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A Criterion for the Minimal Closedness of the Lie Subalgebra Corresponding to a Connected Nonclosed Lie Subgroup

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ABSTRACT. A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be *minimally closed* (after A. Malcev [11]) if the corresponding connected Lie subgroup is closed in the simply connected Lie group determined by $\mathfrak{g}$. The aim of this paper is to prove the following theorem:

Let $H\subset G$ be any connected (not necessarily closed) Lie subgroup of a Lie group G. Denote by $\mathfrak{h}$, $\bar{\mathfrak{h}}$ and $\mathfrak{g}$ the Lie algebras of H, of its closure $\bar{H}$ and of G, respectively. If there exists a Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ such that (a) $\mathfrak{c}+\bar{\mathfrak{h}}=\mathfrak{g}$, (b) $\mathfrak{c}\cap\bar{\mathfrak{h}}=\mathfrak{h}$, then $\mathfrak{h}$ is minimally closed.

As a corollary we obtain that if $\pi_1(G)$ is finite, then no such a Lie subalgebra $\mathfrak{c}$ exists provided that H is nonclosed.

The proof is carried out on the ground of the theory of Lie algebroids and by using some ideas from the theory of transversally complete foliations.

0. INTRODUCTION

A) Let G be any connected Lie group. Assume that $H\subset G$ is any of its connected and nonclosed Lie subgroups. Denote by $\mathfrak{h}$, $\bar{\mathfrak{h}}$, $\mathfrak{g}$ the Lie algebras of H, of its closure $\bar{H}$ and of G, respectively.

0.1. Problem. *Does there exist a Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ such that* (a) $\mathfrak{c}+\bar{\mathfrak{h}}=\mathfrak{g}$, (b) $\mathfrak{c}\cap\bar{\mathfrak{h}}=\mathfrak{h}$?

In work [7], some topological obstructions of the existence of such a Lie algebra $\mathfrak{c}$ were found. Namely, the following theorem was proved.

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0.2. Theorem. *If the following homomorphism of algebras*

$$V(\bar{\mathfrak{h}}/\mathfrak{h})^* \longrightarrow (V\bar{\mathfrak{h}}^*)_I \xrightarrow{h_P} H_{dR}(G/\bar{H}) \qquad (1)$$

(where h_P is the Chern-Weil homomorphism of the $\bar{H}$-principal fibre bundle $P=(G \longrightarrow G/\bar{H})$) is nontrivial, then such a Lie subalgebra $\mathfrak{c}$ *does not exist.* ∎

Next, it was noticed that the case of a compact and semisimple Lie group is a case for which homomorphism (1) is always nontrivial. As a corollary we have.

0.3. Theorem. *If G is a compact and semisimple Lie group, then no Lie subalgebra* $\mathfrak{c}$ *fulfilling* (a) *and* (b) *above exists.* ∎

We add that (1) appears as the Chern-Weil homomorphism of the Lie algebroid of the TC-foliation $\mathscr{F}=\{gH;\ g\in G\}$ of left cosets of G by H, determined by the author [7], [8].

B) In the present paper, a Lie algebroid of a connected (not necessarily closed) Lie subgroup H of a given Lie group G is constructed precisely. It can be noticed that it is the same as the one constructed in the theory of P. Molino [12] for the corresponding TC-foliation $\mathscr{F}$ of left cosets. Next, we get to the core of the structure of this Lie algebroid and prove some strengthening of theorem 0.3 (by weakening the assumptions to the finiteness of $\pi_1(G)$) without using any characteristic classes. This fact is obtained as a corollary from the theorem saying that:

0.4. Theorem. *The existence of a Lie subalgebra* $\mathfrak{c}$ *fulfilling* (a) *and* (b) *above implies the minimal closedness of* $\mathfrak{h}$ *(in the sense of Malcev* [11]). ∎

1. PRELIMINARIES

We give a few elementary facts concerning the theory of Lie algebroids, needed in the sequel. We assume that in our paper all the manifolds considered are of C^∞-class and Hausdorff. By $\Omega^0(M)$ we denote the ring of C^∞ functions on a manifold M, by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M, and by $SecA$ the $\Omega^0(M)$-module of all C^∞ global cross-sections of a given vector bundle A (over M).

1.1. Definition [15], [16]. *By a transitive Lie algebroid on a manifold M we mean a system*

$$A=(A,[\![\cdot,\cdot]\!],\gamma) \qquad (2)$$

consisting of a vector bundle A (over M) and mappings

$$[\![\cdot,\cdot]\!]: SecA \times SecA \longrightarrow SecA, \qquad \gamma: A \longrightarrow TM,$$

such that

(i) $(SecA, [\![\cdot,\cdot]\!])$ is an **R**-Lie algebra,

(ii) γ, called by K. Mackenzie [9] an *anchor*, is an epimorphism of vector bundles,

(iii) $Sec\gamma: SecA \longrightarrow \mathfrak{X}(M)$, $\xi \longmapsto \gamma \circ \xi$, is a homomorphism of Lie algebras,

(iv) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$ for $f \in \Omega^0(M)$, $\xi, \eta \in SecA$.

$\mathbf{g} := Ker\gamma$ is a vector bundle and the short exact sequence

$$O \longrightarrow \mathbf{g} \hookrightarrow A \xrightarrow{\gamma} TM \longrightarrow O \tag{3}$$

is called an *Atiyah sequence of* (2); in each vector space $\mathbf{g}_{|x} = Ker\gamma_{|x}$, $x \in M$, some Lie algebra structure is defined by

$$[v, w] := [\![\xi, \eta]\!](x), \quad \xi, \eta \in SecA, \ \xi(x) = v, \ \eta(x) = w, \ v, w \in \mathbf{g}_{|x}.$$

$\mathbf{g}_{|x}$ is called the *isotropy Lie algebra of* (2) *at* x. $\mathbf{g}$ is a Lie algebra bundle [2], [5], [6], [9] called (after Mackenzie) the *adjoint of* (2).

Let (2) and $(A', [\![\cdot,\cdot]\!]', \gamma')$ be two transitive Lie algebroids on the same manifold M. By a *strong homomorphism*

$$H: (A', [\![\cdot,\cdot]\!]', \gamma') \longrightarrow (A, [\![\cdot,\cdot]\!], \gamma) \tag{4}$$

between them [4], [10, p. 273] we mean a strong homomorphism of vector bundles $H: A' \longrightarrow A$, such that

(i) $\gamma \circ H = \gamma'$,

(ii) $SecH: SecA' \longrightarrow SecA$, $\xi \longrightarrow H \circ \xi$, is a homomorphism of Lie algebras.

