^{-1}[F] \subset A$; see [5; 1.1.3]. Its Atiyah sequence is

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A^F \xrightarrow{\gamma^F} F \longrightarrow 0\,,$$

where $\boldsymbol{g}$ is the Lie algebra bundle adjoint of A, and $\gamma^F := \gamma|A^F$. Any representation $T : A \to A(\mathfrak{f})$ of A on $\mathfrak{f}$ [5; 2.1.1] restricts to the representation

$$T^F = T|A^F : A^F \longrightarrow A(\mathfrak{f})$$

of A^F on $\mathfrak{f}$. Any T^F–invariant cross-section $\nu \in \operatorname{Sec}\mathfrak{f}$ will be called a *partially invariant cross-section with respect to T over F.*

For any leaf $L \subset M$ of the foliation $\mathcal{F}$, we have the restriction $A^F_{|L}$ of A^F and the inverse image

$$T^F_{|L} : A^F_{|L} \longrightarrow A(\mathfrak{f}_{|L})$$

of T^F.

According to this, the following lemma is obvious.

Lemma 2.1 *A cross-section $\nu \in \operatorname{Sec}\mathfrak{f}$ is T^F-invariant if and only if, for each leaf L of $\mathcal{F}$, the restriction $\nu|L \in \operatorname{Sec}\mathfrak{f}_{|L}$ is $T^F_{|L}$ -invariant.* □

Lemma 2.2 *For finitely many $\mathcal{F}$-basic functions $f^i \in \Omega^\circ_b(M, \mathcal{F})$ and T-invariant cross-sections $\nu_i \in \operatorname{Sec}\mathfrak{f}$, $\sum_i f^i \cdot \nu_i$ is a T^F-invariant cross-section, in other words,*

$$\Omega^\circ_b(M, \mathcal{F}) \cdot (\operatorname{Sec}\mathfrak{f})_{I^\circ(T)} \subset (\operatorname{Sec}\mathfrak{f})_{I^\circ(T^F)} \,. \tag{2.1}$$

Proof. For $\xi \in \operatorname{Sec} A^F$, we obtain

$$\mathcal{L}_{T^F\circ\xi}(\sum_i f^i \cdot \nu_i) = \sum_i f^i \cdot \mathcal{L}_{T^F\circ\xi}(\nu_i) + (\gamma\circ\xi)(f^i)\cdot\nu_i = 0$$

because $\mathcal{L}_{T^F\circ\xi}(\nu_i) = \mathcal{L}_{T\circ\xi}(\nu_i) = 0$ by the T-invariance of ν_i, and $(\gamma\circ\xi)(f^i) = 0$ by the fact that f^i ia basic and $\gamma\circ\xi \in \operatorname{Sec} F$. □

Inclusion (2.1) can not always be replaced by the equality, which means that, in general, not every T^F-invariant cross-section is of the form $\sum_i f^i \cdot \nu_i$ for $\mathcal{F}$-basic functions f^i and T-invariant cross-sections ν_i, see example (2.4) below.

An important class of examples in which (2.1) is the equality is described later in Th.3.2.2.

Definition 2.3 Each T^F-invariant cross-section $\nu \in (\operatorname{Sec}\mathfrak{f})_{I^\circ(T^F)}$ not belonging to $\Omega^\circ_b(M, \mathcal{F}) \cdot (\operatorname{Sec}\mathfrak{f})_{I^\circ(T)}$ will be called *singular.* The characteristic class corresponding to any singular cross-section will be also called *singular.*

Example 2.4 Consider the Möbius band M with the foliation $\mathcal{F}$ by "meridians", see the Fig. 1.

Fig. 1

Let F denote the tangent bundle to $\mathcal{F}$. Equip M with a flat Riemannian structure for which the fields $e_1 = \frac{\partial}{\partial x}$ and $e_2 = \frac{\partial}{\partial y}$ form an orthonormal base. The vector field e_1 on M is not continuous at points of the segment AB.

Let P be the $O(2,\mathbb{R})$-principal bundle of orthonormal frames of the tangent bundle TM. P is connected because M is not orientable [2] (therefore the structure Lie group $O(2,\mathbb{R})$ cannot be reduced to $SO(2,\mathbb{R})$).

