1.3 Hilbert space methods

One of the most often used theorems of functional analysis is the Riesz representation theorem.

**Theorem 1.30 (Riesz representation theorem).** If \( x^* \) is a continuous linear functional on a Hilbert space \( H \), then there is exactly one element \( y \in H \) such that

\[
<x^*, x> = (x, y).
\]

(1.32)

1.3.1 To identify or not to identify–the Gelfand triple

Riesz theorem shows that there is a canonical isometry between a Hilbert space \( H \) and its dual \( H^* \). It is therefore natural to identify \( H \) and \( H^* \) and is done so in most applications. There are, however, situations when it cannot be done.

Assume that \( H \) is a Hilbert space equipped with a scalar product \((\cdot, \cdot)_H\) and that \( V \subset H \) is a subspace of \( H \) which is a Hilbert space in its own right, endowed with a scalar product \((\cdot, \cdot)_V\). Assume that \( V \) is densely and continuously embedded in \( H \) that is \( V = H \) and \( \|x\|_H \leq c \|x\|_V \), \( x \in V \), for some constant \( c \). There is a canonical map \( T : H^* \to V^* \) which is given by restriction to \( V \) of any \( h^* \in H^* \):

\[
<T h^*, v >_{V^* \times V} = <h^*, v >_{H^* \times H}, \quad v \in V.
\]

We easily see that

\[
\|T h^*\|_{V^*} \leq C \|h^*\|_{H^*}.
\]

Indeed

\[
\|T h^*\|_{V^*} = \sup_{\|v\|_V \leq 1} |<T h^*, v >_{V^* \times V}| = \sup_{\|v\|_V \leq 1} |<h^*, v >_{H^* \times H}| \\
\leq \|h^*\|_{H^*} \sup_{\|v\|_V \leq 1} \|v\|_H \leq c \|h^*\|_{H^*}.
\]

Further, \( T \) is injective. For, if \( T h_1^* = T h_2^* \), then

\[
0 = <T h_1^* - T h_2^*, v >_{V^* \times V} = <h_1^* - h_2^*, v >_{H^* \times H}
\]

for all \( v \in V \) and the statement follows from density of \( V \) in \( H \). Finally, the image of \( T H^* \) is dense in \( V^* \). Indeed, let \( v \in V^{**} \) be such that \( <v, T h^* >= 0 \) for all \( h^* \in H^* \). Then, by reflexivity,

\[
0 = <v, T h^* >_{V^{**} \times V} = <T h^*, v >_{V^* \times V} = <h^*, v >_{H^* \times H}, \quad h^* \in H^*
\]

implies \( v = 0 \).

Now, if we identify \( H^* \) with \( H \) by the Riesz theorem and using \( T \) as the canonical embedding from \( H^* \) into \( V^* \), one writes
and the injections are dense and continuous. In such a case we say that $H$ is the pivot space. Note that the scalar product in $H$ coincides with the duality pairing $<\cdot,\cdot>_V: \quad (f,g)_H = <f,g>_{V^*\times V}, \quad f \in H, g \in V.$

Remembering now that $V$ is a Hilbert space with scalar product $(\cdot,\cdot)_V$ we see that identifying also $V$ with $V^*$ would lead to an absurd – we would have $V = H = H^* = V^*.$ Thus, we cannot identify simultaneously both pairs. In such situations it is common to identify the pivot space $H$ with its dual $H^*$ but to leave $V$ and $V^*$ as separate spaces with duality pairing being an extension of the scalar product in $H.$

An instructive example is $H = L_2([0,1],dx)$ (real) with scalar product

$$(u,v) = \int_0^1 u(x)v(x)dx$$

and $V = L_2([0,1],wdx)$ with scalar product

$$(u,v) = \int_0^1 u(x)v(x)w(x)dx,$$

where $w$ is a nonnegative bounded measurable function. Then it is useful to identify $V^* = L_2([0,1],w^{-1}dx)$ and

$$<f,g>_{V^*\times V} = \int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)\sqrt{w(x)} \frac{g(x)}{\sqrt{w(x)}} dx \leq \|f\|_V\|g\|_{V^*}.$$

1.3.2 The Radon-Nikodym theorem

Let $\mu$ and $\nu$ be finite nonnegative measures on the same $\sigma$-algebra in $\Omega.$ We say that $\nu$ is absolutely continuous with respect to $\mu$ if every set that has $\mu$-measure 0 also has $\nu$ measure 0.