If homomorphism (4) is a bijection, then H^{-1} is also a homomorphism of Lie algebroids; then H is called an *isomorphism of Lie algebroids*.

1.2. Example. By a *trivial Lie algebroid* [14] we mean any algebroid isomorphic to $(TM\times\mathfrak{g}, [\![\cdot,\cdot]\!], pr_1)$ where $\mathfrak{g}$ is a finitely dimensional Lie algebra and the bracket $[\![\cdot,\cdot]\!]$ is defined by

$$[\![(X,\sigma),(Y,\eta)]\!]=([X,Y],\ \mathcal{L}_X\eta-\mathcal{L}_Y\sigma+[\sigma,\eta]),$$

$X,\ Y\in\mathfrak{X}(M)$, $\sigma,\eta\colon\ M\longrightarrow\mathfrak{g}$ ($[\sigma,\eta]$ is defined point by point: $[\sigma,\eta](x)=[\sigma(x),\eta(x)]$, $x\in M$).

1.3. Example (See [5], [6], [9]). By the *Lie algebroid $A(P)$ of a principal fibre bundle $P=(P,\pi,M;G,\cdot)$* we mean a transitive Lie algebroid on M $(A(P),[\![\cdot,\cdot]\!],\gamma)$ in which $A(P)=TP/G$, $\gamma([v])=\pi_*(v)$ where $[v]$ denotes the equivalence class of v, and the bracket $[\![\xi,\eta]\!]$, $\xi,\eta\in SecA(P)$, is constructed on the basis of the following observation: For each cross-section $\eta\in SecA(P)$, there exists exactly one C^∞ right-invariant vector field $\eta'\in\mathfrak{X}^R(P)$ such that $[\eta'(z)]=\eta(\pi z)$, and the mapping $SecA(P)\longrightarrow\mathfrak{X}^R(P)$, $\eta\longrightarrow\eta'$, is an isomorphism of $\Omega^0(M)-$modules. The bracket $[\![\xi,\eta]\!]$ is a cross-section of $A(P)$ such that $[\![\xi,\eta]\!]':=[\xi',\eta']$.

The Lie algebroid of a trivial principal fibre bundle $P=M\times G$ is canonically isomorphic to the trivial Lie algebroid $A=TM\times\mathfrak{g}$, $\mathfrak{g}$ is the right Lie algebra of G, via

$$A(P)=T(M\times G)/G=TM\times(TG/G)\ni(v,[w])\longrightarrow(v,\Theta^R(w))\in TM\times\mathfrak{g};$$

Θ^R denotes the canonical right-invariant 1-form on G [5], [6].

A transitive Lie algebroid strongly isomorphic to $A(P)$ for some principal fibre bundle is called *integrable* [9]. There exist non-integrable Lie algebroids discovered by R. Almeida and P. Molino [1]. Lie algebroids of some TC-foliations are non-integrable, for example, the Lie algebroid of the foliation of left cosets of any connected and simply connected Lie group by a connected nonclosed Lie subgroup has this property.

1.4. Definition. By a *connection in transitive Lie algebroid* (2), see [5], [9], [15], we mean a homomorphism of vector bundles $\lambda\colon TM\longrightarrow A$ such that $\gamma\circ\lambda=id_{TM}$, i.e. a splitting of Atiyah sequence (3) of A

$$0\longrightarrow\mathbf{g}\hookrightarrow A\underset{\lambda}{\overset{\gamma}{\rightleftarrows}}TM\longrightarrow 0.$$

By a *curvature tensor of a connection* λ in (2) we shall mean a tensor $\Omega_b\in\Omega^2(M;\mathbf{g})$ $(=Sec\Lambda^2T^*M\otimes\mathbf{g})$ defined by

$$\Omega_b(X,Y)=\lambda[X,Y]-[\![\lambda X,\lambda Y]\!],\ X,\ Y\in\mathfrak{X}(M).$$

λ also determines a covariant derivative ∇ in $\mathbf{g}$ by

$$\nabla_X \sigma = [\![\lambda X, \sigma]\!], \quad X \in \mathfrak{X}(M), \quad \sigma \in Sec\mathbf{g},$$

See [5], [9].

It turns out that the Lie algebra structure in $SecA$ is uniquely determined by $\mathbf{g}$, ∇, Ω_b and λ, namely, we have

1.5. Theorem [5], [9]. *The mapping* φ: $TM \oplus \mathbf{g} \longrightarrow A$, $(v, w) \longrightarrow \lambda v + w$, *is an isomorphism of Lie algebroids provided that in* $TM \oplus \mathbf{g}$ *the following Lie algebroid structure is defined:*

(a) *the bracket:*

$[\![(X, \sigma), \ (y, \eta)]\!] = ([X, Y], -\Omega_b(X, Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma, n]), \quad X, Y \in \mathfrak{X}(M)$, σ, $n \in Sec\mathbf{g}$ *(*$[\sigma, \eta]$ *is defined point by point:* $[\sigma, \eta](x) = [\sigma(x), \eta(x)]$, $x \in M$*).*

(b) *the anchor:* $\gamma = pr_1$: $TM \oplus \mathbf{g} \longrightarrow TM$. ∎

2. THE LIE ALGEBROID OF A CONNECTED (NOT NECESSARILY CLOSED) LIE SUBGROUP

Let G be any connected Lie group and $H \subset G$ any connected (not necessarily closed) Lie subgroup of G. H determines the foliation $\mathscr{F} = \{gH;\ g \in G\}$ of left cosets of G by H. $\mathscr{F}$ is a transversally complete foliation [12], [13] because right-invariant vector fields are from the normalizer of $\mathfrak{X}(\mathscr{F})$ and generate the entire tangent space $T_g G$ for any $g \in G$.

Denote by E the tangent bundle to $\mathscr{F}$ and $Q = TG/E \xrightarrow{r} G$ the transversal bundle of $\mathscr{F}$. Let

$$\alpha: TG \longrightarrow Q$$

be the canonical projection and let $\bar{v}$, $v \in TG$, denote the vector $\alpha(v)$. R_t: $TG \longrightarrow TG$ stands for the differential of the right translation by $t \in G$.

2.1. Lemma. (i) R_t, $t \in \bar{H}$ *(*$\bar{H}$ *is the closure of* H*), maps* E *into* E *inducing the isomorphism of vector bundles* $\overline{R}_t$: $Q \longrightarrow Q$, $\bar{v} \longmapsto \overline{R_t(v)}$.

