THE CHERN-WEIL HOMOMORPHISM OF REGULAR LIE ALGEBROIDS

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Abstract

The aim of this paper is to construct the Chern-Weil homomorphism for regular Lie algebroids. This homomorphism, in the case of an arbitrary integrable transitive Lie algebroid A, agrees with the one for any connected principal bundle for which A is its Lie algebroid. Next, it is proved that there exist nonintegrable transitive Lie algebroids having the nontrivial Chern-Weil homomorphism. Lie algebroids of some transversally complete foliations have this property. Some applications to nonclosed Lie subgroups and to vector bundles over foliated manifolds are given.,

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0.1 INTRODUCTION

1. In [24] K.Mackenzie gives the first general and abstract treatment of the algebraic properties of Lie algebroids. The present work belongs to this direction. It is based on:

(a) the observation by the author that the Chern-W eil homomorphism of a connected principal bundle is an invariant of the Lie algebroid of this bundle,

(b) the construction of an equivalent of this homomorphism in a large class of regular (thus, nontransitive in general) Lie algebroids,

(c) the discovery of a class of Lie algebroids which are not integrable, i.e. which do not come from principal bundles, but have nontrivial Chern-Weil homomorphisms.

[Analogous observations, which will be the topic of the next work by the author, concern the characteristic classes of flat (and partially flat) principal bundles]. This enables one to apply this technique to the investigation of some geometric structures defined on objects not being principal bundles but possessing Lie algebroids, such as transversally complete foliations, nonclosed Lie subgroups, vector bundles over foliated manifolds, Poisson manifolds or some complete closed pseudogroups.

This work concerns the Chern-Weil homomorphism and transversally complete foliations, chiefly, foliations of left cosets of Lie groups by nonclosed connected Lie subgroups.

2. The notion of a Lie algebroid comes from J.Pradines [29], [30]. Originally, this notion was invented in connection with the study of differential groupoids (J.Pradines in [29] introduced the so-called Lie functor which assigns a Lie algebroid to any differential groupoid). Since each principal bundle P determines a differential groupoid (the so-called Lie groupoid PP^{-1} of Ehresmann [6]), therefore each principal bundle P defines - in an indirect manner - a Lie algebroid $A(P)$. P.Libermann noticed [21] that the vector bundle of this Lie algebroid is canonically isomorphic to the vector bundle TP/G (G is the structure

Lie group of P). The construction of the Lie functor for principal bundles with the omission of the indirect step of differential groupoids was made independently by K.Mackenzie [23] and by the author [16].

The Chern-Weil homomorphism h_P of a principal bundle P has been known for some forty years [3]. One can ask the question whether this homomorphism is an invariant of the Lie algebroid $A(P)$ of a given principal bundle P. In [17] (see also [19]) the author proved that it is so under the assumption that the structure Lie group G of P is connected. It turns out that this condition can be eliminated entirely (see Chapter 5). More precisely, the Chern-Weil homomorphism of a principal bundle P appears as a characteristic feature of the Lie algebroid $A(P)$ of P in every case (provided only that P is connected). This means that, knowing only the Lie algebroid $A(P)$ of P, one can uniquely reproduce the ring of invariant polynomials $(\bigvee \mathfrak{g}^*)_I$ and the Chern-Weil homomorphism $h_P : (\bigvee \mathfrak{g}^*)_I \to H_{dR}(M)$ ($\mathfrak{g}$ denotes the Lie algebra of G).

We pay our attention to the fact that this holds although in the Lie algebroid $A(P)$ there is no direct information about the structure Lie group of P (which may be disconnected !).

In addition, we must point out two things:

1) A Lie algebroid is - in some sense - a simpler structure than a principal bundle. Namely, nonisomorphic principal bundles can possess isomorphic Lie algebroids. For example, there exists a nontrivial principal bundle for which the Lie algebroid is trivial (the nontrivial Spin (3)-structure of the trivial principal bundle $\mathbb{R}P(5) \times \mathrm{SO}(3)$ [18], [19]).

2) There exist other sources of Lie algebroids than principal bundles, for example, transversally complete foliations [26], [27], Poisson manifolds [4], [5], or some complete closed pseudogroups [31]. Among them there are ones which give "nonintegrable" Lie algebroids, i.e. those which are transitive and cannot be realized as the Lie algebroids of principal bundles . Namely, according to Almeida-Molino theorem [1], [27], Lie algebroids of nondevelopable (and only such) transversally complete foliations have this property. An example of such a foliation is any transversally complete foliation with nonclosed leaves on a simply connected manifold. A more concrete example is any foliation of left cosets of any connected and simply connected Lie group by a Lie subgroup connected and dense in some torus.

3. In connection with the above, it seems important to construct the Chern-Weil homomorphism in some category of Lie algebroids, being a generalization of that for principal bundles. This problem is solved in our paper (chapter 4) in the category of regular Lie algebroids, i.e. of such ones in which the anchor is of constant rank. Namely,

$$h_A : \bigoplus^{k\geq 0} \left(\mathrm{Sec} \bigvee^k \boldsymbol{g}^* \right)_{I^0} \longrightarrow H_E(M)$$
$$\Gamma \longmapsto \left[\frac{1}{k!} \langle \Gamma, \Omega_b \vee ... \vee \Omega_b \rangle \right]$$

serves as this homomorphism for the regular Lie algebroid A (with the adjoint bundle of Lie algebras $\boldsymbol{g}$), where $\Omega_b \in \Omega^2_E(M; \boldsymbol{g})$ is the curvature tensor of any connection in A, whereas $\left(\mathrm{Sec} \bigvee^k \boldsymbol{g}^* \right)_{I^0}$ is the space of invariant cross-sections of $\bigvee^k \boldsymbol{g}^*$ with respect to the adjoint representation of A on $\bigvee^k \boldsymbol{g}^*$, i.e. $\Gamma \in \left(\mathrm{Sec} \bigvee^k \boldsymbol{g}^* \right)_{I^0}$ if and only if

$$\forall_{\xi \in \mathrm{Sec}\, A} \forall_{\sigma_1, ..., \sigma_k \in \mathrm{Sec}\, \boldsymbol{g}} \left((\gamma \circ \xi) \langle \Gamma, \sigma_1 \vee ... \vee \sigma_k \rangle = \sum_{i=1}^{k} \langle \Gamma, \sigma_1 \vee ... \vee [\![\xi, \sigma_i]\!] \vee ... \vee \sigma_k \rangle \right)$$

The nontriviality of h_A means, of course, that in A there is no flat connection.

The existence of a natural isomorphism of algebras ν such that

$$\begin{array}{ccc} \bigoplus^{k\geq 0}\left(\operatorname{Sec}\bigvee^{k}\boldsymbol{g}^{*}\right)_{I^0} & & \\ & \searrow^{h_{A(P)}} & \\ \cong\uparrow\nu & & H_{dR}(M) \\ & \nearrow_{h_P} & \\ \left(\bigvee\mathfrak{g}^{*}\right)_I & & \end{array}$$

for the Lie algebroid $A(P)$ of a principal bundle P (provided only that P is connected) means that the Chern-Weil homomorphism of a Lie algebroid is some generalization of this notion known on the ground of principal bundles. On the other hand, this also means that the Chern-Weil homomorphism of a principal bundle is a characteristic feature of its Lie algebroid (for connected principal bundles).

We give two applications of the homomorphism obtained:

- the transitive case is used for TC-foliations, especialy, for the foliations of left cosets of Lie groups by nonclosed connected Lie subgroups (chapters 6 and 7),
- the nontransitive case - for vector bundles over foliated manifolds (section 5.7).

4. Chapters 6 and 7 concern transversally complete foliations. We start with giving a precise construction of the Lie algebroid $A(M;\mathcal{F})$ of a TC-foliation $(M;\mathcal{F})$. Next, we explain the geometric signification of connections in $A(M;\mathcal{F})$:

Let E and E_b be the distributions tangent to the foliation $\mathcal{F}$ and to the basic foliation $\mathcal{F}_b$, respectively. Connections in A are in the correspondence to the C^∞ distributions $\bar{C}\subset TM$ satisfying the conditions : (1) $\bar{C}+E_b=TM$, (2) $\bar{C}\cap E_b=E$, (3) an arbitrarily taken vector belonging to $\bar{C}$ is the value of some foliate vector field having all values in $\bar{C}$ [in the case of left cosets of a connected Lie group G by a connected Lie subgroup $H\subset G$, condition (3) is equivalent to : (3') $\bar{C}$ is $\bar{H}$-right-invariant].

In particular, such a distribution $\bar{C}$ always exists. A connection in A is flat if and only if the corresponding distribution in TM is completely integrable. Thus the nontriviality of the Chern-Weil homomorphism of $A(M;\mathcal{F})$ means that then there exists no completely integrable distribution $\bar{C}\subset TM$ satisfying conditions (1)-(3) above. In chapter 7 we give a wide class of transversally complete foliations for which the Chern-Weil homomorphisms of the corresponding Lie algebroids are nontrivial. It will be some class of foliations of left cosets of Lie groups by nonclosed connected Lie subgroups. As a preparation in this direction we give (Th.7.4.2):

> *Let $H\subset G$ be any connected Lie subgroup of G and let h, $\bar{h}$ and g be the Lie algebras of H, of its closure $\bar{H}$ and of G, respectively. Let $A(G;H)$ be the Lie algebroid of the foliation of left cosets of G by H. Denote by*

$h_P : \left(\bigvee \bar{\mathfrak{h}}^*\right)_I \to H_{dR}\left(G_{/\bar{H}}\right)$ the Chern-Weil homomorphism of the $\bar{H}$-principal bundle $P = \left(G \to G_{/\bar{H}}\right)$. Then there exists an isomorphism of algebras ρ such that the following diagram commutes:

$$\begin{array}{ccc} \bigoplus^{k\geq 0}\left(\operatorname{Sec}\bigvee^k \boldsymbol{g}^*\right)_{I^0} & \overset{h_{A(G;H)}}{\longrightarrow} & H_{dR}\left(G_{/\bar{H}}\right) \\ \cong\downarrow \rho & & \uparrow h_P \\ \left(\bigvee\left(\bar{\mathfrak{h}}_{/\mathfrak{h}}\right)^*\right) & \rightarrowtail & \left(\bigvee \bar{\mathfrak{h}}^*\right)_I \end{array}$$

Because of the well-known fact that, under the assumption that G is a connected, compact and semisimple Lie group,

$$(h_P)^2 : \left(\bar{\mathfrak{h}}^*\right)_I \overset{\cong}{\longrightarrow} H^2_{dR}\left(G_{/\bar{H}}\right)$$

is an isomorphism, we assert, thanks to the diagram above, that $h_{A(G;H)}$ is nontrivial. This means that then there exists no C^∞ completely integrable distribution $\bar{C} \subset TG$ such that (1) $\bar{C} + E_b = TG$, (2) $\bar{C} \cap E_b = E$, (3) $\bar{C}$ is $\bar{H}$-right-invariant.

As a corollary we also obtain that (Cor.7.4.8) :

No Lie subalgebra $c \subset g$ satisfying (1) $c + \bar{h} = g$, (2) $c \cap \bar{h} = h$ exists. (Such a Lie subalgebra determines some flat connection in $A(G;H)$).

Adding the simple connectedness to the assumption about G, we get, according to the Almeida-Molino theorem, some nonintegrable transitive Lie algebroid having the nontrivial Chern-Weil homomorphism.

0.2 PRELIMINARIES

We assume that in our work all the manifolds considered, are of the C^∞-class and Hausdorff, and that the manifolds M, M', ... over which we have Lie algebroids are, in addition, connected. By $\Omega^0(M)$ we denote the ring of C^∞ functions on a manifold M, by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M, and by $\operatorname{Sec} A$ the $\Omega^0(M)$-module of all C^∞ global cross-sections of a given vector bundle A (over M).

Denote by $\mathfrak{F}$ the category of couples (M, E) consisting of a manifold M and a C^∞ constant dimensional and involutive distribution $E \subset TM$. A morphism $f : (M', E') \to (M, E)$ in $\mathfrak{F}$ from (M', E') to (M, E) is a C^∞ mapping $f : M' \to M$ such that $f_*[E'] \subset E$.

Let (M, E) be an object of the category $\mathfrak{F}$, and $\mathfrak{f}$ any vector bundle on M. Each element of

$$\Omega_E(M;\mathfrak{f}) = \bigoplus^{k\geq 0} \Omega^k_E(M;\mathfrak{f}), \text{ where } \Omega^k_E(M;\mathfrak{f}) := \operatorname{Sec}\bigwedge^k E^* \bigotimes \mathfrak{f}$$

is called a (C^∞) tangential differential form on (M,E) with values in $\mathfrak{f}$, while, for the trivial vector bundle $\mathfrak{f}=M\times\mathbb{R}$, briefly a ($C^\infty$ real) *tangential differential form* on (M,E) (for that, see [28]). The space of tangential differential forms on (M,E) will be denoted by $\Omega^0(M)$.

There is an obvious differential d^E of degree $+1$ in $\Omega^0_E(M)$ which can be defined in an elementary way in terms of local coordinates [28] or, equivalently, by the global formula:

$$d^E(\Theta)(X_0,...,X_k)=\sum_j(-1)^j X_j\left(\Theta\left(X_0,...\hat{j}...X_k\right)\right)+$$
$$+\sum_{i<j}(-1)^{i+j}\Theta\left([X_i,X_j],X_0,...\hat{i}...\hat{j}...X_k\right)$$

(for $\Theta\in\Omega^k_E(M;\mathfrak{f})$). We evidently have $\left(d^E\right)^2=0$. The *tangential cohomology space* $H_E(M)$ of (M,E) is, by definition, the cohomology space of the complex $(\Omega^0(M),d)$. If $E=TM$, then $H_E(M)$ is the de Rham cohomology space $H_{dR}(M)$ of M.

For a morphism $f:(M',E')\to(M,E)$ of $\mathfrak{F}$ and a vector bundle $\mathfrak{f}$ on M, we can define, in a standard way, the pullback of forms $f^*:\Omega_E(M;\mathfrak{f})\to\Omega_{E'}(M';f^*\mathfrak{f})$.

The usual law of the commuting of f^* with the differentiation of real-valued forms holds:

$$f^*\circ d^E=d^{E'}\circ f^*.$$

Let $\mathfrak{f}^1,...,\mathfrak{f}^k,\mathfrak{f}$ be vector bundles over M. An arbitrary k-linear homomorphism of vector bundles $\varphi:\mathfrak{f}^1\times...\times\mathfrak{f}^k\to\mathfrak{f}$ determines the mapping

$$\varphi_*:\Omega_E\left(M;\mathfrak{f}^1\right)\times...\times\Omega_E\left(M;\mathfrak{f}^k\right)\longrightarrow\Omega_E(M;\mathfrak{f})$$

defined by the standard formula

$$\varphi_*(\Theta_1,...,\Theta_k)(x;v_1\wedge...\wedge v_m)$$
$$=\frac{1}{q_1!\cdot...\cdot q_k!}\sum_\sigma \operatorname{sgn}\sigma\cdot\varphi\left(\Theta_1\left(x;v_{\alpha(1)}\wedge...\right),...,\Theta_k\left(x;...v_{\sigma(m)}\right)\right)$$

in which $m=\sum q_i$ where q_i is the degree of $\Theta_i\in\Omega_E\left(M;\mathfrak{f}^i\right)$.

Sometimes, the form $\varphi_*(\Theta_1,...,\Theta_k)$ will be denoted in other ways:

(a) for forms of degree 0 (i.e. for cross-sections of the vector bundles $\mathfrak{f}^i$), by $\varphi(\Theta_1,...,\Theta_k)$

(b) for the standard homomorphisms $\bigotimes^k:\mathfrak{f}\times...\times\mathfrak{f}\to\bigotimes^k\mathfrak{f}$, $\bigvee^k:\mathfrak{f}\times...\times\mathfrak{f}\to\bigvee^k\mathfrak{f}$ by $\Theta_1\otimes...\otimes\Theta_k$ and $\Theta_1\vee...\vee\Theta_k$ respectively;

(c) for the duality $\langle\cdot,\cdot\rangle:\bigvee^k\mathfrak{f}^*\times\bigvee^k\mathfrak{f}\to\mathbb{R}$, [8], by $\langle\Theta_1,\Theta_2\rangle$, etc.

1 THE CATEGORY OF REGULAR LIE ALGEBROIDS

1.1 The category of regular Lie algebroids.

Definition 1.1.1 *(see [29], [30]). By a Lie algebroid on a manifold M we mean a system*

$$A = (A, [\![\cdot,\cdot]\!], \gamma) \tag{1.1}$$

consisting of a vector bundle A (over M) and mappings

$$[\![\cdot,\cdot]\!] : \operatorname{Sec} A \times \operatorname{Sec} A \longrightarrow \operatorname{Sec} A, \quad \gamma : A \longrightarrow TM,$$

such that

(i) $((\operatorname{Sec} A, [\![\cdot,\cdot]\!]))$ *is an $\mathbb{R}$-Lie algebra,*

(ii) γ, *called by K.Mackenzie [23] an A nchor, is a homomorphism of vector bundles,*

(iii) $\operatorname{Sec}\gamma : \operatorname{Sec}\gamma \longrightarrow \mathfrak{X}(M)$, $\xi \longmapsto \gamma\circ\xi$, *is a homomorphism of Lie algebras,*

(iv) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$ *for* $f\in\Omega^0(M)$, $\xi,\eta\in\operatorname{Sec} A$.

Lie algebroid 1.1 is called

(a) *regular* if γ is a constant rank; then $E := \operatorname{Im}\gamma$ is, of course, C^∞ constant dimensional and completely integrable distribution, 1.1 is then also called a *Lie algebroid over* (M, E). $\boldsymbol{g} = \ker\gamma$ is a vector bundle, called the adjoint of 1.1, and the short exact sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\ \gamma\ } E \longrightarrow 0 \tag{1.2}$$

is called the Atiyah sequence of 1.1;

(b) *transitive* if γ is an epimorphism.

The concept of a Lie algebroid enables one to make many generalizations [15], [22].

Let 1.1 be a regular Lie algebroid. In each vector space $\boldsymbol{g}_{|x}$ $(= \ker\gamma_{|x})$, $x\in M$, some Lie algebra structure is defined by

$$[v,w] := [\![\xi,\eta]\!](x), \quad \xi,\eta\in\operatorname{Sec} A,\ \xi(x)=v,\ \eta(x)=w,\ v,w\in\boldsymbol{g}_{|x}.$$

$\boldsymbol{g}_{|x}$ is called the *isotropy Lie algebra* of 1.1 at x. For transitive Lie algebroid 1.1, $\boldsymbol{g}$ is a Lie algebra bundle [2], [19], [23].

Example 1.1.2 *The following are important examples of transitive Lie algebroids:*

(1^0) the Lie algebroid $A(P) = TP_{/G}$ of a G-principal bundle P, see [16], [19], [23],

(2⁰) the Lie algebroid CDO $(\mathfrak{f})$ *of covariant differential operators on a vector bundle* $\mathfrak{f}$ *, see [23],*

(3⁰) the Lie algebroid $i^*(T^\alpha\Phi)$ *of a Lie groupoid* Φ*, see [13], [30],*

(4⁰) the Lie algebroid $A(M;\mathcal{F})$ *of a transversally complete foliation* $(M;\mathcal{F})$*, see [26], [27]; in particular,*

(5⁰) the Lie algebroid $A(G;H)$ *of the foliation of left cosets of a Lie group* G *by a nonclosed connected Lie subgroup* $H\subset G$*, see [20], [27],*

(6⁰) the Lie algebroid of some pseudogroups, see [31].

The following are examples of nontransitive (in general) Lie algebroids:

(1⁰) the Lie algebroid $i^*(T^\alpha\Phi)$ of a differential groupoid Φ, see [12], [29], [30],

(2⁰) the Lie algebroid of a Poisson manifold, see [4], [5],

(3⁰) the regular Lie algebroid $A^E=\gamma^{-1}[E]\subset A$ defined by transitive Lie algebroid 1.1 and an involutive distribution $E\subset TM$ (for example, a Lie groupoid (or a vector bundle) over a foliated manifold determines such an object).

Definition 1.1.3 (24) *Let 1.1 and* $A'=(A',[\![\cdot,\cdot]\!]',\gamma')$ *be two Lie algebroids (even not necessarily regular) on manifolds* M *and* M'*, respectively. By a homomorphism*

$$H:(A',[\![\cdot,\cdot]\!]',\gamma')\longrightarrow(A,[\![\cdot,\cdot]\!],\gamma) \tag{1.3}$$

between them we mean a homomorphism of vector bundles $H:A'\to A$*, say, over* $f:M'\to M$*, such that,*

(a) $\gamma\circ H=f_*\circ\gamma'$*,*

(b) for arbitrary cross-sections $\xi,\xi'\in\operatorname{Sec}A$ *with H-decompositions*

$$H\circ\xi=\sum_i f^i\cdot(\eta_i\circ f),\quad H\circ\xi'=\sum_j f'^j\cdot\left(\eta'_j\circ f\right),$$

$f^i,f'^j\in\Omega^0(M')$, $\eta_i,\eta'_j\in\operatorname{Sec}A$*, we have*

$$H\circ[\![\xi,\xi']\!]'=\sum_{i,j}f^i\cdot f'^j\cdot[\![\eta_i,\eta'_j]\!]\circ f+\sum_i(\gamma'\circ\xi)\left(f'^j\right)\cdot\eta'_j\circ f-$$
$$-\sum_j(\gamma'\circ\xi')\left(f^i\right)\circ\eta_i\circ f.$$

In the case of Lie algebroids A and A' on the same manifold M, a strong homomorphism $H:A'\to A$ of vector bundles is a homomorphism of Lie algebroids if and only if

(1') $\gamma \circ H = \gamma'$,

(2') $\operatorname{Sec} H : \operatorname{Sec} A' \to \operatorname{Sec} A$, $\xi \mapsto H \circ \xi$, is a homomorphism of Lie algebras.

Indeed, " $\Rightarrow$ " is trivial.

" $\Leftarrow$ " Let $H \circ \xi = \sum_i f^i \cdot \eta_i$ and $H \circ \xi' = \sum_j f'^j \cdot \eta'_j$ be H-decompositions of $\xi, \xi' \in \operatorname{Sec} A'$. Then

$$\begin{aligned}
&H \circ [\![\xi, \xi']\!]' \\
&= [\![H \circ \xi, H \circ \xi']\!] = [\![\sum_i f^i \cdot \eta_i, \sum_j f'^j \cdot \eta'_j]\!] \\
&= \sum_{i,j} f^i \cdot f'^j \cdot [\![\eta_i, \eta'_j]\!] + \sum_{i,j} f^i \cdot (\gamma \circ \eta_i) \left(f'^j\right) \cdot \eta'_j - \sum_{i,j} f'^j \cdot \left(\gamma \circ \eta'_j\right) \left(f^i\right) \cdot \eta_i \\
&= \sum_{i,j} f^i \cdot f'^j \cdot [\![\eta_i, \eta'_j]\!] + \sum_i (\gamma' \circ \xi) \left(f'^j\right) \cdot \eta'_j - \sum_j (\gamma' \circ \xi') \left(f^i\right) \circ \eta_i.
\end{aligned}$$

If homomorphism 1.3 is a bijection, then H^{-1} is also a homomorphism of Lie algebroids; then H is called an *isomorphism of Lie algebroids.*

Below, we represent each nonstrong homomorphism 1.3 of regular Lie algebroids over $f : (M', E') \to (M, E)$ as a superposition of some strong homomorphism $\bar{H} : A' \to f\hat{\ } A$ with the canonical nonstrong one $\varkappa : f\hat{\ } A \to A$ where $f\hat{\ } A$ is the so-called inverse-image of A over f. The term "*inverse-image of A over f*" appears in work [24] by K.Mackenzie, but in the sense not quite helpful here (for example, Mackenzie's definition, although it is general enough, ensures neither the existence of the inverse-image of A nor its regularity for a regular Lie algebroid A). For the sake of completness, we add that the two definitions, 1.1.4 below and 1.4 from [24], are equivalent on the ground of transitive Lie algebroids.

Definition 1.1.4 *Let 1.1 be a regular Lie algebroid over (M, E) and let $f : (M', E') \to (M, E)$ be a morphism of the category $\mathfrak{F}$. The inverse-image of A by f is a regular Lie algebroid over (M', E')*

$$\left(f\hat{\ } A, [\![\cdot, \cdot]\!], \mathrm{pr}_1\right) \tag{1.4}$$

in which

(i)

$$f\hat{\ } A = E' \times_{(f_*, \gamma)} A = \{(v, w) \in E' \times A;\ f_*(v) = \gamma(w)\} \subset E' \bigoplus f^* A$$

($f\hat{\ } A$ is a submanifold of $E' \bigoplus f^* A$ because $f_* \times \gamma : E' \times A \to E \times E$ is transverse to the diagonal $\Delta \subset E \times E$, and $f\hat{\ } A = (f_*, \gamma)^{-1} [\Delta]$,

(ii) the bracket $[\![\cdot, \cdot]\!]$ in $\operatorname{Sec} f\hat{\ } A$ is defined in the following way: Let $\left(X_i, \bar{\xi}_i\right) \in \operatorname{Sec} f\hat{\ } A$, $i = 1, 2$ (where $X_i \in \operatorname{Sec} E'$, $\bar{\xi}_i \in \operatorname{Sec} f\hat{\ } A$). Then, locally (say

on $U \subset M'$), $\bar{\xi}_i$ is of the form $\sum_j g_i^j \cdot \xi_i^j \circ f$ for some $g_i^j \in \Omega_0(M')$ and $\xi_i^j \in \operatorname{Sec} A$, and we put

$$\begin{aligned}
&[\![(X_1, \bar{\xi}_1), (X_2, \bar{\xi}_2)]\!]_{|U} \\
&= ([X_1, X_2], \sum_{j,k} g_1^j \cdot g_2^k \cdot [\![\xi_1^j, \xi_2^k]\!] \circ f + \sum_k X_1(g_2^k) \cdot \xi_2^k \circ f - \\
&- \sum_i X_2 (g_1^j) \cdot \xi_1^j \circ f)_{|U}.
\end{aligned}$$

The correctness of this definition. By antisymmetry, it is sufficient to show that $\sum_{j,k} g_1^j \cdot g_2^k \cdot [\![\xi_1^j, \xi_2^k]\!] \circ f + \sum_k X_1(g_2^k) \cdot \xi_2^k \circ f$

is independent of the choice of the decomposition for $\bar{\xi}_2$. Consider simultaneously the 2-linear function $F : \Omega^0(M') \times \operatorname{Sec} A \to \operatorname{Sec} f^{\hat{}} A$ given by

$$F(g, \xi) = \sum_j g_1^j \cdot g \cdot [\![\xi_1^j, \xi]\!] \circ f + X_1(g) \cdot \xi \circ f, \quad g \in \Omega^0(M'), \ \xi \in \operatorname{Sec} A.$$

Clearly $\sum_{j,k} g_1^j \cdot g_2^k \cdot [\![\xi_1^j, \xi_2^k]\!] \circ f + \sum_k X_1(g_2^k) \cdot \xi_2^k \circ f = \sum_k F(g_2^k \cdot \xi_2^k)$. For $t \in \Omega^0(M)$, by standard calculations and thanks to the assumption that $f_*(X_1(x)) = \gamma(\bar{\xi}_1(f(x)))$, one can easily notice that (cf. Lemma 1.4 from [24])

$$F(g, t \cdot \xi) = F(g \cdot (t \circ f), \xi).$$

To prove the examined independence, take two decompositions $\bar{\xi}_2 = \sum_k g_2^k \cdot \xi_2^k \circ f = \sum_r \tilde{g}_2^r \cdot \tilde{\xi}_2^r \circ f$. For a point $x \in M'$, let ν_s be a local basis of the module $\operatorname{Sec} A$ around $f(x)$ and let $\xi_2^k = \sum_s h_k^s \cdot \nu_s$, $\tilde{\xi}_2^r = \sum_s \tilde{h}_r^s \cdot \nu_s$ (around $f(x)$), $h_k^s, \tilde{h}_r^s \in \Omega^0(M)$; then, around x we have $\sum_k g_2^k \cdot h_k^s \circ f = \sum_r \tilde{g}_2^r \cdot \tilde{h}_r^s \circ f$ for each s. Therefore, in the end, we obtain

$$\begin{aligned}
&\sum_k F(g_2^k \cdot \xi_2^k) \\
&= \sum_k F\left(g_2^k, \sum_s h_k^s \cdot \nu_s\right) = \sum_s F\left(\sum_k g_2^k \cdot h_k^s \circ f, \nu_s\right) \\
&= \sum_s F\left(\sum_r \tilde{g}_2^r \cdot \tilde{h}_r^s \circ f, \nu_s\right) = \sum_r F\left(\tilde{g}_2^r, \sum_s \tilde{h}_r^s \cdot \nu_s\right) \\
&= \sum_r F\left(\tilde{g}_2^r, \tilde{\xi}_2^r\right).
\end{aligned}$$

The Atiyah sequence of the inverse-image $f^{\hat{}} A$ of A is

$$0 \longrightarrow f^* \boldsymbol{g} \longrightarrow f^{\hat{}} A \xrightarrow{\operatorname{pr}_1} E' \longrightarrow 0$$

(identify $f^* \boldsymbol{g}$ with $0 \bigoplus f^* \boldsymbol{g}$).