Consider the Atiyah sequence of the Lie algebroid $(A(P), [\![\cdot,\cdot]\!], \gamma)$ of the principal bundle P:

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A(P) \xrightarrow{\ \gamma\ } TM \longrightarrow 0\,,$$

where $\boldsymbol{g} \cong \mathrm{End}_{Sk}(TM, TM)$ is the vector bundle of skew-symmetric endomorphisms (each element $\sigma \in \boldsymbol{g}_{|x}$ with respect to the base e_{1x}, e_{2x} has a matrix of the form $\left[\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right]$ for some real c).

Denote by $A(\boldsymbol{g})$ the Lie algebroid of the vector bundle $\boldsymbol{g}$. Consider the adjoint representation $ad_{A(P)} : A(P) \to A(\boldsymbol{g})$ and the one $ad^F_{A(P)} : A(P)^F \to A(\boldsymbol{g})$ determined by $ad_{A(P)}$.

For a real $c \in \mathbb{R}$, $\tilde{\ }$ denotes the cross-section of $\boldsymbol{g}$ whose value at $x \in M$ is an endomorphism $T_xM \longrightarrow T_xM$ with the matrix $\left[\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right]$. The cross-section $\tilde{c}$ is continuous except at points of the segment AB, and $\tilde{c}$ restricted to any leaf L of the foliation $\mathcal{F}$, $\tilde{c}|L$, is:

1) smooth,

2) invariant with respect to the restricted representation $ad(L) : A(P)^F_{|L} \longrightarrow A(\boldsymbol{g})_{|L}$ (equal to the adjoint representation of the Lie algebroid of the principal bundle $P_{|L}$).

It is evident to see the property 1) holds. As for 2), leaves of $\mathcal{F}$ are diffeomorphic to the segment $(0,1) \subset \mathbb{R}$, therefore the transitive Lie algebroid $A(P)^F_{|L}$ is trivial: $A(P)^F_{|L} \cong TI \times Sk(2,\mathbb{R})$, $\quad \mathbb{I} = (\nvdash, \nVdash) \subset \mathbb{R}$, see 1.1. In this identification, $\tilde{c}$ is a constant cross-section $(0, \left[\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right])$. The equality $[\![\xi, \tilde{c}]\!] = 0$ for any $\xi \in \mathrm{Sec}(TI \times Sk(2,\mathbb{R}))$ now follows trivially from the definition of the bracket in the Lie algebra $\mathrm{Sec}(TI \times Sk(2,\mathbb{R}))$ for the trivial Lie algebroid $TI \times Sk(2,\mathbb{R})$. The property 2) can be also noticed in another way by using Proposition 5.5.2 in [5]: L is contractible, so $P_{|L}$ has a reduction to $SO(2,\mathbb{R})$, say, P'_L. On the other hand, $Sk(2,\mathbb{R})$ is abelian, therefore each of its elements is invariant with respect to the adjoint representation of $Sk(2,\mathbb{R})$ [2], and $\tilde{c}$ corresponds (via the isomorphism described in Prop. 5.6.2 [5], applied to P'_L) to the invariant element $\left[\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right]$.

Define a cross-section $Pf \in \mathrm{Sec}\, \boldsymbol{g}^*$ by the formula $Pf(\left[\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right]) = c$. Of course, Pf is continuous except at points of the segment AB. Besides, Pf has properties analogous to 1) and 2) above (we only need to change the representation $ad_{A(P)}$ by its contragredient). Notice that Pf restricted to L corresponds to the Pfaffian [1] for the principal bundle P'_L.

The last step of our construction will be the smoothing of Pf which consists in multiplying it by a suitable basic function g. First, we notice that the space of leaves of $\mathcal{F}$ is the circle S^1 with a base point x_0 corresponding to the segment AB. Next, g is defined in such a way that it is C^∞ and has only one zero at x_0 and all the derivatives are also zero at x_0. Clearly, $Pf \cdot g$ is the sought-for partially invariant cross-section.

3 The tangential Chern-Weil homomorphism

3.1 The tangential Chern-Weil homomorphism of a transitive Lie algebroid

Definition 3.1.1 By the *tangential Chern-Weil homomorphism of a transitive Lie algebroid A, over a foliated manifold (M,F),* we mean the Chern-Weil homomorphism

$$h_{A^F} : I^\circ(A^F) \longrightarrow H_F(M)$$

of the regular Lie algebroid A^F.

Clearly, h_{A^F} measures the nonexistence of flat connections in A^F.