(ii) *The mapping* $\overline{R}$: $Q \times \bar{H} \longrightarrow Q$, $(\bar{v}, t) \longmapsto \overline{R}_t(\bar{v})$, *is a right strongly free action.*

Proof. Easy calculations. ■

As a corollary we obtain

2.2. *The topological space $A(G;H)$ of orbits of the action $\overline{R}$, i.e.*

$$A(G;H)=Q/\approx \quad \text{where} \quad \overline{v}\approx\overline{w} \Leftrightarrow \exists_{t\in\overline{H}} (\overline{R}_t(\overline{v})=\overline{w})$$

has a uniquely determined structure of a C^∞ manifold, such that the canonical projection $\beta: Q\longrightarrow A(G;H)$ is a submersion.

In the sequel, the vector $\beta(\overline{v})$, $\overline{v}\in Q$, will be denoted by $[\overline{v}]$ and $\pi_b: G\longrightarrow G/\overline{H}$ stands for the canonical projection. Of course, $\overline{r}: A(G;H)\longrightarrow G/\overline{H}$, $[\overline{w}]\mapsto\pi_b(r\overline{w})$, is a correctly defined projection. Its smoothness follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\beta} & A(G;H) \\ \downarrow r & & \downarrow \overline{r} \\ G & \xrightarrow{\pi_b} & G/\overline{H} \end{array} .$$

For the fibre $A(G;H)_{|\overline{g}}$ of $\overline{r}$ over $\overline{g}\in G/\overline{H}$, the mapping $\beta_g: Q_{|g}\longrightarrow A(G;H)_{|\overline{g}}$, $g\in\pi_b^{-1}(\overline{g})$, is a bijection. Via β_g we introduce in $A(G;H)_{|\overline{g}}$ some structure of a real vector space and, clearly, it is independent of the choice of g. We wish to arrange the system $(A(G;H), \overline{r}, G/\overline{H})$ to be a vector bundle. For the purpose, we find local trivializations of this system.

2.3. Definition. *A C^∞ cross-section $\zeta\in SecQ$ is called a transversal field if, for any $g\in G$ and $t\in\overline{H}$,*

$$\zeta(gt)=\overline{R}_t(\zeta(g))$$

(that is, if ζ is $\overline{H}$-right-invariant).

2.4. Example. The C^∞ cross-section $\overline{Y}_w:=\alpha\circ Y_w$ where Y_w stands for the right-invariant vector field on G generated by $w\in\mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of G) is a transversal field. Therefore, transversal fields generate the entire space $Q_{|g}$ for any $g\in G$.

2.5. Remarks. Denote by $l(G;H)$ the space of all transversal fields.

(a) $l(G;H)$ forms a module over the ring $\Omega^0(G/\overline{H})$ under the multiplication $\overline{f}\cdot\zeta:=\overline{f}\circ\pi_b\cdot\zeta$, $\overline{f}\in\Omega^0(G/\overline{H})$, $\zeta\in l(G;H)$.

(b) If transversal fields $\zeta_1,\ldots,\zeta_s$ are linearly independent at a point $g\in G$, then, immediately by the definition, they are linearly independent at each point gt, $t\in\overline{H}$, and, in consequence, at some open $\mathscr{F}_b$-saturated open subset where $\mathscr{F}_b$ is the so-called *basic foliation* $\mathscr{F}_b=\{g\overline{H};\ g\in G\}$.

(c) Let $\zeta, \zeta_i\in l(G;H)$, $i\leq s$. If ζ_i are linearly independent on $U=\pi_b^{-1}[\overline{U}]$ ($\overline{U}$ open in $G/\overline{H}$) and $\zeta=\sum_{i=1}^{s} f^i\zeta_i$ for $f^i\in\Omega^0(U)$, then the functions f^i are of the form $f^i=\overline{f}^i\circ\pi_b|U$ for some $\overline{f}^i\in\Omega^0(\overline{U})$.

2.6. Proposition. *Let $q=\dim G-\dim H$, i.e. $q=\operatorname{codim}\mathscr{F}$. Suppose that $\zeta_1,\ldots,\zeta_q$ are transversal fields linearly independent at each point of a set $U=\pi_b^{-1}[\overline{U}]$, $\overline{U}$ open in $G/\overline{H}$. Then*

$$\varphi\colon \overline{U}\times\mathbf{R}^q\longrightarrow\overline{r}^{-1}[\overline{U}]\subset A(G;H)$$
$$(\overline{g},\alpha)\longrightarrow[\textstyle\sum\alpha^i\zeta_i(g)],\ g\in\pi_b^{-1}(\overline{g}),$$

is a local trivialization of $\overline{r}\colon A(G;H)\longrightarrow G/\overline{H}$.

Proof. Of course, $\varphi_{\overline{g}}\colon \mathbf{R}^q\longrightarrow A(G;H)_{|\overline{g}}$, $\overline{g}\in\overline{U}$, is an isomorphism of vector spaces. This proposition will be proved by showing that φ is a diffeomorphism. For the purpose, take the mapping $\psi\colon U\times\mathbf{R}^q\longrightarrow r^{-1}[U]\subset Q$, $(g,\alpha)\longrightarrow\sum\alpha^i\zeta_i(g)$, being a local trivialization of Q. Our assertion follows now from the commutativity of the diagram

$$\begin{array}{ccccc}
U\times\mathbf{R}^q & \xrightarrow[\cong]{\psi} & r^{-1}[U]\subset Q & \xrightarrow{r} & G\\
\downarrow{\scriptstyle \pi_b\times id} & & \downarrow{\scriptstyle\beta} & & \downarrow\\
\overline{U}\times\mathbf{R}^q & \xrightarrow{\varphi} & \overline{r}^{-1}[\overline{U}]\subset A(G;H) & \xrightarrow{\overline{r}} & G/\overline{H}
\end{array}\ \blacksquare$$

2.7. Remark. The structure of a C^∞ manifold in $A(G;H)$ can be obtained independently by demanding that φ's be diffeomorphisms.

Now, we introduce a structure of a Lie algebroid into the vector bundle $A(G;H)$. Firstly, we define the anchor $\gamma\colon A(G;H)\longrightarrow T(G/\overline{H})$ by $[w]\longrightarrow\pi_{b*}(w)$ (the correctness is easy to obtain). Secondly, we introduce in $Sec\,A(G;H)$ a structure of a Lie algebra in the way described below.