Clearly,

$$\varkappa = \operatorname{pr}_2 : f^{\hat{}} A \longrightarrow A$$

is a homomorphism of regular Lie algebroids.

Proposition 1.1.5 *Let A and A' be regular Lie algebroids over (M,E) and (M',E'), respectively. Let $H : A' \to A$ be a homomorphism of vector bundles over $f : (M',E') \to (M,E)$. Then H is a homomorphism of Lie algebroids if and only if*

(1) $\gamma \circ H = f_* \circ \gamma'$,

(2) $\bar{H} : A' \to f^{\hat{}} A$, $\nu \mapsto (\gamma'(v), H(v))$, is a strong homomorphism of Lie algebroids.

Proof. The very easy proof will be omitted. ■

According to this proposition, each nonstrong homomorphism of regular Lie algebroids is canonically represented as the superposition

$$H : A' \xrightarrow{\bar{H}} f^{\hat{}} A \xrightarrow{\varkappa} A. \tag{1.5}$$

In the case of regular Lie algebroids, each homomorphism 1.3 determines a homomorphism of the associated Atiyah sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{g}' & \hookrightarrow & A' & \xrightarrow{\gamma'} & E' & \longrightarrow & 0 \\ & & \downarrow H^+ & & \downarrow H & & \downarrow f_* & & \\ 0 & \longrightarrow & \boldsymbol{g} & \hookrightarrow & A & \xrightarrow{\gamma} & E & \longrightarrow & 0 \end{array}$$

(H^+ is the restricted homomorphism of the adjoint vector bundles and $H^+_{|x} : \boldsymbol{g}'_{|x} \to \boldsymbol{g}_{|f(x)}$, $x \in M$, is a homomorphism of Lie algebras).

An example of a nonstrong (in general) homomorphism of regular Lie algebroids is the tangent mapping $f_* : E' \to E$ to any C^∞ morphism $f : (M',E') \to (M,E)$ of the category $\mathfrak{F}$, cf. [24].

All regular Lie algebroids and all homomorphisms between them form a category fundamental in our considerations.

Lemma 1.1.6 *Let A and B be two regular Lie algebroids over (M,E), $H : A \to B$ a strong homomorphism, and $f : (M',E') \to (M,E)$ any morphism of $\mathfrak{F}$. Then the mapping $f^{\hat{}} H : f^{\hat{}} A \to f^{\hat{}} B$, $(u,v) \mapsto (u, H(v))$, is a strong homomorphism of regular Lie algebroids.*

Proof. Of course, $\mathrm{pr}_1 \circ f^{\hat{}} H = \mathrm{pr}_1$. To prove that $\mathrm{Sec}\, f^{\hat{}} H$ is a homomorphism of Lie algebras, take two cross-sections $\xi, \eta \in \mathrm{Sec}\, f^{\hat{}} A$, $\xi = \left(X, \sum_i f^i \cdot \xi_i \circ f\right)$, $\eta = \left(Y, \sum_j g^j \cdot \xi_j \circ f\right)$, and calculate

$$\begin{aligned} & f^{\hat{}} H \circ [\![\xi, \eta]\!] \\ & = f^{\hat{}} H \circ ([\![(X, \sum_i f^i \cdot \xi_i \circ f), (Y, \sum_j g^j \cdot \xi_j \circ f)]\!]) \\ & = ([X,Y], \sum_{i,j} f^i \cdot g^j \cdot [\![H \circ \xi_i, H \circ \xi_j]\!] \circ f + \sum_i \left(X\left(g^i\right) - Y\left(f^i\right)\right) \cdot H \circ \xi_i \circ f) \\ & = [\![(X, \sum_i f^i \cdot H \circ \xi_i \circ f), (Y, \sum_j g^j \cdot H \circ \xi_j \circ f)]\!] \\ & = [\![f^{\hat{}} H \circ \xi, f^{\hat{}} H \circ \eta]\!]. \end{aligned}$$

■

$f^{\hat{}}H$ is called the *inverse-image of H over f.*

1.2 The Lie algebroid $A(\mathfrak{f})$ of a vector bundle $\mathfrak{f}$

Definition 1.2.1 *Let $\mathfrak{f}$ be any vector bundle on a manifold M, with a vector space V as the typical fibre. A linear homomorphism $l : \mathrm{Sec}\,\mathfrak{f} \to \mathfrak{f}_{|x}$ is called an $\mathfrak{f}$-vector tangent at x if and only if there exists a vector $u \in T_xM$ such that*

$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu(x)$$

for all $f \in \Omega^0(M)$ and $\nu \in \mathrm{Sec}\,\mathfrak{f}$.

The vector u determined uniquely by l, is called the *anchor* of l and denoted by $q(l)$. All $\mathfrak{f}$-vectors tangent at x form a vector space $A(\mathfrak{f})_{|x}$. Put $A(\mathfrak{f}) = \coprod_{x\in M} A(\mathfrak{f})_{|x}$ (a disjoint sum) and let $p : A(\mathfrak{f}) \to M$ be the canonical projection. Clearly, each $\mathfrak{f}$-vector l is factorized by some linear mapping $\tilde{l}$ from the space of 1-jets at x:

$$\begin{array}{ccc} \mathrm{Sec}\,\mathfrak{f} & \longrightarrow & \left(J^1\mathfrak{f}\right)_{|x} \\ & \searrow l & \downarrow \tilde{l} \\ & & \mathfrak{f}_{|x}, \end{array}$$

and the mapping just obtained $A(\mathfrak{f}) \to \mathrm{Hom}\left(J^1\mathfrak{f};\mathfrak{f}\right)$, $l \mapsto \tilde{l}$, is a monomorphism on each fibre. One can prove [23] that the image of this mapping, equalling $\mathrm{CDO}\,\mathfrak{f}$, is a vector subbundle of $\mathrm{Hom}\left(J^1\mathfrak{f};\mathfrak{f}\right)$. Via this mapping we shall identify $A(\mathfrak{f})$ with $\mathrm{CDO}\,\mathfrak{f}$ to obtain a transitive Lie algebroid with $q : A(\mathfrak{f}) \to TM$, $l \mapsto q(l)$, as the anchor. A cross-section $\xi \in \mathrm{Sec}\,A(\mathfrak{F})$ defines a differential operator $\mathcal{L}_\xi$ in $\mathfrak{f}$ by the formula:

$$\mathcal{L}_\xi(\nu)(x) = \xi_x(\nu), \ \ \nu \in \mathrm{Sec}\,\mathfrak{f},\ x \in M,$$

being a *covariant differential operator in* $\mathfrak{f}$. Besides, each covariant differential operator in $\mathfrak{f}$ is of the form $\mathcal{L}_\xi$ for exactly one cross-section $\xi \in \mathrm{Sec}\,A(\mathfrak{f})$. The bracket $[\![\cdot,\cdot]\!]$ of cross-sections of $A(\mathfrak{f})$ is defined in the classical - for differential operators - manner, i.e. for $\xi,\eta \in \mathrm{Sec}\,A(\mathfrak{f})$, $[\![\xi,\eta]\!]$ is a cross-section of $A(\mathfrak{f})$ such that $\mathcal{L}_{[\![\xi,\eta]\!]} = \mathcal{L}_\xi \circ \mathcal{L}_\eta - \mathcal{L}_\eta \circ \mathcal{L}_\xi$. The Atiyah sequence of $A(\mathfrak{f})$ is

$$0 \longrightarrow \mathrm{End}\,\mathfrak{f} \overset{i}{\hookrightarrow} A(\mathfrak{f}) \overset{q}{\longrightarrow} TM \longrightarrow 0$$

(and $\mathcal{L}_{i\circ\xi}(\nu) = \xi(\nu)$ for $\xi\in(\mathfrak{f})$, where $\xi(\nu)\in\mathrm{Sec}\,\mathfrak{f}$ is defined by $\xi(\nu)(x) = \xi_x(\nu)$, $x \in M$).

Take now a vector bundle $\mathfrak{f}$ on M and a mapping $f : M' \to M$. Consider the inverse-image $f^{\hat{}}(A(\mathfrak{f}))$ $(= TM'_{(f_*,q)}A(\mathfrak{f})\ )$ of $A(\mathfrak{f})$.

Lemma 1.2.2 *For $x \in M'$ and $(u,l) \in f^{\hat{}}(A(\mathfrak{f}))$, there exists exactly one element $w \in A(\mathfrak{f})_{|x}$ with the anchor u, such that $w(\nu\circ f) = l(\nu)$, $\nu\in\mathrm{Sec}\,\mathfrak{f}$. The correspondence $(u,l) \mapsto w$ establishes a strong isomorphism*

$$c_{\mathfrak{f}} : f^{\hat{}}(A(\mathfrak{f})) \longrightarrow A(f^*\mathfrak{f})$$

of transitive Lie algebroids.

Proof. Let $x \in M'$ and $(u,l) \in f^{\hat{}}\,(A(\mathfrak{f}))_{|x}$, i.e. $u \in T_xM'$, $l \in A(\mathfrak{f})_{|f(x)}$ and

$$f_*(u) = q(l). \tag{1.6}$$

The uniqueness of an element $w \in A(\mathfrak{f})_{|x}$ with the anchor u, such that $w(\nu \circ f) = l(\nu)$, $\nu \in \operatorname{Sec}\mathfrak{f}$, is evident. As to the existence of such an element, we notice that any cross-section $\tau \in \operatorname{Sec} f^*\mathfrak{f}$ can be represented (not uniquely) in the form $\tau = \sum_i f^i \cdot \nu_i \circ f$, $f^i \in \Omega^0(M')$, $\nu_i \in \operatorname{Sec}\mathfrak{f}$. Put

$$\tilde{w}(\tau) = \sum_i f^i(x) \cdot l(\nu_i) + u\left(f^i\right) \cdot \nu_i \circ f(x).$$

The correctness of this definition: Let $\tau = \sum_i f^i \cdot \nu_i \circ f = \sum_j g^j \cdot \tau_j \circ f$ (locally in some neighbourhood of x). Take an arbitrary basis $\mu_1, ..., \mu_n$ of cross-sections of $\mathfrak{f}$ around $f(x)$ and let $\nu_i = \sum_s \varphi_i^s \cdot \mu_s$, $\tau_j = \sum_s \psi_j^s \cdot \mu_s$. Therefore in a neighbourhood of x

$$\sum_i f^i \cdot \varphi_i^s \circ f = \sum_j g^j \cdot \psi_j^s \circ f, \quad s = 1, ..., n. \tag{1.7}$$

Equalities 1.6 and 1.7 yield

$$\begin{aligned}
&\sum_i f^i(x) \cdot l(\nu_i) + u\left(f^i\right) \cdot \nu_i \circ f(x) \\
&= \sum_s \left(\sum_i f^i(x) \cdot \varphi_i^s \circ f(x)\right) \cdot l(\mu_s) + \sum_s u\left(\sum_i f^i \cdot \varphi_i^s \circ f\right) \cdot \mu_s \circ f(x) \\
&= \sum_j g^j(x) \cdot l(\tau_j) + u\left(g^j\right) \cdot \tau_j \circ f(x).
\end{aligned}$$

[It is easy to see that $\tilde{w}$ is an $f^*\mathfrak{f}$-vector tangent at x (with the anchor u). Clearly, the mapping obtained $c_{\mathfrak{f}} : f^{\hat{}}\,(A(\mathfrak{f})) \to A(f^*\mathfrak{f})$, $(u,l) \mapsto w$, is a strong homomorphism of vector bundles. The smoothness of $c_{\mathfrak{f}}$ follows from the fact that $c_{\mathfrak{f}}$ maps a smooth cross-section to a smooth one: namely, $\left(X, \sum_i f^i \cdot \xi_i \circ f\right)$ is carried over to a cross-section η such that $\mathcal{L}_\eta(\nu \circ f) = \sum_i f^i \cdot \mathcal{L}_{\xi_i}(\nu) \circ f$, $\nu \in \operatorname{Sec}\mathfrak{f}$.

It remains to show that $c_{\mathfrak{f}}$ is a homomorphism of transitive Lie algebroids. Of course, $q \circ c_{\mathfrak{f}} = \mathrm{pr}_1$. To see that $\operatorname{Sec}(c_{\mathfrak{f}})$ is a homomorphism of Lie algebras, take two cross-sections $\xi, \eta \in \operatorname{Sec} f^{\hat{}}\,(A(\mathfrak{f}))$. They are (locally) of the form $\xi = \left(X, \sum_i f^i \cdot \xi_i \circ f\right)$, $\eta = \left(Y, \sum_j g^j \cdot \xi_j \circ f\right)$ for $f^i, g^j \in \Omega^0(M')$ and $\xi \in_i$

$\operatorname{Sec} A(\mathfrak{f})$. We calculate (for $\nu \in \operatorname{Sec}\mathfrak{f}$)

$$
\begin{aligned}
&\mathcal{L}_{[\![c_{\mathfrak{f}}\circ\xi, c_{\mathfrak{f}}\circ\eta]\!]}(\nu\circ f)\\
&=\mathcal{L}_{c_{\mathfrak{f}}\circ\xi}\circ\mathcal{L}_{c_{\mathfrak{f}}\circ\eta}(\nu\circ f)-\mathcal{L}_{c_{\mathfrak{f}}\circ\eta}\circ\mathcal{L}_{c_{\mathfrak{f}}\circ\xi}(\nu\circ f)\\
&=\sum_{i,j} f^i\cdot g^j\cdot\mathcal{L}_{[\![\xi_i,\eta_j]\!]}(\nu)\circ f+\sum_i\left(X\left(g^i\right)-Y\left(f^i\right)\right)\cdot\mathcal{L}_{\xi_i}(\nu)\circ f\\
&=\mathcal{L}_{c_{\mathfrak{f}}\circ[\![\xi,\eta]\!]}(\nu\circ f).
\end{aligned}
$$

■

2 REPRESENTATIONS OF LIE ALGEBROIDS ON VECTOR BUNDLES

2.1 Definition and fundamental examples.

Definition 2.1.1 (cf. [23, p.106]) *Let $\mathfrak{f}$ and (1) be any vector bundle and Lie algebroid (both over M), respectively. By a representation of A on $\mathfrak{f}$ we mean a strong homomorphism of Lie algebroids*

$$T: A\longrightarrow A(\mathfrak{f}). \tag{2.1}$$

2.1.1 Adjoint representation (defined by Mackenzie [23] for the transitive case)

One can trivially notice that if $\nu\in\operatorname{Sec}\boldsymbol{g}$, then, for $\xi\in\operatorname{Sec}A$, the value of $[\![\xi,\nu]\!]$ at x depends only on the value of ξ at x and belongs to $\boldsymbol{g}_{|x}$. In this way, it is the correctly defined element $[\![v,\nu]\!]\in\boldsymbol{g}_{|x}$ for $v\in A$ and $\nu\in\operatorname{Sec}\boldsymbol{g}$.

A very important representation is the so-called *adjoint representation* of a regular Lie algebroid A

$$\operatorname{ad}_A: A\longrightarrow A(\boldsymbol{g})$$

defined uniquely by the following property:

$$\operatorname{ad}_A(v)(\nu)=[\![v,\nu]\!],\quad v\in A,\quad \nu\in\operatorname{Sec}\boldsymbol{g}.$$

To see the existence of ad_A, we only need to notice that $\operatorname{Sec}\boldsymbol{g}\ni\nu\mapsto[\![v,\nu]\!]\in\boldsymbol{g}_{|x}$ is a $\boldsymbol{g}$-vector. The smoothness of ad_A is evident.

2.1.2 Contragredient representation

The contragredient representation of 2.1 is, by definition,

$$T^\natural: A\longrightarrow A(\mathfrak{f}^*)$$

such that $\langle\mathcal{L}_{T^\natural\circ\xi}(\varphi),\nu\rangle=(\gamma\circ\xi)\langle\varphi,\nu\rangle-\langle\varphi,\mathcal{L}_{T\circ\xi}(\nu)\rangle$, $\xi\in\operatorname{Sec}A$, $\varphi\in\operatorname{Sec}\mathfrak{f}^*$, $\nu\in\operatorname{Sec}\mathfrak{f}$.

2.1.3 Representations induced by a single one

A single representation $T : A \to A(\mathfrak{f})$ determines (as in the case of a representation of a Lie algebra in a vector space) a number of new ones. Among them we shall need the following ones:

- The symmetric product $\bigvee^k T$ of A on $\bigvee^k \mathfrak{f}$ as the one for which

$$\begin{aligned}&\mathcal{L}_{\bigvee^k T\circ\xi}\left(\nu^1 \vee ... \vee \nu^k\right)\\&=\sum_i \nu^1 \vee ... \vee \mathcal{L}_{T\circ\xi}\left(\nu^i\right) \vee ... \vee \nu^k, \quad \nu^i \in \operatorname{Sec}\mathfrak{f},\ \xi \in \operatorname{Sec} A.\end{aligned}$$

- The representation $\operatorname{Hom}^k(T)$ of A on the space of k-linear homomorphisms $\operatorname{Hom}^k(\mathfrak{f};\mathbb{R})$ as the one for which

$$\begin{aligned}&\mathcal{L}_{\operatorname{Hom}^k(T)\circ\xi}(\varphi)\left(\nu^1,...,\nu^k\right)\\&=(\gamma\circ\xi)\left(\varphi\left(\nu^1,...,\nu^k\right)\right)-\sum_i \varphi\left(\nu^1,...,\mathcal{L}_{T\circ\xi}\left(\nu^i\right),...,\nu^k\right),\end{aligned}$$

  for any k-linear homomorphism $\varphi : \mathfrak{f}\times ...\times\mathfrak{f}\to\mathbb{R}$ and for $\nu \in \operatorname{Sec}\mathfrak{f}$, $\xi \in \operatorname{Sec} A$.

Via the above, the given representation 2.1 determines $\bigvee^k T^\natural$ of A on the space $\bigvee^k f^*$.

Lemma 2.1.2 *The representation* $\bigvee^k T^\natural$ *is defined by the following formula:*

$$\begin{aligned}&\langle\mathcal{L}_{\bigvee^k T^\natural\circ\xi}\Gamma,\nu_1\vee...\vee\nu_k\rangle\\&=(\gamma\circ\xi)\langle\Gamma,\nu_1\vee...\vee\nu_k\rangle-\sum_i\langle\Gamma,\nu_1\vee...\vee\mathcal{L}_{T\circ\xi}(\nu_i)\vee...\vee\nu_k\rangle\end{aligned}$$

for $\Gamma\in\operatorname{Sec}\left(\bigvee^k f^*\right)$ *and* $\nu\in\operatorname{Sec}\mathfrak{f}$.

Proof. We need the following

Sublemma. Let, for a given matrix B, the symbol $\operatorname{perm}_i^j(B)$ denote the permanent of the matrix which arises from B by the eliminating of the i^{th} row and j^{th} column (for the definition of a permanent, see [8]). The following properties of the permanent of the matrix $B=\left[f_i^j; i,j\le k\right]$ hold:

- The expansion formula with respect to the i_o^{th} row or j_o^{th} column:

$$\operatorname{perm} B=\sum_{j=1}^k f_{i_o}^j\cdot\operatorname{perm}_{i_o}^j(B)=\sum_{i=1}^k f_i^{j_o}\cdot\operatorname{perm}_i^{j_o}(B),$$

- the law of differentiation :

$$X\left(\operatorname{perm} B\right)=\sum_{i,j}X\left(f_i^j\right)\cdot \operatorname{perm}_i^j\left(B\right)$$

where f_i^j and X are C^∞ functions and a vector field on a given manifold, respectively.

The very easy proof will be omitted.

Proof of Lemma (2.1.2). It is sufficient to show the equality for a cross-section Γ of the form $\Gamma = u^{*1}\vee ...\vee u^{*k}$, $u^{*j}\in \operatorname{Sec}\boldsymbol{g}$. Using the above sublemma, we obtain, for $\nu\in \operatorname{Sec}\mathfrak{f}$,

$$\begin{aligned}
&\langle \mathcal{L}_{\bigvee^k T^\natural\circ\xi}u^{*1}\vee ...\vee u^{*k},\nu_1\vee ...\vee\nu_k\rangle\\
&=\langle\sum_j u^{*1}\vee ...\vee \mathcal{L}_{T^\natural\circ\xi}u^{*j}\vee ...\vee u^{*k},\nu_1\vee ...\vee\nu_k\rangle\\
&=\sum_j \operatorname{perm}\begin{bmatrix}\langle u^{*1},\nu_1\rangle & ... & \langle \mathcal{L}_{T^\natural\circ\xi}u^{*j},\nu_1\rangle & ... & \langle u^{*k},\nu_1\rangle\\ \vdots & & \vdots & & \vdots\\ \langle u^{*1},\nu_k\rangle & ... & \langle \mathcal{L}_{T^\natural\circ\xi}u^{*j},\nu_k\rangle & ... & \langle u^{*k},\nu_k\rangle\end{bmatrix}\\
&=\sum_j \operatorname{perm}\begin{bmatrix}\langle u^{*1},\nu_1\rangle & ... & (\gamma\circ\xi)\,\langle u^{*j},\nu_1\rangle & ... & \langle u^{*k},\nu_1\rangle\\ \vdots & & \vdots & & \vdots\\ \langle u^{*1},\nu_k\rangle & ... & (\gamma\circ\xi)\,\langle u^{*j},\nu_k\rangle & ... & \langle u^{*k},\nu_k\rangle\end{bmatrix}\\
&-\sum_j \operatorname{perm}\begin{bmatrix}\langle u^{*1},\nu_1\rangle & ... & \langle u^{*j},\mathcal{L}_{T\circ\xi}\nu_1\rangle & ... & \langle u^{*k},\nu_1\rangle\\ \vdots & & \vdots & & \vdots\\ \langle u^{*1},\nu_k\rangle & ... & \langle u^{*j},\mathcal{L}_{T\circ\xi}\nu_k\rangle & ... & \langle u^{*k},\nu_k\rangle\end{bmatrix}\\
&=\sum_{i,j}\left(\gamma\circ\xi\right)\langle u^{*j},\nu_i\rangle\cdot\operatorname{perm}_i^j-\sum_{i,j}\langle u^{*j},\mathcal{L}_{T\circ\xi}\nu_i\rangle\cdot\operatorname{perm}_i^j\\
&=\left(\gamma\circ\xi\right)\operatorname{perm}\begin{bmatrix}\langle u^{*1},\nu_1\rangle & ... & \langle u^{*k},\nu_1\rangle\\ \vdots & & \vdots\\ \langle u^{*1},\nu_k\rangle & & \langle u^{*k},\nu_k\rangle\end{bmatrix}\\
&-\sum_i \operatorname{perm}\begin{bmatrix}\langle u^{*1},\nu_1\rangle & ... & \langle u^{*k},\nu_1\rangle\\ \vdots & & \vdots\\ \langle u^{*1},\mathcal{L}_{T\circ\xi}\nu_i\rangle & ... & \langle u^{*k},\mathcal{L}_{T\circ\xi}\nu_i\rangle\\ \vdots & & \vdots\\ \langle u^{*1},\nu_k\rangle & & \langle u^{*k},\nu_k\rangle\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&=\left(\gamma\circ\xi\right)\langle u^{*1}\vee ...\vee u^{*k},\nu_1\vee ...\vee\nu_k\rangle\\
&-\sum_i\langle u^{*1}\vee ...\vee u^{*k},\nu_1\vee ...\vee\mathcal{L}_{T\circ\xi}\nu_i\vee ...\vee\nu_k\rangle.
\end{aligned}$$

■

2.2 The inverse-image of a representation

Definition 2.2.1 *Let A be any regular Lie algebroid over (M,E), and $\mathfrak{f}$ any vector bundle over M, whereas $f:(M',E')\to(M,E)$ – any morphism of the category $\mathfrak{F}$. By the inverse-image of a representation*

$$T:A\to A(\mathfrak{f})$$

over f we mean the representation $f^{*}T:f^{\hat{}}(A)\to A(f^{*}\mathfrak{f})$ defined as the superposition

$$f^{*}T:f^{\hat{}}(A)\xrightarrow{f^{\hat{}}T}f^{\hat{}}A(\mathfrak{f})\xrightarrow{c_{\mathfrak{f}}}A(f^{*}\mathfrak{f})$$

where $c_{\mathfrak{f}}$ is the isomorphism described in Lemma 1.2.2 whereas $f^{\hat{}}T$ is the inverse-image of T over f, see Lemma 1.1.6.