For any leaf L of the foliation $\mathcal{F}$, the Atiyah sequence of the transitive Lie algebroid $A^F_{|L}$ is

$$0 \longrightarrow \boldsymbol{g}_{|L} \hookrightarrow A^F_{|L} \xrightarrow{\gamma_{|L}} TL \longrightarrow 0\,.$$

Without any difficulties we can verify the following

Proposition 3.1.2 *The Chern-Weil homomorphisms h_A of A, h_{A^F} of A^F and $h_{A^F_{|L}}$ of $A^F_{|L}$ are connected with one another via the commuting diagram*

$$\begin{tikzcd}
I^{\circ}(A) \arrow[rr, "h_A"] \arrow[d, "\Gamma \mapsto 1\cdot\Gamma"'] & & H(M) \arrow[d, "\alpha \mapsto \alpha^F"] \\
\Omega^{\circ}_b(M,\mathcal{F})\cdot I^{\circ}(A) \arrow[r, "i"] & I^{\circ}(A^F) \arrow[r, "h_{A^F}"] \arrow[d, "\Gamma \mapsto \Gamma|L"'] & H_F(M) \arrow[d, "\theta \mapsto \theta|L"] \\
& I^{\circ}(A^F_{|L}) \arrow[r, "h_{A^F_{|L}}"] & H_{dR}(L)
\end{tikzcd}$$

where α^F is the tangential cohomology class whose representantive comes from some representantive of α by restricting it to vectors from the distribution F. □

Notice that the triviality of h_A implies the same assertion for $h_{A^F}\circ i$:

$$h_{A^F}\circ i(\sum_i f^i\cdot\Gamma_i) = \sum_i f^i\cdot(h_A(\Gamma_i)) = 0.$$

3.2 The tangential Chern-Weil homomorphism of a principal bundle

Let P be a G-principal bundle on M, whereas $F\subset TM$ and $\mathcal{F}$ are as above.

Definition 3.2.1 By the *tangential Chern-Weil homomorphism of P over (M,F)* we mean the Chern-Weil homomorphism $h_{A(P)^F}$ of the regular Lie algebroid $A(P)^F$.

Let $\mathfrak{g}$ be the right Lie algebra of G and $(\bigvee\mathfrak{g}^{\star})_{\mathfrak{I}(\mathfrak{G})}$ — the algebra of G-invariant elements [2].

Theorem 3.2.2 *Let $G_{\circ}$ be the connected component containing the unit of G. If each $G_{\circ}$-invariant element of $\bigvee\mathfrak{g}^{\star}$ is G-invariant, then the domain of the homomorphism $h_{A(P)^F}$ is equal to $\Omega^{\circ}_b(M,\mathcal{F})\cdot I^{\circ}(A(P))$ ($\cong\Omega^{\circ}_b(M,\mathcal{F})\cdot(\bigvee\mathfrak{g}^{\star})_{\mathfrak{I}(\mathfrak{G})}$ when P is connected).*

Remark 3.2.3 The assumptions of the above theorem are satisfied, for example, when G is connected or when $G = GL(V)$.

Proof of Theorem 3.2.2. Since the representation $ad_{A(P)}$ is the differential of the representation Ad_P and the same holds for the symmetric powers of their contragredients (see Th. 5.4.3 in [5]), the theorem clearly follows from the following Proposition. □

Proposition 3.2.4 *Let $A = A(P)$ for a G-principal bundle P on a manifold M, and let $\mathfrak{f}$ be a vector bundle on M with a typical fibre V. Assume that $T : A(P) \to A(\mathfrak{f})$ is the differential of some $(\mu : G \to GL(V))$-representation $H : P \to L(\mathfrak{f})$ of P on $\mathfrak{f}$ (see 1.6). If each $(\mu|G_\circ : G_\circ \to GL(V))$-invariant vector $v \in V$ is μ-invariant, then, for any involutive distribution F on M, each T^F-invariant cross-section of $\mathfrak{f}$ is of the form of a finite sum $\sum_i f^i \cdot \Psi_i$ for $\mathcal{F}$-basic functions f^i and T-invariant cross-sections Ψ_i.*

Proof. According to Propositions 5.5.2–3 in [5] we have

$$V_I \cong (\operatorname{Sec}\mathfrak{f})_{I(H)} \subset (\operatorname{Sec}\mathfrak{f})_{I^\circ(T)}$$

where V_I is the space of μ-invariant vectors, whereas $(\operatorname{Sec}\mathfrak{f})_{I(H)}$ and $(\operatorname{Sec}\mathfrak{f})_{I^\circ(T)}$ are the spaces of invariant cross-sections with respect to H and T, respectively.