Take a homomorphism of $\Omega^0(G/\overline{H})$ − modules

$$c\colon l(G;H) \longrightarrow Sec\, A(G;H),\ \zeta \longrightarrow c_\zeta, \tag{5}$$

where c_ζ is a C^∞ cross-section of $A(G;H)$ defined by $c_\zeta(\bar{g}) = [\zeta(g)]$, $g \in \pi_b^{-1}(\bar{g})$.

2.8. Lemma. *c is an isomorphism of $\Omega^0(G/\overline{H})$ − modules.*

Proof. We check at once that (5) is a monomorphism. To see that it is also an epimorphism, take an arbitrary C^∞ cross-section $\xi \in Sec\, A(G;H)$ and define a cross-section ζ of Q in such a way that the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\beta} & A(G;H) \\ \zeta\uparrow & & \uparrow\xi \\ G & \xrightarrow{\pi_b} & G/\overline{H} \end{array}$$

commutes, i.e. $c_\zeta = \xi$. The smoothness of ζ is the last thing to notice. In order to get this, take transversal fields $\zeta_1, \ldots, \zeta_q$ being a basis on $U = \pi_b^{-1}[\overline{U}]$ ($\overline{U}$ is open in $G/\overline{H}$ and contains an arbitrarily taken point of $G/\overline{H}$). Then $c_{\zeta_1}, \ldots, c_{\zeta_q}$ forms a basis of $A(G;H)$ on $\overline{U}$. Therefore, $\xi = \sum \bar{f}^i c_{\zeta_i}$ on $\overline{U}$ for some $\bar{f}^i \in \Omega^0(\overline{U})$. Of course, $\zeta = \sum \bar{f}^i \circ \pi_b \cdot \zeta_i$ on U, which ends the proof. ■

2.9. The space $l(G;H)$ has a natural structure of a real Lie algebra. Indeed, let $\zeta, v \in l(G;H) \subset Sec Q$. Take arbitrary vector fields $X, Y \in \mathbf{X}(G)$ such that $\zeta = \overline{X}$ $(:= \alpha \circ X)$ and, analogously, $v = \overline{Y}$. Put.

$$[\zeta, v] := \overline{[X, Y]}. \tag{6}$$

We need notice that

(a) $\overline{[X, Y]} \in l(G;H)$,

(b) definition (6) is correct.

Let us first observe

2.10. Lemma. *If $\zeta \in l(G; H)$ is of the form $\zeta = \overline{X}$ for a vector field $X \in \mathbf{X}(G)$, then X belongs to the normalizer of $\mathbf{X}(\mathscr{F})$, that is,*

$$[X, Y] \in \mathbf{X}(\mathscr{F}) \text{ for all } Y \in \mathbf{X}(\mathscr{F}) \tag{7}$$

[*i.e. X is the so-called foliate vector field for $\mathscr{F}$, see* [13]].

Proof. Of course, it is sufficient to show relation (7) for left-invariant vector fields $Y = X_h$, $h \in \mathfrak{h}$, only. To this end, take an arbitrary $g \in G$ and express ζ locally on a set $U = \pi_b^{-1}[\overline{U}]$ containing g ($\overline{U}$ open in $G/\overline{H}$), in the form $\zeta_{|U} = \sum \overline{f}^i \circ \pi_{b|U} \cdot \overline{Y}_{w_i|U}$, $\overline{f}^i \in \Omega(\overline{U})$, $w_i \in \mathfrak{g}$ (for $\overline{Y}_w$, see 2.4). Then

$$Z := \sum \overline{f}^i \circ \pi_{b|U} \cdot Y_{w_i|U} - X_{|U} \in \mathbf{X}(\mathscr{F}_U)$$

and, furthermore, we have

$$[X, X_h]_{|U} = [\sum \overline{f}^i \circ \pi_{b|U} \cdot Y_{w_i|U} - Z, X_{h|U}]$$

$$= -[Z, X_h]_{|U} \in \mathbf{X}(\mathscr{F}_U),$$

thus $[X, X_h] \in \mathbf{X}(\mathscr{F})$. ■

2.11. Remark. It can be proved that condition (7) is equivalent to the fact that $\zeta := \overline{X}$ is a transversal field; however, the sufficiency of this condition will not be used in the sequel.

Now, we are able to prove (a) and (b) from 2.9.

(a): To get the equality $\overline{R}_t(\overline{[X, Y]}(g)) = \overline{[X, Y]}(gt)$, $g \in G$, $t \in \overline{H}$, take the vector fields $Z_1 = R_t X - X$ and $Z_2 = R_t Y - Y$ tangent to $\mathscr{F}$. Applying 2.10, we deduce that

$$\overline{R}_t(\overline{[X, Y]}(g)) = \overline{R_t([X, Y](g))} = \overline{R_t([X, Y])(gt)}$$

$$= \overline{[R_t X, R_t Y](gt)} = \overline{[X + Z_1, Y + Z_2]}(gt)$$

$$= \overline{[X\, Y]}(gt).$$

(b): Immediately from 2.10.

2.12. In $SecA(G; H)$ we introduce the bracket $[\![\cdot, \cdot]\!]$ (forming a Lie algebra) by demanding that (5) be an isomorphism of Lie algebras, i.e. $[\![c_\zeta, c_v]\!] := c_{[\zeta, v]}$, $\zeta, v \in l(G; H)$.

2.13. Theorem. *The system*

$$A(G;H)=(A(G;H),[\![\cdot,\cdot]\!],\gamma) \tag{8}$$

is a transitive Lie algebroid on $G/\overline{H}$.

Proof. (1) $Sec\gamma\colon SecA(G;H)\longrightarrow \mathfrak{X}(G/\overline{H})$ is a homomorphism of Lie algebras. To see this, take $\xi,\eta\in SecA(G;H)$. Find vector fields $X, Y\in\mathfrak{X}(G)$ such that $\xi=c_{\overline{X}}$, $\eta=c_{\overline{Y}}$. By the definition of γ,

$$(Sec\gamma)(c_{\overline{X}})(\overline{g})=\pi_{b*g}(X_g) \text{ for } \overline{g}=\pi_b(g),\ g\in G,$$

from which we obtain that X is π_b-related to $\gamma\circ\xi$ and, analogously, Y to $\gamma\circ\eta$. Therefore $[X, Y]$ is π_b-related to $[\gamma\circ\xi,\gamma\circ\eta]$ and to $\gamma\circ[\![\xi,\eta]\!]$ simultaneously, which confirms our assertion.

