NON-EXISTENCE OF EXCHANGE TRANSFORMATIONS OF ITERATED JET FUNCTORS

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Abstract. We study the problem of the non-existence of natural transformations \( J^r J^s Y \to J^s J^r Y \) of iterated jet functors depending on some geometric object on the base of \( Y \).

1. Introduction. It is well known that for every couple \( F \) and \( G \) of product preserving bundle functors defined on the category \( Mf \) of smooth manifolds and all smooth maps there is an exchange natural equivalence \( \kappa_{F,G} : FG \to GF \), [5]. Moreover, denoting by \( p_F^F : FM \to M \) and \( p_G^G : GM \to M \) the bundle projections, we have \( p_F^G \circ \kappa_{F,G} \circ p_F^{FM} = F(p_G^G) \), i.e. \( \kappa_{F,G} \) interchanges the projections \( p_F^{GM} \) and \( F(p_G^G) \). We remark that this property generalizes the classical involution \( \kappa_{TT} : TTM \to TTM \) of the iterated tangent bundle to every pair \( F \) and \( G \) of product preserving functors on \( Mf \).

In [2] we applied this point of view to natural equivalences \( A_{F,G} : FG \to GF \) of iterated bundle functors defined on the category \( FM_m \) of fibered manifolds with \( m \)-dimensional bases and of fibered manifold morphisms covering local diffeomorphisms. In particular, we have extended the concept of the canonical involution in the following way. Given an arbitrary fibered manifold \( Y \to M \), we denote by \( p_Y^F : FY \to Y \) and \( p_Y^G : GY \to Y \) the bundle projections. Then the natural equivalence \( A_{F,G} \) is called the

2000 Mathematics Subject Classification: 58A05, 58A20.
Key words and phrases: natural transformation, jet prolongation, symplectic structure.
The first author was supported by a grant of the GA ČR No 201/05/0523.
The paper is in final form and no version of it will be published elsewhere.
involution, if \( p_{FG}^G \circ A_{FG}^Y = F(p_{FG}^G) \) for an arbitrary fibered manifold \( Y \to M \). By [2], [6] and [9], involutions can be applied in the prolongation of connections.

An important example of a bundle functor on \( \mathcal{FM}_m \) is the \( r \)-th jet functor \( J^r \), which associates to an arbitrary fibered manifold \( Y \to M \) its \( r \)-th jet prolongation \( J^rY \to M \). For such functors we have proved that for any \( r \) and \( s \) there is no involution \( J^rJ^s \to J^sJ^r \), [2]. On the other hand, M. Modugno [9] has introduced the involution \( \text{ex}_\Lambda : J^1J^1 \to J^1J^1 \) depending on a classical linear connection \( \Lambda \) on the base manifold \( M \).

In this paper we study the more general problem on the non-existence of natural transformations (not necessarily involutions)

\[
(A_\sigma)_Y : J^rJ^sY \to J^sJ^rY
\]

depending on some geometric object \( \sigma \) on the base of \( Y \). The main result will be proved in Section 1. Further, in Section 2 we prove that for \( r \neq s \) there is no natural transformation \( (A_\omega)_Y : J^rJ^sY \to J^sJ^rY \) depending on a symplectic form \( \omega \) on the base of \( Y \). As a direct consequence we obtain that for \( r \neq s \) there is no natural transformation \( J^rJ^sY \to J^sJ^rY \).

Finally, Section 3 is devoted to the problem of the non-existence of non-trivial natural transformations \( (A_\omega)_Y : J^rJ^sY \to J^sJ^rY \).

We remark that higher order jet functors play an important role in differential geometry, see e.g. [3], [4] and [7]. In what follows we use the terminology and notation from the book [5]. We denote \( \mathcal{MF}_m \subset \mathcal{M} \) the subcategory of \( m \)-dimensional manifolds and their local diffeomorphisms and by \( \mathcal{FM}_{m,n} \subset \mathcal{FM}_m \) the subcategory of fibered manifolds with \( n \)-dimensional fibres and their local fibered diffeomorphisms. All manifolds and maps are assumed to be infinitely differentiable.