Lemma 2.2.2 *The inverse-image of the adjoint representation is adjoint, i.e.*

$$f^{*}(\mathrm{ad}_{A})=\mathrm{ad}_{f^{\hat{}}A}.$$

Proof. It is enough to check the equality

$$f^{*}(\mathrm{ad}_{A})(u)(\nu\circ f)=\mathrm{ad}_{f^{\hat{}}A}(u)(\nu\circ f)$$

for $\nu\in\mathrm{Sec}\,\boldsymbol{g}$ and $u\in f^{\hat{}}A$. Write $u=(v,w)$ for $v\in E'$ and $w\in A$, see Def. 1.1.4. Then

$$\begin{aligned}f^{*}(\mathrm{ad}_{A})(u)(\nu\circ f)&=c_{\mathfrak{f}}\circ f^{\hat{}}\,\mathrm{ad}_{A}(v,w)(\nu\circ f)\\&=c_{\mathfrak{f}}(v,\mathrm{ad}_{A}(w))(\nu\circ f)=(\mathrm{ad}_{A}(w))(\nu)\\&=[\![w,\nu]\!]=[\![(v,w),(0,\nu\circ f)]\!]=\mathrm{ad}_{f^{\hat{}}A}(u)(\nu\circ f).\end{aligned}$$

■

Lemma 2.2.3 *Under the canonical identifications $f^{*}(\mathfrak{f}^{*})\cong(f^{*}\mathfrak{f})^{*}$, $f^{*}\left(\bigvee^{k}\mathfrak{f}\right)\cong\bigvee^{k}(f^{*}\mathfrak{f})$, the following equalities of representations hold:*

(a) $f^{*}\left(T^{\natural}\right)=(f^{*}T)^{\natural}$,

(b) $f^{*}\left(\bigvee^{k}T\right)=\bigvee^{k}(f^{*}T)$.

Proof. (a): Let $x\in M'$ and $(v,w)\in\left(f^{\hat{}}A\right)_{|x}$, i.e. $v\in E'_{|x}$, $w\in A_{|f(x)}$ and $f_{*}(v)=\gamma(w)$. Of course (by the uniqueness considered in Lemma 1.2.2), it is sufficient to show the equality

$$f^{*}\left(T^{\natural}\right)(v,w)(\nu^{*}\circ f)=(f^{*}T)^{\natural}(v,w)(\nu^{*}\circ f)$$

for $\nu^* \in \mathrm{Sec}\,\mathfrak{f}^*$. Both sides of the equality are elements of the space $f^*_{|f(x)}$ $(= (f^*\mathfrak{f}^*)_{|x})$, therefore, to show this, take arbitrary $u \in \mathfrak{f}$ and a cross-section $\nu \in \mathrm{Sec}\,\mathfrak{f}$, such that $\nu(f(x)) = u$.

$$
\begin{aligned}
&\langle f^* \left(T^\natural\right)(v,w)(\nu^* \circ f), u\rangle \\
&= \langle f^* \left(T^\natural\right)(v,w)(\nu^* \circ f), \nu(f(x))\rangle \\
&= \langle c_{(\mathfrak{f}^*)} \left(v, T^\natural(w)\right)(\nu^* \circ f), \nu \circ f(x)\rangle \\
&= \langle T^\natural(w)(\nu^*), \nu \circ f(x)\rangle \\
&= (\gamma(w))\langle \nu^*, \nu\rangle - \langle \nu^*(f(x)), T(w)(\nu)\rangle \\
&= f_*(v)\langle \nu^*, \nu\rangle - \langle \nu^*(f(x)), c_{\mathfrak{f}}(v, T(w))(\nu \circ f)\rangle \\
&= v(\langle \nu^* \circ f, \nu \circ f\rangle) - \langle \nu^*(f(x)), f^*T(v,w)(\nu \circ f)\rangle \\
&= \langle (f^*T)^\natural(v,w)(\nu^* \circ f), \nu \circ f(x)\rangle \\
&= \langle (f^*T)^\natural(v,w)(\nu^* \circ f), u\rangle.
\end{aligned}
$$

(b): Under the canonical identification $f^*\left(\bigvee^k \mathfrak{f}\right) = \bigvee^k (f^*\mathfrak{f})$, we have $\nu_1 \circ f \vee \ldots \vee \nu_k \circ f = (\nu_1 \vee \ldots \vee \nu_k) \circ f$ for $\nu_i \in \mathrm{Sec}\,\mathfrak{f}$. Since a cross-section $\nu \in \mathrm{Sec} \bigvee^k \mathfrak{f}$ is (locally) a linear combination of cross-sections of the form $\nu_1 \vee \ldots \vee \nu_k$, $\nu_i \in \mathrm{Sec}\,\mathfrak{f}$, we see (by the same argument as in (a) above) that it is sufficient to notice the following:

$$
\begin{aligned}
&f^*\left(\bigvee^k T\right)(v,w)((\nu_1 \vee \ldots \vee \nu_k) \circ f) \\
&= \bigvee^k T(w)(\nu_1 \vee \ldots \vee \nu_k) \\
&= \sum_i \nu_i(f(x)) \vee \ldots \vee T(w)(\nu_i) \vee \ldots \vee \nu_k(f(x)) \\
&= \bigvee^k (f^*T)(v,w)(\nu_1 \circ f \vee \ldots \vee \nu_k \circ f).
\end{aligned}
$$

■

2.3 Invariant cross-sections (cf. Mackenzie, [23, p.195])

Definition 2.3.1 *Let 2.1 be any representation of a regular Lie algebroid A over (M,E) on $\mathfrak{f}$. A cross-section $\nu \in \mathrm{Sec}\,\mathfrak{f}$ will be called invariant (or, more precisely, T-invariant , or, after Mackenzie, A-parallel) if $T(v)(\nu) = 0$ for all $v \in A$ and $\nu \in \mathrm{Sec}\,\mathfrak{f}$.*

Denote by $(\mathrm{Sec}\,\mathfrak{f})_{I^o(T)}$ (or briefly by $(\mathrm{Sec}\,\mathfrak{f})$ if it does not lead to confusion) the space of all T-invariant cross-sections of $\mathfrak{f}$. $(\mathrm{Sec}\,\mathfrak{f})_{I^o(T)}$ is an $\Omega^0_b(M,\mathcal{F})$-module where $\mathcal{F}$ is the foliation having E as its tangent bundle [$\Omega^0_b(M,\mathcal{F})$ being the ring of $\mathcal{F}$-basic functions].

One can prove (cf. [23]) that each invariant cross-section $\nu \in \mathrm{Sec}\,\mathfrak{f}$ with respect to a representation $T : A \to A(\mathfrak{f})$ of a transitive Lie algebroid A is uniquely determined by the value at one of the points of M.

Lemma 2.3.2 *Let $T : A \to A(\mathfrak{f})$ be a given representation of A on $\mathfrak{f}$. An element $\varphi \in \operatorname{Sec} \bigvee^k \mathfrak{f}$ determines a k-linear homomorphism $\tilde{\varphi} : \mathfrak{f} \times ... \times \mathfrak{f} \to \mathbb{R}$ by the formula : $\tilde{\varphi}(v_1, ..., v_k) = \langle \varphi, v_1, ..., v_k \rangle$. We have that φ is $\bigvee^k T^\natural$-invariant if and only if $\tilde{\varphi}$ is $\operatorname{Hom}^k(T)$-invariant.*

Proof. Follows directly from Lemma 2.1.2 and the definitions. ■

Lemma 2.3.3 *Let $T : A \to A(\mathfrak{f})$ be a given representation of A on $\mathfrak{f}$. Let $\Gamma_1 \in \operatorname{Sec} \bigvee^k \mathfrak{f}$ and $\Gamma_2 \in \operatorname{Sec} \bigvee^l \mathfrak{f}$ be $\bigvee^k T$- and $\bigvee^l T$-invariant cross-sections, respectively. Then the symmetric product $\Gamma_1 \vee \Gamma_2 \in \operatorname{Sec} \bigvee^{k+l} \mathfrak{f}$ is $\bigvee^{k+l} T$-invariant.*

Proof. Follows trivially from the equality

$$\left(\bigvee^{k+l} T\right)(v)(\Gamma_1 \vee \Gamma_2) = \left(\bigvee^k T(v)\right)(\Gamma_1) \vee \Gamma_2(x) + \Gamma_1(x) \vee \left(\bigvee^l T\right)(v)(\Gamma_2)$$

for $v \in A_{|x}$, $x \in M$; which can easily be checked by considering simple tensors $\Gamma_1 = \nu_1 \vee ... \vee \nu_k$, $\Gamma_2 = \nu_{k+1} \vee ... \vee \nu_{k+l}$, $\nu_i \in \operatorname{Sec} \mathfrak{f}$, only. ■

Theorem 2.3.4 *Let A be any regular Lie algebroid over (M, E), and $\mathfrak{f}$ any vector bundle over M, whereas $f : (M', E') \to (M, E)$ – any morphism of the category $\mathcal{F}$. For a representation $T : A \to A(\mathfrak{f})$, the linear mapping $f^* : \operatorname{Sec} \mathfrak{f} \to \operatorname{Sec} \mathfrak{f}^*$, $\nu \mapsto \nu \circ f$, can be restricted to the spaces of cross-sections invariant under T and f^*T, respectively:*

$$f^*_{I^o} : (\operatorname{Sec} \mathfrak{f})_{I^o(T)} \longrightarrow (\operatorname{Sec} f^* \mathfrak{f})_{I^o(f^*T)}.$$

Proof. Let $\nu \in (\operatorname{Sec} \mathfrak{f})_{I^o(T)}$ and $(v, w) \in f\hat{\ } A$. Then

$$\begin{aligned} f^*T(v, w)(\nu \circ f) &= c_{\mathfrak{f}} \circ F\hat{\ } T(v, w)(\nu \circ f) \\ &= c_{\mathfrak{f}}(v, T(w))(\nu \circ f) = T(w)(\nu) = 0. \end{aligned}$$

■

3 CONNECTIONS IN REGULAR LIE ALGEBROIDS

In this chapter we fix a regular Lie algebroid (1.1) over $(M, E) \in \mathfrak{F}$ with the Atiyah sequence 1.2.

3.1 Connections, curvature and partial exterior covariant derivatives.

Definition 3.1.1 *By a connection in A we mean a homomorphism of vector bundles $\lambda : E \to A$ such that $\gamma \circ \lambda = \mathrm{id}_E$. The uniquely determined homomorphism $\omega : A \to \boldsymbol{g}$ such that $\omega|\boldsymbol{g} = \mathrm{id}$ and $\omega|\operatorname{Im}\lambda = 0$ is called the connection* *form of* λ. *The projection* $H : A \to A$ *onto the second component with respect to the decomposition* $A = \boldsymbol{g} \bigoplus C$, $C := \operatorname{Im}\lambda$, *is the horizontal projection. By the curvature tensor of a connection* λ *we shall mean the form* $\Omega_b \in \Omega_E^2(M, \boldsymbol{g})$ *defined by*

$$\Omega_b(X_1, X_2) = -\omega([\![\lambda \circ X_1, \lambda \circ X_2]\!]), \quad X_i \in \operatorname{Sec} E,$$

or, equivalently, by

$$\Omega_b(X_1, X_2) = \lambda \circ [X_1, X_2] - [\![\lambda \circ X_1, \lambda \circ X_2]\!], \quad X_i \in \operatorname{Sec} E. \tag{3.1}$$

A given connection λ in A determines the so-called *partial exterior covariant derivative* $\nabla : \Omega_E(M, \boldsymbol{g}) \to \Omega_E(M, \boldsymbol{g})$ by the formula

$$(\nabla\Theta)(X_0, ..., X_k) = \sum_{j=0}^{k} (-1)^j [\![\lambda \circ X_j, \Theta(X_0, ...\hat{\jmath}..., X_k)]\!] +$$
$$+ \sum_{i<j} (-1)^{i+j} \Theta([X_i, X_j], X_0, ...\hat{\imath}...\hat{\jmath}..., X_k),$$

$X_i \in \operatorname{Sec} E$, for $\Theta \in \Omega_E^k(M, \boldsymbol{g})$. Without difficulties we assert that

$$\nabla(\nu \cdot \theta) = \nabla\nu \wedge \theta + \nu \cdot d^E\theta \tag{3.2}$$

for $\nu \in \operatorname{Sec}\boldsymbol{g}$ and $\theta \in \Omega_E(M)$; besides, the linear operator

$$\nabla | \operatorname{Sec}\boldsymbol{g} : \operatorname{Sec}\boldsymbol{g} \to \Omega_E^1(M, \boldsymbol{g})$$

is a *partial covariant derivative* (in the sense of [11], compare [13], [14]).

Proposition 3.1.2 *(1) If* φ*:*$\boldsymbol{g} \times ... \times \boldsymbol{g} \to \mathbb{R}$ *is a* $\operatorname{Hom}^k(\operatorname{ad}_A)$*-invariant k-linear homomorphism, then, for* $\Theta_i \in \Omega_E^{q_i}(M, \boldsymbol{g})$*, we have*

$$d^E(\varphi_*(\Theta_1, ..., \Theta_k)) = \sum_{i=1`k} (-1)^{q_1 + ... + q_{i-1}} \varphi_*(\Theta_1, ..., \nabla\Theta_i, ..., \Theta_k).$$

(2) $\nabla\Omega_b = 0$ *(The Bianchi identity).*

Proof. (1): We begin with the following lemma: ■

Lemma For a $\operatorname{Hom}^k(\operatorname{ad}_A)$-invariant k-linear homomorphism φ:$\boldsymbol{g} \times ... \times \boldsymbol{g} \longrightarrow \mathbb{R}$ and $\nu_i \in \operatorname{Sec}\boldsymbol{g}$, we have

$$d(\varphi(\nu_1, ..., \nu_k)) = \sum_i \varphi_*(\nu_1, ..., \nabla\nu_i, ..., \nu_k).$$

Proof of the lemma. According to definitions 2.1.3 and 2.3.1, we have, for $X \in \operatorname{Sec} E$,

$$\begin{aligned} d^E(\varphi(\nu_1, ..., \nu_k))(X) &= X(\varphi(\nu_1, ..., \nu_k)) \\ &= (\gamma \circ \lambda \circ X)(\varphi(\nu_1, ..., \nu_k)) \\ &= \sum_i \varphi(\nu_1, ..., \mathcal{L}_{\mathrm{ad}_A \circ \lambda \circ X}(\nu_i), ..., \nu_k) \\ &= \sum_i \varphi(\nu_1, ..., [\![\lambda \circ X, \nu_i]\!], ..., \nu_k) \\ &= \sum_i \varphi(\nu_1, ..., (\nabla \nu_i)(X), ..., \nu_k) \\ &= \sum_i \varphi_*(\nu_1, ..., \nabla \nu_i, ..., \nu_k)(X). \end{aligned}$$

To continue the proof of our Proposition, we notice that since both sides of the examined equality are $\mathbb{R}$-linear with respect to each Θ_i, and each $\boldsymbol{g}$-valued form Θ is (locally) a linear combination of forms $\nu \cdot \theta$ where $\nu \in \operatorname{Sec} \boldsymbol{g}$ and θ is a real form, therefore it is sufficient to show the equality for $\Theta_i = \nu_i \cdot \theta_i$, $\nu_i \in \operatorname{Sec} \boldsymbol{g}$, $\theta_i \in \Omega^{q_i}(M)$. From the lemma above and 3.2 we obtain

$$\begin{aligned} & d^E(\varphi_*(\nu_1 \cdot \theta_1, ..., \nu_k \cdot \theta_k)) \\ &= d^E(\varphi(\nu_1, ..., \nu_k) \cdot \theta_1 \wedge ... \wedge \theta_k) \\ &= d^E(\varphi(\nu_1, ..., \nu_k)) \wedge \theta_1 \wedge ... \wedge \theta_k + \varphi(\nu_1, ..., \nu_k) \cdot d^E(\theta_1 \wedge ... \wedge \theta_k) \\ &= \sum_i \varphi_*(\nu_1, ..., \nabla \nu_i, ..., \nu_k) \wedge \theta_1 \wedge ... \wedge \theta_k + \\ &\quad + \varphi(\nu_1, ..., \nu_k) \cdot \sum_{i=1}^k (-1)^{q_1 + ... + q_{i-1}} \, \theta_1 \wedge ... \wedge d^E\theta_i \wedge ... \wedge \theta_k \\ &= \sum_{i=1}^k (-1)^{q_1 + ... + q_{i-1}} \, \varphi_*(\nu_1 \cdot \theta_1, ..., \nabla \nu_i \wedge \theta_i, ..., \nu_k \cdot \theta_k) + \\ &\quad + \sum_{i=1}^k (-1)^{q_1 + ... + q_{i-1}} \, \varphi_*(\nu_1 \cdot \theta_1, ..., \nu_1 \cdot d^E\theta_i, ..., \nu_k \cdot \theta_k) \\ &= \sum_{i=1}^k (-1)^{q_1 + ... + q_{i-1}} \, \varphi_*(\nu_1 \cdot \theta_1, ..., \nabla \nu_i \wedge \theta_i + \nu_i \cdot d^E\theta_i, ..., \nu_k \cdot \theta_k) \\ &= \sum_{i=1}^k (-1)^{q_1 + ... + q_{i-1}} \, \varphi_*(\nu_1 \cdot \theta_1, ..., \nabla(\nu_i \cdot \theta_i), ..., \nu_k \cdot \theta_k). \end{aligned}$$

(2): From the definitions, equality 3.1 and the Jacobi identity in $\operatorname{Sec} E$ we obtain:

$$\begin{aligned}
&\nabla\Omega_b(X_0, X_1, X_2) \\
&= [\![\lambda \circ X_0, \Omega_b(X_1, X_2)]\!] - [\![\lambda \circ X_1, \Omega_b(X_0, X_2)]\!] + [\![\lambda \circ X_2, \Omega_b(X_0, X_1)]\!] \\
&\quad - \Omega_b([X_0, X_1], X_2) + \Omega_b([X_0, X_2], X_1) - \Omega_b([X_1, X_2], X_0) \\
&= [\![\lambda \circ X_0, \lambda \circ [X_1, X_2] - [\![\lambda \circ X_1, \lambda \circ X_2]\!]]\!] - [\![\lambda \circ X_1, \lambda \circ [X_0, X_2] - \\
&- [\![\lambda \circ X_0, \lambda \circ X_2]\!]]\!] + [\![\lambda \circ X_2, \lambda \circ [X_0, X_1] - [\![\lambda \circ X_0, \lambda \circ X_1]\!]]\!] - \\
&- \lambda \circ [[X_0, X_1], X_2] + [\![\lambda \circ [X_0, X_1], \lambda \circ X_2]\!] + \lambda \circ [[X_0, X_2], X_1] - \\
&- [\![\lambda \circ [X_0, X_2], \lambda \circ X_1]\!] - \lambda \circ [[X_1, X_2], X_0] + [\![\lambda \circ [X_1, X_2], \lambda \circ X_0]\!] \\
&= 0.
\end{aligned}$$

3.2 Inverse-image of a connection

Definition 3.2.1 *Let λ be a connection in A. Take a morphism $f : (M', E') \to (M, E)$ of the category $\mathfrak{F}$ (see Preliminaries). By the inverse-image of λ over f we mean the connection $\bar{\lambda}$ in the inverse-image of A over f, (4), defined by $\lambda(v) = (v, \lambda(f(v)))$, $v \in E'$. Notice the commuting of the diagram*

$$\begin{array}{ccc}
f\hat{\ } A & \xrightarrow{\mathrm{pr}_2} & A \\
\bar{\lambda} \uparrow & & \lambda \uparrow \\
E' & \xrightarrow{f_*} & E
\end{array}$$

and the equality $\bar{\lambda} \circ X = (X, \lambda \circ f_* \circ X)$ for $X \in \operatorname{Sec} E'$. The connection form of $\bar{\lambda}$ is $\omega : f\hat{\ } A \longrightarrow f^*\boldsymbol{g}$, $(v, w) \mapsto \omega(w)$, where ω is such a form for λ.

Proposition 3.2.2 *Let λ be a connection in A, and Ω_b - its curvature tensor. Then $\bar{\Omega}_b$, the curvature tensor of the inverse-image $\bar{\lambda}$ of λ over f, is equal to $\bar{\Omega}_b(X, Y) = (f^*\Omega_b)(X, Y)$, $X, Y \in \operatorname{Sec} E'$.*

Proof. We start with the following
Lemma (1) *For $X \in \operatorname{Sec} E'$, we have $(X, \lambda \circ f_* \circ X) \in \operatorname{Sec} f\hat{\ } A$.* ■
(2) For $X, Y \in \operatorname{Sec} E'$, we have

$$[\![(X, \lambda \circ f_* \circ X), (Y, \lambda \circ f_* \circ Y)]\!] = ([X, Y]), \lambda \circ f_* \circ [X, Y] - (f^*\Omega_b)(X, Y)).$$

Proof of the Lemma. (1) is evident. To prove (2) we establish the equality in some neighbourhood of an arbitrary point $x \in M'$. For the purpose, take any commuting vector fields $Y^1, ..., Y^n \in \operatorname{Sec}(E)$ being a local basis in some neighbourhood U of $y := f(x)$. Then, on $U' := f[U] \subset M'$, we may write

$$(f_* \circ X)_{|U'} = (\sum_i g^i \cdot Y^i \circ f)_{|U'}, \quad (f_* \circ Y)_{|U'} = (\sum_j h^j \cdot Y^j \circ f)_{|U'}$$

for some $g^i, h^j \in \Omega^0(M)$. Therefore, by 3.1,

$$
\begin{aligned}
&[\![(X, \lambda \circ f_* \circ X), (Y, \lambda \circ f_* \circ Y)]\!]_{|U'} \\
&= [\![(X, \sum_i g^i \cdot \lambda \circ Y^i \circ f), (Y, \sum_j h^j \cdot \lambda \circ Y^j \circ f)]\!]_{|U'} \\
&= ([X,Y], \sum_{i,j} g^i \cdot h^j \cdot [\![\lambda \circ Y^i, \lambda \circ Y^j]\!] \circ f + \sum_j X(h^j) \cdot \lambda \circ Y^j \circ f - \\
&\qquad - \sum_i Y(g^i) \cdot \lambda \circ Y^i \circ f)_{|U'} \\
&= ([X,Y], -\sum_{i,j} g^i \cdot h^j \cdot \Omega_b(Y^i, Y^j) \circ f + \sum_j X(h^j) \cdot \lambda \circ Y^j \circ f - \\
&\qquad - \sum_i Y(g^i) \cdot \lambda \circ Y^i \circ f)_{|U'} \\
&= ([X,Y], -(f^*\Omega_b)(X,Y) + \sum_j X(h^j) \cdot \lambda \circ Y^j \circ f - \sum_i Y(g^i) \cdot \lambda \circ Y^i \circ f)_{|U'}.
\end{aligned}
$$

It remains to prove that

$$(f_* \circ [X,Y])_{|U'} = \sum_j X(h^j) \cdot Y^j \circ f - \sum_i Y(g^i) \cdot Y^i \circ f)_{|U'}.$$

Let $\alpha \in \Omega^0(M)$; then $(f_* \circ X)(\alpha)_{|U'} = X(\alpha \circ f)_{|U'} = (\sum_i g^i \cdot Y^i(\alpha) \circ f)_{|U'}$, analogously - for Y; so,

$$
\begin{aligned}
f_* \circ [X,Y](\alpha)_{|U'} &= [X,Y](\alpha \circ f)_{|U'} \\
&= (X(Y(\alpha \circ f)) - Y(X(\alpha \circ f)))_{|U'} \\
&= (X(\sum_j h^j \cdot Y^j(\alpha) \circ f) - Y(\sum_i g^i \cdot Y^i(\alpha) \circ f))_{|U'} \\
&= (\sum_j X(h^j) \cdot Y^j(\alpha) \circ f - \sum_i Y(g^i) \cdot Y^i(\alpha) \circ f)_{|U'}
\end{aligned}
$$

because

$$
\begin{aligned}
&(\sum_j h^j \cdot X(Y^j(\alpha) \circ f) - \sum_i g^i \cdot Y(Y^i(\alpha) \circ f))_{|U'} \\
&= (\sum_j h^j \cdot (f_* \circ X)(Y^j \alpha) - \sum_i g^i \cdot (f_* \circ Y)(Y^i \alpha))_{|U'} \\
&= (\sum_{i,j} h^j \cdot g^i \cdot Y^i(Y^j \alpha) \circ f - \sum_{i.,j} g^i \cdot h^j \cdot Y^j(Y^j \alpha) \circ f)_{|U'} \\
&= \sum_{i,j} h^j \cdot g^i \cdot [Y^i, Y^j](\alpha) \circ f_{|U'} \\
&= 0.
\end{aligned}
$$

Proof of the proposition. Let $X, Y \in \operatorname{Sec} E'$ and $x \in M'$. By the lemma above, we have

$$\begin{aligned}
\bar{\Omega}_b(X,Y)(x) &= -\bar{\omega}(\llbracket \bar{\lambda} \circ X, \bar{\lambda} \circ Y \rrbracket)(x) \\
&= -\bar{\omega}(\llbracket (X, \lambda \circ f_* \circ X), (Y, \lambda \circ f_* \circ Y) \rrbracket)(x) \\
&= -\bar{\omega}(([X,Y], \lambda \circ f_* \circ [X,Y] - (f^*\Omega_b)(X,Y)))(x) \\
&= -\omega_{f(x)}((\lambda \circ f_* \circ [X,Y] - (f^*\Omega_b)(X,Y))(x)) \\
&= (f^*\Omega_b)(X,Y)(x).
\end{aligned}$$

Proposition 3.2.3 *Let $H : A' \to A$ be an arbitrary homomorphism (say, over $f : (M', E') \to (M, E)$) of regular Lie algebroids. Let $\lambda : E \to A$ and $\lambda' : E' \to A'$ be connections in A and A', respectively, such that $H \circ \lambda' = \lambda \circ f$; then the curvature tensors Ω_b and Ω'_b of λ and λ', respectively, are related to each other via*

$$(f^*\Omega_b)_x = H^+_{|x}(\Omega'_{bx}), \quad x \in M'.$$

Proof. Represent canonically H in the form of superposition 1.5. Let $\bar{\lambda}$ be the inverse-image of λ over f and denote by $\bar{\Omega}$ the curvature tensor of $\bar{\lambda}$. Consider the following diagram

$$\begin{array}{ccccc}
\boldsymbol{g}'_{|x} & & \xrightarrow{H^+_{|x}} & & \boldsymbol{g}_{|f(x)} \\
& \searrow \bar{H}^+_{|x} & & // & \\
\uparrow \Omega'_{bx} & & \boldsymbol{g}_{|f(x)} & & \uparrow \Omega_{bf(x)} \\
& & \uparrow \bar{\Omega}_{bx} & & \\
E'_{|x} \times E'_{|x} & = & E'_{|x} \times E'_{|x} & \xrightarrow{f_{*x} \times f_{*x}} & E_{|f(x)} \times E_{|f(x)}.
\end{array}$$

By Prop. 3.2.2, we have the commutativity of the right square. Thus the proposition reduces to the case of a strong homomorphism, say, $\bar{H} : A' \to \bar{A}'$:

$$\begin{aligned}
\bar{\Omega}_b(X,Y) &= -\bar{\omega}(\llbracket \bar{\lambda} \circ X, \bar{\lambda} \circ Y \rrbracket) = -\bar{\omega}(\llbracket \bar{H} \circ \lambda' \circ X, \bar{H} \circ \lambda' \circ Y \rrbracket) \\
&= -\bar{\omega} \circ \bar{H} \circ \llbracket \lambda' \circ X, \lambda' \circ Y \rrbracket = -\bar{H}^+ \circ \omega' \llbracket \lambda' \circ X, \lambda' \circ Y \rrbracket \\
&= \bar{H}^+_* \Omega'_b(X,Y).
\end{aligned}$$

■

4 THE CHERN-WEIL HOMOMORPHISM OF A REGULAR LIE ALGEBROID

4.1 Definition of the homomorphism.

Let 1.1 be an arbitrary but fixed regular Lie algebroid over $(M, E) \in \mathfrak{F}$ and let 1.2 be its Atiyah sequence. Assume also that a connection λ in A is given, and that $\Omega_b \in \Omega^2_E(M, \boldsymbol{g})$ is its curvature tensor. Let us fix a point $x \in M$. By the commutativity of the algebra $\bigoplus^{k\geq 0}\bigwedge^{2k} E^*_{|x}$, there exists [8,p.192] exactly one homomorphism of algebras

$$\tilde{\mathcal{X}}_{(A,\lambda),x} : \bigvee \boldsymbol{g}^*_{|x} \longrightarrow \bigoplus^{k\geq 0} \bigwedge^{2k} E^*_{|x}$$

such that $\tilde{\mathcal{X}}_{(A,\lambda),x}(1) = 1$ and $\tilde{\mathcal{X}}_{(A,\lambda),x}(\Gamma) = \langle \Gamma, \Omega_{bx}\rangle$, $\Gamma \in \bigvee^1 \boldsymbol{g}^*_{|x} = \boldsymbol{g}^*_{|x}$.