**Theorem 1.31.** If $\nu$ is absolutely continuous with respect to $\mu$ then there is an integrable function $g$ such that

$$\nu(E) = \int_E g d\mu, \quad (1.33)$$

for any $\mu$-measurable set $E \subset \Omega.$
1.3 Hilbert space methods

Proof. Assume for simplicity that $\mu(\Omega), \nu(\Omega) < \infty$. Let $H = L_2(\Omega, d\mu + d\nu)$ on the field of reals. Schwarz inequality shows that if $f \in H$, then $f \in L_1(d\mu + d\nu)$. The linear functional

$$< x^*, f > = \int f d\mu$$

To be completed

1.3.3 Projection on a convex set

Corollary 1.32. Let $K$ be a closed convex subset of a Hilbert space $H$. For any $x \in H$ there is a unique $y \in K$ such that

$$\|x - y\| = \inf_{z \in K} \|x - z\|. \quad (1.34)$$

Moreover, $y \in K$ is a unique solution to the variational inequality

$$(x - y, z - y) \leq 0 \quad (1.35)$$

for any $z \in K$.

Proof. Let $d = \inf_{z \in K} \|x - z\|$. We can assume $x \notin K$ and so $d > 0$. Consider $f(z) = \|x - z\|$, $z \in K$ and consider a minimizing sequence $(z_n)_{n \in \mathbb{N}}$, $z_n \in K$ such that $d \leq f(z_n) \leq d + 1/n$. By the definition of $f$, $(z_n)_{n \in \mathbb{N}}$ is bounded and thus it contains a weakly convergent subsequence, say $(\zeta_n)_{n \in \mathbb{N}}$. Since $K$ is closed and convex, by Corollary 1.24, $\zeta_n \rightharpoonup y \in K$. Further we have

$$\|h - x - y\| \leq \|h - x - \zeta_n\| \leq \|h\| \liminf_{n \to \infty} \|x - \zeta_n\| \leq \|h\| \liminf_{n \to \infty} d + \frac{1}{n} = \|h\|d$$

for any $h \in H$ and thus, taking supremum over $\|h\| \leq 1$, we get $f(y) \leq d$ which gives existence of a minimizer.

To prove equivalence of (1.35) and (1.34) assume first that $y \in K$ satisfies (1.34) and let $z \in K$. Then, from convexity, $v = (1 - t)y + tz \in K$ for $t \in [0, 1]$ and thus

$$\|x - y\|^2 \leq \|x - ((1 - t)y + tz)\|^2 = \|(x - y) - t(z - y)\|^2$$

and thus

$$\|x - y\|^2 \leq \|x - y\|^2 - 2t(x - y, z - y) + t^2 \|z - y\|^2.$$  

Hence

$$t \|z - y\|^2 \geq 2(x - y, z - y)$$

for any $t \in (0, 1]$ and thus, passing with $t \to 0$, $(x - y, z - y) \leq 0$. Conversely, assume (1.35) is satisfied and consider
\[ \|x - y\|^2 - \|x - z\|^2 = (x - y, x - y) - (x - z, x - z) = 2(x, z) - 2(x, y) + 2(y, y) - 2(y, z) - (y, y) \]

so

\[ \|x - y\| \leq \|x - z\| \]

for any \( z \in K \).

For uniqueness, let \( y_1, y_2 \) satisfy

\[ (x - y_1, z - y_1) \leq 0, \quad (x - y_2, z - y_2) \leq 0, \quad z \in H. \]

Choosing \( z = y_2 \) in the first inequality and \( z = y_1 \) in the second and adding them, we get \( \|y_1 - y_2\|^2 \leq 0 \) which implies \( y_1 = y_2 \).

We call the operator assigning to any \( x \in K \) the element \( y \in K \) satisfying (1.34) the projection onto \( K \) and denote it by \( P_K \).