2. The main result. Let \( F \) be a natural bundle on \( \mathcal{MF}_m \). Given a manifold \( M \) we denote by \( \Gamma_{\text{loc}}(FM) \) the set of local smooth sections of \( FM \). Further, suppose that for all \( \mathcal{MF}_m \)-objects \( M \) we have \( D(M) \subset \Gamma_{\text{loc}}(FM) \) such that from \( \sigma \in D(M_1) \) it follows that \( F\psi \circ \sigma \circ \psi^{-1} \in D(M_2) \) for any \( \mathcal{MF}_m \)-map \( \psi : M_1 \to M_2 \).

**Definition 1.** An \( \mathcal{FM}_{m,n} \)-natural operator \( A : D \leadsto (J^rJ^s, J^sJ^r) \) is an invariant family of functions

\[
A : D(M) \to C_M^\infty(J^rJ^sY, J^sJ^rY)
\]

into the space \( C_M^\infty(J^rJ^sY, J^sJ^rY) \) of all base preserving maps \( J^rJ^sY \to J^sJ^rY \) for any \( \mathcal{FM}_{m,n} \)-object \( Y \to M \). The invariance means that for any \( \mathcal{FM}_{m,n} \)-objects \( Y_1 \to M_1 \) and \( Y_2 \to M_2 \) and any \( \mathcal{FM}_{m,n} \)-map \( \Psi : Y_1 \to Y_2 \) covering \( \psi : M_1 \to M_2 \) and any sections \( \sigma_1 \in D(M_1) \) and \( \sigma_2 \in D(M_2) \) from \( \sigma_2 \circ \psi = F\psi \circ \sigma_1 \) it follows that \( A(\sigma_2) \circ J^rJ^s\Psi = J^sJ^r\Psi \circ A(\sigma_1) \).

In the special case \( D(M) = \Gamma_{\text{loc}}(FM) \) we write \( A : F \leadsto (J^rJ^s, J^sJ^r) \) instead of \( A : D \leadsto (J^rJ^s, J^sJ^r) \). The main result of the present paper is the following non-existence theorem.

**Theorem 1.** Let \( r \) and \( s \) be natural numbers such that \( r > s \). Let \( F \) and \( D \) be as above. Suppose that there exists a section \( \rho \in D(\mathbb{R}^m) \) and an \( \mathcal{MF}_m \)-map \( \varphi = (\varphi^i) : \mathbb{R}^m \to \mathbb{R}^n \) such that
(a) $F\varphi \circ \rho \circ \varphi^{-1} = \rho$ near $0 \in \mathbb{R}^m$,
(b) $j_0^{\rho+s-1}\varphi = \text{id}$,
(c) $\varphi^1$ depends only on $x^1$ and
(d) $\frac{d^{x^1}}{dx^1} \varphi^1(0) \neq 0.$
Then there is no $\mathcal{FM}_{m,n}$-natural operator $A : D \leadsto (J^rJ^s, J^sJ^r)$.