Lemma 4.1.1 $\tilde{\mathcal{X}}_{(A,\lambda),x}(\Gamma) = \frac{1}{k!}\cdot \langle \Gamma, \underbrace{\Omega_{bx} \vee ... \vee \Omega_{bx}}_{k\ \ times}\rangle$ *for* $\Gamma \in \bigvee^k \boldsymbol{g}^*_{|x}$.

Proof. Define auxiliarily a mapping ■

$$\tilde{\beta}_x : \bigotimes \boldsymbol{g}^*_{|x} \longrightarrow \bigoplus^{k\geq 0} \bigwedge^{2k} E^*_{|x},\ \Gamma \longmapsto \langle \Gamma, \Omega_{bx}\otimes ... \otimes \Omega_{bx}\rangle,\ \text{for}\ \ \Gamma \in \bigotimes^{k} \boldsymbol{g}^*_{|x}.$$

Thanks to the simplicity of the nature of the duality $\bigotimes \boldsymbol{g}^*_{|x} \times \bigotimes \boldsymbol{g}_{|x} \longrightarrow \mathbb{R}$ (see [8]), we state (analogously as in Lemma III in [9,p.261]) that $\tilde{\beta}_x$ is a homomorphism of algebras. Take the canonical projection π_x:$\bigotimes \boldsymbol{g}^*_{|x} \longrightarrow \bigvee \boldsymbol{g}^*_{|x}$, $w \otimes_1 ... \otimes w_k \mapsto w_1 \vee ... \vee w_k$. The following diagram

$$\begin{array}{ccc} \bigotimes \boldsymbol{g}^*_{|x} & \xrightarrow{\tilde{\beta}_x} & \bigoplus^{k\geq 0}\bigwedge^{2k} E^*_{|x} \\ \pi_x \downarrow & \nearrow \tilde{\mathcal{X}}_{(A,\lambda),x} & \\ \bigvee \boldsymbol{g}^*_{|x} & & \end{array}$$

commutes, which can easily be seen by checking on simple tensors $w_1 \otimes ... \otimes w_k \in \bigotimes^k \boldsymbol{g}^*_{|x}$. Let $\kappa_x : \bigvee \boldsymbol{g}^*_{|x} \longrightarrow \bigotimes \boldsymbol{g}^*_{|x}$ denote a mapping defined by $\kappa(w_1 \vee ... \vee w_k) = \frac{1}{k!}\cdot \sum_\sigma w_{\sigma(1)} \otimes ... \otimes w_{\sigma(k)}$. Then $\pi_x \circ \kappa_x = \mathrm{id}$ and (see [8, pp.91,193]), for $\Gamma \in \bigvee^k \boldsymbol{g}^*_{|x}$ and $u_i \in \boldsymbol{g}_{|x}$,

$$\langle \kappa_x(\Gamma), u_1 \otimes ... \otimes u_k\rangle = \frac{1}{k!}\cdot \langle \Gamma, u_1 \vee ... \vee u_k\rangle.$$

Therefore

$$\tilde{\mathcal{X}}_{(A,\lambda),x}(\Gamma) = \tilde{\mathcal{X}}_{(A,\lambda),x}(\pi_x \circ \kappa_x \Gamma) = \tilde{\beta}_x(\kappa\Gamma) = \langle \kappa_x\Gamma, \Omega_{bx}\otimes ... \otimes \Omega_{bx}\rangle,$$

so, for $v_i \in E_{|x}$,

$$\begin{aligned} \tilde{\mathcal{X}}_{(A,\lambda),x}(\Gamma)(v_1 \wedge ... \wedge v_{2k}) &= \langle \kappa_x(\Gamma), \Omega_{bx}\otimes ... \otimes \Omega_{bx}(v_1 \wedge ... \wedge v_{2k})\rangle \\ &= \frac{1}{k!}\cdot \langle \Gamma, \Omega_{bx}\vee ... \vee \Omega_{bx}\rangle(v_1 \wedge ... \wedge v_{2k}). \end{aligned}$$

Fix an integer $k \geq 0$. The family of homomorphisms $\tilde{\mathcal{X}}^k_{(A,\lambda),x} : \bigvee^k \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge^{2k} E^*_{|x}$, $x \in M$, gives rise to a strong homomorphism of vector bundles $\tilde{\mathcal{X}}^k_{(A,\lambda)} : \bigvee \boldsymbol{g}^* \longrightarrow \bigwedge^{2k} E^*$ and, by the Lemma above, we have the equality

$$\tilde{\mathcal{X}}^k_{(A,\lambda)} \circ \Gamma = \frac{1}{k!}\cdot \langle \Gamma, \Omega_b \vee ... \vee \Omega_b\rangle$$

for $\Gamma \in \operatorname{Sec}\bigvee^k \boldsymbol{g}^*$, from which we obtain that $\tilde{\mathcal{X}}^k_{(A,\lambda)}$ is a C^∞ homomorphism of vector bundles. The homomorphism of $\Omega^0(M)$-moduli

$$\mathcal{X}_{(A,\lambda)} : \bigoplus^{k\geq 0} \operatorname{Sec} \bigvee^k \boldsymbol{g}^* \longrightarrow \bigoplus^{k\geq 0} \Omega^{2k}_E(M) \subset \Omega(M),$$

induced on the cross-sections, is, of course, a homomorphism of algebras. The adjoint representation ad_A gives rise to a representation $\bigvee^k \mathrm{ad}^\natural_A$ of A on $\bigvee^k \boldsymbol{g}^*$, see 2.1.3. Denote by $(\operatorname{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o}$ the space of invariant (under $\bigvee^k \mathrm{ad}^\natural_A$) cross-sections of $\bigvee^k \boldsymbol{g}^*$ and restrict $\mathcal{X}_{(A,\lambda)}$ to invariant cross-sections to obtain

$$\mathcal{X}_{(A,\lambda),I^o} : \bigoplus^{k\geq 0} (\operatorname{Sec} \bigvee^k \boldsymbol{g}^*)_{I^o} \longrightarrow \Omega_E(M).$$

According to Lemma 2.3.3 $\bigoplus^{k\geq 0}(\operatorname{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o}$ forms an algebra.

Proposition 4.1.2 *The forms from the image of $\mathcal{X}_{(A,\lambda),I^o}$ are closed.*

Proof. Let $\Gamma \in (\operatorname{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o}$. Then, by Lemma 2.3.2 and Prop. 3.1.2,

$$\begin{aligned} d^E(\mathcal{X}_{(A,\lambda),I^o}(\Gamma)) &= \frac{1}{k!} \cdot d^E(\langle \Gamma, \Omega_b \vee ... \vee \Omega_b \rangle) = \frac{1}{k!} \cdot d^E(\tilde{\Gamma}_*(\Omega_b, ..., \Omega_b)) \\ &= \frac{1}{k!} \cdot \sum_j (\tilde{\Gamma}_*(\underbrace{\Omega_b, ..., \Omega_b}_{k-1 \text{ times}}, \nabla\Omega_b, ..., \Omega_b)) = 0. \end{aligned}$$

Define the superposition

$$h_{(A,\lambda)} : \bigoplus^{k\geq 0} (\operatorname{Sec} \bigvee^k \boldsymbol{g}^*)_{I^o} \overset{\mathcal{X}_{(A,\lambda),I^o}}{\longrightarrow} \ker d^E \longrightarrow H_E(M).$$

■

4.2 The functoriality of the homomorphism $h_{(A,\lambda)}$

Let $H : A' \to A$ be an arbitrary homomorphism (say, over $f : (M', E') \to (M, E)$) of regular Lie algebroids. Define the pullback $H^{+*} : \bigoplus^{k\geq 0} \operatorname{Sec}\bigvee^k \boldsymbol{g}^* \longrightarrow \bigoplus^{k\geq 0} \operatorname{Sec}\bigvee^k \boldsymbol{g}^{*\prime}$ by the formula:

$$\langle (H^{+*}(\Gamma))_x, v_1 \vee ... \vee v_k \rangle = \langle \Gamma_{f(x)}, H^+_{|x}(v_1) \vee ... \vee H^+_{|x}(v_k) \rangle, \ \ x \in M', v_i \in \boldsymbol{g}'_{|x}.$$

It is easy to see that H^{+*} is a homomorphism of algebras.

Proposition 4.2.1 *The pullback H^{+*} maps invariant cross-sections into invariant ones.*

Proof. Represent H in the form of superposition 1.5 and notice that $H^{+*}(\Gamma) = \bar{H}^{+*}((\mathcal{X}^+)^*(\Gamma))$; therefore we see that it is enough to consider two cases: (a) a strong homomorphism and, (b) the canonical homomorphism $\mathcal{X} : f\,\hat{}\,A \to A$, see §1.1.

(a) Consider the case of a strong homomorphism $H : A' \to A$ of regular Lie algebroids (both over (M, E)). Let $\Gamma \in (\operatorname{Sec} \bigvee^k \boldsymbol{g}^*)_{I^o}$. For $\xi \in \operatorname{Sec} A'$, $\sigma_i \in \operatorname{Sec} \boldsymbol{g}'$,

$$\begin{aligned}
&(\gamma' \circ \xi)\langle H^{+*}\Gamma, \sigma_1 \vee ... \vee \sigma_k\rangle \\
&= (\gamma \circ H \circ \xi)\langle \Gamma, H^+ \circ \sigma_1 \vee ... \vee H^+ \circ \sigma_k\rangle \\
&= \sum_i \langle \Gamma, H^+ \circ \sigma_1 \vee ... \vee H^+ \circ [\![\xi, \sigma_i]\!] \vee ... \vee H^+ \circ \sigma_k\rangle \\
&= \sum_i \langle H^{+*}\Gamma, \sigma_1 \vee ... \vee [\![\xi, \sigma_i]\!] \vee ... \vee \sigma_k\rangle .
\end{aligned}$$

■

(b) Consider the canonical homomorphism $\mathcal{X} : f\,\hat{}\,A \to A$. Identify $f^*(\bigvee^k \boldsymbol{g}^*) \cong \bigvee^k (f^*\boldsymbol{g})^*$. Then $(\mathcal{X}^+)^*\Gamma = f^*\Gamma$ and, applying Lemmas 2.2.2 and 2.2.3, we get

$$f^*(\bigvee^k \operatorname{ad}^\natural_A) = \bigvee^k f^* \operatorname{ad}^\natural_A = \bigvee^k \operatorname{ad}^\natural_{f\,\hat{}\,A} .$$

Our assertion now follows from Theorem 2.3.4.

Theorem 4.2.2 (The functoriality property) *Let $H : A' \to A$ be a homomorphism (say, over $f : (M', E') \to (M, E)$) of regular Lie algebroids. Then, for arbitrarily taken connections λ' and λ in A' and A, respectively, such that $H \circ \lambda' = \lambda \circ f_*$, the following diagram commutes:*

$$\begin{array}{ccc}
\bigoplus^{k\geq 0} (\operatorname{Sec} \bigvee^k \boldsymbol{g}^*)_{I^o} & \overset{h_{(A,\lambda)}}{\longrightarrow} & H_E(M) \\
H^{+*} \downarrow & & \downarrow f^\# \\
\bigoplus^{k\geq 0} (\operatorname{Sec} \bigvee^k \boldsymbol{g}'^*)_{I^o} & \overset{h_{(A',\lambda')}}{\longrightarrow} & H_{E'}(M') .
\end{array}$$

Proof. Of course, it is enough to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
\bigoplus^{k\geq 0} (\operatorname{Sec} \bigvee^k \boldsymbol{g}^*) & \overset{\mathcal{X}_{(A,\lambda)}}{\longrightarrow} & \Omega_E(M) \\
H^{+*} \downarrow & & \downarrow f^* \\
\bigoplus^{k\geq 0} (\operatorname{Sec} \bigvee^k \boldsymbol{g}'^*) & \overset{\mathcal{X}_{(A',\lambda')}}{\longrightarrow} & \Omega_{E'}(M') .
\end{array}$$

Let Ω_b be the curvature tensor of λ . Take $\Gamma \in \operatorname{Sec} \bigvee^k \boldsymbol{g}$. By Prop. 3.2.3, we have, for $x \in M'$ and $v_i \in E'_{|x}$,

$$
\begin{aligned}
&f^* \circ \mathcal{X}_{(A,\lambda)}(\Gamma)(x; v_1 \wedge \dots \wedge v_{2k}) \\
&= \frac{1}{k!} \cdot \langle \Gamma, \Omega_b \vee \dots \vee \Omega_b \rangle (f(x); f_* (v_1) \wedge \dots \wedge f_* (v_{2k})) \\
&= \frac{1}{k!} \cdot \langle \Gamma_{f(x)}, \Omega_b \vee \dots \vee \Omega_b (f(x); f_* (v_1) \wedge \dots \wedge f_* (v_{2k})) \rangle \\
&= \frac{1}{k!} \cdot \langle \Gamma_{f(x)}, \frac{1}{2^k} \cdot \sum_\sigma \operatorname{sgn} \sigma \cdot \Omega_b (f(x); f_* \left(v_{\sigma(1)}\right) \wedge f_* \left(v_{\sigma(2)}\right)) \vee \dots \\
&\qquad \dots \vee \Omega_b (f(x); f_* \left(v_{\sigma(2k-1)}\right) \wedge f_* \left(v_{\sigma(2k)}\right)) \rangle \\
&= \frac{1}{k!} \cdot \langle \Gamma_{f(x)}, \frac{1}{2^k} \cdot \sum_\sigma \operatorname{sgn} \sigma \cdot H^+_{|x} (\Omega_b' (x; v_{\sigma(1)} \wedge v_{\sigma(2)})) \vee \dots \\
&\qquad \dots \vee H^+_{|x} (\Omega_b' (x; v_{\sigma(2k-1)} \wedge v_{\sigma(2k)})) \rangle \\
&= \frac{1}{k!} \cdot \langle H^{+*}(\Gamma)_x, \frac{1}{2^k} \cdot \sum_\sigma \operatorname{sgn} \sigma \cdot \Omega_b' (x; v_{\sigma(1)} \wedge v_{\sigma(2)}) \vee \dots \\
&\qquad \dots \vee \Omega_b' (x; v_{\sigma(2k-1)} \wedge v_{\sigma(2k)}) \rangle \\
&= \frac{1}{k!} \cdot \langle H^{+*}(\Gamma)_x, (\Omega_b' \vee \dots \vee \Omega_b')(x; v_1 \wedge \dots \wedge v_{2k}) \rangle \\
&= \mathcal{X}_{(A',\lambda')} \circ H^{+*}(\Gamma)(x; v_1 \wedge \dots \wedge v_{2k}).
\end{aligned}
$$

■

4.3 The independence on the choice of a connection

Theorem 4.3.1 *Let 1.1 be an arbitrary regular Lie algebroid over (M, E). Then, the homomorphism his independent of the choice of a connection λ.*

Proof. Let $\lambda_1 : E \to A$, $i = 0, 1$, be two arbitrarily taken connections in A and let $\omega_i : A \to \boldsymbol{g}$ be their connection forms. Take the regular Lie algebroid $T\mathbb{R} \times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times E)$ [24] being the product of the trivial Lie algebroid $T\mathbb{R}$ with A and take in it the connection form $\omega : T\mathbb{R} \times E \to 0 \times \boldsymbol{g}$ defined by

$$\omega_{(t,x)}(v, w) = (0, \omega_{0x}(w) \cdot (1 - t) + \omega_{1x}(w) \cdot t).$$

The following

$$
\begin{gathered}
G : T\mathbb{R} \times A \longrightarrow A, \ (v, w) \longmapsto w, \\
F_t : A \longrightarrow T\mathbb{R} \times A, \ \ w \longmapsto (\theta_t, w),
\end{gathered}
$$

(θ_t is the null tangent vector at $t \in \mathbb{R}$), $t \in \mathbb{R}$, are homomorphisms of regular Lie algebroids over $\mathrm{pr}_2 : (\mathbb{R} \times M, T\mathbb{R} \times E) \to (M, E)$ and $j_t : (M, E) \to (\mathbb{R} \times M, T\mathbb{R} \times E)$ $(x \mapsto (t, x))$, respectively. Notice the equality $\omega \circ F_i = F_i^+ \circ \omega_i$, $i = 0, 1$. Let $\lambda : T\mathbb{R} \times E \longrightarrow T\mathbb{R} \times A$ be the connection in $T\mathbb{R} \times A$, corresponding to ω. We see that $\lambda \circ j_{i*} = F_i \circ \lambda_i$. Functoriality property 4.2.2 yields the commutativity of the diagram for$i = 0, 1$:

$$\begin{array}{ccc} \bigoplus^{k\geq 0}(\operatorname{Sec}\bigvee^k (0\times \boldsymbol{g})^*)_{I^o} & \overset{h_{(T\mathbb{R}\times A,\lambda)}}{\longrightarrow} & H_{T\mathbb{R}\times E}(\mathbb{R}\times M) \\ F_i^{+*}\downarrow & & \downarrow j_i^{\#} \\ \bigoplus^{k\geq 0}(\operatorname{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o} & \overset{h_{(A,\lambda_i)}}{\longrightarrow} & H_E(M)\,. \end{array}$$

Consider the homotopy $H = \mathrm{id}_{\mathbb{R}\times M}$ joining j_0 to j_1. Since $H : (\mathbb{R}\times M, T\mathbb{R}\times E) \to (\mathbb{R}\times M, T\mathbb{R}\times E)$ is a morphism of the category $\mathfrak{F}$, therefore H implies the equality $j_0^{\#} = j_1^{\#}$ ($h : \Omega^*_{T\mathbb{R}\times E}(\mathbb{R}\times M) \to \Omega_E^{*-1}(M)$ defined by $(h\Theta)(x; v_1\wedge ... \wedge v_{q-1}) = \int_0^1 \Theta_{(t,x)}(\frac{\partial}{\partial t}\wedge v_1 \wedge ... \wedge v_{q-1})dt$ is a cochain homotopy operator, i.e. the condition $j_0^* - j_1^* = h\circ d^E + d^E \circ h$ holds, cf. [28]). From the fact that $G\circ F_i = \mathrm{id}_A$, $i = 0, 1$, we have

$$F_i^{+*}\circ G^{+*} = \mathrm{id}\,.$$

Therefore ■

$$\begin{aligned} h_{(A,\lambda_0)} &= h_{(A,\lambda_0)}\circ F_i^{+*}\circ G^{+*} = j_0^{\#}\circ h_{(T\mathbb{R}\times A,\lambda)}\circ G^{+*} \\ &= j_1^{\#}\circ h_{(T\mathbb{R}\times A,\lambda)}\circ G^{+*} = h_{(A,\lambda_1)}. \end{aligned}$$

The theorem just proved means that the examined homomorphism $h_{(A,\lambda)}$ is, in fact, a characteristic feature of the regular Lie algebroid A and justifies its being denoted by h_A. It will be called (traditionally) the Chern-Weil homomorphism of A, whereas its image $\operatorname{Im} h_A \subset H_E(M)$ will be called the Pontryagin algebra of A and denoted by $\operatorname{Pont} A$. Clearly,

$$h_A(\Gamma) = [\frac{1}{k!}\cdot\langle\Gamma, \Omega_b \vee ... \vee \Omega_b\rangle] \ \text{ if } \ \Gamma \in (\operatorname{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o}, \tag{4.1}$$

where Ω_b is the curvature tensor of any connection in A. As a simple corollary from Theorem 4.3.1 we obtain

Corollary 4.3.2 *If the Chern-Weil homomorphism h_A of a regular Lie algebroid A is nontrivial (i.e. $h_A^+ \neq 0$), then there exists no flat connection in A.*

In the nearest chapter we compare this homomorphism with the well-known homomorphism for principal bundles, whereas in the next ones we examine this homomorphism more precisely for Lie algebroids called into existence by other objects such as TC-foliations or nonclosed Lie subgroups.

5 COMPARISON WITH PRINCIPAL BUNDLES

5.1 The Lie algebroid of a principal bundle [16], [19], [23].

Let us fix a G-principal bundle $(P = (P, \pi, M, G, \cdot))$. By a Lie algebroid $A(P)$ of a P we mean a transitive Lie algebroid $(A(P), [\![\cdot,\cdot]\!], \gamma)$ on a manifold M, in which

$A(P) = TP/G$ (i.e. the vectors v and $(R_a)_* v$, $v \in TP$, are identified for each $a \in G$), $\gamma([v]) = \pi_* (v)$, $v \in TP$, where $[v]$ denotes the equivalence class of v, and the bracket is constructed on the basis of the following observation (see [16] [19]): For each cross-section $\eta \in \operatorname{Sec} A(P)$, there exists exactly one C^∞ right-invariant vector field $\eta' \in \mathfrak{X}^R(P)$ such that $[\eta'(z)] = \eta(\pi z)$, and the mapping $\operatorname{Sec} A(P) \longrightarrow \mathfrak{X}^R(P)$, $\eta \mapsto \eta'$, is an isomorphism of $\Omega^0(M)$-modules. The bracket $[\![\xi, \eta]\!]$ for $\xi, \eta \in \operatorname{Sec} A(P)$ is defined in such a way that $[\![\xi, \eta]\!]' = [\xi', \eta']$. [The Lie algebroid of a principal bundle can also be constructed in some other ways [16], [19]].

The Lie algebra bundle $\boldsymbol{g}$ adjoint of $A(P)$ is canonically isomorphic to the Ad-associated Lie algebra bundle $P \times_G \mathfrak{g}$ ($\mathfrak{g}$ denotes the right! Lie algebra of G) via $\tau : P \times_G \mathfrak{g} \to \boldsymbol{g}$, $(z, v) \mapsto \hat{z}(v)$, where

$$\hat{z} : \mathfrak{g} \longrightarrow \boldsymbol{g}_{|x}, \;\; v \longmapsto [(A_z)_{*e}(v)], \;\;\; x := \pi(z), \tag{5.1}$$

is an isomorphism of Lie algebras, $A_z : G \longrightarrow P$, $a \mapsto z \cdot a$ (see [16], [19]). Notice that

$$(za) = z \circ \operatorname{Ad}_G(a), \;\; z \in P, \;\; a \in G.$$

Let $(P', \pi', M, G', \cdot\prime)$ and $(P, \pi, M, G, \cdot)$ be two principal bundles (on the same manifold M) and $\mu : G' \longrightarrow G$ - a homomorphism of Lie groups. By a (μ-)homomorphism of principal bundles

$$F : (P', \pi', M, G', \cdot\prime) \longrightarrow (P, \pi, M, G, \cdot)$$

we shall mean a mapping $F : P' \longrightarrow 3P$ such that $\pi \circ F = \pi'$ and $F(z \cdot\prime a) = F(z) \cdot \mu(a)$, $z \in P'$, $a \in G'$. F determines a homomorphism of Lie algebroids $dF : A(P') \to A(P)$, $[v] \mapsto [F_*(v)]$ (see [16], [19]).

5.2 The Lie algebroid of a principal bundle of repers

With a vector bundle $\mathfrak{f}$ we associate the Lie algebroid $A(\mathfrak{f})$, see 1.2. Of course, with $\mathfrak{f}$ we can also associate the Lie algebroid $A(L\mathfrak{f})$ of the principal bundle $L\mathfrak{f}$ of repers of $\mathfrak{f}$. Both of them are isomorphic [23] which can be proved by considerably simpler means than those of K.Mackenzie [23]. We begin by giving some simple

Example 5.2.1 *For the right Lie algebra $T_{\mathrm{id}}(GL(V))$ of the Lie group $GL(V)$, V being any finite dimensional $\mathbb{R}$-vector space, the following linear homomorphism*

$$\rho : T_{\mathrm{id}}(GL(V)) \longrightarrow \operatorname{End} V, \;\; v \longmapsto (w \longmapsto v(\tilde{w})),$$

where $\tilde{w} : GL(V) \longrightarrow V$, $a \mapsto a^{-1}(w)$, is an isomorphism of Lie algebras provided that $\operatorname{End} V$ is equipped with the canonical Lie algebra structure $[l_1, l_2] := l_1 \circ l_2 - l_2 \circ l_1$. Of course, thanks to the fact that $GL(V)$ can be considered as an open subset of $\operatorname{End} V$, we have the canonical identification $c : T_{\mathrm{id}}(GL(V)) \longrightarrow \operatorname{End} V$. Then, $\rho_V = -\operatorname{id}$.

Now, we apply this idea to vector bundles. Let $\mathfrak{f}$ be any vector bundle over M with the typical fibre V and let $L\mathfrak{f}$ be the $GL(V)$-principal bundle of all repers of $\mathfrak{f}$ interpreted as linear isomorphisms $V \longrightarrow \mathfrak{f}$, $x \in M$. For a cross-section $\nu \in \operatorname{Sec}\mathfrak{f}$, define the C^∞ mapping

$$\nu : L\mathfrak{f} \longrightarrow V,\ u \longmapsto u^{-1}(\nu(\pi u)). \tag{5.2}$$

It is easy to see that, for $\xi \in \operatorname{Sec} A(L\mathfrak{f})$ and $\nu \in \operatorname{Sec}\mathfrak{f}$,

$$\mathcal{L}_\xi(\nu) : M \longrightarrow \mathfrak{f},\ x \longmapsto u(\xi'_u(\tilde{\nu})),\ u \in (L\mathfrak{f})_{|x},$$

is a correctly defined C^∞ cross-section of $\mathfrak{f}$. By a simple calculation we assert that

(i^0) $\mathcal{L}_\xi(f \cdot \nu) = f \cdot \mathcal{L}_\xi(\nu) + (\gamma \circ \xi)(f) \cdot \nu$, $f \in \Omega^0(M)$, which means that $\mathcal{L}_\xi$:$\operatorname{Sec}\mathfrak{f} \longrightarrow \operatorname{Sec}\mathfrak{f}$ is a *covariant differential operator*, [23],

(ii^0) $\mathcal{L}_{f\cdot\xi} = f \cdot \mathcal{L}_\xi$,

(iii^0) $\mathcal{L}_{[\![\xi,\eta]\!]} = \mathcal{L}_\xi \circ \mathcal{L}_\eta - \mathcal{L}_\eta \circ \mathcal{L}_\xi$.