Let $\nu \in \operatorname{Sec}\mathfrak{f}$ be T^F-invariant. In particular, $\nu_x \in \mathfrak{f}_{|x}$ is $(T^F)^+_{|x}$ $(= T^+_{|x})$-invariant (for the index "+", see section 1.1). We recall, see section 1.2 in [5], that the LAB adjoint of $A(\mathfrak{f})$ is isomorphic to End $\mathfrak{f}$. Via this isomorphism, the following diagram

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{d\mu} & \operatorname{End}(V) \\
\Big\downarrow{\scriptstyle \hat{z}} & & \Big\downarrow \quad \begin{matrix} a \\ \downarrow \\ {\scriptstyle H(z)\circ a\circ H(z)^{-1}} \end{matrix} \\
\boldsymbol{g}_{|x} & \xrightarrow{T^+_{|x}} & \operatorname{End}(\mathfrak{f}_{|x})
\end{array}$$

commutes for each $z \in P$ ($\hat{z}$ is defined in section 1.6). This easily implies that, for $z \in P$, the vector $w_x := H(z)^{-1}(\nu_x) \in V$ is $d\mu$-invariant. By the assumption with G, w_x is μ-invariant [2]. Let $(w_1, \ldots, w_k)$ be a base of the space V_I of μ-invariant vectors. Then there exists real numbers $f^1(x), \ldots, f^k(x)$ such that $w_x = \sum_i f^i(x) \cdot w_i$; on the other hand, we have the equality $\nu = \sum_i f^i \cdot \nu_{w_i}$ where ν_{w_i} are H-invariant (therefore T-invariant) cross-sections defined in 1.6. The linear independence of ν_{w_i} proves the smoothness of f^i. Finally, we prove that $f^i \in \Omega_b^\circ(M, \mathcal{F})$. Let $X \in \operatorname{Sec} F$. There exists $\xi \in \operatorname{Sec} A(P)^F$ such that $\gamma \circ \xi = X$. Therefore we obtain

$$0 = \mathcal{L}_{T^F\circ\xi}(\sum_i f^i \cdot \nu_{w_i}) = \sum_i (f^1 \cdot \mathcal{L}_{T\circ\xi}(\nu_{w_i}) + (\gamma \circ \xi)(f^i) \cdot \nu_{w_i}) = \sum_i X(f^i) \cdot \nu_{w_i},$$

which implies $X(f^i) = 0$. The free choice of X proves that f^i are basic. □

3.3 The tangential Chern-Weil homomorphism of a vector bundle

Let $\mathfrak{f} \xrightarrow{p} M$ be a vector bundle with typical fibre V and let $G \subset GL(V)$ be a Lie subgroup of $GL(V)$. Assume that $\mathcal{A}$ is any maximal family of distinguished local trivializations $\psi : U_\psi \times V \longrightarrow p^{-1}[U_\psi]$, U_ψ being an open subset of M and $\bigcup U_\psi = M$ such that the transition functions have values in G. The pair $(\mathfrak{f}, \mathcal{A})$, called a *vector bundle with the structure Lie group* G (or in the sequel, briefly, a *G-vector bundle*), determines

a) the G-principal bundle $L(\mathfrak{f}, \mathcal{A})$ of distinguished frames

$$L(\mathfrak{f}, \mathcal{A}) = \{\psi_{|x} : V \longrightarrow \mathfrak{f}_{|x}\,;\ \psi \in \mathcal{A},\, x \in U_\psi\},$$

b) the transitive Lie algebroid denoted by $A(\mathfrak{f}, \mathcal{A})$ (being a Lie subalgebroid of $A(\mathfrak{f})$), putting

$$A(\mathfrak{f}, \mathcal{A}) = \Phi_{\mathfrak{f}}[A(L(\mathfrak{f}, \mathcal{A}))]$$

(we recall that $\Phi_{\mathfrak{f}} : A(L\mathfrak{f}) \longrightarrow A(\mathfrak{f})$ is the canonical isomorphism of transitive Lie algebroids, see section 1.6).

Definition 3.3.1 By the *tangential Chern-Weil homomorphism of a G-vector bundle* $(\mathfrak{f}, \mathcal{A})$*, over a foliated manifold* (M, F)*,* we mean the Chern-Weil homomorphism

$$h_{A(\mathfrak{f},\mathcal{A})^{\mathcal{F}}} : I^\circ(A(\mathfrak{f}, \mathcal{A})^{\mathcal{F}}) \longrightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{M})$$

of the regular Lie algebroid $A(\mathfrak{f}, \mathcal{A})^{\mathcal{F}}$.