**Proof.** Denoting by $\mathbb{R}^{m,n}$ the product fibered manifold $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, we identify sections of $\mathbb{R}^{m,n}$ with maps $\mathbb{R}^m \to \mathbb{R}^n$. Further, we use the notation
$$j_0 J^s(f(x, \bar{x}) = j_0(x \to j_2(\bar{x} \to f(x, \bar{x}))) \in J_0^s J^s R^{m,n}.$$ 
Suppose that there exists an operator $A$ in question. Consider first the restriction $\tilde{A} : J_0 J^s R^{m,n} \to J_0^s J^r R^{m,n}$ of $A(\rho)$ to the fibers over $0 \in \mathbb{R}^m$. Using the invariance of $\tilde{A}$ with respect to the fiber homotheties of $\mathbb{R}^{m,n}$ and the homogeneous function theorem from [5] we see that $\tilde{A}$ is linear. Taking into account the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{m,n}$-maps
$$(x^1, \ldots, x^m, y^1, ty^2, \ldots, ty^n)$$
for $t \neq 0$ we can write
$$\tilde{A}(j_0^r J^s(x^1,0,\ldots,0)) = j_0^s J^r(\sum_{|\alpha| \leq s} \sum_{|\beta| \leq r} a_{\alpha\beta} x^\alpha (x-x)^\beta,0,\ldots,0)$$
for some uniquely determined $a_{\alpha\beta} \in \mathbb{R}$ for $m$-tuples $\alpha, \beta$ with $|\alpha| \leq s, |\beta| \leq r$. Further, considering the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{m,n}$-map
$$(x^1, \ldots, x^m, y^1 - x^1, y^2, \ldots, y^n)$$
we get from (1)
$$\tilde{A}(j_0^r J^s(x^1 - x^1,0,\ldots,0)) = j_0^s J^r(\sum_{|\alpha| \leq s} \sum_{|\beta| \leq r} a_{\alpha\beta} x^\alpha (x-x)^\beta - x^1,0,\ldots,0).$$
Then the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{m,n}$-map
$$(x^1, \ldots, x^m, y^1 + (y^1)^{s+1}, y^2, \ldots, y^n)$$
and the linearity of $\tilde{A}$ yield
$$\tilde{A}(j_0^r J^s((x^1 - x^1)^{s+1},0,\ldots,0)) = j_0^s J^r(\sum_{|\alpha| \leq s} \sum_{|\beta| \leq r} a_{\alpha\beta} x^\alpha (x-x)^\beta - x^1)^{s+1},0,\ldots,0).$$
But $j_0^r J^s(x^1 - x^1)^{s+1} = 0$ as $j_2^s(x^1 - x^1)^{s+1} = 0$. Then from (2) we get
$$j_0^s J^r(\sum_{|\alpha| \leq s} \sum_{|\beta| \leq r} a_{\alpha\beta} x^\alpha (x-x)^\beta - x^1)^{s+1} = 0,$$
which reads $a_{(0)(0)} = 0$ and
$$j_0^s J^r((a_{(0)e_1} - 1)^{s+1}(x^1 - x^1)^{s+1} + \sum_{l=1}^m (a_{(0)e_l})^{s+1}(x^l - x^l)^{s+1} + \ldots) = 0,$$
where the dots denote the linear combination of terms $x^\gamma (x - x)^\eta$ for other $(\gamma, \eta)$, $|\gamma| \leq s$, $|\eta| \leq r$. Since $r \geq s + 1$ (because of the assumption $r > s$), we have

$$a_{(0)e_1} = 1 \text{ and } a_{(0)e_l} = 0 \text{ for } l = 2, \ldots, m,$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ is the $m$-tuple with 1 in the $j$-th position. Then (by (4) and the assumptions (b) and (c)), the map $\varphi^{-1} \times \text{id}_{\mathbb{R}^n}$ sends

$$j_0^s j^r \left( \sum_{|\alpha| \leq s} \sum_{|\beta| \leq r} a_{\alpha\beta} x^\alpha (x - x)^\beta \right)$$

into

$$(5) \quad j_0^s j^r \left( \sum_{|\alpha| \leq s} \sum_{|\beta| \leq r} a_{\alpha\beta} x^\alpha (x - x)^\beta + \frac{1}{(r + s)!} \frac{d^{r+s}}{d(x^1)^{r+s}} \varphi^1(0)(x^1)^{r+s} + \ldots \right),$$

where the dots denote an expansion of terms $(x^1)^{r+s}$ and $x^\gamma x^\eta$ with $|\gamma + \eta| > r + s$. Of course,

$$j_0^s j^r (x^1)^{r+s} \neq 0 \text{ and } j_0^s j^r (x^1)^{r+s} = 0.$$

By the Newton formula, $j_0^s j^r (x^\gamma x^\eta) = j_0^s j^r (x^\gamma (x - x)^\eta)$ is the linear combination of terms $j_0^s j^r (x^2 (x - x)^2)$ with $|\gamma + \eta| = |\gamma + \eta|$ (which are zero if $|\gamma + \eta| > r + s$). Then the dots in (5) are zero. Therefore $\varphi^{-1} \times \text{id}_{\mathbb{R}^n}$ does not preserve the right hand side of (1) because $\frac{d^{r+s}}{d(x^1)^{r+s}} \varphi^1(0) \neq 0$ (assumption (d)). On the other hand $\varphi^{-1} \times \text{id}_{\mathbb{R}^n}$ preserves the left hand side of (1) because it preserves both the section $\rho$ (assumption (a)) and $j_0^s j^r (x^1, 0, \ldots, 0)$ (as $j_0^r \varphi = \text{id}$ because of assumption (b)). This is a contradiction. ■

