Series Logo
Volume **00**, Number 00, Xxxx 19xx

CHARACTERISTIC HOMOMORPHISMS OF REGULAR LIE ALGEBROIDS

JAN KUBARSKI

(Poster, European Congress of Mathematics, Paris, July 6-10, 1992)

1. Introduction

Differential Geometry has discovered many objects which determine a Lie algebroid [fulfilling a role analogous to that of Lie algebras for Lie groups] as, for example, *differential groupoids* (and, in consequence, *principal bundles*) (J.Pradines 1967), *transversally complete foliations* (and, in consequence, *nonclosed Lie subgroups*) (P.Molino 1977), *Poisson manifolds* (A.Coste, P.Dazord, A.Weinstein 1987), and some complete closed pseudogroups (A.Silva 1988).

The author has constructed characteristic homomorphisms for regular Lie algebroids:

(a) the Chern-Weil homomorphism,
(b) the characteristic homomorphisms of flat (and partially flat) regular Lie algebroids.

These homomorphisms for integrable Lie algebroids (i.e. transitive ones coming from connected principal bundles) agree with the classical ones of these bundles. We pay our attention to the fact that this holds although in the Lie algebroid of a principal bundle there is no direct information about the structure Lie group of this bundle (which may be disconnected !).

There exist non-integrable transitive Lie algebroids which have the non-trivial Chern-Weil homomorphism. Lie algebroids of some transversally complete foliations have this property (for example, Lie algebroids of the foliations of left cosets of connected compact and semisimple Lie groups by nonclosed connected Lie subgroups).

2. Definitions and examples

A *Lie algebroid on a manifold* M is (J.Pradines 1966) a vector bundle A on M together with

1) a Lie algebra structure $[\![\cdot,\cdot]\!]$ in the space $\operatorname{Sec} A$ of all C^∞ global cross-sections of A,
2) a linear homomorphism $\gamma : A \to TM$ (called an *anchor*) such that $\operatorname{Sec}\gamma : \operatorname{Sec} A \to \mathcal{X}(M)$, $\xi \mapsto \gamma\circ\xi$, is a homomorphism of Lie algebras and the following equality holds: $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$, $\sigma,\eta\in\operatorname{Sec} A$, $f\in C^\infty(M)$.

Received by the editors July 1992 .
This author was granted with a fellowship by the Organizing Committee of the ECM, Paris, July 6-10 1992 .

A Lie algebroid A is called *regular* if γ is a constant rank; then $E := \mathrm{Im}\ \gamma$ is a C^∞ constant dimensional and completely integrable distribution. A is then called a *Lie algebroid over a foliated manifold* (M, E), too. A Lie algebroid A is called *transitive* if γ is an epimorphism.

Example 1. The following are simple fundamental examples of transitive Lie algebroids:

(1^0) Finitely dimensional Lie algebra.

(2^0) Tangent bundle TM to a manifold M with the bracket $[\cdot,\cdot]$ of vector fields and id_{TM} as an anchor.

(3^0) Trivial Lie algebroid $TM\times\mathfrak{g}$ (Ngo-Van-Que) where $\mathfrak{g}$ is as in (1^o). The bracket is defined by the formula,

$$[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - \mathcal{L}_Y\sigma + [\sigma,\eta]),$$

$X, Y \in \mathcal{X}(M),\ \sigma,\eta : M \to \mathfrak{g}$, and the anchor is the projection $TM\times\mathfrak{g} \to TM$.

(4^0) Bundle of jets J^kTM (P.Libermann).

(5^0) General form (K.Mackenzie, J.Kubarski). Let a system $(\mathbf{g}, \nabla, \Omega_b)$ be given, consisting of a Lie algebra bundle $\mathbf{g}$ on a manifold M, a covariant derivative ∇ in $\mathbf{g}$ and a 2-form $\Omega_b \in \Omega^2(M,\mathbf{g})$ on M with values in $\mathbf{g}$, fulfilling the conditions:

(i) $\nabla^2\sigma = -[\Omega_b,\sigma], \quad \sigma\in \mathrm{Sec}\,\mathbf{g}$,

(ii) $\nabla_X[\sigma,\eta] = [\nabla_X\sigma,\eta] + [\sigma,\nabla_X\eta], \quad X\in X(M), \quad \sigma,\eta\in\mathrm{Sec}\,\mathbf{g}$,

(iii) $\nabla\Omega_b = 0$.