Now, we describe elements of the Lie algebroids $A(\mathfrak{f}, \mathcal{A})$ in terms of the vector bundle $(\mathfrak{f}, \mathcal{A})$.

Theorem 3.3.2 *Let $l \in A(\mathfrak{f})_{|x}$, $x \in M$, then $l \in A(\mathfrak{f}, \mathcal{A})_{|x}$ if and only if, for any $\psi \in \mathcal{A}$ such that $x \in U_\psi$, the endomorphism*

$$V \ni u \longmapsto \psi_{|x}^{-1}(l(\psi(\cdot, u))) \in V$$

belongs to the Lie algebra of G.

We begin with the following lemma.

Lemma 3.3.3 *For an arbitrary local trivialization $\psi : U \times V \longrightarrow p^{-1}[U]$ of a vector bundle $\mathfrak{f}$, the following diagram*

$$\begin{array}{ccc} A(L\mathfrak{f}) & \xrightarrow{\Phi_{\mathfrak{f}}} & A(\mathfrak{f}) \\ {\scriptstyle \hat{\psi}^A}\big\uparrow & & \big\uparrow{\scriptstyle \bar{\psi}} \\ T(U) \times T_{id}(GL(V)) & \xrightarrow{id \times \rho_V} & T(U) \times \mathrm{End}(V) \end{array}$$

is commutative when

1) $\hat{\psi}^A$ is a local trivialization determined by $\hat{\psi} : U \times GL(V) \to L\mathfrak{f}$, $(x, a) \mapsto \psi(x, \cdot) \circ a$, via the formula (see [4; 1.1]) $\hat{\psi}^A(v, w) = [(\hat{\psi})_{\star(x,e)}(v, w)]$, *2)* $\bar{\psi}$ *is described in Th. 1.3.1,*

3) $\rho_V : T_{id}(GL(V)) \to \mathrm{End}\,(V)$ *is an isomorphism of Lie algebras (*$T_{id}(GL(V))$ *is meant as a right Lie algebra), defined by [5; 5.2.1]*

$$\rho_V(w) : V \longrightarrow V,\ u \mapsto \partial_v(\tilde{u}),$$

where $\tilde{u} : GL(V) \to V,\ a \mapsto a^{-1}(u)$.

Proof. For $v \in T_xU,\ w \in T_{id}(GL(V))$ and $\nu \in \mathrm{Sec}\,\mathfrak{f}$, we obtain (see 1.6)

$$\begin{aligned}
\Phi_{\mathfrak{f}}\circ\hat{\psi}^A(v,w)(\nu) &= \Phi_{\mathfrak{f}}([\hat{\psi}_*(v,w)])(\nu)\\
&= \psi(x,e)(\partial_{\hat{\psi}_*(v,w)}(\tilde{\nu}))\\
&= \psi_{|x}(\partial_{(v,w)}(\tilde{\nu}\circ\hat{\psi}))\\
&= \psi_{|x}(\partial_v(\tilde{\nu}\circ\hat{\psi}(\cdot,e)) + \partial_w(\tilde{\nu}\circ\hat{\psi}(x,\cdot)))\\
&= \psi_{|x}(\partial_v(\nu_\psi)) + \psi_{|x}(\rho_V(w)(\nu_\psi(x)))\\
&= \bar{\psi}(v,\rho_V(w))(\nu)\\
&= \bar{\psi}(id\times\rho_V)(v,w)(\nu). \quad \square
\end{aligned}$$

Proof of Theorem 3.3.2. Suppose that $x \in M$ and take an arbitrary local trivialization $\psi \in \mathcal{A}$ such that $x \in U_\psi$. Clearly, the restriction $\hat{\psi}_G$ of $\hat{\psi}$ to $U \times G$,

$$\hat{\psi}_G : U \times G \longrightarrow L(\mathfrak{f},\mathcal{A}),\ (y,a) \longmapsto \psi(y,\cdot)\circ a\,,$$

is a local trivialization of the G-principal bundle $L(\mathfrak{f},\mathcal{A})$. Passing to algebroids, we have the following diagram:

$$\begin{array}{ccccc}
A(L(\mathfrak{f},\mathcal{A})) & \hookrightarrow & A(L\mathfrak{f}) & \xrightarrow{\Phi_{\mathfrak{f}}} & A(\mathfrak{f})\\
\uparrow \hat{\psi}_G^A & & \uparrow \hat{\psi}^A & & \uparrow \bar{\psi}\\
TU \times T_{id}G & \hookrightarrow & TU \times T_{id}(GL(V)) & \xrightarrow{id\times\rho_V} & TU \times \mathrm{End}(V)\,.
\end{array}$$