**Corollary 1.** Let $F$ be a natural vector bundle. Then there is no $\mathcal{FM}_{m,n}$-natural operator $A : F \leadsto (J^r J^s, J^s J^r)$ for $r > s$.

**Proof.** It follows from Theorem 1 with $\rho = 0$. ■

**Corollary 2.** For $r > s$ there is no $\mathcal{FM}_{m,n}$-natural transformation $B : J^r J^s \rightarrow J^s J^r$.

**Proof.** Any such $B : J^r J^s \rightarrow J^s J^r$ can be treated as the corresponding constant $\mathcal{FM}_{m,n}$-natural operator $B : T \leadsto (J^r J^s, J^s J^r)$. ■

**Open Problem.** Clearly, for $r = s$ Theorem 1 does not hold (we have the identity map $J^r J^r Y \rightarrow J^r J^r Y$). On the other hand, we do not know whether Theorem 1 is true in the case $r < s$.

### 3. Natural transformations $J^r J^s \rightarrow J^s J^r$ depending on a symplectic structure.

From Theorem 1 we obtain easily

**Proposition 1.** For $r > s$ there is no $\mathcal{FM}_{2m,n}$-natural operator $A : \text{SYMP} \leadsto (J^r J^s, J^s J^r)$ transforming symplectic structures $\omega$ on $M$ into natural transformations $A(\omega) : J^r J^s Y \rightarrow J^s J^r Y$. 
Proof. In Theorem 1 we put \( m = 2m \), \( F = \wedge^2 T^* \),

\[
D(M) = \text{SYMP}(M) = \text{the space of local symplectic structures on } M,
\]

\[
\rho = \sum_{i=1}^{m} dx^i \wedge dx^{m+i} = \text{the standard symplectic structure on } \mathbb{R}^{2m},
\]

\[
\varphi = \left( x^1 + \frac{1}{r+s} (x^1)^{r+s}, x^2, \ldots, x^m, \frac{x^{m+1}}{1 + (x^1)^{r+s-1}}, x^{m+2}, \ldots, x^{2m} \right).
\]

We can see that \( \varphi \) and \( \rho \) satisfy assumptions (a)(d) of Theorem 1. \( \blacksquare \)

Now we prove

**Proposition 2.** For \( s > r \) there is no \( \mathcal{FM}_{2m,n} \) -natural operator \( A : \text{SYMP} \rightarrow (J^r J^s, J^s J^r) \) transforming symplectic structures \( \omega \) on \( M \) into natural transformations \( A(\omega) : J^r J^s Y \rightarrow J^s J^r Y \).

**Proof.** Suppose that there exists \( A \) in question. Let

\[
\bar{A} : J_0^s J^r \mathbb{R}^{2m,n} \rightarrow J_0^s J^r \mathbb{R}^{2m,n}
\]

be the restriction of \( A(\omega) \) to the fiber over \( 0 \in \mathbb{R}^{2m} \), where \( \omega^o = \sum_{i=1}^{m} dx^i \wedge dx^{m+i} \)

is the standard symplectic structure on \( \mathbb{R}^{2m} \). Using the invariance of \( \bar{A} \) with respect to the fiber homotheties of \( \mathbb{R}^{2m,n} \) and the homogeneous function theorem we see that \( \bar{A} \) is linear. Next, considering the invariance of \( \bar{A} \) with respect to the \( \mathcal{FM}_{2m,n} \) -maps

\[
\left( \tau_1 x^1, \ldots, \tau_m x^m, \frac{1}{\tau_1} x^{m+1}, \ldots, \frac{1}{\tau_m} x^{2m}, y^1, \tau y^2, \ldots, \tau y^n \right)
\]