Then $TM\oplus\mathbf{g}$ forms a transitive Lie algebroid with the bracket defined by

$$[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], -\Omega_b(X,Y) + \nabla_X\eta - \nabla_Y\sigma + [\sigma,\eta]),$$

the anchor being the projection onto the first component.

Every transitive Lie algebroid is — up to an isomorphism — of this form.

Example 2. The following are important examples of transitive Lie algebroids:

(6^0) The Lie algebroid $A(P) = TP/G$ of a G-principal bundle P (K.Mackenzie, J.Kubarski).

(7^0) The Lie algebroid CDO($\mathfrak{f}$) of covariant differential operators on a vector bundle $\mathfrak{f}$ (K.Mackenzie). Another isomorphic construction of this object is the Lie algebroid $A(\mathfrak{f})$ of a vector bundle $\mathfrak{f}$ (J.Kubarski), here the fibre $A(\mathfrak{f})_{|x}$ is the space of linear homomorphisms $l : \mathrm{Sec}\,\mathfrak{f} \to \mathfrak{f}_{|x}$ such that there exists a vector $u\in T_xM$ for which $l(f\cdot\nu) = f(x)\cdot l(\nu) + u(f)\cdot\nu(x),\quad f\in C^\infty(M)$, $\nu\in\mathrm{Sec}(\mathfrak{f})$.

(8^0) The Lie algebroid $i^*T^\alpha\Phi$ of a Lie groupoid Φ (J.Pradines).

(9^0) The Lie algebroid $A(M,\mathcal{F})$ of a transversally complete foliation $(M,\mathcal{F})$ (P.Molino); in particular,

(10^0) the Lie algebroid $A(G;H)$ of the foliation of left cosets of a Lie group G by a nonclosed connected Lie subgroup $H\subset G$ (for the construction independent of the theory of transversally complete foliations, see J.Kubarski).

(11^0) The Lie algebroids of some pseudogroups (A.Silva).

Example 3. The following are examples of nontransitive (in general) Lie algebroids:

(12^0) A Lie algebra bundle is a totally nontransitive Lie algebroid (K.Mackenzie).

(13^0) Any C^∞-constant dimensional involutive distribution $E \subset TM$ forms a nontransitive Lie algebroid.

(14^0) If A is a transitive Lie algebroid on M and (M, E) is a foliated manifold, then $A^E := \gamma^{-1}[E] \subset A$ forms, in an evident manner, a regular Lie algebroid over (M, E). For example, a Lie groupoid (or a vector bundle) over a foliated manifold determines such an object.

(15^0) Any vector subbundle $R \subset J^k(TM)$ for which $\mathrm{Sec}R$ is a Lie subalgebra of $\mathrm{Sec}(J^k(TM))$ (called the Lie equation) forms a Lie algebroid, in general, nontransitive one.

(16^0) Differential groupoids and, more generally some groupoids defined in the category of Sikorski differential spaces are the source of nontransitive Lie algebroids. For example, such groupoids can be constructed by coming from the so-called $\sum$*-bundles over foliated manifolds (J.Kubarski)* defined as triples $(\mathfrak{f}, E, \sum)$ in which $\mathfrak{f}$ is a vector bundle on M, E is a foliation on M and $\sum = \{\sigma_1, \ldots, \sigma_n\}$ is a set of cross-sections $\sigma_i \in \mathfrak{f}^{p_i, q_i}$ fulfilling the axiom:

— for any points x, y lying on the same leaf of E, there exists an isomorphism $\alpha : \mathfrak{f}_{|x} \to \mathfrak{f}_{|y}$, called a distinguished one, for which $\alpha^{p_i, q_i}(\sigma_i(x)) = \sigma_i(y)$.