Understanding $GL(V)$ as an open subset of $\mathrm{End}(V)$, we have the canonical isomorphism $T_{id}(GL(V)) \cong \mathrm{End}(V)$; then $\varrho_V = -id$ (see 5.2.1 in [5]). Let $\mathfrak{g} \subset \mathrm{End}(\mathfrak{V})$ be the Lie algebra of G. Clearly, $\varrho_V[T_{id}G] = \mathfrak{g}$, therefore, according to the above diagram, the restriction $\bar{\psi}_{\mathfrak{g}}$ of $\bar{\psi}$ to $TU \times \mathfrak{g}$ has values in $A(\mathfrak{f},\mathcal{A})$ and $\bar{\psi}_{\mathfrak{g}} : TU \times \mathfrak{g} \to \mathfrak{A}(\mathfrak{f},\mathcal{A})$ is a local trivialization of the Lie algebroid $A(\mathfrak{f},\mathcal{A})$.

Take $l \in A(\mathfrak{f})_{|x}$. From the above we obtain:

$$l \in A(\mathfrak{f},\mathcal{A})_{|x} \Longleftrightarrow \bar{\psi}_{|x}^{-1}(l) \in TU \times \mathfrak{g}.$$

To prove the " $\Longrightarrow$ " part of our theorem, assume that $l \in A(\mathfrak{f},\mathcal{A})_{|x}$ and write $l = \bar{\psi}_{|x}(q(l), a)$ for $a \in \mathfrak{g}$ ($q(l)$ is the anchor of l, see 1.3) and notice that we must show the relation $a(u) = \psi_{|x}^{-1}(l(\psi(\cdot,u)))$, i.e. equivalently, that $\psi_{|x}(a(u)) = l(\psi(\cdot,u))$. But from the fact $\nu_{\psi(\cdot,u)} \equiv u$, we obtain

$$\begin{aligned} \psi_{|x}(a(u)) &= \psi_{|x}(\partial_{q(l)}(\nu_{\psi(\cdot,u)}) + a(\nu_{\psi(\cdot,u)}(x))) \\ &= \bar{\psi}(q(l),a)(\psi(\cdot,u)) \\ &= l(\psi(\cdot,u)). \end{aligned}$$

To prove the " $\Longleftarrow$ " part, assume that $a := (u \mapsto \psi_{|x}^{-1}(l(\psi(\cdot,u))))$ belongs to $\mathfrak{g}$. It only remains to notice that $l = \bar{\psi}_{|x}(q(l),a)$. However, l and $\bar{\psi}_{|x}(q(l),a)$ are $\mathfrak{f}$-vectors with the same anchors, so, to show their identity, it is sufficient to check their behaviour on the cross-sections generating the module $\operatorname{Sec}\mathfrak{f}$ near x, for example, on $\psi(\cdot,u)$, $u \in V$:

$$\begin{aligned} \bar{\psi}_{|x}(q(l),a)(\psi(\cdot,u)) &= \psi_{|x}(\partial_{q(l)}(\psi(\cdot,u)_\psi) + \psi_{|x}^{-1}(l(\psi(\cdot,\psi(\cdot,u)_\psi(x)))) \\ &= l(\psi(\cdot,u)). \quad \square \end{aligned}$$

If ∇ is a covariant derivative in $\mathfrak{f}$, then, for $v \in T_xM, \;\; x \in M$, the linear mapping

$$\nabla_v : \operatorname{Sec}\mathfrak{f} \longrightarrow \mathfrak{f}_{|x}, \quad \nu \longmapsto \nabla_v\nu,$$

is clearly an $\mathfrak{f}$-vector, i.e. $\nabla_v \in A(\mathfrak{f})_{|x}$. Besides, the mappimg

$$TM \ni v \longmapsto \nabla_v \in A(\mathfrak{f})$$

is a connection in $A(\mathfrak{f})$. Conversely, any connection $\lambda : TM \to A(\mathfrak{f})$ determines a covariant derivative ∇ in $\mathfrak{f}$ by the formula

$$\nabla_v(\nu) = \lambda(v)(\nu), \quad v \in TM,\, \nu \in \operatorname{Sec}\mathfrak{f}.$$