preserving \( \omega^o \) one can easily show that

\[
\bar{A}(j_0^s j^r (x^1, 0, \ldots, 0)) = j_0^s j^r (ax^1 + bx^1 + \ldots, 0, \ldots, 0),
\]

where the dots mean some combination of monomials in \( x, x \) of degree \( \geq 2 \). Taking into account the invariance of \( \bar{A} \) with respect to the \( \mathcal{FM}_{2m,n} \) -map

\[
\left( x^1 + (x^1)^{r+1}, x^2, \ldots, x^m, \frac{x^{m+1}}{1 + (r+1)(x^1)^r}, x^{m+2}, \ldots, x^{2m}, y^1, \ldots, y^n \right)^{-1}
\]

preserving \( \omega^o \) and \( j_0^s j^r x^1 \) we deduce

(6) \[
\bar{A}(j_0^s j^r (a(x^1)^{r+1} + b(x^1)^{r+1} + \ldots, 0, \ldots, 0)) = 0,
\]

where the dots denote some expression of monomials of degree \( \geq r+2 \). Applying \( (\text{id}_{\mathbb{R}^{2m}} \times \text{id}_{\mathbb{R}^n}) \) to both sides of (6) we get

(7) \[
\bar{A}(j_0^s j^r (a(x^1)^{r+1} + b(x^1)^{r+1}, 0, \ldots, 0)) = 0.
\]

But

\[
\bar{A}(j_0^s j^r (x^1)^{r+1}) = \bar{A}(j_0^s j^r (x^1 - x^1 + x^1)^{r+1}) = j_0^s j^r \left( \sum_{k=0}^{r} C_{r+1}^{r+1} (x^1 - x^1)^k (x^1)^{r+1-k} \right).
\]

From (7) and the assumption \( s > r \) we get \( a + b = 0 \) and \( b = 0 \). Then we have

(8) \[
\bar{A}(j_0^s j^r (x^1, 0, \ldots, 0)) = j_0^s j^r (*, 0, \ldots, 0),
\]
where \(*\) denote some linear combination of monomials in \(x, \bar{x}\) of degree \(\geq 2\). Further, using the invariance of \(\tilde{A}\) with respect to the \(\mathcal{FM}_{2m,n}\)-map
\[
(x^1, \ldots, x^{2m}, y^1 - x^1, y^2, \ldots, y^n)
\]
we get from (8)
\[
(9) \quad \tilde{A}(j_0^s j^s(x^1 - x^1, 0, \ldots, 0)) = j_0^s j^r(-x^1, *, 0, \ldots, 0).
\]
Then using the invariance of \(\tilde{A}\) with respect to the \(\mathcal{FM}_{2m,n}\)-map
\[
(x^1, \ldots, x^{2m}, y^1 + (y^1)^{s+1}, y^2, \ldots, y^n)
\]
we obtain from (9)
\[
(10) \quad \tilde{A}(j_0^s j^s((x^1 - x^1)^{s+1}, 0, \ldots, 0)) = j_0^s j^r((x^1)^{s+1} + **, 0, \ldots, 0),
\]
where ** is some linear combination of monomials in \(x, \bar{x}\) of degree \(\geq s+2\). But \(j_0^s j^s((x^1 - x^1)^{s+1}) = 0\). So from (10) we have
\[
\tilde{A}(x^1)_{s+1} = 0.
\]
This is a contradiction as \(j_0^s j^r((x^1)^{s+1}) = J_0^{r+s}((x^1)^{s+1}) \neq 0\). □

From Corollary 2 and Proposition 3 we obtain

**Proposition 3.** For \(r \neq s\) there is no natural transformation \(A : J^r J^s \to J^r J^r\).

*Proof.* It suffices to prove the case \(s > r\). Obviously, such \(A\) can be treated as a natural operator \(A : \text{SYMP} \rightsquigarrow (J^r J^s, J^r J^r)\) constant with respect to elements from SYMP. By Proposition 3 the proof is complete. □

It is interesting to point out that the only natural transformation \(J^r J^s \to J^r J^s\) is the identity, [1].