By (i^0), $\mathcal{L}_\xi$ can be interpreted as a C^∞ cross-section of $A(\mathfrak{f})$ with $q \circ \mathcal{L}_\xi = \gamma \circ \xi$, see 1.2, and, by (ii^0), $\operatorname{Sec} A(L\mathfrak{f}) \longrightarrow \operatorname{Sec} A(\mathfrak{f})$, $\xi \mapsto \mathcal{L}_\xi$, is a $\Omega^0(M)$-homomorphism. Therefore we see the existence and the uniqueness of a homomorphism of vector bundles

$$\Phi_{\mathfrak{f}} : A(L\mathfrak{f}) \to A(\mathfrak{f})$$

such that $\Phi_{\mathfrak{f}} \circ \xi$ is the cross-section of $A(\mathfrak{f})$ corresponding to a covariant differential operator $\mathcal{L}_\xi$. By (iii^0), $\Phi_{\mathfrak{f}}$ is a homomorphism of Lie algebroids. $\Phi_{\mathfrak{f}}$ is defined by the formula:

$$\Phi_{\mathfrak{f}}([v])(\nu) = u(v(\tilde{\nu})),\ \text{ where }\ v \in T_u(L\mathfrak{f}),\ \ u \in L\mathfrak{f}.$$

Proposition 5.2.2 $\Phi_{\mathfrak{f}}$ *is an isomorphism of transitive Lie algebroids.*

Proof. Look at the homomorphism of associated Atiyah sequences induced by $\Phi_{\mathfrak{f}}$. By the 5-Lemma, it is clear that it suffices to see that $\Phi_{\mathfrak{f}}^+ : \boldsymbol{g} \longrightarrow \operatorname{End}\mathfrak{f}$ is an isomorphism of vector bundles ($\boldsymbol{g}$ being the adjoint Lie algebra bundle of $A(L\mathfrak{f})$). For the purpose, take $x \in M$, $u \in (L\mathfrak{f})$ and notice the commutativity of the diagram

$$\begin{array}{cccc} \boldsymbol{g}_{|x} & \overset{\Phi^+_{|x}}{\longrightarrow} & \operatorname{End}\left(\mathfrak{f}_{|x}\right) & u \circ a \circ u^{-1} \\ \hat{u} \uparrow\cong & & \cong\uparrow & \uparrow \\ T_{\mathrm{id}}(GL(V)) & \overset{\rho}{\longrightarrow} & \operatorname{End} V & a \end{array}$$

■

5.3 Representations of principal bundles on vector bundles

Let $\mathfrak{f}$ be any fixed vector bundle over M, with a vector space V as a typical fibre. Denote by $L\mathfrak{f}$ the $GL(V)$-principal bundle of all repers $z : V \xrightarrow{\cong} \mathfrak{f}_{|x}$, $x \in M$.

Definition 5.3.1 *Let $\mu : G \to GL(V)$ be a homomorphism of Lie groups. By a μ-representation of a principal bundle $(P, \pi, M, G, \cdot)$ on $\mathfrak{f}$ we mean a μ-homomorphism of principal bun-dles*

$$F : P \longrightarrow L\mathfrak{f}. \tag{5.3}$$

Example 5.3.2 *(a). By the adjoint representation of P we mean the Ad_G-repre-sentation*

$$\mathrm{Ad}_P : P \longrightarrow L\boldsymbol{g},\ z \longmapsto \hat{z},$$

where $\hat{z}$ is defined by (5.1).

(b). The contragredient representation of (5.3) is

$$F^{\natural} : P \longrightarrow L(\mathfrak{f}^*),\ z \longmapsto (F(z)^{-1})^*.$$

(c). The symmetric product of (5.3) is

$$\bigvee^k F : P \longrightarrow L(\bigvee^k \mathfrak{f}),\ z \longmapsto \bigvee^k F(z).$$

5.4 Differential of a representation

Definition 5.4.1 *By the differential of a representation $F : P \longrightarrow L\mathfrak{f}$ we mean the representation $F' : A(P) \to A(\mathfrak{f})$ defined as the superposition $F' = \Phi_{\mathfrak{f}} \circ dF$.*

Example 5.4.2 *Consider a Lie group G as a G-principal bundle. Its Lie algebroid $A(G)$ (on a one-point manifold) can be canonically identified with the right Lie algebra $\mathfrak{g}$ of G (see [19]) via the isomorphism*

$$\varphi = \varphi_G : A(G) \longrightarrow \mathfrak{g},\ \ [v] \longmapsto \Theta^R(v),$$

where Θ^R denotes the canonical right-invariant 1-form on G. Therefore $[\![v, w]\!] = [v, w]^R$ ($[\cdot,\cdot]^R$ is the right Lie algebra structure on $\mathfrak{g}$). The Atiyah sequence of $A(G)$ equals

$$0 \longrightarrow \mathfrak{g} = \mathfrak{g} \longrightarrow 0 \longrightarrow 0$$

($\mathfrak{g}$ is treated here as a vector bundle over a one-point manifold), whereas the principal bundle $L\mathfrak{g}$ of repers of the vector bundle $\mathfrak{g}$ is the same as the Lie group $GL(\mathfrak{g})$ of all automorphisms of the vector space $\mathfrak{g}$. Besides, the following two isomorphisms

$$A(GL(\mathfrak{g})) = A(L\mathfrak{g}) \xrightarrow{\Phi_{\mathfrak{g}}} A(\mathfrak{g}) = \mathfrak{g}^* \bigotimes \mathfrak{g} \cong \mathrm{End}\,\mathfrak{g},$$

and

$$A(L\mathfrak{g}) = A(GL(\mathfrak{g})) \cong T_{\mathrm{id}}(GL(\mathfrak{g})) \xrightarrow{\rho_{\mathfrak{g}}} \operatorname{End}\mathfrak{g},$$

are identical (which is not difficult to prove). Also, after the identifications $A(G) \cong \mathfrak{g}$ *and* $A(GL(\mathfrak{g})) \cong T_{\mathrm{id}}(GL(\mathfrak{g}))$*, the adjoint representation* Ad_G *of the principal bundle* G *is simply the adjoint representation of the Lie group* G*. Therefore* $d(\mathrm{Ad}_G) = (\mathrm{Ad}_G)_{*e}$*.*

Seeing the following commuting diagram

$$\begin{array}{ccccc} \mathfrak{g} & \overset{(\mathrm{Ad}_G)_{*e}}{\longrightarrow} & T_{\mathrm{id}}(GL(\mathfrak{g})) & \overset{c}{\cong} & \operatorname{End}\mathfrak{g} \\ & \searrow^{(\mathrm{Ad}_G)'} & \Phi_{\mathfrak{g}} \downarrow\cong & \searrow^{\rho_{\mathfrak{g}}} & \downarrow -\mathrm{id} \\ & & A(\mathfrak{g}) & \cong & \operatorname{End}\mathfrak{g} \end{array}$$

and recalling that $(c \circ (\mathrm{Ad}_G)_{*e})(v)(w) = [v,w]^L$ *(*$[\cdot,\cdot]^L$ *is the left Lie algebra structure on* $\mathfrak{g}$*), we assert that, for* $v, w \in \mathfrak{g}$*,*

$$\begin{aligned}(\mathrm{Ad}_G)'(v)(w) &= -(\mathrm{Ad}_G)(v)(w) = -[v,w]^L \\ &= [v,w]^R = [\![v,w]\!] = \mathrm{ad}_{A(G)}(v)(w),\end{aligned}$$

which means that $(\mathrm{Ad}_G)' = -\,\mathrm{ad}_{A(G)}$*.*

Theorem 5.4.3 *(a)* $(\mathrm{Ad}_P)' = \mathrm{ad}_{A(P)}$*,*

(b) $(F^\natural)' = (F')^\natural$ *and* $(\bigvee^k F)' = \bigvee^k (F')$ *for any representation 5.3.*

We start with the following

Lemma 5.4.4 *Let* $\psi : U \times V \longrightarrow p^{-1}[U]$ *be a local trivialization of a vector bundle* $\mathfrak{f}$ *(with* V *as a typical fibre). For* $\nu \in \operatorname{Sec}\mathfrak{f}$*, denote by* ν_ψ *the function* $U \ni x \mapsto \psi_{|x}^{-1}(\nu) \in V$*. Then the mapping*

$$\psi : TU \times \operatorname{End}V \longrightarrow A(\mathfrak{f})_{|U},$$

such that $\bar{\psi}(v,a)(\nu) = \psi_{|x}(v(\nu_\psi) + a(\nu_\psi(x)))$ *when* $v \in T_xU$ *and* $a \in \operatorname{End}V$*, is an isomorphism of Lie algebroids.*

Proof. It is immediate that $\bar{\psi}(v,a)$ is an $\mathfrak{f}$-vector with v as the anchor, which means that $q \circ \bar{\psi} = \mathrm{pr}_1$. First, we notice that $\bar{\psi}$ is a bijection such that $\bar{\psi}_{|x} : T_xU \times \operatorname{End}V \to A(\mathfrak{f})_{|x}$ is a linear isomorphism. The fact that $\bar{\psi}_{|x}$ is a monomorphism is clear. To see that it is an epimorphism, take an arbitrary $l \in A(\mathfrak{f})_{|x}$ and notice that the element $\bar{\psi}_{|x}^{-1}(l(\nu)) - q(l)(\nu_\psi)$ of V depends only on the value of $\nu \in \operatorname{Sec}\mathfrak{f}$ at x. Denote by $a(u)$ the element where ν is a cross-section of $\mathfrak{f}$ such that $\nu(x) = \psi_{|x}(u), u \in V$. Put $a = (u \mapsto a(u)) \in \operatorname{End}V$. One can trivially assert that $\bar{\psi}(q(l), a) = l$. It remains to verify that $\operatorname{Sec}\bar{\psi}$ is a homomorphism of suitable Lie algebras. To this end, take $X, Y \in \mathfrak{X}(U)$ and

$\sigma, \eta \in \Omega^0(U; \operatorname{End} V)$. For $x \in U$ and $\nu \in \operatorname{Sec} \mathfrak{f}$, we have

$$
\begin{aligned}
&[\![\bar{\psi} \circ (X, \sigma), \bar{\psi} \circ (Y, \eta)]\!]_x(\nu) \\
&= \bar{\psi}_{|x}(X_x, \sigma_x)(\bar{\psi} \circ (Y, \eta)(\nu)) - \bar{\psi}_{|x}(Y_x, \eta_x)(\bar{\psi} \circ (X, \sigma)(\nu)) \\
&= \psi_{|x}(X_x(Y(\nu_\psi) + \eta(\nu_\psi)) + \sigma_x(Y_x(\nu_\psi) + \eta_x(\nu_\psi(x))))- \\
&- \psi_{|x}(Y_x(X(\nu_\psi) + \sigma(\nu_\psi)) + \eta_x(X_x(\nu_\psi) + \sigma_x(\nu_\psi(x)))) \\
&= \psi_{|x}([X, Y]_x(\nu_\psi) + X_x(\eta))(\nu_\psi(x)) - Y_x(\sigma)(\nu_\psi(x)) + [\sigma_x, \eta_x](\nu_\psi(x))) \\
&= (\bar{\psi} \circ ([X, Y], \mathcal{L}_X(\eta) - \mathcal{L}_Y(\sigma) + [\sigma, \eta]))_x(\nu) \\
&= (\bar{\psi} \circ [\![(X, \sigma), (Y, \eta)]\!])_x(\nu).
\end{aligned}
$$

Proof of theorem 5.4.3 (a): The case $P = G$ was considered in Example 5.4.2. To prove (a) in all its generality, take an arbitrary local trivialization $\varphi : U \times G \longrightarrow P$. φ determines a local trivialization $\varphi^A : TU \times \mathfrak{g} \to A(P)$, $(v, w) \mapsto [\varphi_*(v, w)]$, of the Lie algebroid $A(P)$, (see [19]), especially, a local trivialization $\psi := \varphi_0^A : U \times \mathfrak{g} \to \boldsymbol{g}$, $(x, w) \mapsto \varphi_{|x}^A(\theta_x, w)$,of the vector bundle $\boldsymbol{g}$. Next, according to the Lemma above, we obtain a local trivialization $\bar{\psi} : TU \times \operatorname{End} \mathfrak{g} \to A(\boldsymbol{g})$ of the Lie algebroid $A(\boldsymbol{g})$. To prove that $(\operatorname{Ad}_P)' = \operatorname{ad}_{A(P)}$, it is sufficient to show (taking account of the classical equality $(Ad)' = \operatorname{ad}$) that two following diagrams commute for any φ:

$$
\begin{array}{ccc}
A(P) & \xrightarrow{(\operatorname{Ad}_P)'} & A(\boldsymbol{g}) \\
\varphi^A \uparrow & & \uparrow \bar{\psi} \\
TU \times \mathfrak{g} & \underset{\operatorname{id} \times (\operatorname{Ad}_G)'}{\longrightarrow} & TU \times \operatorname{End} \boldsymbol{g}
\end{array}
\qquad
\begin{array}{ccc}
A(P) & \xrightarrow{\operatorname{ad}_{A(P)}} & A(\boldsymbol{g}) \\
\varphi^A \uparrow & & \uparrow \bar{\psi} \\
TU \times \mathfrak{g} & \underset{\operatorname{id} \times (-\operatorname{ad}_{\mathfrak{g}})}{\longrightarrow} & TU \times \operatorname{End} \boldsymbol{g}
\end{array}
$$

in which $(\operatorname{Ad}_G)' : \mathfrak{g} \xrightarrow{\operatorname{Ad}_{G*e}} T_{\operatorname{id}}(GL(\mathfrak{g})) \cong \operatorname{End} \mathfrak{g}$ (c as in Example 5.4.2). For the purpose, take $\nu \in \operatorname{Sec} \boldsymbol{g}$, $v \in T_x U$, $w \in \mathfrak{g}$ and notice that $\psi_{|x} = \varphi(x, e)\hat{}$, whereas $\tilde{\nu} \circ \operatorname{Ad}_P \circ \varphi : U \times G \longrightarrow \mathfrak{g}$ is given by

$$
\begin{aligned}
\tilde{\nu} \circ \operatorname{Ad}_P \circ \varphi(x, a) &= \tilde{\nu}(\varphi(x, a)\hat{}\,) = \tilde{\nu}(\varphi(x, e)\hat{} \circ \operatorname{Ad}_G a) = \operatorname{Ad}_G(a^{-1})(\nu_\psi(x)) \\
&= \nu_\psi(x)\tilde{}\,(\operatorname{Ad}_G a).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(\operatorname{Ad}_P)' \circ \varphi^A(v, w)(\nu) \\
&= \Phi_{\boldsymbol{g}} \circ d(\operatorname{Ad}_P)([\varphi_*(v, w)])(\nu) \\
&= \Phi_{\boldsymbol{g}}([\operatorname{Ad}_{P*}(\varphi_*(v, w))])(\nu) = \psi_{|x}((\operatorname{Ad}_P \circ \varphi)_*(v, w)(\tilde{\nu})) \\
&= \psi_{|x}((v, w)(\tilde{\nu} \circ \operatorname{Ad}_P \circ \varphi)) = \psi_{|x}(v(\nu_\psi) + w(\nu_\psi(x)\tilde{} \circ \operatorname{Ad}_G(\cdot))) \\
&= \psi_{|x}(v(\nu_\psi) + \rho_{\boldsymbol{g}}(\operatorname{Ad}_{G*} w)(\nu_\psi(x))) = \psi_{|x}(v(\nu_\psi) - \operatorname{Ad}'_G(w)(\nu_\psi(x))) \\
&= \bar{\psi} \circ (\operatorname{id} \times - \operatorname{Ad}'_G)(v, w)(\nu).
\end{aligned}
$$

Since $\varphi^A \circ (0, \nu_\psi) = \nu$, we have, following the fact that φ^A is an isomorphism of Lie algebroids, that

$$\begin{aligned}
\mathrm{ad}_{A(P)} \circ \varphi^A(v, w)(\nu) &= [\![\varphi^A(v, w), \nu]\!] \\
&= [\![\varphi^A(v, w), \varphi^A \circ (0, \nu_\psi)]\!] = \varphi^A_{|x}[\![(v, w), (0, \nu_\psi)]\!] \\
&= \varphi^A_{|x}(\theta_x, v(\nu_\psi) + [w, \nu_\psi(x)]^R) = \psi_{|x}(v(\nu_\psi) - [w, \nu_\psi(x)]^L) \\
&= \psi_{|x}(v(\nu_\psi) - \mathrm{ad}_{\mathfrak{g}}(w)(\nu_\psi(x))) \\
&= \bar{\psi} \circ (\mathrm{id} \times - \mathrm{ad}_{\mathfrak{g}})(v, w)(\nu).
\end{aligned}$$

(b): Consider the identical representation $\mathrm{id}_{L\mathfrak{f}} : L\mathfrak{f} \longrightarrow L\mathfrak{f}$. Of course, $\Phi_{\mathfrak{f}} : A(L\mathfrak{f}) \to A(\mathfrak{f})$ is its differential. First, we notice that

(1) $F^\natural = \mathrm{id}^\natural_{L\mathfrak{f}} \circ F$,

(2) $T^\natural = \mathrm{id}^\natural_{A(\mathfrak{f})} \circ T$ for any representation $T : A \to A(\mathfrak{f})$, in particular, $(F')^\natural = \mathrm{id}^\natural_{A(\mathfrak{f})} \circ (F')$,

(3) $(\mathrm{id}'_{L\mathfrak{f}})^\natural = (\mathrm{id}^\natural_{L\mathfrak{f}})'$ or, equivalently,

$$\mathrm{id}^\natural_{A(\mathfrak{f})} \circ \Phi_{\mathfrak{f}} = \Phi_{\mathfrak{f}*} \circ d(\mathrm{id}^\natural_{L\mathfrak{f}}).$$

(1) and (2) follow directly from the definitions. (3): Let $\varphi \in \mathrm{Sec}\,\mathfrak{f}^*$, $\nu \in \mathrm{Sec}\,\mathfrak{f}$, $u \in (L\mathfrak{f})_{|x}$ and $v \in T_u(L\mathfrak{f})$. Then

$$\begin{aligned}
&\langle \Phi_{\mathfrak{f}*} \circ d(\mathrm{id}^\natural_{L\mathfrak{f}})([v])(\varphi), \nu_x \rangle \\
&= \langle u^{-1*}((\mathrm{id}^\natural_{L\mathfrak{f}})_* (v) (\tilde{\varphi})), \nu_x \rangle \\
&= \langle v(\tilde{\varphi} \circ \mathrm{id}^\natural_{L\mathfrak{f}}), \tilde{\nu}(u) \rangle.
\end{aligned}$$

On the other hand (for $\pi : L\mathfrak{f} \longrightarrow M$ being the projection),

$$\begin{aligned}
&\langle (\mathrm{id}^\natural_{A(\mathfrak{f})} \circ \Phi_{\mathfrak{f}}([v]))(\varphi), \nu_x \rangle \\
&= \pi_* (v) \langle \varphi, \nu \rangle - \langle \varphi_x, \Phi_{\mathfrak{f}}([v])\nu \rangle \\
&= v(\langle \varphi, \nu \rangle \circ \pi) - \langle \varphi_x \circ u, v(\tilde{\nu}) \rangle.
\end{aligned}$$

To end the proof of (3), notice that $\langle \varphi, \nu \rangle \circ \pi = \langle \tilde{\varphi} \circ \mathrm{id}^\natural_{L\mathfrak{f}}, \tilde{\nu} \rangle$ and apply the Leibniz formula for $v\langle \tilde{\varphi} \circ \mathrm{id}^\natural_{L\mathfrak{f}}, \tilde{\nu} \rangle$.

From (1)-(3) above we obtain

$$\begin{aligned}
(F^\natural)' = \Phi_{\mathfrak{f}} \circ d(F^\natural) &= \Phi_{\mathfrak{f}} \circ d(\mathrm{id}^\natural_{L\mathfrak{f}} \circ F) = \mathrm{id}^\natural_{A(\mathfrak{f})} \circ \Phi_{\mathfrak{f}} \circ dF \\
&= \mathrm{id}^\natural_{A(\mathfrak{f})} \circ F' = (F')^\natural.
\end{aligned}$$

(c): First, we notice that

(1) $\bigvee^k F = (\bigvee^k \mathrm{id}_{L\mathfrak{f}}) \circ F$,

(2) $\bigvee^k T = (\bigvee^k \mathrm{id}_{A(\mathfrak{f})}) \circ T$ for any representation $T{:}A \to A(\mathfrak{f})$, in particular, $\bigvee^k (F') = (\bigvee^k \mathrm{id}_{A(\mathfrak{f})}) \circ (F')$,

(3) $\bigvee^k (\mathrm{id}'_{L\mathfrak{f}}) = (\bigvee^k \mathrm{id}_{L\mathfrak{f}})'$, or equivalently,

$$(\bigvee^k \mathrm{id}_{A(\mathfrak{f})}) \circ \Phi_{\mathfrak{f}} = \Phi_{\bigvee^k \mathfrak{f}} \circ d(\bigvee^k \mathrm{id}_{L\mathfrak{f}}).$$

(1) and (2) follow directly by the definitions. (3): Let $\nu_i \in \mathrm{Sec}\,\mathfrak{f}$, $u \in (L\mathfrak{f})_{|x}$, $v \in T_u(L\mathfrak{f})$. Then

$$\begin{aligned}
&\Phi_{\bigvee^k \mathfrak{f}} \circ d(\bigvee^k \mathrm{id}_{L\mathfrak{f}})([v])(\nu_1 \vee ... \vee \nu_k)\\
&= u \vee ... \vee u((\bigvee^k \mathrm{id}_{L\mathfrak{f}})_* (v)\, (\nu_1 \vee ... \vee \nu_k)\tilde{}\,)\\
&= u \vee ... \vee u(v((\nu_1 \vee ... \vee \nu_k)\tilde{} \circ \bigvee^k \mathrm{id}_{L\mathfrak{f}}))\\
&= u \vee ... \vee u(v(\tilde{\nu}_1 \vee ... \vee \tilde{\nu}_k))\\
&= u \vee ... \vee u(\sum_i \tilde{\nu}_1(u) \vee ... \vee v(\tilde{\nu}_i) \vee ... \vee \tilde{\nu}_k(u))\\
&= \sum_i \nu_{1x} \vee ... \vee u(v(\tilde{\nu}_i)) \vee ... \vee \nu_{kx})\\
&= \sum_i \nu_{1x} \vee ... \vee \mathrm{id}_{A(\mathfrak{f})} \circ \Phi_{\mathfrak{f}}([v])\nu_i \vee ... \vee \nu_{kx})\\
&= (\bigvee^k \mathrm{id}_{A(\mathfrak{f})}) \circ \Phi_{\mathfrak{f}}([v])(\nu_1 \vee ... \vee \nu_k).
\end{aligned}$$

From (1)-(3) above we obtain

$$\begin{aligned}
&(\bigvee^k F)'\\
&= \Phi_{\bigvee^k \mathfrak{f}} \circ d(\bigvee F) = \Phi_{\bigvee^k \mathfrak{f}} \circ d((\bigvee \mathrm{id}) \circ F) = \Phi_{\bigvee^k \mathfrak{f}} \circ d(\bigvee \mathrm{id}) \circ dF\\
&= (\bigvee^k \mathrm{id}_{L\mathfrak{f}}) \circ \Phi_{\mathfrak{f}} \circ dF = (\bigvee^k \mathrm{id}_{L\mathfrak{f}}) \circ F' = \bigvee^k (F').
\end{aligned}$$

■

Problem 5.4.5 *Prove part (a) of the above theorem immediately without using this fact for a single Lie group.*

5.5 Invariant cross-sections

Definition 5.5.1 *Let 5.3 be any representation of a principal bundle P on $\mathfrak{f}$. A cross-section $\nu \in \operatorname{Sec}\mathfrak{f}$ will be called invariant (or, more precisely, F-invariant) if there exists a vector $v \in V$ such that $F(z)(v) = \nu_{\pi z}$ for all $z \in P$ (equivalently, if the function $\tilde{\nu} \circ F$ is constant, where $\tilde{\nu}$ is defined by 5.2. Denote by $(\operatorname{Sec}\mathfrak{f})_{I(F)}$ the space of all invariant (with respect to F) cross-sections of $\mathfrak{f}$.*

Proposition 5.5.2 *Let 5.3 be a μ-representation of P on $\mathfrak{f}$. Denote by V_I the subspace of V of μ-invariant vectors (see [9,p.39]). Then, for $v \in V_I$, the function*

$$\nu_v : M \longrightarrow \mathfrak{f},\ x \mapsto F(z)(v),$$

where $z \in P_{|x}$, is a correctly defined C^∞ cross-section of $\mathfrak{f}$, and

$$V_I \to (\operatorname{Sec}\mathfrak{f})_{I(F)},\ v \longmapsto \nu_v,$$

is an isomorphism of vector spaces.

Proposition 5.5.3 *The spaces of invariant cross-sections $(\operatorname{Sec}\mathfrak{f})_{I(F)}$ and $(\operatorname{Sec}\mathfrak{f})_{I^o(F')}$ under a representation $F : P \longrightarrow L\mathfrak{f}$ and its differential $F' : A(P) \to A(\mathfrak{f})$ are related by*

(a) $(\operatorname{Sec}\mathfrak{f})_{I(F)} \subset (\operatorname{Sec}\mathfrak{f})_{I^o(F')}$,

(b) if P is connected (nothing is assumed about the connectedness of G !), then $(\operatorname{Sec}\mathfrak{f})_{I(F)} = (\operatorname{Sec}\mathfrak{f})_{I^o(F')}$.

Proof. (a). Let $\nu \in (\operatorname{Sec}\mathfrak{f})_{I(F)}$; this means that $\tilde{\nu} \circ F$ is constant. Thus, for $[w] \in A(P)_{|x}$, $w \in T_zP$, we have

$$\begin{aligned} F'([w])(\nu) &= \Phi_{\mathfrak{f}} \circ dF([w])(\nu) \\ &= \Phi_{\mathfrak{f}}[F_*(w)](\nu) = F(z)(F_*(w)(\tilde{\nu})) \\ &= F(z)(w(\tilde{\nu} \circ F)) = 0. \end{aligned}$$

(b). Let $\nu \in (\operatorname{Sec}\mathfrak{f})_{I^o(F')}$; this means that $F'(v)(\nu) = 0$ for all $v \in A(P)$. Let $w \in T_zP$, then

$$\begin{aligned} w(\tilde{\nu} \circ F) &= F_*(w)(\tilde{\nu}) = F(z)^{-1}(\Phi_{\mathfrak{f}}([F_*(w)])(\nu)) \\ &= F(z)^{-1}F'([w])(\nu) = 0. \end{aligned}$$

From the assumption about the connectedness of P it follows that $\tilde{\nu} \circ F$ is constant. ■

5.6 The Chern-Weil homomorphism

Consider the representation $\operatorname{Ad}_G^\vee : G \to \operatorname{GL}\left(\bigvee^k \mathfrak{g}^*\right)$ induced by Ad_G on the k-symmetric power of the dual vector space $\mathfrak{g}^*$. According to 5.3.2(b)(c), Ad_P :

$P \longrightarrow L\boldsymbol{g}$ determines the $\mathrm{Ad}_G^{\vee}$-representation $\mathrm{Ad}_P^{\vee}$ $(:= \bigvee^k \mathrm{ad}_P^{\natural})$: $P \longrightarrow L(\bigvee^k \boldsymbol{g}^*)$. Theorem 5.4.3 yields that the differential of $\mathrm{Ad}_P^{\vee}$ is equal to $\mathrm{ad}_{A(P)}^{\vee}$ $(:= \bigvee^k \mathrm{ad}_{A(P)}^{\natural})$: $A(P) \to A(\bigvee^k \boldsymbol{g}^*)$; therefore Propositions 5.5.2 and 5.5.3 give rise to a monomorphism of vector spaces

$$\nu : (\bigvee^k \mathfrak{g}^*)_I \rightarrowtail (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^o}, \quad w \longmapsto \nu_w,$$

where $\nu_w(x) = \bigvee^k (\hat{z}^{-1})^*(w)$, $x \in M$, $z \in P_{|x}$, and next, assert that φ is an isomorphism if P is connected.