Lemma 3.3.4 *If ∇ is the covariant derivative in $\mathfrak{f}$ corresponding to a connection λ in $A(\mathfrak{f})$, then the curvature tensor $R \in \Omega^2(M;\operatorname{End}\mathfrak{f})$ of ∇ is equal to the curvature tensor Ω of λ multiplied by -1.*

Proof. For $X, Y \in \mathcal{X}(M)$ and $\nu \in \operatorname{Sec}\mathfrak{f}$, we have (see 1.3):

$$\begin{aligned} \Omega(X,Y)(\nu) &= \mathcal{L}_{\lambda\circ[X,Y]}(\nu) - \mathcal{L}_{[\![\lambda\circ X,\lambda\circ Y]\!]}(\nu) \\ &= \nabla_{[X,Y]}(\nu) - [\nabla_X,\nabla_Y](\nu) \\ &= -R_{X,Y}(\nu). \quad \square \end{aligned}$$

Corollary 3.3.5 *A connection λ in $A(\mathfrak{f})$ is flat over an involutive distribution $F \subset TM$ if and only if the corresponding covariant derivative in $\mathfrak{f}$ is flat over F.* $\square$

Definition 3.3.6 A covariant derivative ∇ in $\mathfrak{f}$ will be called a *covariant derivative in the G-vector bundle* $(\mathfrak{f},\mathcal{A})$ if the corresponding connection ∇ in $A(\mathfrak{f})$ has values in $A(\mathfrak{f},\mathcal{A})$.

Clearly, the correspondence (described above) between connections in $A(\mathfrak{f},\mathcal{A})$ and covariant derivatives in $(\mathfrak{f},\mathcal{A})$ is one-to-one.

According to Th. 3.3.2, ∇ is a covariant derivative in $(\mathfrak{f},\mathcal{A})$ if and only if, for any $\psi\in\mathcal{A}$, $x\in U_\psi$ and $v\in T_xM$, the endomorphism

$$V\ni u\longmapsto \psi_{|x}^{-1}(\nabla_v(\psi(\cdot,u)))\in V$$

belongs to the Lie algebra $\mathfrak{g}$ of G.

Remark 3.3.7 Important examples of a reduction of the structure Lie group $GL(V)$ in $\mathfrak{f}$ can be obtained via the so-called Σ-bundles [2; Ch.VIII]. Let $(\mathfrak{f},\Sigma_\mathfrak{f})$ be an arbitrary Σ-bundle, $\Sigma_\mathfrak{f}$ being a finite ordered set of cross-sections ν_i of $\mathfrak{f}^{p_i,q_i}$ $(:=\otimes^{p_i}\mathfrak{f}^\star\bigotimes\otimes^{q_i}\mathfrak{f})$, $i\leq m$, subject to the following condition:

> *there is a finite ordered system $\Sigma_V=\{v_1,\ldots,v_m\}$ of tensors $v_i\in V^{p_i,q_i}$ (V is the typical fibre of $\mathfrak{f}$) and there is a system of local trivializations $\mathcal{A}$ of $\mathfrak{f}$ such that, for each $\psi\in\mathcal{A}$, $\psi_{|x}^{p_i,q_i}(v_i)=\nu_i(x)$, $i\leq m$, $x\in U_\psi$.*

The pair $(\mathfrak{f},\mathcal{A})$ is then a vector bundle with a reduction to the closed Lie subgroup $G\subset GL(V)$ consisting of those and only those linear isomorphisms $\varphi: V\to V$ for which $\varphi^{p_i,q_i}(v_i)=v_i$, $i\leq m$. The Lie algebra $\mathfrak{g}$ of G is the subalgebra of $\mathrm{End}(V)$ consisting of the linear transformations φ of V which satisfy $\theta_\varphi^{p_i,q_i}(v_i)=0$, $i\leq m$, where θ^{p_i,q_i} is the canonical representation of the Lie algebra $\mathrm{End}(V)$ on V^{p_i,q_i} [2]. Denote also by θ^{p_i,q_i} the canonical representation of the Lie algebroid $A(\mathfrak{f})$ on $\mathfrak{f}^{p_i,q_i}$, generated by the identical one $id_{A(\mathfrak{f})}: A(\mathfrak{f})\to A(\mathfrak{f})$; see [5; 2.2].