(2) The equality $[\![\xi,\overline{f}\cdot\eta]\!]=\overline{f}\cdot[\![\xi,\eta]\!]+(\gamma\circ\xi)(\overline{f})\cdot\eta$, $\overline{f}\in\Omega^{\circ}(G/\overline{H})$, $\xi,\eta\in Sec\,A(G;H)$, follows easily from

$$[\overline{X},\overline{f}\circ\pi_b\cdot\overline{Y}]=\overline{f}\circ\pi_b\cdot[\overline{X},\overline{Y}]+(\gamma\circ c_{\overline{X}})(\overline{f})\cdot\overline{Y}. \blacksquare$$

Lie algebroid (8) will be called the *Lie algebroid of a Lie subgroup H of G*. It can be interesting only in the case of a nonclosed H because the closedness of H implies the triviality of $A(G;H)$: $A(G;H)\cong T(G/\overline{H})$.

2.14. Remark. One can prove [cf. [7]] that Lie algebroid (8) is equal to the one constructed by P. Molino [12], [13] for the TC-foliation $\mathscr{F}$.

3. STRUCTURE THEOREMS

Let (8) be the Lie algebroid of a connected Lie subgroup H of a connected Lie group G and

$$O\longrightarrow \mathbf{g}\hookrightarrow A(G;H)\xrightarrow{\gamma} T(G/\overline{H})\longrightarrow O$$

its Atiyah sequence. In this section we prove three fundamental facts concerning $A(G;H)$:

- *The adjoint Lie algebra bundle* **g** *of* $A(G;H)$ *is a trivial bundle of abelian Lie algebras.*
- *If the Lie algebroid* $A(G;H)$ *admits a flat connection (i.e. a connection with the zero curvature tensor), then it is trivial.*

• *Let* $\mathfrak{h}$, $\bar{\mathfrak{h}}$, $\mathfrak{g}$ *denote the Lie algebras of* H, $\bar{H}$ *and* G, *respectively. Suppose that there exists a Lie subalgebra* $\mathfrak{c} \subset \mathfrak{g}$ *such that* (a) $\mathfrak{c} + \bar{\mathfrak{h}} = \mathfrak{g}$, (b) $\mathfrak{c} \cap \bar{\mathfrak{h}} = \mathfrak{h}$. *Then* $A(G; H)$ *admits a flat connection.*

The crucial role in the proving of the first fact is played by the following Malcev theorem (for a short "foliated" proof of it, see [7]).

3.1. The Malcev Theorem [11], [17]. *If* H *is a dense connected Lie subgroup of a Lie group* T, *then* H *is a normal subgroup of* T *and* T/H *is abelian.* ∎

By this, according to our notations, $\mathfrak{h}$ is an ideal of $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}/\mathfrak{h}$ is an abelian Lie algebra.

3.2. Theorem. *For a vector* $w \in \bar{\mathfrak{h}}$, *the cross-section* $\bar{X}_w$ *of the transversal bundle* Q, *induced by the left-invariant vector field* X_w, *is a transversal field, and the mapping*

$$\varphi\colon G/\bar{H} \times \bar{\mathfrak{h}}/\mathfrak{h} \longrightarrow \mathfrak{g}, (\bar{g}, [w]) \longrightarrow [\bar{X}_w(g)],\ g \in \pi_b^{-1}(\bar{g}), \tag{9}$$

is a global trivialization of the Lie algebra bundle $\mathfrak{g}$.

Proof. It is sufficient to show that $\bar{X}_w$, $w \in \bar{\mathfrak{h}}$, is a transversal field; the rest is easy. Clearly, for $t \in \bar{H}$ and $g \in G$ $\bar{R}_t(\bar{X}_w(g)) = \bar{L}_g(\bar{R}_t(w))$ and $\bar{X}_w(gt) = \bar{L}_g(\bar{L}_t(w))$ where $\bar{L}_g\colon Q \longrightarrow Q$ is an automorphism of the vector bundle Q, determined by the differential L_g of the left-translation by g. Therefore, it remains to prove that $R_t(w) - L_t(w) \in E_{|t}$, which means that the vector field $X := Y_w - X_w$ is tangent to the foliation $\mathscr{F}$ at each point of $\bar{H}$.

Firstly, we notice that X is foliate; to see this, we calculate: Let $h \in \mathfrak{h}$, then $[X, X_h] = [Y_w - X_w, X_h] = X_{[h,w]} \in \mathbf{X}(\mathscr{F})$ because $[h, w] \in \mathfrak{h}$ according to the Malcev theorem.

Secondly, any foliate vector field X (for a foliation $\mathscr{F}$) in any distinguished local coordinates $x = (x^1, \ldots, x^p, y^1, \ldots, y^q)$ ($p = \dim \mathscr{F}$, $q = \operatorname{codim} \mathscr{F}$) is of the form $X(x, y) = \sum a^i(x, y) \frac{\partial}{\partial x^i} + \sum b^j(y) \frac{\partial}{\partial y^j}$ [13]; therefore, which is easy to see, if it is tangent to $\mathscr{F}$ at a point z, then it is tangent to $\mathscr{F}$ at each point of the closure of the leaf through z. In our situation, $X(e) = [Y_w - X_w](e) = O \in E_{|e}$, so — by the above — our theorem is proved. ∎

Now, we proceed to the second problem.

3.3. Theorem. *If the Lie algebroid $A(G;H)$ is flat, then it is trivial.*

Proof. Let $\lambda\colon T(G/\overline{H})\longrightarrow A(G;H)$ be a flat connection in $A(G;H)$. Then, taking account of 1.4 and isomorphism (9) of Lie algebra bundles, we have an isomorphism of Lie algebroids

$$\rho\colon T(G/\overline{H})\times\overline{\mathfrak{h}}/\mathfrak{h}\longrightarrow A(G;H),\ (v,[w])\longrightarrow\lambda v+[\overline{X}_w(g)],$$

$v\in T_{\bar g}G/\overline{H}$, $\bar g=\pi_b(g)$, $g\in G$, provided that in $T(G/\overline{H})\times\overline{\mathfrak{h}}/\mathfrak{h}$ the Lie algebroid structure is defined by the following formula

$$[\![(X,\sigma),(Y,\eta)]\!]=([X,Y],\nabla^{\circ}_X\eta-\nabla^{\circ}_Y\sigma+[\sigma,\eta]),$$

$X, Y\in\mathfrak{X}(G/\overline{H})$, $\sigma,\eta\colon G/\overline{H}\longrightarrow\overline{\mathfrak{h}}/\mathfrak{h}$, where ∇° is a covariant derivative in the trivial vector bundle $T(G/\overline{H})\times\overline{\mathfrak{h}}/\mathfrak{h}$, such that φ maps ∇° onto ∇, i.e.