Theorem 5.6.1 *(cf. [17], [19]) The Chern-Weil homomorphism h_P of P and $h_{A(P)}$ of $A(P)$ are related by the following commutative diagram*

$$\begin{array}{ccc} \bigoplus^{k\geq 0}(\mathrm{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o} & & \\ & \searrow^{h_{A(P)}} & \\ \uparrow \nu & & H_{dR}(M) \\ & \nearrow_{h_P} & \\ (\bigvee \mathfrak{g}^*)_I & & \end{array}$$

Proof. To see this, we only need to observe the equality

$$\pi^*(\frac{1}{k!} \cdot \langle \nu_w, \Omega_b \vee ... \vee \Omega_b \rangle) = \frac{1}{k!} \cdot \langle w, \Omega \vee ... \vee \Omega \rangle \tag{5.4}$$

where Ω and Ω_b are, respectively: the curvature form of some connection $H \subset TP$ in P and the curvature tensor of the corresponding connection λin the Lie algebroid $A(P)$ $(H_{|z} = (\pi^A_{|z})^{-1}[\mathrm{Im}\,\lambda_{|z}],\ z \in P$, where $\pi^A : TP \to A(P)$ is the canonical projection). Both sides of 5.4 are horizontal forms, so we must notice the equality on the horizontal vectors only. Let $\lambda : TM \to A(P)$ be any connection in $A(P)$ and let $v^z \in T_zP$ denote the horizontal lifting of $v \in T_{\pi z}M$. By the relationship between Ω_b and Ω,

$$\Omega_b(x; v \wedge w) = \hat{z}(\Omega(z, v^z \wedge w^z)), \quad z \in P_{|x}, \quad v, w \in T_xM,$$

we have, for $w \in (\bigvee^k \mathfrak{g}^*)_I$, $z \in P$ and $v_i \in T_{\pi z} M$,

$$\begin{aligned}
&\pi^*(\frac{1}{k!} \cdot \langle \nu_w, \Omega_b \vee \ldots \vee \Omega_b \rangle)(z; v_1^z \wedge \ldots \wedge v_{2k}^z) \\
&= \frac{1}{k!} \cdot \langle (\bigvee^k (\hat{z}^{-1})^*)(w), \frac{1}{2^k} \cdot \sum_\sigma \operatorname{sgn}\sigma \cdot \Omega_b(x; v_{\sigma(1)} \wedge v_{\sigma(2)}) \vee \ldots \\
&\qquad \ldots \vee \Omega_b(x; v_{\sigma(2k-1)} \wedge v_{\sigma(2k)}) \rangle \\
&= \frac{1}{k!} \cdot \langle w, \frac{1}{2^k} \cdot \sum_\sigma \operatorname{sgn}\sigma \cdot (\hat{z}^{-1}) \Omega_b(x; v_{\sigma(1)} \wedge v_{\sigma(2)}) \vee \ldots \\
&\qquad \ldots \vee (\hat{z}^{-1}) \Omega_b(x; v_{\sigma(2k-1)} \wedge v_{\sigma(2k)}) \rangle \\
&= \frac{1}{k!} \cdot \langle w, \frac{1}{2^k} \cdot \sum_\sigma \operatorname{sgn}\sigma \cdot \Omega(z; v_{\sigma(1)}^z \wedge v_{\sigma(2)}^z) \vee \ldots \vee \Omega(z; v_{\sigma(2k-1)}^z \wedge v_{\sigma(2k)}^z) \rangle \\
&= \frac{1}{k!} \cdot \langle w, \Omega \vee \ldots \vee \Omega \rangle (z; v_1^z \wedge \ldots \wedge v_{2k}^z).
\end{aligned}$$

■

Remark 5.6.2 *In [19] it is proved that the Chern-Weil homomorphism of a principal bundle is an invariant of the so-called "local isomorphisms" between principal bundles, fulfilling an additional condition (the Ch-W property) which is satisfied, for example, in the case of principal bundles with connected structure Lie groups. By 5.6.1 above, we can assert more, namely, that the Chern-Weil homomorphism of a principal bundle is a characteristic feature of the Lie algebroid of this bundle provided only that it is connected. In consequence, the Chern-Weil homomorphism of a principal bundle is an invariant of all local isomorphisms between connected principal bundles. More precisely, we have:*

Proposition 5.6.3 *Let $\mathfrak{F} : P' \longrightarrow P$ be a local homomorphism of principal bundles (see [16], [19]). Assume that P' is connected. Then, for an arbitrary partial homomorphism $F : P' \supset D_F \longrightarrow P$ belonging to $\mathfrak{F}$ and the corresponding local homomorphism $\mu : G' \supset D_\mu \longrightarrow G$ of Lie groups, we have*

(1) $\bigvee (d\mu)^* [(\bigvee \mathfrak{g}^*)_I] \subset (\bigvee \mathfrak{g}'^*)_I$ *and* $\bigvee (d\mu)^* : (\bigvee \mathfrak{g}^*)_I \to (\bigvee \mathfrak{g}'^*)_I$ *is independent of the choice of $F \in \mathfrak{F}$,*

(2) the diagram

$$\begin{array}{ccc}
(\bigvee \mathfrak{g}^*)_I & & \\
 & \searrow^{h_P} & \\
\bigvee (d\mu)^* \downarrow & & H_{dR}(M) \\
 & \nearrow_{h_{P'}} & \\
(\bigvee \mathfrak{g}'^*)_I & &
\end{array}$$

commutes.

Proof. Let $d\mathfrak{F} : A(P') \to A(P)$ be the homomorphism of Lie algebroids induced by $\mathfrak{F}$. By functoriality property 4.2.2, we obtain the commutative diagram

$$\begin{array}{ccccc}
(\bigvee \mathfrak{g}^*)_I & & \xrightarrow{h_P} & & H_{dR}(M) \\
& \searrow^{\nu} & & \nearrow^{h_{A(P)}} & \\
& & \bigoplus^{k\geq 0}(\mathrm{Sec}\bigvee^k \boldsymbol{g}^*)_{I^o} & & \\
& & \downarrow (d\mathfrak{F})^{+*} & & \| \\
& & \bigoplus^{k\geq 0}(\mathrm{Sec}\bigvee^k \boldsymbol{g}'^*)_{I^o} & & \\
& \cong \nearrow_{\nu'} & & \searrow_{h_{A(P')}} & \\
(\bigvee \mathfrak{g}'^*)_I & & \xrightarrow{h_{P'}} & & H_{dR}(M)
\end{array}$$

To end the proof, it is enough to check that $\bigvee(d\mu)^* = \nu'^{-1} \circ (d\mathfrak{F})^{+*} \circ \nu$. Let F be an arbitrary partial homomorphism belonging to $\mathfrak{F}$. Take $x \in U_F$ and $z \in P'_{|x}$. By the obvious equality $F(z)\hat{} \circ d\mu = dF^+_{|x} \circ \hat{z}$, we have the commutative diagram

$$\begin{array}{ccc}
\bigvee \mathfrak{g}^* & \xrightarrow{\bigvee^k \left(F(z)\hat{}^{-1}\right)^*} & \bigvee^k \boldsymbol{g}^*_{|x} \\
{\scriptstyle \bigvee^k (d\mu)^*} \downarrow & & \downarrow {\scriptstyle \bigvee^k (dF^+_{|x})^*} \\
\bigvee \mathfrak{g}'^* & \xrightarrow{\bigvee^k \left(\hat{z}^{-1}\right)^*} & \bigvee^k \boldsymbol{g}'^*_{|x}
\end{array}$$

Notice also that $(d\mathfrak{F}^+)^*(\Gamma)_x = \bigvee^k (dF^+_{|x})^*(\Gamma_x)$, and that $\nu_w(x) = \bigvee^k (F(z)\hat{}^{-1})^*(w)$. The result is now trivial:

$$\nu'^{-1}((d\mathfrak{F}^+)^*(\nu_w)) = \bigvee^k \hat{z}^* \circ \bigvee^k (dF^+_{|x})^* \circ \bigvee^k (F(z)\hat{}^{-1})^*(w) = \bigvee^k (d\mu)^*(w).$$

■

5.7 Remarks on the tangential Chern-Weil homomorphism

Let P be a connected H-principal bundle on a manifold M, and $F \subset TM$ a C^∞ constant dimensional involutive distribution. Let $\mathcal{F}$ denote the foliation of M determined by F. We recall that the transitive Lie algebroid $A(P)$ and the distribution F give rise to a regular Lie algebroid over (M, F) equalling $A(P)^F := \gamma^{-1}[F] \subset A(P)$, see 1.1.2. By the *tangential Chern-Weil homomorphism* of P over the foliated manifold $(M, \mathcal{F})$ we mean the Chern-Weil homomorphism

$$h_{A(P)^F} : \bigoplus^k (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^o(\mathrm{ad}_{A(P)^F})} \longrightarrow H_F(M)$$

of the regular Lie algebroid $A(P)^F$ ($\boldsymbol{g}$ is the Lie algebra bundle adjoint of $A(P)$). $h_{A(P)^F}$ measures the nonexistence of a partial (over F) flat connection in P. In the case of P equalling to the G-principal bundle $L_G\mathfrak{f}$ of G-repers of some

G-vector bundle $\mathfrak{f}$ $(G \subset \mathrm{GL}(n, \mathbb{R}), n = \mathrm{rank}\,\mathfrak{f})$, the tangential Chern-Weil homomorphism measures the nonexistence of (suitable) flat partial covariant derivatives. Notice that the superposition

$$(\bigvee \mathfrak{g}^*)_I \cong \bigoplus^k (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I(\mathrm{ad}_A)} \subset \bigoplus^k (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^\circ(\mathrm{ad}_{A^F})} \xrightarrow{h_{A^F}} H_F(M)$$

(in which $A := A(L_G \mathfrak{f})$) agree for $G = \mathrm{GL}(n, \mathbb{R})$ with the homomorphism obtained by Moore and Schochet [28] to investigate vector bundles over foliated manifolds. However, the domain of our homomorphism h_{A^F} contains, in general, more elements. To further consideration of the matter, the author will devote an individual paper.

In the end, we add that the generalization of the Bott Vanishing Theorem from [14] can be formulated in our language as follows:

- *Let* $\{F, F'\}$ $(F' \subset F \subset TM)$ *be a flag of foliations on* M. *If* $F = F' \bigoplus f$, *then* $\mathrm{Pont}^k(A(f)^F) = 0$ *for* $k \geq 2 \cdot \mathrm{rank}\, f$.

This theorem follows easily from the existence of a flat partial covariant derivative in $\mathfrak{f}$ over F.

6 THE LIE ALGEBROID OF A TC-FOLIATION

6.1 TC-foliations. Basic properties [26], [27]

A foliation $(M, \mathcal{F})$is said to be *transversally complete* [*TC-foliation* for short] (see P.Molino [26], [27]) if, at each point $x \in M$, the family $L_c(M, \mathcal{F})$ of complete global ($\mathcal{F}$-)foliate vector fields generates the entire tangent space $T_x M$.

For an arbitrary TC-foliation, we adopt the following notations:

- $\mathcal{F}_b$- the basic foliation,
- E, E_b - the vector bundles tangent to $\mathcal{F}$ and $\mathcal{F}_b$, respectively,
- L_x, L_{bx} - the leaves of $\mathcal{F}$ and $\mathcal{F}_b$, respectively, passing through $x \in M$,
- $r : Q \to M$ $(Q = TM/E)$ - the transverse bundle,
- $\pi_b : M \to W$ - the basic fibration,
- $\alpha : TM \to Q$ - the canonical projection,
- $\bar{X} := \alpha \circ X$ - the cross-section of Q corresponding to a (local) vector field X on M,
- $l(M, \mathcal{F})$ - the Lie algebra (and the $\Omega^0(W)$-module, as well) of transverse fields.

Recall that by a *transverse field* we mean a cross-section $\zeta \in \operatorname{Sec} Q$ such that, in any simple distinguished open set U equipped with distinguished local coordinates $\left(x^1, ..., x^p, y^1, ..., y^q\right)$ $(p = \dim \mathcal{F},\ q = \operatorname{codim} \mathcal{F})$, ζ is of the form $\zeta = \sum_j b^j \cdot \overline{\frac{\partial}{\partial y^j}}$ for the functions b^j constant on the plaques. If $\zeta = \bar{X}$, then $\xi \in l\,(M, \mathcal{F})$ if and only if $X \in L\,(M, \mathcal{F})$.

Besides, the foliation $\mathcal{F}$ is simple and defined by a locally trivial basic fibration $\pi_b : M \to W$ with a Hausdorff manifold W.

A fundamental role in the construction of the Lie algebroid of $(M, \mathcal{F})$ is played by the following properties:

(A) If $\zeta, \nu \in l\,(M, \mathcal{F})$ and, for some $x \in M$, $\zeta\,(x) = \nu\,(x)$, then $\zeta\,(y) = \nu\,(y)$ for all $y \in L_{bx}$.

(B) Every foliate vector field X projects onto W, giving a vector field X_W, and the homomorphism of Lie algebras $L\,(M, \mathcal{F}) \longrightarrow \mathfrak{X}\,(W)\,,\ X \longmapsto X_W$, factorizes to a homomorphism of Lie algebras $\bar{\gamma} : l\,(M, \mathcal{F}) \longrightarrow \mathfrak{X}\,(W)\,,$ $\bar{X} \longmapsto X_W$. The following equality holds:

$$[\bar{X}, \bar{f} \circ \pi_b \cdot \bar{Y}] = \bar{f} \circ \pi_b \cdot [\bar{X}, \bar{Y}] + X_W(\bar{f}) \cdot \bar{Y}, \quad \bar{f} \in \Omega^0\,(W)\,,\ X, Y \in L\,(M, \mathcal{F})\,.$$

6.2 Construction of the Lie algebroid of a TC-foliation

Let $(M, \mathcal{F})$ be an arbitrary TC-foliation. In the transverse bundle $r : Q \to M$ of $(M, \mathcal{F})$ we introduce the equivalence relation " $\approx$" as follows :

For $\bar{v}, \bar{w} \in Q$ we put

$$\bar{v} \approx \bar{w} \iff \{\pi_b(r\bar{v}) = \pi_b(r\bar{w}) \text{ and } \exists_{\zeta \in l(M,\mathcal{F})}\ (\zeta(r\bar{v}) = \bar{v} \text{ and } \zeta(r\bar{w}) = \bar{w})\}.$$

(A) and (B) above makes the following lemma obvious.

Lemma 6.2.1 *Take x and y lying on the same leaf of the basic foliation $\mathcal{F}_b$. Then, for each vector $\bar{v} \in Q_{|x}$, there exists exactly one vector $\bar{w} \in Q_{|y}$ such that $\bar{v} \approx \bar{w}$. The correspondence $\bar{v} \longmapsto \bar{w}$ establishes a linear isomorphism $\alpha_x^y : Q_{|x} \longrightarrow Q_{|y}$.*

Clearly, two vectors $\bar{v}, \bar{w} \in Q$ are in the equivalence relation $\approx$ if and only if they corresponds to each other via one of the isomorphisms α_x^y. In the sequel, $[\bar{v}]$ denotes the equivalence class of $\bar{v}$ and $A\,(M, \mathcal{F}) := Q/\approx$ denotes the set of all equivalence classes (with the quotient topology) and

$$\bar{r} : A\,(M, \mathcal{F}) \longrightarrow W, \quad [\bar{v}] \longmapsto \pi(r\bar{v}),$$

the projection.

Each fibre $A\,(M, \mathcal{F})_{|\bar{x}} := \bar{r}^{-1}\,(\bar{x})\,,\ \bar{x} \in W$, possesses a structure of a vector space, defined uniquely by demanding that for each $x \in \pi_b^{-1}\,(\bar{x})$ the canonical bijection $\beta_{|x} : Q_{|x} \longrightarrow A\,(M, \mathcal{F})_{|x}\,,\ \bar{v} \longmapsto [\bar{v}]$, be a linear isomorphism. The family $\beta_{|x},\ x \in M$, determines the canonical projection $\beta : Q \longrightarrow A\,(M, \mathcal{F})$

being a homomorphism of vector bundles over the basic fibration π_b. We equip $A(M,\mathcal{F})$ with a structure of a C^∞ manifold as follows: For any $\bar{x} \in W$, we find (as a consequence of (A)) its open neighbourhood $\bar{U}$ and transverse fields $\zeta_1, ..., \zeta_q \in l(M,\mathcal{F})$ which are linearly independent on $U := \pi_b^{-1}[\bar{U}]$, and we put

$$\begin{aligned} \varphi : \bar{U}\times\mathbb{R}^q &\longrightarrow \bar{r}^{-1}[\bar{U}]\subset A \\ (\bar{x}, a) &\longmapsto [\sum_i a^i \zeta_{ix}], \ \ x \in \pi_b^{-1}(\bar{x}). \end{aligned}$$

φ is a bijection such that $\varphi_{\bar{x}} : \mathbb{R}^q \longrightarrow A(M,\mathcal{F})_{|\bar{x}}$ is an isomorphism of vector spaces. It is easy to see that $\bar{r}^{-1}[\bar{U}]$ is open and φ is a hemeomorphism, see the following diagram:

$$\begin{array}{ccccccc} U\times\mathbb{R}^q & \xrightarrow{\cong} & r^{-1}[U] & \subset & Q & & (x,a) \longmapsto \sum_i a^i \zeta_{ix} \\ \downarrow \pi_b \times \mathrm{id} & & & & \downarrow \beta & & \\ \bar{U}\times\mathbb{R}^q & \xrightarrow{\varphi} & \bar{r}^{-1}[\bar{U}] & \subset & A(M;\mathcal{F}) & & \end{array}$$

In $A(M,\mathcal{F})$ there is exactly one C^∞ manifold structure (compatible with the topology) for which the φ's are diffeomorphisms. To see this, we must only notice that, for another φ' (defined on $\bar{U}' \times \mathbb{R}^q$ via $\zeta'_1, ..., \zeta'_q \in l(M,\mathcal{F})$), $\varphi'^{-1} \circ \varphi$ is C^∞. Clearly, for a point $x_o \in \bar{U} \cap \bar{U}'$ there exists its neighbourhood $\bar{U}''\subset\bar{U} \cap \bar{U}'$ and functions $\bar{f}_i^j \in \Omega^0(W)$ such that $\zeta_i = \sum_j \bar{f}_i^j \circ \pi_b \cdot \zeta'_j$ on $U'' := \pi_b^{-1}[\bar{U}'']$. Therefore we have

$$\varphi'^{-1} \circ \varphi(\bar{x}, a) = (\bar{x}, (\sum_i a^i \bar{f}_i^1(\bar{x}), ..., \sum_i a^i \bar{f}_i^q(\bar{x}))), \quad \bar{x} \in U'', \ a \in \mathbb{R}^q,$$

which proves the smoothness of $\varphi'^{-1} \circ \varphi$. Of course, $\bar{r} : A(M,\mathcal{F}) \longrightarrow W$ is C^∞ and $(A(M,\mathcal{F}), \bar{r}, W)$ is a vector bundle with φ's as local trivializations.

The mapping

$$\gamma : A(M,\mathcal{F}) \longrightarrow TW, \ \ [\bar{v}] \longmapsto \pi_{b*}(v),$$

is a correctly defined epimorphism of vector bundles.

Proposition 6.2.2 *(1) A cross-section $\zeta \in \operatorname{Sec} Q$ is a transverse field if and only if there exists a cross-section $\xi \in \operatorname{Sec} A(M,\mathcal{F})$ such that the following diagram*

$$\begin{array}{ccc} Q & \xrightarrow{\beta} & A \\ \uparrow \zeta & & \uparrow \xi \\ M & \xrightarrow{\pi_b} & W \end{array} \tag{6.1}$$

commutes. Such a ξ is at most one.

(2) The correspondence $\zeta \longmapsto \xi$ establishes an isomorphism of $\Omega^0(W)$-\3modules

$$c : l(M,\mathcal{F}) \longrightarrow \operatorname{Sec} A(M,\mathcal{F}),$$

Proof. (1): Necessity is evident.

Sufficiency. Let $\zeta \in \operatorname{Sec} Q$ be a cross-section of Q for which there exists $\xi \in \operatorname{Sec} A(M,\mathcal{F})$ making diagram 6.1 commute. Equivalently,

$$\alpha_x^y(\zeta_x) = \zeta_y$$

for any points x and y lying on the same leaf of $\mathcal{F}_b$. To prove that ζ is a transverse field, we first observe that if $(x^1,...,x^p,y^1,...,y^q)$ are distinguished local coordinates in U, then, for any points x and y lying on the same plaque, we have $\overline{\frac{\partial}{\partial y^i}}_{|x} \approx \overline{\frac{\partial}{\partial y^i}}_{|y}$, $i \leq q$. Indeed, $(M,\mathcal{F})$is TC, therefore there exists a transverse field $\nu \in l(M,\mathcal{F})$ such that $\nu_x = \overline{\frac{\partial}{\partial y^i}}_{|x}$. Locally on U, $\nu = \sum_k b^k \cdot \overline{\frac{\partial}{\partial y^k}}$ with the functions b^k constant on plaques. Since $b^k(x) = \delta_i^k$, therefore $b^k(y) = \delta_i^k$; in consequence, $\nu_y = \overline{\frac{\partial}{\partial y^i}}_{|y}$, so, by the definition of the equivalence relation $\approx$, we have that $\overline{\frac{\partial}{\partial y^i}}_{|x} \approx \overline{\frac{\partial}{\partial y^i}}_{|y}$.

Passing to the proof of sufficiency, write locally $\zeta = \sum_k b^k \cdot \overline{\frac{\partial}{\partial y^k}}$. Take x and y belonging to one of the plaques. We have, by the above,

$$\sum_k b^k(y) \cdot \overline{\frac{\partial}{\partial y^k}}_{|y} = \zeta_y = \alpha_x^y(\zeta_x) = \alpha_x^y(\sum_k b^k(x) \cdot \overline{\frac{\partial}{\partial y^k}}_{|x})$$
$$= \sum_k b^k(x) \cdot \alpha_x^y(\overline{\frac{\partial}{\partial y^k}}_{|x}) = \sum_k b^k(x) \cdot \overline{\frac{\partial}{\partial y^k}}_{|y}.$$

Thus $b^k(x) = b^y(y)$, which confirms (1).

(2): c is a monomorphism of $\Omega^0(W)$-modules, as is easy to check. The surjectivity follows from (1) and the observation indicating that, for a cross-section $\xi \in \operatorname{Sec} A(M,\mathcal{F})$, there exists a cross-section $\zeta \in \operatorname{Sec} Q$ making diagram 6.1 commute. ■

In $\operatorname{Sec} A(M,\mathcal{F})$ we introduce the bracket $[\![\cdot,\cdot]\!]$ (forming a Lie algebra) by demanding that c be an isomorphism of Lie algebras, i.e. $[\![c(\zeta),c(\nu)]\!] = c([\zeta,\nu])$, $\zeta,\nu \in l(M,\mathcal{F})$. The system $(A(M,\mathcal{F}), [\![\cdot,\cdot]\!], \gamma)$ is a transitive Lie algebroid (over the basic manifold W), which is clear from (B). It is called the Lie algebroid of the TC-foliation $(M,\mathcal{F})$. Let $\boldsymbol{g} = \ker\gamma$ be the adjoint Lie algebra bundle of $A(M,\mathcal{F})$. We have the following isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & l^+(M,\mathcal{F}) & \hookrightarrow & l(M,\mathcal{F}) & \overset{\bar{\gamma}}{\longrightarrow} & \mathfrak{X}(W) & \longrightarrow & 0 \\ & & \cong\downarrow c^+ & & \cong\downarrow c & & \| & & \\ 0 & \longrightarrow & \operatorname{Sec}\boldsymbol{g} & \hookrightarrow & \operatorname{Sec} A(M,\mathcal{F}) & \longrightarrow & \mathfrak{X}(W) & \longrightarrow & 0 \end{array}$$

6.3 Connections and the Chern-Weil homomorphism

Let $(M,\mathcal{F})$ be an arbitrary TC-foliation and $(A(M,\mathcal{F}), [\![\cdot,\cdot]\!], \gamma)$ - its Lie algebroid. Notice that, for any $x \in M$, the isomorphism $\beta_{|x} : Q_{|x} \longrightarrow A(M,\mathcal{F})_{|\bar{x}}$ maps $Q'_{|x} := E_{b|x}/E_{|x}$ onto $\boldsymbol{g}_{|\bar{x}}$, $\bar{x} := \pi_b(x)$. A connection λ in $A(M,\mathcal{F})$ determines the so-called horizontal subbundle $C^\lambda := \operatorname{Im}\lambda \subset A(M,\mathcal{F})$ (i.e. the condition $A(M,\mathcal{F}) = \boldsymbol{g} \bigoplus C^\lambda$ holds), and next, the distribution $\bar{C}^\lambda \subset TM$ on the manifold M by $\bar{C}^\lambda_{|x} := \alpha^{-1}_{|x}[\beta^{-1}_{|x}[C^\lambda_{|\bar{x}}]]$, $x \in M$.

Lemma 6.3.1 *The correspondence $\lambda \longmapsto \bar{C}^\lambda$ establishes a bijection between connections in $A(M,\mathcal{F})$ and distributions $\bar{C} \subset TM$ such that*

(1) $E_b \cap \bar{C} = E$,

(2) $E_b + \bar{C} = TM$,

(3) $\bar{C}_{|x} = \{X(x);\, X \in L(M,\mathcal{F}) \cap \operatorname{Sec}\bar{C}\}$ *for each point $x \in M$. In particular, such a distribution $\bar{C}$ always exists (and is C^∞).*

Proof. "$\Longrightarrow$" Let $\bar{C} = \bar{C}^\lambda$ for some connection λ.

(1): $(E_b \cap \bar{C})_{|x} = \alpha^{-1}_{|x}[\beta^{-1}_{|x}[\boldsymbol{g}_{|\bar{x}} \cap C^\lambda_{|\bar{x}}]] = \ker((\beta \circ \alpha)_{|x}) = E_{|x}$.

(2): $(E_b + \bar{C})_{|x} = \alpha^{-1}_{|x}[\beta^{-1}_{|x}[\boldsymbol{g}_{|\bar{x}} + C^\lambda_{|\bar{x}}]] = \alpha^{-1}_{|x}[\beta^{-1}_{|x}[A(M,\mathcal{F})_{|\bar{x}}]] = T_xM$.