Let $\Phi \subset GL(\mathfrak{f})$ be the subgroupoid of all distinguished isomorphisms for a given $\sum$-bundle. Φ (however, not being, in general, a submanifold) is (always) equipped with a structure of a Sikorski differential subspace of $GL(\mathfrak{f})$, determining a grupoid in the category of differential spaces. More concrete examples:

(1) the vector bundle $\mathfrak{g} = \mathrm{Ker}\,\gamma$ of any regular Lie algebroid, with natural structures of Lie algebras in the fibres;

(2) let (M, E) be a foliated manifold and $\omega \in \Omega(M)$ — any basic form. Then $(TM, E, \{\omega\})$ is a $\sum$-bundle over (M, E).

(17^0) The Lie algebroid of a Poisson manifold (A.Coste, P.Dazord, A.Weinstein).

3. The construction of the Chern-Weil homomorphism of a regular Lie algebroid

A) Let A be any regular Lie algebroid over a foliated manifold (M, E). $\mathbf{g} = \mathrm{Ker}\,\gamma$ is a vector bundle whose any fibre possesses a structure of a Lie algebra. Any splitting $\lambda : E \to A$ of the Atiyah sequence $0 \to \mathbf{g} \hookrightarrow A \to E \to 0$ is called a *connection* in A. λ defines a tangential differential form $\Omega_b \in \Omega^2_E(M; \mathbf{g})$ $(= \mathrm{Sec} \bigwedge^2 E^* \otimes \mathbf{g})$, called the *curvature tensor* of λ by $\Omega_b(X, Y) = \lambda[X, Y] - [\![\lambda X, \lambda Y]\!]$.

A connection λ is said to be *flat* if $\Omega_b = 0$.

Theorem 1. *If $A = A(P)$, P being a principal bundle, then there is a bijection between connections in A and P.*

Theorem 2. *If $A = A(P)^E$ (see example (14^0)), P being as above and E — an involutive distribution on M, then there is a bijection between connections in A and partial connections in P over E.*

Theorem 3. *If $A = A(M, \mathfrak{F})$, $(M, \mathfrak{F})$ being a transversally complete foliation, then there is a bijection between connections in A and C^∞ distributions $C \subset TM$ fulfilling the conditions*

(1) $C + E_b = TM$,

(2) $C \cap E_b = E$,

(3) $C_{|x} = \{X(x); X \in \operatorname{Sec} C \cap L(M, \mathfrak{F})\}$, $x \in M$, *where* $L(M,\mathfrak{F})$ *is the Lie algebra of foliate vector fields and* E_b *is the vector bundle tangent to the basic foliation* $\mathfrak{F}_b$.

In the case of $\mathfrak{F}$ *being a foliation of left cosets of* G *by* H *(see example* (10^0)*), condition* (3) *is equivalent to:*

(3′) C *is* $\bar{H}$*-right-invariant* ($\bar{H}$ *is the closure of* H*).*

Such a distribution C *always exists and a connection in* $A(M,\mathfrak{F})$ *is flat if and only if the corresponding distribution in* TM *is complete integrable.*

By a *representation of* A *on a vector bundle* $\mathfrak{f}$ (both over the same manifold) we mean a strong homomorphism of Lie algebroids $T : A \to A(\mathfrak{f})$. A cross-section $\nu \in \operatorname{Sec}(\mathfrak{f})$ is called *invariant* if $T(v)(\nu) = 0$ for all $v \in A$. An important example is the *adjoint representation* $ad_A : A \to A(\mathbf{g})$ of regular Lie algebroid A defined by $ad_A(v)(\nu) = [v, \nu] := [\![\xi, \nu]\!](x)$ where $\xi(x) = v \in A_{|x}$ and $\nu \in \operatorname{Sec}\mathbf{g}$. ad_A induces a representation of A on the symmetric power $\bigvee^k \mathbf{g}^*$. A cross section $\Gamma \in \operatorname{Sec}\bigvee^k \mathbf{g}^*$ is invariant with respect to this representation if and only if, for any $\xi \in \operatorname{Sec} A$ and $\sigma_1, \ldots, \sigma_k \in \operatorname{Sec}\mathbf{g}$,

$$(\gamma \circ \xi)\langle \Gamma, \sigma_1 \vee \cdots \vee \sigma_k \rangle = \sum_{j=1}^{k} \langle \Gamma, \sigma_1 \vee \cdots \vee [\![\xi, \sigma_j]\!] \vee \cdots \vee \sigma_k \rangle.$$