$$\nabla^{\circ}_X\sigma=\varphi^{-1}\circ\nabla_X(\varphi\circ\sigma)=\varphi^{-1}\circ[\![\lambda X,\varphi\circ\sigma]\!].$$

Looking at example 1.2, we see that to end the proof, it is sufficient to show the equality

$$\nabla^{\circ}_X=\mathcal{L}_X,\qquad X\in\mathfrak{X}(G/\overline{H}),$$

which is equivalent to the fact that the covariant derivative ∇°_X of any constant function $\hat w\colon G/\overline{H}\longrightarrow\overline{\mathfrak{h}}/\mathfrak{h}$, $\bar g\longrightarrow[w]$, $w\in\overline{\mathfrak{h}}$, is zero, i.e. that $[\![\lambda X,c_{\overline{X}_w}]\!]=0$. The cross-section λX is locally of the form $\lambda X=\sum\bar f^i c_{\overline{Y}_{w_i}}$, $\bar f^i\in\Omega^{\circ}(G/\overline{H})$, thus

$$\begin{aligned}[\![\lambda X,c_{\overline{X}_w}]\!]&=[\![\textstyle\sum\bar f^i c_{\overline{Y}_{w_i}},c_{\overline{X}_w}]\!]\\&=\textstyle\sum(\bar f^i[\![c_{\overline{Y}_{w_i}},c_{\overline{X}_w}]\!]-\gamma\circ c_{\overline{X}_w}(\bar f^i)\cdot c_{\overline{Y}_{w_i}})\\&=0\end{aligned}$$

because $\gamma\circ c_{\overline{X}_w}=0$ and $[\![c_{\overline{Y}_{w_i}},c_{\overline{X}_w}]\!]=c_{\overline{[Y_{w_i},X_w]}}=0$. ∎

It remains to consider the third problem.

3.4. Theorem. *Suppose that there exists a Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ such that* (a) $\mathfrak{c}+\overline{\mathfrak{h}}=\mathfrak{g}$, (b) $\mathfrak{c}\cap\overline{\mathfrak{h}}=\mathfrak{h}$. *Then $A(G;H)$ admits a flat connection.*

Proof. The construction of a flat connection in $A(G;H)$ has four steps.

Step 1. Denote by $\overline{C} \subset TG$ the left-invariant distribution generated by $\mathfrak{c}$, i.e. the vector bundle tangent to the foliation $\{gF;\ g \in G\}$ where F is the connected Lie subgroup with the Lie algebra equalling $\mathfrak{c}$. $\overline{C}$ fulfils the following conditions (in which E_b is the vector bundle tangent to the foliation $\mathscr{F}_b = \{g\overline{H};\ g \in G\}$):

($\bar{1}$) $\overline{C} + E_b = TG$,

($\bar{2}$) $\overline{C} \cap E_b = E$,

($\bar{3}$) $\overline{C}$ is $\overline{H}$-right-invariant [i.e. $\overline{C}_{|gt} = R_t[\overline{C}_{|g}]$, $g \in G$, $t \in \overline{H}$],

($\bar{4}$) $\overline{C}$ is involutive.

Clearly, ($\bar{1}$), ($\bar{2}$) and ($\bar{4}$) hold. To see ($\bar{3}$), take an arbitrary vector $v \in \overline{C}_{|g}$. we have $v = L_g(w)$ for some $w \in \mathfrak{c}$. Since $R_t(v) = L_g(R_t(w))$, we need only to observe that $R_t(w) \in \overline{C}_{|t}$ for $t \in \overline{H}$. Write $t = \lim t_n$, $t_n \in H$; then, by the closedness of $\overline{C}$ in TG, we obtain that $R_t(w) = \lim R_{t_n}(w) \in \overline{C}$ because $R_{t_n}[\overline{C}] = \overline{C}$.

Step 2. Let $\overline{C} \subset TG$ be a distribution realizing conditions ($\bar{1}$)$\div$($\bar{4}$) above. Via the epimorphism $\alpha\colon TG \longrightarrow Q$ we define a subbundle $C' \subset Q$ by $C'_{|g} = \alpha_g[\overline{C}_{|g}]$, $g \in G$. [The fact that C' is a subbundle is obtained from the relation $E \subset \overline{C}$ which holds by ($\bar{2}$)]. C' fulfils the following conditions:

(1′) $Q' \oplus C' = Q$ where $Q' = E_b / E \subset Q$,

(2′) C' is $\overline{H}$-right-invariant [i.e. $C'_{|gt} = \overline{R}_t[C'_{|g}]$, $g \in G$, $t \in \overline{H}$],

(3′) $l_i(G; H) := Sec\, C' \cap l(G; H)$ is a Lie subalgebra of $l(G; H)$.

(1′) and (2′) are obvious. To check (3′), take arbitrary $\zeta, \nu \in l_i(G; H)$ and write $\zeta = \overline{X}$, $\nu = \overline{Y}$ for some vector fields $X, Y \in \mathfrak{X}(\overline{C})$. According to ($\bar{4}$), $[X, Y] \in \mathfrak{X}(\overline{C})$, which gives the relation $\overline{[X, Y]} \in Sec\, C'$. On the other hand (see 2.9), $[\zeta, \nu] = \overline{[X, Y]} \in l(G; H)$.

Step 3. Let $C' \subset Q$ be any vector subbundle realizing conditions (1′)$\div$(3′) above. Via the linear homomorphism $\beta\colon Q \longrightarrow A(G; H)$ we define a subbundle $C \subset A(G; H)$ by $C_{|\bar{g}} = \beta_g[C'_{|g}]$, $g \in \pi_b^{-1}(\bar{g})$, $\bar{g} \in G/\overline{H}$. Thanks to the equality $\beta \circ \overline{R}_t = \beta$, $t \in \overline{H}$, the correctness of this definition is evident. To see that C is a C^∞ vector subbundle of $A(G; H)$, it is sufficient to notice that a local C^∞ cross-section of $A(G; H)$ lying in C and passing through an arbitrarily taken vector from C exists. Let $\bar{v} \in C'_{|g}$ and $\bar{g} = \pi_b(g)$. Take a local

C^∞ cross-section $\varphi\colon U \longrightarrow G$ of the submersion $\pi_b\colon G \longrightarrow G/\overline{H}$, such that $\varphi(\overline{g}) = g$, and consider the diagram

$$\begin{array}{ccccc}
i^*C' & \xrightarrow{\ \bar{i}\ } & C' \subset Q & \xrightarrow{\ \beta\ } & A(G;H) \\
\downarrow & & \downarrow & & \downarrow \\
Im\varphi & \xrightarrow{\ i\ } & G & \xrightarrow{\ \pi_b\ } & G/\overline{H} \\
\uparrow & & & & \uparrow \\
\lfloor & & \varphi & & U.
\end{array}$$

Diminishing U if necessary, we may assume that the vector bundle $i^*C' \longrightarrow Im\varphi$ has a global C^∞ cross-section Z passing through $\overline{v}$. Put $\xi = \beta \circ \bar{i} \circ Z \circ \varphi\colon U \longrightarrow A(G;H)$; ξ is, of course, a C^∞ cross-section of $A(G;H)$ over U such that $\xi(\overline{g}) = [\overline{v}]$. The vector bundle C fulfils the conditions

(1) $\mathbf{g} \oplus C = A(G;H)$,

(2) $SecC$ is a Lie subalgebra of $SecA(G;H)$.