(3): Let $v \in \bar{C}_{|x}$. We have to find a foliate vector field X lying in the distribution $\bar{C}$ and such that $X_x = v$. For the purpose, take arbitrarily a cross-section $\xi \in \operatorname{Sec}(C^\lambda)$ such that $\xi_{|\bar{x}} = [\bar{v}]$, and next, the cross-section $\zeta \in \operatorname{Sec} Q$ defined by $\zeta_y = \beta^{-1}_{|y}(\xi_{\bar{y}})$, $y \in \pi_b^{-1}(\bar{y})$, $\bar{y} \in W$. ζ is a transverse field, see Proposition 6.2.2. Let $\zeta = \bar{Y}$ for a foliate vector field Y. Then $v - Y_x \in E_{|x}$. Taking an arbitrary vector field $X \in \mathfrak{X}(\mathcal{F})$ such that $X_x = v - Y_x$, we obtain that $X + Y \in L(M,\mathcal{F}) \cap \operatorname{Sec}\bar{C}$ and $(X+Y)_x = v$.

"$\Longleftarrow$" Let $\bar{C} \subset TM$ be any distribution on M satisfying (1)-(3). There exists a subbundle $C \subset A(M,\mathcal{F})$ such that $C_{|\bar{x}} = \beta_{|x} \circ \alpha_{|x}[\bar{C}_{|x}]$, $x \in \pi_b^{-1}(\bar{x})$, $\bar{x} \in W$. To see this this formula, i.e. the independence of the right-hand side of the choice of a point $x \in \pi_b^{-1}(\bar{x})$. In order to get this, it is sufficient to notice the inclusion $\beta_{|x} \circ \alpha_{|x}[\bar{C}_{|x}] \subset \beta_{|y} \circ \alpha_{|y}[\bar{C}_{|y}]$ for $x, y \in \pi_b^{-1}(\bar{x})$. For $v \in \bar{C}_{|x}$ and $X \in L(M,\mathcal{F}) \cap \operatorname{Sec}\bar{C}$ such that $X_x = v$, we have $X_y \in \bar{C}_{|y}$. Since $\bar{X}$ is a transverse field, according to the definition of the equivalence relation $\approx$ in Q, we have $[\bar{v}] = [\bar{X}_y] \in \beta_{|y} \circ \alpha_{|y}[\bar{C}_{|y}]$. C is easily seen to be C^∞ and complementing $\boldsymbol{g}$, thus, in consequence, determining some connection λ for which the property $\bar{C}^\lambda = \bar{C}$ is obvious by the construction. ∎

Definition 6.3.2 ***(a)***. *A distribution $\bar{C} \subset TM$ fulfilling (1)-(3) from Lemma 6.3.1 will be called a connection for the TC-foliation $(M,\mathcal{F})$.*

(b). *If $\bar{C} = \bar{C}^\lambda$ for a connection λ in $A(M,\mathcal{F})$ and if ω and Ω_b are the connection form and the curvature tensor of λ, respectively, then the tensors $\bar{\omega} \in \Omega^1(M;Q')$ and $\bar{\Omega} \in \Omega^2(M;Q')$ defined in such a way that the following diagrams*

$$
\begin{array}{ccc}
T_xM & & \\
\alpha_{|x}\downarrow & \searrow \bar{\omega}_x & \\
Q_{|x} & & Q'_{|x} \\
\beta_{|x}\downarrow\cong & & \cong\downarrow \beta^+_{|x} \\
A_{|\bar{x}} & \xrightarrow{\omega_{|\bar{x}}} & \boldsymbol{g}_{|\bar{x}}
\end{array}
\qquad
\begin{array}{ccc}
T_xM \times T_xM & & \\
& \searrow \bar{\Omega}_x & \\
\pi_{b*} \times \pi_{b*}\downarrow & & Q'_{|x} \\
& & \cong\downarrow \beta^+_{|x} \\
T_{\bar{x}} \times T_{\bar{x}} & \xrightarrow{\Omega_{b\bar{x}}} & \boldsymbol{g}_{\bar{x}}
\end{array}
$$

are commutative will be called the connection form and the curvature form of the connection $\bar{C}$, respectively.

$\bar{\omega}$ may be defined immediately in the following way: $\bar{\omega}(v) = \bar{v}_1$ $(= \alpha(v_1))$ if $v = v_1 + v_2$ is an arbitrary decomposition such that $v_1 \in E_b$, $v_2 \in \bar{C}$. Lemma 6.3.3 below gives an independent definition of $\bar{\Omega}$. Let $C_u \subset \bar{C}$ be any complement of E (i.e. $\bar{C} = E \bigoplus C_u$). Of course, $TM = E_b \bigoplus C_u$. Put $\bar{H} : TM \longrightarrow TM$ as the projection onto the second component. $\bar{H}$ plays a role of the horizontal projection for $\bar{C}$ (although it is not uniquely determined by $\bar{C}$), giving the equality

$$H \circ (\beta \circ \alpha) = (\beta \circ \alpha) \circ \bar{H}$$

in which H is the horizontal projection for the connection λ.

Lemma 6.3.3 *(a) The vector field $\bar{H} \circ Y$ is foliate if $Y \in \mathfrak{X}(M)$ is such a field,*
(b) For $Y_1, Y_2 \in L(M, \mathcal{F})$ $\bar{\Omega}(Y_1, Y_2) = -\bar{\omega}([\bar{H} \circ Y_1, \bar{H} \circ Y_2])$.

Proof. (a): Let $Y \in L(M, \mathcal{F})$. To prove that $\bar{H} \circ Y$ is a foliate vector field, it is sufficient to show that $\alpha \circ (\bar{H} \circ Y)$ is a transverse field. Since

$$\beta \circ (\alpha \circ \bar{H} \circ Y) = H \circ \beta \circ \alpha \circ Y = H \circ \beta \circ \bar{Y} = (H \circ c(\bar{Y})) \circ \pi_b,$$

proposition 6.2.2(1) yields our assertion.

(b): Let $Y_1, Y_2 \in L(M, \mathcal{F})$ and $x \in M$. We have

$$\begin{aligned}
\bar{\Omega}_x(Y_{1x}, Y_{2x}) &= \beta^{+-1}_{|x} \circ \beta^{+}_{|x}(\bar{\Omega}_x(Y_{1x}, Y_{2x})) \\
&= \beta^{+-1}_{|x}(\Omega_{b\bar{x}}(\pi_{b*}Y_{1x}, \pi_{b*}Y_{2*})) \\
&= \beta^{+-1}_{|x}(\Omega_{b\bar{x}}(\gamma_{|\bar{x}}(c(\bar{Y}_1)_{\bar{x}}), \gamma_{|\bar{x}}(c(\bar{Y}_2)_{\bar{x}}))) \\
&= \beta^{+-1}_{|x}(\Omega_b(\gamma \circ c(\bar{Y}_1), \gamma \circ c(\bar{Y}_2))_{\bar{x}}) \\
&= -\beta^{+-1}_{|x}\omega_{\bar{x}}([\![\lambda \circ \gamma \circ c(\bar{Y}_1), \lambda \circ \gamma \circ c(\bar{Y}_2)]\!]_{\bar{x}}) \\
&= -\beta^{+-1}_{|x}\omega_{\bar{x}}([\![H \circ c\left(\bar{Y}_1\right), H \circ c\left(\bar{Y}_2\right)]\!]_{\bar{x}}) \\
&= -\beta^{+-1}_{|x}\omega_{\bar{x}}([\![c(\alpha \circ \bar{H} \circ Y_1), c(\alpha \circ \bar{H} \circ Y_2)]\!]_{\bar{x}}) \\
&= -\beta^{+-1}_{|x}\omega_{\bar{x}}(c([\alpha \circ \bar{H} \circ Y_1, \alpha \circ \bar{H} \circ Y_2])_{\bar{x}}) \\
&= -\beta^{+-1}_{|x}\omega_{\bar{x}}(c(\alpha \circ [\bar{H} \circ Y_1, \bar{H} \circ Y_2]_x)) \\
&= -\beta^{+-1}_{|x}\omega_{\bar{x}}(\beta_{|x} \circ \alpha_{|x}([\bar{H} \circ Y_1, \bar{H} \circ Y_2]_x)) \\
&= -\bar{\omega}_x([\bar{H} \circ Y_1, \bar{H} \circ Y_2]_x).
\end{aligned}$$

■

Proposition 6.3.4 *The following conditions are equivalent:*
(1) $\Omega_b = 0$,
(2) $\bar{\Omega} = 0$,
(3) $L(M, \mathcal{F}) \cap \mathrm{Sec}\, \bar{C}$ is a Lie subalgebra of $L(M, \mathcal{F})$,
(4) the distribution $\bar{C}$ is completely integrable.

Proof. The equivalence (1) $\Longleftrightarrow$ (2) is evident.

(2) $\Longrightarrow$ (3): Let $Y_1, Y_2 \in L(M, \mathcal{F}) \cap \operatorname{Sec} \bar{C}$. It is sufficient to prove that $[Y_1, Y_2] \in \operatorname{Sec} \bar{C}$ (because $L(M, \mathcal{F})$ is a Lie algebra). Using the decomposition $\bar{C} = E \bigoplus C_b$, we write $Y_1 = X_1 + Y_{1u}$ where $X_1 \in \operatorname{Sec} E$ and $Y_{1u} \in \operatorname{Sec} C_u$, $i = 1, 2$. Then

$$[Y_1, Y_2] = [X_1, X_2] + [X_1, Y_{2u}] - [X_2, Y_{1u}] + [Y_{1u}, Y_{2u}].$$

We have

(a) $[X_1, X_2] \in \operatorname{Sec} E$ $(\subset \operatorname{Sec} \bar{C})$,

(b) $[X_1, Y_{2u}], [X_2, Y_{1u}] \in \operatorname{Sec} E$ because $X_i \in \operatorname{Sec} E$ and the vector fields $Y_{iu} = Y_i - X_i$, $i = 1, 2$, are foliate,

(c) $[Y_{1u}, Y_{2u}] \in \operatorname{Sec} \bar{C}$ by Lemma 6.3.3(b) and the equalities $\bar{C} = \ker \bar{\omega}$ and $[Y_{1u}, Y_{2u}] = [\bar{H} \circ Y_1, \bar{H} \circ Y_2]$.

3) $\Longrightarrow$ (4): Take $Z_1, Z_2 \in \operatorname{Sec} \bar{C}$, $x \in M$, and put $\bar{x} = \pi_b(x) \in W$. Take also cross-sections $\xi_1, ..., \xi_q \in \operatorname{Sec} C$ being a local basis of the vector bundle C on a neighbourhood $W' \subset W$ of $\bar{x}$. The cross-sections $\zeta_1, ..., \zeta_q \in \operatorname{Sec} Q$ for which the equalities $\beta \circ \zeta_i = \xi_i \circ \pi_b$, $i \leq q$, hold exist and are linearly independent transverse fields (see Proposition 6.2.2). Besides, any vector fields X_i representing ζ_i are (by the definition of $\bar{C}$) from $\operatorname{Sec} \bar{C}$ and linearly independent on $W'' := \pi_b^{-1}[W']$. Adding any vector fields $X_{q+1}, ..., X_{q+p} \in \operatorname{Sec} E$ forming a local basis of E on some neighbourhood U of x, we obtain a system $(X_1, ..., X_{q+p})$ of foliate vector fields being a local basis of $\bar{C}$ on $U \cap W''$. Let $Z_i = \sum_j a_i^j \cdot X_j$, $i = 1, 2$ $(a_i^j \in \Omega^0(U \cap W''))$. Then, on $U \cap W''$, we have

$$[Z_1, Z_2] = \sum_{j,k} (a_1^j \cdot a_2^k \cdot [X_j, X_k] + a_1^j \cdot X_j(a_2^k) \cdot X_k - a_2^k \cdot X_k(a_1^J) \cdot X_J) \in \operatorname{Sec} \bar{C}$$

according to assumption (3).

(4) $\Longrightarrow$ (2) - trivial by Lemma 6.3.3(b). ■

As a consequence of the above proposition and Corollary 4.3.2 we obtain the aim of this chapter:

Theorem 6.3.5 ***(The geometric signification of the Chern-Weil homomorphism for TC-foliations)*** *If the Chern-Weil homomorphism of the Lie algebroid $A(M, \mathcal{F})$ of a TC-foliation $(M, \mathcal{F})$ is nontrivial, then there exists no completely integrable distribution $\bar{C}$ on the manifold M satisfying conditions (1)-(3) from Lemma 6.3.1.*

In chapter 7 we describe a wide class of TC-foliations for which there exists no completely integrable connection $\bar{C}$.

7 THE LIE ALGEBROID OF A NONCLOSED CONNECTED LIE SUBGROUP

7.1 Dense connected Lie subgroups and the Malcev Theorem [7], [25], [32].

Let $H \subset T$ be a connected and dense Lie subgroup of a Lie group T and let $\mathcal{F} = \{tH; t \in T\}$ be the foliation of left cosets of G by H. E, as usual, denotes the tangent bundle to $\mathcal{F}$, whereas $\mathfrak{h}$ and $\mathfrak{t}$ the (left) Lie algebras of H and T, respectively. In the sequel, R_t is the tangent mapping to the right translation by t.

Lemma 7.1.1 *If $\mathfrak{t} = \mathfrak{h} \bigoplus K$ for some linear subspace $K \subset \mathfrak{h}$, then, for each $t \in T$,*

$$E_{|t} \cap R_{t|e}[K] = 0.$$

Proof. Let $v \in E_{|t} \cap R_{t|e}[K]$. Then, v is the value at t of the right-invariant vector field Y_w generated by some vector $w \in K$. Since Y_w is an ($\mathcal{F}$-)foliate vector field belonging to the distribution E at t, it belongs to E for each point of the closure $(tH)^{cl}$ of the leaf tH of $\mathcal{F}$ through t; however, $(tH)^{cl} = T$, therefore $w = Y_w(e) \in E_{|e} \cap K = 0$; in consequence, $v = 0$. ■

Lemma 7.1.2 *Every foliate vector field $Y \in L(T, \mathcal{F})$ is of the form $Y = X + Y_w$ for the uniquely determined vector field $X \in \mathfrak{X}(\mathcal{F})$ (i.e. tangent to $\mathcal{F}$) and vector $w \in K$.*

Proof. As a corollary from 7.1.1, we see that the system $\{\bar{Y}_{w_1}, ..., \bar{Y}_{w_q}\}$ of transverse fields, where $(w_1, ..., w_q)$ is a basis of K, forms a *transverse parallelism* on $(T, \mathcal{F})$. Therefore any vector field $Y \in \mathfrak{X}(T)$ is of the form $Y = X + \sum_j f^j \cdot Y_{w_j}$ where $X \in \mathfrak{X}(\mathcal{F})$ and $f^j \in \Omega^0(G)$. Now, let Y be foliate. Then f^j are constant. Indeed, for an arbitrarily taken vector field $X' \in \mathfrak{X}(\mathcal{F})$, we have: $[X', Y] \in \mathfrak{X}(\mathcal{F})$. However,

$$[X', Y] = [X', X] + \sum_j f^j \cdot [X', Y_{w_j}] + \sum_j X'(f^j) \cdot Y_{w_j},$$

therefore $\sum_j X'(f^j) \cdot Y_{w_j} = 0$, which implies that $X'(f^j) = 0$. The free choice of X' gives the result: f^j are $\mathcal{F}$-basic functions, i.e. in our situation, f^j are constant; $f^j = b^j \in \mathbb{R}$. In the end, we assert that $Y = X + Y_w$ for $w = \sum_j b^j \cdot w_j$. ■

Proposition 7.1.3 *If H is a connected and dense Lie subgroup of a Lie group T, then:*

(i) H is a normal subgroup of T,

(ii) each left-invariant vector field X_w, $w \in \mathfrak{t}$, is foliate, and $X_w = X + Y_w$ for some $X \in \mathfrak{X}(\mathcal{F})$,

(iii) T/H is abelian.

Proof. (i) and (iii) follow from the *Malcev Theorem* [7], [25], [32]. Here we give new, "*foliated proofs*" of these facts.

(i): Equivalently, we need to notice that $\mathfrak{h}$ is invariant under the isomorphisms $\mathrm{Ad}(t)$, $t \in T$. Let $t \in T$ and $u \in \mathfrak{h}$. Put $w := \mathrm{Ad}(t)(u)$; then $E_{|t} \ni L_t(u) = R_t(w)$ (L_t denotes here the tangent mapping to the left translation by t), so the foliate vector field Y_w is tangent to $\mathcal{F}$ at t, which implies that $w = Y_w(e) \in E_{|e} = \mathfrak{h}$.

(ii): The module $\mathfrak{X}(\mathcal{F})$ is generated by the left-invariant vector fields X_u, $u \in \mathfrak{h}$, therefore we need only to check that $[X_u, X_w] \in \mathfrak{X}(\mathcal{F})$ for $w \in \mathfrak{t}$ and $u \in \mathfrak{h}$. But, by virtue of (i), $\mathfrak{h}$ is an ideal in $\mathfrak{t}$, thus $[u,w]^L \in \mathfrak{h}$, which gives $[X_u, X_w] = X_{[u,w]^L} \in \mathfrak{X}(\mathcal{F})$. The second part follows from the observation that $X = X_w - Y_w$ is foliate and $X(e) = 0 \in E_{|e}$.

(iii): T/H is connected, thus it is sufficient to show that $\mathfrak{t}/\mathfrak{h}$ is abelian. Let $u, w \in \mathfrak{t}$. On account of (ii), we have:

$$[u,w]^L = [X_u, X_w](e) = [X + Y_u, X_w](e) = [X, X](e) \in \mathfrak{h}.$$

■

7.2 A structure of the Lie algebra bundle, adjoint of the Lie algebroid $A(G;H)$

Here, we give a more detailed description of the Lie algebroid $A(G;H)$ of the foliation $\mathcal{F} = \{aH; a \in G\}$ of a connected Lie group G by left cosets of a connected and nonclosed (in general) Lie subgroup $H \subset G$. [The fact that $\mathcal{F}$ is TC follows from the observation that all right-invariant vector fields are foliate and generate the entire tangent space T_gG for each $g \in G$]. $A(G;H)$ is called the *Lie algebroid of a connected Lie subgroup H*. Denote by $\mathfrak{h}$ and $\mathfrak{g}$ the Lie algebras of H and G, respectively. In the sequel, Y_w and X_w stand for the right-invariant and left-invariant vector fields on G, respectively, generated by the vector $w \in \mathfrak{g}$. Assume that $T \subset G$ is the closure of H. Then $\mathcal{F}_b := \{gT; g \in G\}$ is the basic foliation and the projection $\pi_b : G \longrightarrow G/T$ is the basic fibration.

Lemma 7.2.1 *The isomorphism $R_{t|g} : T_gG \longrightarrow T_{gt}G$, $t \in T$, $g \in G$, maps $E_{|g}$ onto $E_{|gt}$, thus induces an isomorphism $\bar{R}_{t|g} : Q_{|g} \longrightarrow Q_{|gt}$, and, furthermore, the right free action $\bar{R} : Q \times T \longrightarrow Q$, $(\bar{v}, t) \longmapsto \bar{R}_t(\bar{v})$.*

Proof. Since $\mathrm{Ad}_t[\mathfrak{h}] \subset \mathfrak{h}$ for $t \in T$, $E_{|t} = L_{t|e}[\mathfrak{h}] = R_{t|e}[\mathfrak{h}]$. Thus for $g \in G$,

$$R_{t|g}[E_{|g}] = R_{t|g}[L_{g|e}[\mathfrak{h}]] = L_{g|t}[R_{t|e}[\mathfrak{h}]] = L_{g|t}[E_{|t}] = E_{|gt}.$$

Clearly, $\bar{R}$ is a right smooth free action. ■

Lemma 7.2.2 *(a). For a cross-section $\zeta \in \mathrm{Sec}\, Q$, we have: $\zeta \in l(G, \mathcal{F})$ if and only if, for any $g \in G$ and $t \in T$,*

$$\zeta(gt) = \bar{R}_t(\zeta(g)), \tag{7.1}$$

in other words, if and only if ζ is T-right-invariant (with respect to the action $\bar{R}$).

(b). For $\bar{v}$, $\bar{w} \in Q$,

$$\bar{v} \approx \bar{w} \iff \exists_{t\in T}\ \bar{w} = \bar{R}_t(\bar{v}),$$

i.e. $\bar{v} \approx \bar{w}$ if and only if they belong to the same orbit of the action $\bar{R}$.

Proof. (a). " $\Longrightarrow$ " Let $\zeta \in l(M, \mathcal{F})$ and $g \in G$. Then there exists a vector $w \in \mathfrak{g}$ such that $\zeta(g) = \bar{Y}_w(g)$. According to 6.1(A), ζ and $\bar{Y}_w$ agree on the leaf gT of $\mathcal{F}_b$. So, for $t \in T$,

$$\zeta(gt) = \bar{Y}_w(gt) = \bar{R}_t(\bar{Y}_w(g)) = \bar{R}_t(\zeta(g)).$$

" $\Longleftarrow$ " Let $\zeta \in \operatorname{Sec} Q$ satisfy 7.1. Take vectors $w_1, ..., w_q \in \mathfrak{g}$ in such a way that transverse fields $\bar{Y}_{w_1}, ..., \bar{Y}_{w_2}$ form a base of Q over some $\mathcal{F}_b$-saturated open subset $U \subset G$ containing g. Then $\zeta = \sum_i f^i \cdot \bar{Y}_{w_i}$ for some $f^i \in \Omega^0(U)$. Therefore, property 7.1 of ζ and of $\bar{Y}_{w_i}$ yields that, for $g \in U$ and $t \in T$, $\zeta(gt) = \sum_i f^i(gt) \cdot \bar{Y}_{w_i}(gt)$ and, simultaneously,

$$\begin{aligned} \zeta(gt) = \bar{R}_t(\zeta(g)) &= \bar{R}_t(\sum_i f^i(g) \cdot \bar{Y}_{w_i}(g)) \\ &= \sum_i f^i(g) \cdot \bar{R}_t(\bar{Y}_{w_i}(g)) = \sum_i f^i(g) \cdot \bar{Y}_{w_i}(gt). \end{aligned}$$

These give $f^i(gt) = f^i(g)$, which means that f^i are $\mathcal{F}_U$-basic functions. The assertion follows now trivially (namely, the coefficients with respect to any distinguished local coordinates after multiplying them by basic functions remain constant on plaques).

(b). " $\Longrightarrow$ " Results from (a).

" $\Longleftarrow$ " Let $\bar{v}, \bar{w} \in Q$ and let $\bar{w} = \bar{R}_t(\bar{v})$ for some $t \in T$. Clearly, $\bar{v} = \bar{Y}_u(g)$ for a vector $u \in \mathfrak{g}$ where $g = r(\bar{v})$. So, since $\bar{w} \in Q_{|gt}$ and $\bar{Y}_u \in l(G,\mathcal{F})$ and $\bar{w} = \bar{R}_t(\bar{v}) = \bar{R}_t(\bar{Y}_u(g)) = \bar{Y}_u(gt)$, we assert that $\bar{v} \approx \bar{w}$. ■

Remark 7.2.3 *The above two lemmas enable us to define the Lie algebroid $A(G; H)$ immediately as the space of orbits of the action $\bar{R}$. Such a principle is adopted by the author in [20].*

By the same reasoning as in 7.1.3(ii), we assert that each left-invariant vector field X_w, $w \in \mathfrak{t}$, is foliate.

Proposition 7.2.4 *The Lie algebra bundle $\boldsymbol{g}$ of the transitive Lie algebroid $A(G; H)$ of $\mathcal{F}$ is a trivial bundle of abelian Lie algebras, with the global trivialization*

$$\begin{aligned} G/T \times \mathfrak{t}/\mathfrak{h} &\longrightarrow g \\ (x, [w]) &\longmapsto (c(\bar{X}_w))(x), \end{aligned} \tag{7.2}$$

Proof. Since

$$\psi_g : \mathfrak{t} \longrightarrow E_{b|g}, \quad w \longmapsto L_{g|e}(w), \tag{7.3}$$

is an isomorphism, we see that $\mathfrak{t}/\mathfrak{h} \longrightarrow Q'_{|g}$, $[w] \longmapsto \bar{X}_w(g)$, is also an isomorphism. Hence - mapping 7.2 (whose correctness of the definition is easy to check) is an isomorphism of vector bundles. To verify that $\boldsymbol{g}_{|x}$ is abelian, take $u_1, u_2 \in \boldsymbol{g}_{|x}$. By the above, there exist w_1 and w_2 belonging to $\mathfrak{t}$ such that $u_1 = (c(\bar{X}_{w_1}))(x)$ for $i = 1, 2$. So, we have

$$\begin{aligned}[] [u_1, u_2] &= [c(\bar{X}_{w_1})(x), c(\bar{X}_{w_2})(x)] = [\![c(\bar{X}_{w_1}), c(\bar{X}_{w_2})]\!](x) \\ &= c([\bar{X}_{w_1}, \bar{X}_{w_2}])(x) = c(\bar{X}_{[w_1,w_2]^L})(x) = 0 \end{aligned}$$

(because the relation $w := [w_1, w_2]^L \in \mathfrak{h}$ implies $X_w \in \operatorname{Sec} E$). ■

7.3 Connections in $A(G;H)$

Proposition 7.3.1 *Any distribution $\bar{C} \subset TG$ is a connection for the TC-foliation $\mathcal{F}$ [see Def.6.3.2] if and only if it is C^∞, satisfies (1) and (2) from Lemma 6.3.1, and*

(3′) $\bar{C}$ is T-right-invariant, i.e. $\bar{C}_{|gt} = R_t[\bar{C}_{|g}]$, $g \in G$, $t \in T$.

Proof. " $\Longrightarrow$ " Let $\bar{C}$ fulfil conditions (1)-(3) from Lemma 6.3.1 and take $v \in \bar{C}_{|g}$. By condition (3), $v = X(g)$ for some $X \in L(G, \mathcal{F}) \cap \operatorname{Sec} \bar{C}$. Since $\bar{X} \in l(G, \mathcal{F})$, 7.2.2(a) shows that

$$\overline{X(gt)} = \bar{X}(gt) = \bar{R}_t(\bar{X}(g)) = \bar{R}_t(\bar{v}) = \overline{R_t(v)},$$

which yields $R_t(v) - X(gt) \in E_{|gt}$. Condition (1) and the relation $X \in \operatorname{Sec} \bar{C}$ give now $R_t(v) \in \bar{C}_{|gt}$. So, $R_t[\bar{C}_{|g}] \subset \bar{C}_{|gt}$, therefore the equality of the dimensions gives the examined T-right-invariance of $\bar{C}$.