**Proposition 1.33.** Let \( K \) be a nonempty closed and convex set. Then \( P_K \) is non expansive mapping.

**Proof.** Let \( y_i = P_K x_i, i = 1, 2 \). We have

\[ (x_1 - y_1, z - y_1) \leq 0, \quad (x_2 - y_2, z - y_2) \leq 0, \quad z \in H. \]

so choosing, as before, \( z = y_2 \) in the first and \( z = y_1 \) in the second inequality and adding them together we obtain

\[ \|y_1 - y_2\|^2 \leq (x_1 - x_2, y_1 - y_2), \]

hence \( \|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\| \).

**1.3.4 Theorems of Stampacchia and Lax-Milgram**

**1.3.5 Motivation**

Consider the Dirichlet problem for the Laplace equation in \( \Omega \subset \mathbb{R}^n \)

\[ \begin{align*}
-\Delta u &= f \quad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega, \quad (1.36)
\end{align*} \]

Assume that there is a solution \( u \in C^2(\Omega) \cap C(\Omega) \). If we multiply (1.36) by a test function \( \phi \in C_0^\infty(\Omega) \) and integrate by parts, then we obtain the problem

\[ \iint_{\Omega} \Delta u \phi \, d\Omega = -\iint_{\Omega} \nabla u \cdot \nabla \phi \, d\Omega = \int_{\partial \Omega} \phi \frac{\partial u}{\partial \nu} \, d\Omega. \]

Conversely, if $u$ satisfies (1.38), then it is a distributional solution to (1.37).

Moreover, if we consider the minimization problem for

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx$$

over $K = \{u \in C^2(\Omega); u|_{\partial \Omega} = 0\}$ and if $u$ is a solution to this problem then for any $\epsilon > 0$ and $C_0^\infty(\Omega)$ we have

$$\epsilon < \epsilon \int_{\Omega} |\nabla u + \epsilon \nabla \phi|^2 + \epsilon^2 \int_{\Omega} \nabla u \cdot \nabla \phi + \epsilon \int_{\Omega} \phi^2 \nabla u \cdot \nabla \phi$$

then we also obtain (1.38). The question is how to obtain the solution.

In a similar way, we consider the obstacle problem, to minimize $J$ over $K = \{u \in C^2(\Omega); u|_{\partial \Omega} = 0, u \geq g\}$ over some continuous function $g$ satisfying $g|_{\partial \Omega} < 0$. Note that $K$ is convex. Again, if $u \in K$ is a solution then for any $\epsilon > 0$ and $\phi \in K$ we obtain that $u + \epsilon(\phi - u) = (1 - \epsilon)u + \epsilon\phi$ is in $K$ and therefore

$$J(u + \epsilon(\phi - u)) \geq J(u).$$

Here, we obtain only

$$\int_{\Omega} \nabla u \cdot \nabla(\phi - u) dx \geq \int_{\Omega} f(\phi - u) dx.$$

for any $\phi \in K$. For twice differentiable $u$ we obtain

$$\int_{\Omega} \Delta u(\phi - u) dx \geq \int_{\Omega} f(\phi - u) dx$$

and choosing $\phi = u + \psi$, $\psi \in C_0^\infty(\Omega)$ we get

$$-\Delta u \geq f$$

almost everywhere on $\Omega$. As $u$ is continuous, the set $N = \{x \in \Omega; u(x) > g(x)\}$ is open. Thus, taking $\psi \in C_0^\infty(N)$, we see that for sufficiently small $\epsilon > 0$, $u \pm \epsilon\phi \in K$. Then, on $N$

$$-\Delta u = f$$

Summarizing, for regular solutions the minimizer satisfies

$$-\Delta u \geq f$$

$$u \geq g$$

$$(\Delta u + f)(u - g) = 0$$

on $\Omega$. 

Hilbert space theory

We begin with the following definition.

**Definition 1.34.** Let $H$ be a Hilbert space. A bilinear form $a : H \times H \to \mathbb{R}$ is said to be

(i) continuous if there is a constant $C$ such that
$$|a(x, y)| \leq C \|x\| \|y\|, \quad x, y \in H;$$

coercive if there is a constant $\alpha > 0$ such that
$$a(x, x) \geq \alpha \|x\|^2.$$

Note that in the complex case, coercivity means $|a(x, x)| \geq \alpha \|x\|^2$.