(1) is evident by the observation that β_g maps isomorphically $Q'_{|g}$ onto $\mathbf{g}_{|\overline{g}}$. To see (2), take arbitrary $\xi, \eta \in SecC$. According to 2.8, there exist transversal fields ζ, v such that $c_\zeta = \xi$ and $c_v = \eta$. Of course, $\beta_g(\zeta_g) = \xi_{\overline{g}}$ and $\beta_g(v_g) = \eta_{\overline{g}}$, $g \in \pi_b^{-1}(\overline{g})$. From the definition of C we obtain that ζ and v belong to $l_i(G;H)$. By (3′), $[\zeta, v] \in SecC' \cap l(G;H)$, therefore $[\![\xi, \eta]\!] = c_{[\zeta, v]} \in SecC$.

Step 4. Let $C \subset A(G;H)$ be a vector subbundle realizing conditions (1) and (2) above. Then, of course, a splitting λ of the Atiyah sequence of $A(G;H)$, see the diagram

$$O \longrightarrow \mathbf{g} \hookrightarrow A(G;H) = \mathbf{g} \oplus C \underset{\lambda}{\overset{\gamma}{\rightleftarrows}} T(G/\overline{H}) \longrightarrow O.$$

such that $Im\lambda = C$, is a flat connection in $A(G;H)$. ■

Combining the above theorems we get

3.5. Corollary. *The existence of a Lie subalgebra* $\mathfrak{c} \subset \mathfrak{g}$ *fulfilling* $\mathfrak{c} + \overline{\mathfrak{h}} = \mathfrak{g}$ *and* $\mathfrak{c} \cap \overline{\mathfrak{h}} = \mathfrak{h}$ *implies the triviality of the Lie algebroid* $A(G;H)$.

4. MAIN RESULTS

Let the symbols H, $\overline{H}$, G, $\mathfrak{h}$, $\overline{\mathfrak{h}}$, $\mathfrak{g}$ have the same meaning as in the previous two sections.

4.1. Theorem. *If there is a Lie subalgebra* $\mathfrak{c}\subset\mathfrak{g}$ *such that* (a) $\mathfrak{c}+\overline{\mathfrak{h}}=\mathfrak{g}$, (b) $\mathfrak{c}\cap\overline{\mathfrak{h}}=\mathfrak{h}$, *then the Lie algebra* $\mathfrak{h}$ *is minimally closed.*

Proof. Corollary 3.5 states that the Lie algebroid $A(G;H)$ of the Lie subgroup $H\subset G$ is trivial, i.e. there exists a Lie algebroid isomorphism Φ: $A(G;H)\longrightarrow A_o := T(G/\overline{H})\times\overline{\mathfrak{h}}/\mathfrak{h}$. Such a Lie algebroid is, of course, integrable: A_o is the Lie algebroid of the trivial principal fibre bundle $P=G/\overline{H}\times F$ for an arbitrarily taken Lie group F with the abelian Lie algebra $\overline{\mathfrak{h}}/\mathfrak{h}$, see 1.3. The following reasoning is due to R. Almeida and P. Molino, see the proof of their theorem [13, p. 138]. Consider the Lie algebroid $(TG\times\overline{\mathfrak{h}}/\mathfrak{h}, [\![\cdot,\cdot]\!], pr_1)$ of the trivial principal fibre bundle $G\times F$. The linear homomorphism of vector bundles.

$$\lambda\colon TG\longrightarrow TG\times\overline{\mathfrak{h}}/\mathfrak{h},\quad v\longmapsto (v, pr_2\circ\Phi([\overline{v}]))$$

is a connection in this Lie algebroid. λ is flat. Indeed, it is sufficient to show the equality $[\![\lambda X,\lambda Y]\!]=\lambda[X,Y]$ only for $X, Y\in\mathbf{X}(G)$ such that the corresponding cross-sections $\overline{X}$, $\overline{Y}$ of Q are transversal fields. However, the equality is then easy to obtain by using the fact that Φ is a homomorphism of Lie algebras, namely, writing $\lambda X=(X, pr_2\circ\Phi\circ c_{\overline{X}}\circ\pi_h)$ (and, analogously, for λY), we have

$$\begin{aligned}
[\![\lambda X,\lambda Y]\!] &= [\![(X, pr_2\circ\Phi\circ c_{\overline{X}}\circ\pi_h),(Y, pr_2\circ\Phi\circ c_{\overline{Y}}\circ\pi_h)]\!]\\
&= ([X,Y],\ \mathcal{L}_X(pr_2\circ\Phi\circ c_{\overline{Y}}\circ\pi_h)-\mathcal{L}_Y(pr_2\circ\Phi\circ c_{\overline{X}}\circ\pi_h)\\
&\qquad\qquad +[pr_2\circ\Phi\circ c_{\overline{X}}\circ\pi_h,\ pr_2\circ\Phi\circ c_{\overline{Y}}\circ\pi_h])\\
&= ([X,Y],\ \mathcal{L}_{\gamma\circ c_{\overline{X}}}(pr_2\circ\Phi\circ c_{\overline{Y}})\circ\pi_h-\mathcal{L}_{\gamma\circ c_{\overline{Y}}}(pr_2\circ\Phi\circ c_{\overline{X}})\circ\pi_h\\
&\qquad\qquad +[pr_2\circ\Phi\circ c_{\overline{X}},\ pr_2\circ\Phi\circ c_{\overline{Y}}]\circ\pi_h)\\
&= ([X,Y],\ pr_2\circ[\![(\gamma\circ c_{\overline{X}}, pr_2\circ\Phi\circ c_{\overline{X}}),(\gamma\circ c_{\overline{Y}}, pr_2\circ\Phi\circ c_{\overline{Y}})]\!]\circ\pi_h)\\
&= ([X,Y],\ pr_2\circ[\![\Phi\circ c_{\overline{X}},\ \Phi\circ c_{\overline{Y}}]\!]\circ\pi_h)\\
&= ([X,Y],\ pr_2\circ\Phi\circ[\![c_{\overline{X}},\ c_{\overline{Y}}]\!]\circ\pi_h)\\
&= ([X,Y],\ pr_2\circ\Phi\circ c_{\overline{[X,Y]}}\circ\pi_h)\\
&= \lambda[X,Y].
\end{aligned}$$