" $\Longleftarrow$ " Assume that $\bar{C} \subset TG$ is a C^∞ distribution satisfying (1) and (2) from Lemma 6.3.1 and (3′) above. For each point $x \in G/T$, we define

$$C_{|x} = \beta_{|g}[\alpha_{|g}[\bar{C}_{|g}]], \quad g \in \pi_b^{-1}(x).$$

(3′), 7.2.1 and 7.2.2(b) imply the correctness of this definition: for $t \in T$ and $g \in G$, we have

$$\begin{aligned} \beta_{|gt}[\alpha_{|gt}[C_{|gt}]] &= \beta_{|gt}[\alpha_{|gt}[R_{t|g}[\bar{C}_{|g}]]] \\ &= \beta_{|gt}[\bar{R}_{t|g}[\alpha_{|g}[\bar{C}_{|g}]]] = \beta_{|g}[\alpha_{|g}[\bar{C}_{|g}]]. \end{aligned}$$

Put $C = \bigcup_{x \in G/T} C_{|x} \subset A(G;H)$. It is a standard calculation to prove that C is a C^∞ subbundle of $A(G;H)$. By assumptions (1) and (2) , C is a horizontal subbundle of $A(G;H)$ [i.e. $C + \boldsymbol{g} = T(G/T)$ and $C \cap \boldsymbol{g} = 0$ hold], therefore it is determined by some connection λ. Clearly, $\bar{C} = \bar{C}^\lambda$ (see 6.3). Thereby, (3) is satisfied according to Lemma 6.3.1. ■

7.4 The Chern-Weil homomorphism of $A(G;H)$

Notice that the situation when $H = T$ or $T = G$ is not interesting from our point of view because then, in the first case, the Lie algebroid $A(G;H)$ is trivial, $A(G;H) \xrightarrow[\cong]{\gamma} T(G/T)$, which implies $\boldsymbol{g} = 0$ and, in consequence, $(h_{A(G;H)})^{+} = 0$; in the second case, the basic manifold W is one-point, so also $(h_{A(G;H)})^{+} = 0$. Therefore we can consider the case $H \neq T \neq G$.

The Proposition 7.2.4 sets up a global trivialization

$$\varphi : \boldsymbol{g} \xrightarrow{\cong} G/T \times \mathfrak{t}/\mathfrak{h} \tag{7.4}$$

therefore any cross-section $\nu \in \operatorname{Sec} \boldsymbol{g}$ determines some $\mathfrak{t}/\mathfrak{h}$-valued function $\overline{\overline{\nu}} : G/T \longrightarrow \mathfrak{t}/\mathfrak{h}$, namely $\overline{\overline{\nu}} := \mathrm{pr}_2 \circ \varphi \circ \nu$. Analogously, via the canonically induced global trivialization $\bigvee^k \boldsymbol{g}^* \cong G/T \times \bigvee^k (\mathfrak{t}/\mathfrak{h})^*$, any cross-section $\Gamma \in \operatorname{Sec} \bigvee^k \boldsymbol{g}^*$ determines some $\bigvee^k (\mathfrak{t}/\mathfrak{h})^*$-valued function $\overline{\overline{\Gamma}} : G/T \longrightarrow \bigvee^k (\mathfrak{t}/\mathfrak{h})^*$. Let $\langle \cdot, \cdot \rangle : \bigvee^k \boldsymbol{g}^* \times \bigvee^k \boldsymbol{g} \longrightarrow \mathbb{R}$ be the canonical duality [8]. It is easily seen that, for $x \in G/T$ and $w_i \in \mathfrak{t}$, $i \leq k$ ($k = \operatorname{rank} \boldsymbol{g} = \dim \mathfrak{t} - \dim \mathfrak{h}$),

$$\langle \Gamma, c(\bar{X}_{w_1}) \vee ... \vee c(\bar{X}_{w_k}) \rangle (x) = \langle \overline{\overline{\Gamma}}(x), [w_1] \vee ... \vee [w_k] \rangle. \tag{7.5}$$

Proposition 7.4.1 *Let $\Gamma \in \operatorname{Sec} \bigvee^k \boldsymbol{g}^*$, then Γ is $\bigvee^k \mathrm{ad}^{\natural}_{A(G;H)}$-invariant [i.e. $\Gamma \in (\operatorname{Sec} \bigvee^k \boldsymbol{g}^*)_{I^0}$] if and only if $\overline{\overline{\Gamma}}$ is constant.*

Proof. " $\Longrightarrow$ " Let Γ be invariant. This means, in particular, that

$$\begin{aligned}&(\gamma \circ c(\bar{Y}_w)) \langle \Gamma, c(\bar{X}_{w_1}) \vee ... \vee c(\bar{X}_{w_k}) \rangle \\ &= \sum_j \langle \Gamma, c(\bar{X}_{w_1}) \vee ... \vee [\![c(\bar{Y}_w), c(\bar{X}_{w_j})]\!] \vee ... \vee c(\bar{X}_{w_k}) \rangle,\end{aligned}$$

for $w \in \mathfrak{g}$, $w \in \mathfrak{t}$; but $[\![c(\bar{Y}_w), c(\bar{X}_{w_j})]\!] = c_{[\bar{Y}_w, \bar{X}_{w_j}]} = 0$, so

$$(\gamma \circ c(\bar{Y}_w)) \langle \Gamma, c(\bar{X}_{w_1}) \vee ... \vee c(\bar{X}_{w_k}) \rangle = 0.$$

The values of vector fields $\gamma \circ c(\bar{Y}_w)$, $w \in \mathfrak{g}$, generate at each point $x \in G/T$ the entire tangent space $T_x(G/T)$; therefore the function $\langle \Gamma, c(\bar{X}_{w_1}) \vee ... \vee c(\bar{X}_{w_k}) \rangle$, thanks to the connectedness of G/T, is constant, so the same holds for the function $G/T \ni x \longmapsto \langle \overline{\overline{\Gamma}}(x), [w_1] \vee ... \vee [w_k] \rangle$. Equivalently, $\overline{\overline{\Gamma}} : G/T \longrightarrow \bigvee^k (\mathfrak{t}/\mathfrak{h})$ is constant.

" $\Longleftarrow$ " Assume now that Γ is such that the function $\overline{\overline{\Gamma}}$ is constant. Thus 7.5 implies the same for the function $\langle \Gamma, c(\bar{X}_{w_1}) \vee ... \vee c(\bar{X}_{w_k}) \rangle$, $w_j \in \mathfrak{t}$. To prove the invariance of Γ, take arbitrarily cross-sections $\nu_i \in \operatorname{Sec} \boldsymbol{g}$, $i \leq k$, and $\xi \in \operatorname{Sec} A(G;H)$. They can be written as follows:

$$\nu_i = \sum_j f_i^j \cdot c(\bar{X}_{w_j}) \quad \text{(globally)}, \quad \xi = \sum_j g^j \cdot c(\bar{Y}_{u_j}) \quad \text{(locally)},$$

for some $f_i^j, g^j \in \Omega^0(G/T)$, $w_i \in \mathfrak{t}$, $u_j \in \mathfrak{g}$. Therefore $\gamma \circ \xi = \sum_j g^j \cdot \gamma \circ c(\bar{Y}_{u_j})$ and

$$\begin{aligned}
[\![\xi, \nu_s]\!] &= \sum_{j,j_s} (g^j \cdot f_s^{j_s} \cdot [\![c(\bar{Y}_{u_j}), c(\bar{X}_{w_{j_s}})]\!] - f_s^{j_s} \cdot \gamma \circ c(\bar{X}_{w_{j_s}})(g^j) \cdot c(\bar{Y}_{u_j} + \\
&\quad + g^j \cdot \gamma \circ c(\bar{Y}_{u_j})(f_s^{j_s}) \cdot c(\bar{X}_{w_{j_s}})) \\
&= \sum_{j,j_s} g^j \cdot \gamma \circ c(\bar{Y}_{u_j})(f_s^{j_s}) \cdot c(\bar{X}_{w_{j_s}})
\end{aligned}$$

because $\gamma \circ c(\bar{X}_{w_{j_s}}) = 0$ and $[\![c(\bar{Y}_{u_j}), c(\bar{X}_{w_{j_s}})]\!] = c(\overline{[Y_{u_j}, X_{w_{j_s}}]}) = 0$. Next, we have

$$\begin{aligned}
&(\gamma \circ \xi)\langle \Gamma, \nu_1 \vee ... \vee \nu_k \rangle \\
&= \sum_{j,j_1,...,j_k} g^j \cdot \gamma \circ c(\bar{Y}_{u_j}) \langle \Gamma, f_1^{j_1} \cdot c(\bar{X}_{w_{j_1}}) \vee ... \vee f_k^{j_k} \cdot c(\bar{X}_{w_{j_k}}) \rangle \\
&= \sum_j g^j \cdot \gamma \circ c(\bar{Y}_{u_j})(f_1^{j_1} \cdot ... \cdot f_k^{j_k} \cdot \langle \Gamma, c(\bar{X}_{w_{j_1}}) \vee ... \vee c(\bar{X}_{w_{j_k}}) \rangle) \\
&= \sum_j g^j \cdot \gamma \circ c(\bar{Y}_{u_j})(f_1^{j_1} \cdot ... \cdot f_k^{j_k}) \cdot \langle \Gamma, c(\bar{X}_{w_{j_1}}) \vee ... \vee c(\bar{X}_{w_{j_k}}) \rangle \\
&= \sum_{j,s,j_1,...,j_k} g^j \cdot f_1^{j_1} \cdot ... \cdot \gamma \circ c(\bar{Y}_{u_j})(f_s^{j_s}) \cdot ... \cdot f_k^{j_k} \cdot \langle \Gamma, c(\bar{X}_{w_{j_1}}) \vee ... \vee c(\bar{X}_{w_{j_k}}) \rangle \\
&= \sum_s \langle \Gamma, \sum_{j_1} f_1^{j_1} \cdot c(\bar{X}_{w_{j_1}}) \vee ... \vee \sum_j g^j \cdot \gamma \circ c(\bar{Y}_{u_j})(f_s^{j_s}) \cdot c(\bar{X}_{w_{j_s}}) \vee ... \\
&\qquad ... \vee \sum_{j_k} f_k^{j_k} \cdot c(\bar{X}_{w_{j_k}}) \rangle \\
&= \sum_s \langle \Gamma, \nu_1 \vee ... \vee [\![\xi, \nu_s]\!] \vee ... \vee \nu_k \rangle,
\end{aligned}$$

which means that Γ is $\bigvee^k \mathrm{ad}^\natural_{A(G;H)}$-invariant. ■

By the above proposition, the value of the function $\overline{\overline{\Gamma}}$ at any point $x \in G/T$ does not depend on x for $\Gamma \in (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^0}$. Denote it by $\hat{\Gamma}$. Clearly,

$$\rho : \bigoplus^{k \geq 0} (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^0} \longrightarrow \bigvee (\mathfrak{t}/\mathfrak{h})^*, \quad \Gamma \longmapsto \hat{\Gamma},$$

is an isomorphism of algebras.

Theorem 7.4.2 *The Chern-Weil homomorphism hof the Lie algebroid $A(G;H)$ makes the following diagram*

$$\begin{array}{ccc}
\bigoplus^{k \geq 0} (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^0} & \overset{h_{A(G;H)}}{\longrightarrow} & H_{dR}(G/T) \\
\downarrow \rho & & \uparrow h_P \\
\bigvee (\mathfrak{t}/\mathfrak{h})^* & \overset{\bigvee j^*}{\rightarrowtail} & (\bigvee \mathfrak{t}^*)_I
\end{array}$$

commute, in which $\bigvee j^*$ *is the monomorphism of algebras induced by the canonical projection* $j : \mathfrak{t} \longrightarrow \mathfrak{t}/\mathfrak{h}$*, whereas* $h_P : (\bigvee \mathfrak{t}^*)_I \longrightarrow H_{dR}(G/T)$ *is the Chern-Weil homomorphism of the* T*-principal bundle* $P = (G \longrightarrow G/T)$*.*

Proof. First, we notice that $\operatorname{Im} \bigvee j^* \subset (\bigvee \mathfrak{t}^*)_I$. Indeed, $\operatorname{Im} \bigvee j^* = \{\bar\Gamma \in \bigvee \mathfrak{t}^*;\ \iota_w \bar\Gamma = 0$ for all $w \in \mathfrak{h}\}$. Since $\mathfrak{h}$ is an ideal in $\mathfrak{t}$ and $\mathfrak{t}/\mathfrak{h}$ is an abelian Lie algebra, see 7.1.3, therefore $[u,w]^L \in \mathfrak{h}$ for all $u, w \in \mathfrak{t}$. Thus, for any $u \in \mathfrak{h}$ and $\bar\Gamma \in \operatorname{Im} \bigvee^k j^*$, we have:

$$\sum_j \langle \bar\Gamma, w_1 \vee ... \vee [u, w_j]^L \vee ... \vee w_k \rangle = 0, \quad w_i \in \mathfrak{t},$$

which means [because T is connected] the Ad_T-invariance of $\bar\Gamma$. That $\bigvee^k j^*$ is a monomorphism follows from the fact that j^* is a monomorphic (see [8,p.108]). By the independence of $h_P(\bar\Gamma)$ and $h_{A(G;H)}(\Gamma)$ on the choice of a connection, we may set an arbitrary connection $C \subset TG$ in the principal bundle P. Then $\bar C := E \bigoplus C_u$ is a connection in TG for $\mathcal{F}$ because $\bar C$ is a C^∞ distribution and requirements (1), (2) and (3′) from 7.3.1 are satisfying:

(1) Clearly, $E \subset \bar C \cap E_b$. To see the opposite inclusion, take arbitrarily $v \in \bar C \cap E_b$ and write $v = v_1 + v_2$ for $v_1 \in E$, $v_2 \in C_u$. Of course, the vector $v_2 = v - v_1 \in C_u \cap E_b = 0$ is null. Therefore $v = v_1 \in E$.

(2) $\bar C + E_b = (E \bigoplus C_u) + E_b \supset C_u \bigoplus E_b = TG$ (E_b is the vertical bundle of P).

(3') For $t \in T$ and $g \in G$, we have, by 7.2.1 and the T-right-invariance of C_u in P,

$$\begin{aligned} R_{t|g}[\bar C_{|g}] &= R_{t|g}[E_{|g} \bigoplus C_{u|g}] = R_{t|g}[E_{|g}] \bigoplus R_{t|g}[C_{u|g}] \\ &= E_{|gt} \bigoplus C_{u|gt} = \bar C_{|gt}. \end{aligned}$$

Let $\omega_u \in \Omega^1(G;\mathfrak{t})$ be the connection form of C_u. Denote by $\bar V : TG \longrightarrow E_b$ the vertical projection. Since $A_g : T \longrightarrow G$, $g \longmapsto gt$, is the restriction to T of the left translation by t, we have $\psi_g \circ \omega_{ug} = \bar V_{|g}$, $g \in G$, where ψ_g is defined by 7.3. According to the definitions of the connection form $\bar\omega$ of $\bar C$ (Def. 6.3.2(b)) and of the isomorphism φ of vector bundles 7.4, we obtain the commuting diagram:

$$\begin{array}{ccccc} T_gG & & \xrightarrow{\omega_{ug}} & & \mathfrak{t} \\ & \searrow \bar V_{|g} & & \psi_g \nearrow & \downarrow j \\ \bar\omega_g \downarrow & & E_{b|g} & & \mathfrak{t}/\mathfrak{h} \\ & \swarrow \alpha_{|g} & & & \uparrow \varphi_{|x} \\ Q'_{|g} & & \xrightarrow{\beta^+_{|g}} & & \boldsymbol{g}_{|x} \end{array} \qquad (7.6)$$

for $g \in G$ and $x = \pi_b(g)$. Let $\Omega_u \in \Omega^2(G;\mathfrak{t})$ and $\bar\Omega \in \Omega^2(G;Q')$ be the curvature forms of C_u and of $\bar C$, respectively, while $\Omega_b \in \Omega^2(G/T;\boldsymbol{g})$ the curvature tensor of the connection λ in $A(G;H)$ for which $\bar{C} = \bar{C}^{\lambda}$. Define auxiliarily the form $\Omega^0 \in \Omega^2(G;\mathfrak{t}/\mathfrak{h})$ by $\Omega^0(g;v_1 \wedge v_2) = \varphi_{|x} \circ \beta^+_{|g}(\bar{\Omega}(g;v_1 \wedge v_2))$, $x = \pi_b(g)$ as above. We prove the equality

$$\Omega^0 = j \circ \Omega_u. \tag{7.7}$$

To this end, take $v_1, v_2 \in T_gG$ and find foliate vector fields $Y_1, Y_2 \in L(G, \mathcal{F})$ such that $Y_i(g) = v_i$, $i = 1, 2$. By Lemma 6.3.3(b) and diagram 7.6, we assert that

$$\begin{aligned}
\Omega^0(g;v_1 \wedge v_2) &= \varphi_{|x} \circ \beta^+_{|g}(\bar{\Omega}(g;v_1 \wedge v_2)) \\
&= \varphi_{|x} \circ \beta^+_{|g}(-\bar{\omega}(g;[\bar{H} \circ Y_1, \bar{H} \circ Y_2](g))) \\
&= -j \circ \omega_u(g;[\bar{H} \circ Y_1, \bar{H} \circ Y_2](g)) \\
&= j(\Omega_u(g;v_1 \wedge v_2)).
\end{aligned}$$

For $\bar{\Gamma} \in (\bigvee^k \mathfrak{t}^*)_I$, the class $h_p(\bar{\Gamma})$ is represented by the form $\theta \in \Omega^{2k}(G/T)$ whose π_b-lifting equals $\frac{1}{k!} \cdot \langle \bar{\Gamma}, \Omega_u \vee ... \vee \Omega_u \rangle$. Let $\bar{\Gamma} = (\bigvee^k j^*)(\hat{\Gamma})$ for $\hat{\Gamma} = \rho(\Gamma)$ where $\Gamma \in (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^0}$; then we have

$$\langle \bar{\Gamma}, \Omega_u \vee ... \vee \Omega_u \rangle = \langle \hat{\Gamma}, \Omega^0 \vee ... \vee \Omega^0 \rangle. \tag{7.8}$$

Indeed, using the fact that homomorphisms of algebras $\bigvee^k j^*$and $\bigvee^k j$ are dual [8,p.108], we obtain, by 7.7, that for $g \in G$ and $v_i \in T_gG$:

$$\begin{aligned}
&\langle \bar{\Gamma}, \Omega_u \vee ... \vee \Omega_u \rangle(g;v_1 \wedge ... \wedge v_{2k}) \\
&= \frac{1}{2^k} \cdot \sum_{\sigma} \mathrm{sgn}\,\sigma \cdot \langle (\bigvee^k j^*)(\hat{\Gamma}), \Omega_u(g;v_{\sigma(1)} \wedge v_{\sigma(2)}) \vee ... \\
&\qquad ... \vee \Omega_u(g;v_{\sigma(2k-1)} \wedge v_{\sigma(2k)}) \rangle \\
&= \frac{1}{2^k} \cdot \sum_{\sigma} \mathrm{sgn}\,\sigma \cdot \langle \hat{\Gamma}, j(\Omega_u(g;v_{\sigma(1)} \wedge v_{\sigma(2)})) \vee ... \vee j(\Omega_u(g;v_{\sigma(2k-1)} \wedge v_{\sigma(2k)})) \rangle \\
&= \frac{1}{2^k} \cdot \sum_{\sigma} \mathrm{sgn}\,\sigma \cdot \langle \hat{\Gamma}, \Omega^0(g;v_{\sigma(1)} \wedge v_{\sigma(2)}) \vee ... \vee \Omega^0(g;v_{\sigma(2k-1)} \wedge v_{\sigma(2k)})) \rangle \\
&= \langle \hat{\Gamma}, \Omega^0 \vee ... \vee \Omega^0 \rangle(g : v_1 \wedge ... \wedge v_{2k}).
\end{aligned}$$

On the other hand, $h_{A(G;H)}(\Gamma)$ is represented by the form $\frac{1}{k!} \cdot \langle \Gamma, \Omega_b \vee ... \vee \Omega_b \rangle$, see 4.1. Put $\Omega^0_b \in \Omega^2(G/T, \mathfrak{t}/\mathfrak{h})$ as follows:

$$\Omega^0_b(x;\bar{v}_1 \wedge \bar{v}_2) = \varphi_{|x}(\Omega_b(x;\bar{v}_1 \wedge \bar{v}_2)), \quad \bar{v}_i \in T_x(G/T).$$

We check that

$$\Omega^0 = \pi^*_b(\Omega^0_b), \tag{7.9}$$

$$\langle \Gamma, \Omega_b \vee ... \vee \Omega_b \rangle = \langle \hat{\Gamma}, \Omega^0_b \vee ... \vee \Omega^0_b \rangle. \tag{7.10}$$

Seeing Def. 6.3.2 of the tensor $\bar{\Omega}$, we assert 7.9 trivially. Using the duality between the homomorphisms $\bigvee \varphi_{|x}^{*}$ and $\bigvee \varphi_{|x}$ of symmetric algebras, we notice that, for $x \in G/T$ and $\bar{v}_i \in T_x(G/T)$,

$$\begin{aligned}
&\langle \Gamma, \Omega_b \vee \ldots \vee \Omega_b \rangle (x; \bar{v}_1 \wedge \ldots \wedge \bar{v}_{2k}) \\
&= \langle \Gamma_x, (\Omega_b \vee \ldots \vee \Omega_b)(x; \bar{v}_1 \wedge \ldots \wedge \bar{v}_{2k}) \rangle \\
&= \langle \bigvee \varphi_{|x}^{*}(\hat{\Gamma}), (\Omega_b \vee \ldots \vee \Omega_b)(x; \bar{v}_1 \wedge \ldots \wedge \bar{v}_{2k}) \rangle \\
&= \langle \hat{\Gamma}, \bigvee \varphi_{|x}((\Omega_b \vee \ldots \vee \Omega_b)(x; \bar{v}_1 \wedge \ldots \wedge \bar{v}_{2k})) \rangle \\
&= \langle \hat{\Gamma}, \Omega_b^0 \vee \ldots \vee \Omega_b^0 \rangle (x; \bar{v}_1 \wedge \ldots \wedge \bar{v}_{2k}),
\end{aligned}$$

which confirms 7.10.

Now, we are able to prove our theorem: Taking $\Gamma \in (\mathrm{Sec} \bigvee^k \boldsymbol{g}^*)_{I^0}$ and keeping the notations above, we assert, by 7.8 and 7.10, that the cohomology classes $h_P(\bigvee^k j^*(\rho(\Gamma)))$ and $h_{A(G;H)}(\Gamma)$ are represented by the forms whose π_b-liftings are equal to $\frac{1}{k!} \cdot \langle \hat{\Gamma}, \Omega^0 \vee \ldots \vee \Omega^0 \rangle$ and $\pi_b^*(\frac{1}{k!} \cdot \langle \hat{\Gamma}, \Omega_b^0 \vee \ldots \vee \Omega_b^0 \rangle)$, respectively. But, these two last forms are identical according to 7.9, which ends the proof. ■

Here is the aim of this section:

Theorem 7.4.3 *If G is any connected, compact and semisimple Lie group and $H \subset G$ is its arbitrary connected nonclosed Lie subgroup, then the Chern-Weil homomorphism his nontrivial.*

Proof. Let T be the closure of H. T is, of course, compact. Applying Th.XI from [10, Ch.IX, p.392] to the principal bundle $P = (G \longrightarrow G/T)$, we get the equivalence of the conditions:

(1) the Chern-Weil homomorphism h_P is m-regular [understanding in $(\bigvee \mathfrak{t}^*)_I$ the natural even gradation],

(2) $H_{dR}^0(G) = \mathbb{R}$ and $H_{dR}^p(G) = 0$, $1 \leq p \leq m$.

Since G is compact and semisimple, it follows that $H_{dR}^0(G) = \mathbb{R}$, $H_{dR}^1(G) = H_{dR}^2(G) = 0$ $[H_{dR}^3(G) \neq 0]$. Combining this with the above-mentioned theorem, we obtain that the Chern-Weil homomorphism h_P is 2-regular, in particular, this yields that

$$(h_P)^{(2)} : (\mathfrak{t}^*)_I \longrightarrow H_{dR}^2(G/T)$$

is an isomorphism. In view of Theorem 7.4.2, we get that

$$(h_{A(G;H)})^{(2)} \circ \rho^{-1} : (\mathfrak{t}/\mathfrak{h})^* \rightarrowtail (\mathfrak{t}^*)_I \xrightarrow{\cong} H_{dR}^2(G/T)$$

is a monomorphism. The assumption $H \neq T$ implies $\mathfrak{t}/\mathfrak{h} \neq 0$, whence we obtain that $(h_{A(G;H)})^{(2)} \neq 0$, and so, $h_{A(G;H)}$ is nontrivial. ■

Remark 7.4.4 *Here is the more concrete example of a nonclosed Lie subgroup: Let T be an arbitrary, not necessarily maximal, torus of G and $H \subset T$ any of its dense connected Lie subgroups.*

Remark 7.4.5 *Adding the simple connectedness of G to the assumptions of Theorem 7.4.3, we get, according to Almeida-Molino's Theorem, see [1], [27], some nonintegrable transitive Lie algebroid having the nontrivial Chern-Weil homomorphism.*

Therefore we can formulate the important

Corollary 7.4.6 *There exists a nonintegrable transitive Lie algebroid having the nontrivial Chern-Weil homomorphism.*

Return to Theorem 7.4.3. As its consequence as well as that of Theorem 6.3.5 and Prop.7.3.1 we obtain that, under the assumptions of Theorem 7.4.3, there exists no completely integrable T-right-invariant distribution $\bar{C} \subset TG$ such that $\bar{C} + E_b = TG$ and $\bar{C} \cap E_b = E$. Now, we give a simple situation in which such a completely integrable distribution exists.

Example 7.4.7 *Assume that the symbols G, H, T, $\mathcal{F}$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{t}$ have the same meaning as before. If there is a Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ such that*

(1) $\mathfrak{c} + \mathfrak{t} = \mathfrak{g}$,

(2) $\mathfrak{c} \cap \mathfrak{t} = \mathfrak{h}$,

then the G-left-invariant distribution $\bar{C}$ determined by $\mathfrak{c}$ (i.e. the one tangent to the foliation $\{gF; g \in G\}$ where F is the connected Lie subgroup with its Lie algebra equalling $\mathfrak{c}$) is a completely integrable connection in TG for $\mathcal{F}$. Indeed, it is clear that the conditions $\bar{C} \cap E_b = E$ and $\bar{C} + E_b = TG$ hold. Therefore it is enough to verify the T-right-invariance of $\bar{C}$ only, i.e. the equality $R_t[\bar{C}_{|g}] = \bar{C}_g$, $t \in T$, $g \in G$. Let $v \in \bar{C}_g$, then $v = L_g(w)$ for some vector $w \in \mathfrak{c}$. Since $R_t(v) = R_t(L_g(w)) = L_g(R_t(w))$, we need observe that $R_t(w) \in \bar{C}_{|t}$. Since T is the closure of H, we have $t = \lim h_n$, $h_n \in H$. In virtue of the closedness of $\bar{C}$, we obtain that the fact that the element $R_t(w)$ (equalling $\lim R_{h_n}(w)$) belongs to $\bar{C}$ follows from the relation $R_h[\mathfrak{c}] \subset \bar{C}_{|h}$ for $h \in H$ which is evident by the relation $r_h[H] \subset H$ where r_h is the left translation by h.

As a simple corollary of 7.4.3 and 7.4.7 we obtain

Corollary 7.4.8 *Under the assumptions of Theorem 7.4.3, no Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ fulfilling (1), (2) from 7.4.7 exists.*

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