4. **Non-identical natural transformations** \(J^r J^r Y \to J^r J^r Y\) **depending on a symplectic structure** Clearly, for \(r = s\) we have the trivial \(\mathcal{FM}_{2m,n}\)-natural operator \(A^0 : \text{SYMP} \rightsquigarrow (J^r J^r, J^r J^r)\) such that \(A^0(\omega) = \text{id}_{J^r J^r Y}\) for any \(\mathcal{FM}_{2m,n}\)-object \(Y \to M\) and any symplectic form \(\omega\) on \(M\). On the other hand, in the case \(r = s\) we formulate the following hypothesis.

**HYPOTHESIS.** There is no non-trivial \(\mathcal{FM}_{2m,n}\)-natural operator \(A : \text{SYMP} \rightsquigarrow (J^r J^r, J^r J^r)\) transforming symplectic structures \(\omega\) on \(M\) into natural transformations \(A(\omega) : J^r J^r Y \to J^r J^r Y\).

It seems that the verification of this hypothesis will be technically complicated. Bellow we prove only

**Proposition 4.** The hypothesis is true for \(r = s = 1\).

*Proof.* Suppose that there exists \(A\) in question. Let
\[
\tilde{A} : J_0^1 J^1 \mathbb{R}^{2m,n} \to J_0^1 J^1 \mathbb{R}^{2m,n}
\]
be the restriction of \(A(\omega^0)\) to the fiber over \(0 \in \mathbb{R}^{2m}\), where \(\omega^0 = \sum_{i=1}^{m} dx^i \wedge dx^{2m+i}\) is the standard symplectic structure on \(\mathbb{R}^{2m}\). Quite similarly to the proof of Proposition 3,
\( \tilde{A} \) is linear and we can write

\[
\tilde{A}(j_0^1 j^1(x^1, 0, \ldots, 0)) = j_0^1 j^1(ax^1 + bx^1 + *, 0, \ldots, 0),
\]

where * is some linear combination of \( x^i x^j \) for \( i, j = 1, \ldots, 2m \). Using the invariance of \( \tilde{A} \) with respect to the \( \mathcal{FM}_{2m,n} \)-maps \((\text{id}_m \times \frac{1}{\beta} \text{id}_m \times \text{id}_{R^n})\) preserving \( \omega^\circ \) we deduce that \( * = 0 \). Analogously to the proof of Proposition 3 we deduce

\[
j_0^1 j^1(b(x^1)^2) = 0,
\]

which reads \( b = 0 \). Further, using the invariance of \( \tilde{A} \) with respect to the \( \mathcal{FM}_{2m,n} \)-map

\[(x^1, \ldots, x^{2m}, y^1 - x^1, y^2, \ldots, y^n)\]

preserving \( \omega^\circ \) we get

\[
\tilde{A}(j_0^1 j^1(x^1 - \bar{x}^1, 0, \ldots, 0)) = j_0^1 j^1(ax^1 - \bar{x}^1).
\]

Taking into account the \( \mathcal{FM}_{2m,n} \)-map

\[(x^1, \ldots, x^{2m}, y^1 + (y^1)^2, y^2, \ldots, y^n)\]

preserving \( \omega^\circ \) we obtain

\[
\tilde{A}(j_0^1 j^1((x^1 - \bar{x}^1)^2, 0, \ldots, 0)) = j_0^1 j^1((ax^1 - \bar{x}^1)^2, 0, \ldots, 0).
\]

Then as \( j_0^1 j^1(\bar{x}^1 - x^1)^2 = 0 \) and \( j_0^1 j^1(x^1)^2 = 0 \), we have

\[
0 = \tilde{A}(j_0^1 j^1(0, \ldots, 0)) = \tilde{A}(j((x^1 - \bar{x}^1)^2, 0, \ldots, 0))
\]

and

\[
j_0^1 j^1(ax^1 - \bar{x}^1)^2 = -2(a - 1)j_0^1 j^1(x^1(\bar{x}^1 - x^1)).
\]

This yields \( a = 1 \), i.e.