By the above, there are no essential difficulties to prove

Proposition 3.3.8 *Let $l\in A(\mathfrak{f})_{|x}$, $x\in M$; then $l\in A(\mathfrak{f},\Sigma_\mathfrak{f})_{|x}$ if and only if $\theta^{p_i,q_i}(l)(\nu_i)=0$, $i\leq m$.* $\square$

A covariant derivative ∇ in $\mathfrak{f}$ detrmines, in the standard way, the covariant derivative ∇^{p_i,q_i} in $\mathfrak{f}^{p_i,q_i}$. Clearly, we have $\nabla^{p_i,q_i}=\theta^{p_i,q_i}\circ\nabla$. According to this and the last proposition, we obtain

Corollary 3.3.9 *A covariant derivative ∇ is in $(\mathfrak{f},\Sigma_\mathfrak{f})$ if and only if $\nabla^{p_i,q_i}(\nu_i)=0$, $i\leq m$, i.e. if and only if ∇ is a Σ-connection.* $\square$

As an example consider a Riemannian bundle $(\mathfrak{f},\{G\})$, G being a Riemannian tensor in $\mathfrak{f}$. In this case, covariant derivatives in $(\mathfrak{f},\{G\})$ are simply Riemannian connections.

Now on the base M let us define a foliation $\mathcal{F}$ having $F\subset TM$ as its tangent bundle. If λ is a connection in the G-vector bundle $(\mathfrak{f},\mathcal{A})$, then the operator ∇ defined by $\nabla_v(\nu)=\lambda(v)(\nu)$, $v\in F$, $\nu\in\mathrm{Sec}\,\mathfrak{f}$, is a partial covariant derivative in $\mathfrak{f}$ over F in the sense of [3]. Conversely, a partial covariant derivative in $\mathfrak{f}$ over F such that $\nabla_v\in A(\mathfrak{f},\mathcal{A})$ for $v\in F$ (called a *partial covariant derivative in* $(\mathfrak{f},\mathcal{A})$ *over* F)

determines a connection $F \ni v \mapsto \nabla_v \in A(\mathfrak{f}, \mathcal{A})$ in $A(\mathfrak{f}, \mathcal{A})^{\mathcal{F}}$. This correspondence is one-to-one. Clearly, in the case of a Σ-bundle $(\mathfrak{f}, \Sigma_{\mathfrak{f}})$, a partial covariant derivative ∇ in $\mathfrak{f}$ over F is in the bundle $(\mathfrak{f}, \Sigma_{\mathfrak{f}})$ if and only if $\nabla\nu = 0$ for $\nu \in \Sigma_{\mathfrak{f}}$.

Now, pass to the investigation of the Chern-Weil homomorphism of the regular Lie algebroid $A(\mathfrak{f}, \mathcal{A})^{\mathcal{F}}$. According to the above, its nontriviality means that in $(\mathfrak{f}, \mathcal{A})$ there is no flat partial covariant derivative over F.

3.4 Open problem

— *Find some G-vector bundle $(\mathfrak{f}, \mathcal{A})$ over a foliated manifold, possessing nontrivial singular characteristic classes.*

Let $\mathrm{Pont}(A) := \mathrm{Im}(h_A)$ be the Pontryagin algebra of a regular Lie algebroid A. Consider a nonorientable Riemannian vector bundle $\mathfrak{f}$ of rank $2m$ and a connected $O(2m;\mathbb{R})$-principal bundle P of orthonormal frames of $\mathfrak{f}$, and the transitive Lie algebroid $A = A(P)$. We have $\mathrm{Pont}^{2m}(P) = \mathrm{Pont}^{2m}(A) = 0$ (and, of course, $\mathrm{Pont}^k(P) = 0$ for $k > 2m$).

Conjecture 3.4.1 There exists an example of a nonorientable Riemannian vector bundle $\mathfrak{f}$ and an involutive distribution F with orientable leaves on the base M of $\mathfrak{f}$, for which

$$\mathrm{Pont}^{2m}(A^F) \neq 0.$$

(Stronger conjecture: $\mathfrak{f}$ can be taken as the tangent bundle TM to some manifold M).

Singular partial invariant cross-sections from the domain of $h^{2m}(A^F)$ can be obtained by "gluing" and "smoothing" the cross-sections (defined on leaves) coming from the Pfaffians (as in Example 2.1).

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