The space of all invariant cross-sections is denoted by $(\bigvee^k \mathbf{g}^*)_I$. $\bigoplus^{k\geq 0}(\operatorname{Sec}\bigvee^k \mathbf{g}^*)_I$ forms an algebra.

Theorem 4. *Let* λ *be any connection in a regular Lie algebroid* A *and* $\Omega_b \in \Omega^2_E(M;\mathbf{g})$ *— its curvature tensor. Then, for* $\Gamma \in (\operatorname{Sec}\bigvee^k \mathbf{g}^*)_I$, *the real tangential form* $\beta(\Gamma) = \langle \Gamma, \Omega_b \vee \cdots \vee \Omega_b \rangle \in \Omega_E(M)$ *is closed. The mapping*

$$h_A : \bigoplus^k (\operatorname{Sec}\bigvee^k \mathbf{g}^*)_I \to H_E(M), \quad \Gamma \mapsto [\beta(\Gamma)],$$

called the Chern-Weil homomorphism of A, *is a homomorphism of algebras independent of the choice of a connection. If the Chern-Weil homomorphism* h_A *of* A *is non-trivial (i.e.* $h_A^+ \neq 0$*), then there exists no flat connection in* A.

Theorem 5. *If* $A = A(P)$, $P = P(M,G)$ *being a connected principal bundle, then there exists an isomorphism of algebras* $\alpha : \bigoplus^k (\operatorname{Sec}\bigvee^k \mathbf{g}^*)_I \xrightarrow{\cong} (\bigvee \mathfrak{g}^*)_{I(G)}$ *(*$\mathfrak{g}$ *being the Lie algebra of the structure Lie group* G *of* P *) such that* $h_P \circ \alpha = h_{A(P)}$.

Theorems 3 and 4 above yield the following

Corollary 1. *If* $A = A(M,\mathfrak{F})$, $(M,\mathfrak{F})$ *being a transversally complete foliation, and* h_A *is non-trivial, then there exists no completely integrable distribution* $C \subset TM$ *fulfilling conditions (1)÷(3) from Theorem 3.*

Theorem 6. *If* $A = A(G;H)$ *(see example* (10^o)*), then there exists an isomorphism of algebras* $\alpha : \bigoplus^k (\operatorname{Sec}\bigvee^k \mathbf{g}^*)_I \xrightarrow{\cong} \bigvee(\bar{h}/\mathfrak{h})^*$ *(* $\bar{h}$ *being the Lie algebra of the closure* $\bar{H}$ *of* H*), such that* h_A *is equal to the superposition*

$$h_A : \bigoplus^k (\operatorname{Sec}\bigvee^k \mathbf{g}^*)_I \xrightarrow{\cong} \bigvee(\bar{h}/\mathfrak{h})^* \rightarrowtail \left(\bigvee \mathfrak{h}^*\right)_{I(\bar{H})} \xrightarrow{h_P} H\left(G/\bar{H}\right)$$

where h_P *is the Chern-Weil homomorphism of the* $\bar{H}$*-principal bundle* $P = (G \to G/\bar{H})$.

Keep the assumptions of Theorem 6. Since $h_P^{(2)} : (\bar{h}^*)_{I(\bar{H})} \to H^2(G/\bar{H})$ is an isomorphism for G being a connected, compact and semisimple Lie group, we obtain that h_A is non-trivial in this case.