Let D be the connection in $G\times F$ determined by λ, i.e. the right-invariant distribution $D\subset T(G\times F)$ for which

$$D_{|(g,e)}=\{(v,\, pr_2\circ\Phi([\bar{v}]);\quad v\in TG\},\quad g\in G,$$

where e denotes the neutral element of F. The flatness of λ implies the involutivity of D. Consider the diagram

$$\begin{array}{ccc} G\times F & \xrightarrow{\;p_2=\pi_h\times id\;} & G/\bar{H}\times F \\ \Big\downarrow p_1 & & \Big\downarrow \\ G & \xrightarrow{\;\pi_h\;} & G/\bar{H}. \end{array}$$

Let $\tilde{G}\subset G\times F$ be any leaf of the distribution D. Of course, $\tilde{p}_1=p_1|\tilde{G}$: $\tilde{G}\longrightarrow G$ is a covering and, which is easy to obtain, $\tilde{p}_2=p_2|\tilde{G}$: $\tilde{G}\longrightarrow G/\bar{H}\times F$ is a submersion. Denote by $\tilde{\mathscr{F}}$ the lifting (by $\tilde{p}_1$) of the foliation $\mathscr{F}$ in $\tilde{G}$. Let $(g,a)\in\tilde{G}$. For $v\in T_gG$, the following conditions are equivalent:

(1) v is tangent to $\mathscr{F}$,

(2) $\tilde{p}_{1*(g,a)}{}^{-1}(v)$ is tangent to $\tilde{\mathscr{F}}$,

(3) $\tilde{p}_{2*(g,a)}(\tilde{p}_{1*(g,a)}{}^{-1}(v))=0.$

From this we obtain that $\tilde{\mathscr{F}}$ is defined by the submersion $\tilde{p}_2$: $\tilde{G}\longrightarrow G/\bar{H}\times F$, in particular, the leaves of $\tilde{\mathscr{F}}$ are closed. Introducing in $\tilde{G}$ a structure of a group in the standard way we obtain: $\tilde{G}$ is a Lie group and $\tilde{p}_1$ is a local isomorphism of Lie groups. It is a standard calculation to obtain that $\tilde{\mathscr{F}}$ is then the foliation of left cosets of $\tilde{G}$ by $\tilde{F}$ where $\tilde{F}$ is a connected Lie subgroup of $\tilde{G}$ with the Lie algebra equalling $\tilde{\mathfrak{h}}=\tilde{p}_{1*\tilde{e}}{}^{-1}[\mathfrak{h}]$ ($\tilde{e}$ being the neutral element of $\tilde{G}$). Therefore $\tilde{F}$ is a closed Lie subgroup. Of course, F, being closed after the lifting to some covering, is also closed after lifting it to the universal one, which means that $\mathfrak{h}$ is minimally closed. ∎

4.2. Theorem. *If $\pi_1(G)$ is finite and $H\neq\bar{H}$, then there exists no Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ fulfilling the conditions $\mathfrak{c}+\bar{\mathfrak{h}}=\mathfrak{g}$ and $\mathfrak{c}\cap\bar{\mathfrak{h}}=\mathfrak{h}$.*

Proof. Let $\pi_1(G)$ be finite. Then, the universal covering is finite, which implies the nonclosedness of the lifting $\tilde{H}$ of H. Our assertion follows now trivially from the previous theorem. ∎

To finish with, we can ask

Can Theorem 4.1 *be inverted ?*

It turns out that the answer is no.

4.3. Example. Let $G=U(2)$. Suppose that $\overline{H}:=T$ is a maximal torus in G. It is well known that $dim T=2$ and the lifting of T to the universal covering $\mathbf{R}\times SU(2)\longrightarrow U(2)$ is isomorphic to the cylinder $\mathbf{R}\times S^1$. Therefore, any Lie subalgebra $\mathfrak{h}$ of $\overline{\mathfrak{h}}$ ($\overline{\mathfrak{h}}$ – the Lie algebra of $\overline{H}$) is minimally closed. We prove, using theorem 0.2, that there exists some 1-dimensional Lie subalgebra $\mathfrak{h}$ of $\overline{\mathfrak{h}}$ for which.

(i) no Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ fulfilling (a) and (b) from 0.1 exists.

(ii) the corresponding connected Lie subgroup of T is dense in T.

Let $h_p\colon V(\overline{\mathfrak{h}}^*)\longrightarrow H_{\mathrm{d}\mathbf{R}}(G/T)$ be the Chern-Weil homomorphism of the T-principal fibre bundle $P=(G\to G/T)$. G and T have the same rank, therefore, according to [3; Th.VII, p. 467], we have that

$$h_P^{(2)}\colon \overline{\mathfrak{h}}^*\longrightarrow H^2_{\mathrm{d}\mathbf{R}}(G/T)$$

is surjective. Moreover, $dim\overline{\mathfrak{h}}^*=2$ and $dim H^2_{\mathrm{d}\mathbf{R}}(G/T)=1$, thus $dim\ Ker h_P^{(2)}=1$. Then it is obvious that there exists a covector $O\neq\beta\in\overline{\mathfrak{h}}^*$ such that (1) $h_P^{(2)}(\beta)\neq 0$, (2) $\mathfrak{h}:=Ker\beta\subset\overline{\mathfrak{h}}$ is a subspace such that the corresponding Lie subgroup $H\subset T$ is dense in T. Of course, the superposition

$$(\overline{\mathfrak{h}}/\mathfrak{h})^*\xrightarrow{\ j\ }\overline{\mathfrak{h}}^*\xrightarrow{\ h_P^{(2)}\ }H^2_{\mathrm{d}\mathbf{R}}(G/T)$$

is nontrivial: $h_P^{(2)}\circ j(\overline{\beta})\neq 0$ where $\overline{\beta}\in(\overline{\mathfrak{h}}/\mathfrak{h})^*$ is a linear homomorphism determined by β. Theorem 0.2 implies the nonexistence of a Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ fulfilling (a) and (b) above.

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