\[(11) \quad \tilde{A}(j_0^1 j^1(x^1, 0, \ldots, 0)) = j_0^1 j^1(x^1, 0, \ldots, 0).
\]

Considering the invariance of \( \tilde{A} \) with respect to the \( \mathcal{FM}_{2m,n} \)-map

\[(x^1 + x^{m+1}, x^2, \ldots, x^{2m}, y^1, \ldots, y^n)^{-1}\]

preserving \( \omega^\circ \) and \( j_0^1 j^1 x^1 \) we deduce from (11)

\[(12) \quad \tilde{A}(j_0^1 j^1(x^{m+1}, 0, \ldots, 0)) = j_0^1 j^1(x^{m+1}, 0, \ldots, 0).
\]

Next, applying the invariance of \( \tilde{A} \) with respect to permutations of first \( m \) base coordinates and respective second \( m \) base coordinates (preserving \( \omega^\circ \)) we deduce from (11) and (12)

\[(13) \quad \tilde{A}(j_0^1 j^1(x^i, 0, \ldots, 0)) = j_0^1 j^1(x^i, 0, \ldots, 0)
\]

for \( i = 1, \ldots, 2m \). Then the invariance of \( \tilde{A} \) with respect to the \( \mathcal{FM}_{2m,n} \)-map

\[(x^1, \ldots, x^{2m}, y^1 + x^j y^1, y^2, \ldots, y^n)\]

preserving \( \omega^\circ \) yields

\[(14) \quad \tilde{A}(j_0^1 j^1(\bar{x}^j x^i, 0, \ldots, 0)) = j_0^1 j^1(\bar{x}^j x^i, 0, \ldots, 0)
\]

for \( i, j = 1, \ldots, 2m \). Taking into account the invariance of \( \tilde{A} \) with respect to the \( \mathcal{FM}_{2m,n} \) maps

\[(x^1, \ldots, x^{2m}, y^1 - x^\beta, y^2, \ldots, y^n)\]
preserving $\omega^\alpha$, where $\beta$ are $2m$-tuples of non-negative integers, from the clear equality
\[ \tilde{A}(j^1_0 j^1_1(0, \ldots, 0)) = j^1_0 j^1_1(1, 0, \ldots, 0) \]
we get
\[ \tilde{A}(j^1_0 j^1_1(1, 0, \ldots, 0)) = j^1_0 j^1_1(1, 0, \ldots, 0) \]
and
\[ \tilde{A}(j^1_0 j^1_1(\bar{x}^i, 0, \ldots, 0)) = j^1_0 j^1_1(\bar{x}^i, 0, \ldots, 0) \]
for $i = 1, \ldots, 2m$. Finally, using the linearity of $\tilde{A}$ and the invariance of $\tilde{A}$ with respect to permutations of fiber coordinates from (14), (15) and (16) we obtain $\tilde{A} = \text{id}$ (these elements generate the vector space $J^1_0 J^1_1 \mathbb{R}^{2m-n}$). Then $A$ is trivial because of the Darboux theorem, which is contradiction.

Denote by $\text{ex}_A : J^1 J^1 \to J^1 J^1$ the involution depending on a classical linear connection $\Lambda$ on the base of $Y$, which was constructed by M. Modugno [9]. Clearly, $\text{ex}_A$ is a non-identic natural equivalence. From Proposition 4 it follows directly the following result, which was also proved in [8].

**Proposition 5.** Symplectic structures do not induce canonically classical linear connections.

**Remark 1.** Let $\Gamma : Y \to J^1 Y$ be a connection on a fibered manifold $Y \to M$. Using the exchange transformation $\text{ex}_A : J^1 J^1 \to J^1 J^1$, one can construct a connection $J^1(\Gamma, \Lambda)$ on $J^1 Y \to M$ by means of a classical linear connection $\Lambda$ on $M$ by
\[ J^1(\Gamma, \Lambda) : = (\text{ex}_A)_Y \circ J^1 \Gamma, \]
see [9]. By propositions 1, 2 and 4, it is impossible to construct in a similar way a connection on $J^r Y \to M$ from a connection $\Gamma$ by means of a symplectic form on $M$.

**References**