Corollary 2. *Taking G connected, compact, semisimple and, in addition, simple connected, we obtain a non-integrable transitive Lie algebroid whose Chern-Weil homomorphism is non-trivial.*

Remark 1. Some Lie subalgebras of $\mathfrak{g}$ determine flat connections in $A(G;H)$. Namely, if $\mathfrak{c} \subset \mathfrak{g}$ is a Lie subalgebra containing $\mathfrak{h}$ and such that $\mathfrak{g}/\mathfrak{h} = \bar{\mathfrak{h}}/\mathfrak{h} \oplus \mathfrak{c}/\mathfrak{h}$, then the G-left-invariant distribution C on G determined by $\mathfrak{c}$ is C^∞ completely integrable and satisfies conditions (1), (2) and (3′) from Theorem 3; therefore C induces a flat connection in $A(G;H)$. The existence of such a Lie subalgebra implies then the triviality of the Chern-Weil homomorphism h_A of A. According to the above, we obtain

Corollary 3. *If G is a connected, compact and semisimple Lie group, then no Lie subalgebra $\mathfrak{c}$ as above exists.*

Bott's phenomenon holds on the ground of regular Lie algebroids, which enables us to use it on the ground of nonclosed Lie subgroups:

Theorem 7. *Let $\mathrm{Pont}(G;\, H) := \mathrm{Im}\; h_{A\,(G;\,H)}$ be the Pontryagin algebra of $A(G;H)$. If $\mathrm{Pont}^p(G;H) \neq 0$, then there exist no Lie subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that (1) $\bar{h} \cap \mathfrak{c} = \mathfrak{h}$, (2) $\mathfrak{f} := \bar{\mathfrak{h}} + \mathfrak{c}$ is a Lie subalgebra of $\mathfrak{g}$ whose codimension is $\leq \frac{p}{2} - 1$ or [in the case of a basic connection] $\leq p-1$ provided that $\mathfrak{f}$ is a Lie subalgebra of a compact Lie subgroup of G.*

The case of G being a connected, compact and semisimple Lie group yields the non-triviality of $h^2_{A(G;H)}$ and, in consequence, the non-existence of a Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ such that (1) $\bar{h} \cap \mathfrak{c} = \mathfrak{h}$, (2) $\mathfrak{f} := \bar{\mathfrak{h}} + \mathfrak{c}$ is a Lie subalgebra of a closed Lie subgroup of G whose codimension is 1.

The significance of the Chern-Weil homomorphism on the ground of regular Poisson manifolds is not known up to now.

B) As for the nontransitive case, consider the Lie algebroid $A(P)^E$ from example (14^0) induced by a connected principal bundle $P = P(M,G)$ over a foliated manifold (M,E). The nontriviality of the Chern-Weil homomorphism $h_{A(P)^E}$ implies the non-existence of flat partial connections in this bundle. The domain of $h_{A(P)^E}$ always contains a subalgebra $\Omega^0_b(M;E)\cdot D$, $\Omega^0_b(M;E)$ being the space of basic functions, whereas D - the domain of $h_{A(P)}$ ($D \cong (\bigvee \mathfrak{g}^*)_{I(G)}$). These two algebras are equal to each other if each G_0-invariant element of $(\bigvee \mathfrak{g}^*)$ is G-invariant (G_0 being the connected component of the unit of G).

I suggest that, for $G = O(2m,\mathbb{R})$, there exists such an example of a G-principal bundle P for which these two algebras are not equal to each other, and that there exists an element Γ belonging to the domain of $h_{A(P)^E}$, such that it is not of the form $\sum f^i \cdot \Gamma_i$ ($f^i \in \Omega^0_b$, $\Gamma_i \in D$) but $h_{A(P)^E}(\Gamma) \neq 0$.

4. The characteristic homomorphism of a flat regular Lie algebroid

In a given regular Lie algebroid A over a foliated manifold (M,E) consider two geometric structures:

(1) a flat connection $\lambda : E \to A$,
(2) a subalgebroid $B \subset A$ over (M, E).

The system (A, B, λ) will then be called an *FS-regular Lie algebroid over* (M, E).

Example 4. *Let (P, P', ω') be any foliated G-principal bundle on a manifold M, with an H-reduction P' and a flat partial connection ω' over an involutive distribution $E \subset TM$. ω determines a flat connection λ in the regular Lie algebroid $A(P)^E$ over (M, E), and the system $(A(P)^E, A(P')^E, \lambda)$ is an FS-regular Lie algebroid.*

There are some characteristic classes of an FS-regular Lie algebroid (A, B, λ), measuring the independence of λ and B, i.e. to what extent $\operatorname{Im}\lambda$ is not contained in B. Let γ and γ_1 denote the anchors in A and B, respectively, and put $\mathbf{g} = \operatorname{Ker}\gamma$, $\mathbf{h} = \operatorname{Ker}\gamma_1$. By the *characteristic homomorphism* of (A, B, λ) we mean

$$\Delta_{\#} : H(\mathbf{g}, B) \to H_E(M)$$

in which $H(\mathbf{g}, B) = H((\operatorname{Sec}\bigwedge(\mathbf{g}/\mathbf{h})^*)_I, \bar{\delta})$ where

(1) $(\operatorname{Sec}\bigwedge(\mathbf{g}/\mathbf{h})^*)_I$ is the space of invariant cross-sections with respect to the canonical representation $B \to A(\bigwedge(\mathbf{g}/\mathbf{h})^*)$ induced by $ad_A|B : B \to A(\mathbf{g})$. Precisely, $\Psi \in (\operatorname{Sec}\bigwedge^k(\mathbf{g}/\mathbf{h})^*)_I$ if and only if, for any $\xi \in \operatorname{Sec} B$ and $\nu_1, \ldots, \nu_k \in \operatorname{Sec}\mathbf{g}$, we have

$$(\gamma_1 \circ \xi)\langle\Psi, [\nu_1] \wedge \cdots \wedge [\nu_k]\rangle = \sum_j \langle\Psi, [\nu_1] \wedge \cdots \wedge [[\![\xi, \nu_j]\!]] \wedge \cdots \wedge [\nu_k]\rangle$$

where $[\nu_j] = \mathbf{s} \circ \nu_j \in \operatorname{Sec}\mathbf{g}/\mathbf{h}$ and $\mathbf{s} : \mathbf{g} \to \mathbf{g}/\mathbf{h}$ is the canonical projection.

(2) $\bar{\delta}$ is a differential in $(\operatorname{Sec}\bigwedge(\mathbf{g}/\mathbf{h})^*)_I$ defined by the formula

$$\langle\bar{\delta}\Psi, [\nu_0] \wedge \cdots \wedge [\nu_k]\rangle = -\sum_{i<j}(-1)^{i+j}\langle\Psi, [[\![\nu_i, \nu_j]\!]] \wedge [\nu_0] \wedge \cdots \hat{\imath} \cdots \hat{\jmath} \cdots \wedge [\nu_k]\rangle.$$

$\Delta_{\#}$ on the level of forms is a homomorphism of algebras $\Delta_* : \operatorname{Sec}\bigwedge(\mathbf{g}/\mathbf{h})^* \to \Omega_E(M)$, such that

$$\Delta_*\Psi(x; w_1, \ldots, w_k) = \Psi(x, [\omega(x; \tilde{w}_1)] \wedge \cdots \wedge [\omega(x; \tilde{w}_k)])$$

for $w_j \in E_{|x}$ where $\tilde{w}_j \in B_{|x}$ satisfy $\gamma_1(\tilde{w}_j) = w_j$. Δ_* restricted to the invariant cross-sections commutes with the differentials $\bar{\delta}$ and d^E, giving a homomorphism $\Delta_{\#}$ on cohomologies.

The fundamental properties of $\Delta_{\#}$ are: (1) $\Delta_* = 0$ if $\operatorname{Im}\lambda \subset B$, (2) the functoriality of $\Delta_{\#}$, (3) the independence of $\Delta_{\#}$ of the choice of homotopic subalgebroids B.

Technical University, Łódź, Poland