**Theorem 1.35.** Assume that $a(\cdot, \cdot)$ is a continuous coercive bilinear form on a Hilbert space $H$. Let $K$ be a nonempty closed and convex subset of $H$. Then, given any $\phi \in (H^*)$, there exists a unique element $x \in K$ such that for any $y \in K$
$$a(x, y - x) \geq <\phi, y - x>_{H^* \times H} \quad (1.40)$$
Moreover, if $a$ is symmetric, then $x$ is characterized by the property
$$x \in K \quad \text{and} \quad \frac{1}{2} a(x, x) - <\phi, x>_{H^* \times H} = \min_{y \in K} \frac{1}{2} a(y, y) - <\phi, y>_{H^* \times H}. \quad (1.41)$$

**Proof.** First we note that from Riesz theorem, there is $f \in H$ such that
$$<\phi, y>_{H^* \times H} = (f, y) \quad \text{for all} \quad y \in H. \quad \text{Now, if we fix} \quad x \in H, \quad \text{then} \quad y \rightarrow a(x, y) \quad \text{is a continuous linear functional on} \quad H. \quad \text{Thus, again by the Riesz theorem, there is an operator} \quad A : H \to H \quad \text{satisfying} \quad a(x, y) = (Ax, y). \quad \text{Clearly,} \quad A \quad \text{is linear and satisfies}
\begin{equation}
\|Ax\| \leq C \|x\|, \quad (1.42)
\end{equation}
and
\begin{equation}
\frac{1}{2} a(x, x) - <\phi, x>_{H^* \times H} = \min_{y \in K} \frac{1}{2} a(y, y) - <\phi, y>_{H^* \times H}. \quad (1.41)
\end{equation}

Indeed,
$$\|Ax\| = \sup_{\|y\|=1} |(Ax, y)| \leq C \|x\| \sup_{\|y\|=1} \|y\|,$$
and (1.43) is obvious.

Problem (1.40) amounts to finding $x \in K$ satisfying, for all $y \in K$,
$$a(x, y - x) \geq <f, y - x>. \quad (1.44)$$
Let us fix a constant $\rho$ to be determined later. Then, multiplying both sides of (1.44) by $\rho$ and moving to one side, we find that (1.44) is equivalent to
Here we recognize the equivalent formulation of the projection problem (1.35), that is, we can write
\[ x = P_K(\rho f - \rho Ax + x) \]  
(1.46)

This is a fixed point problem for \( x \) in \( K \). Denote \( S y = P_K(\rho f - \rho Ay + y) \). Clearly \( S : K \to K \) as it is a projection onto \( K \) and \( K \), being closed, is a complete metric space in the metric induced from \( H \). Since \( P_K \) is nonexpansive, we have
\[ ||S y_1 - S y_2|| \leq ||(y_1 - y_2) - \rho(Ay_1 - Ay_2)|| \]
and thus
\[ ||S y_1 - S y_2||^2 = ||y_1 - y_2||^2 - 2\rho(Ay_1 - Ay_2, y_1 - y_2) + \rho^2||Ay_1 - Ay_2||^2 \]
\[ \leq ||y_1 - y_2||^2(1 - 2\rho \alpha + \rho^2 C^2) \]

We can choose \( \rho \) in such a way that \( k^2 = 1 - 2\rho \alpha + \rho^2 C^2 < 1 \) so that \( S \) has a unique fixed point in \( K \).

Assume now that \( a \) is symmetric. Then \( (x, y)_1 = a(x, y) \) defines a new scalar product which defines an equivalent norm \( ||x||_1 = \sqrt{a(x, x)} \) on \( H \).
Indeed, by continuity and coerciveness
\[ ||x||_1 = \sqrt{a(x, x)} \leq \sqrt{C}||x|| \]
and
\[ ||x|| = \sqrt{a(x, x)} \geq \sqrt{\alpha}||x||. \]

Using again Riesz theorem, we find \( g \in H \) such that
\[ <\phi, y>_{H^* \times H} = a(g, y) \]
and then (1.40) amounts to finding \( x \in K \) such that
\[ (\langle g - x, g - y \rangle)_1 = a(g - x, y - x) \leq 0 \]
for all \( y \in K \) but this is nothing else but finding projection \( x \) onto \( K \) with respect to the new scalar product. Thus, there is a unique \( x \in K \)
\[ \sqrt{a(g - x, g - x)} = \min_{y \in K} \sqrt{a(g - g, g - y)} = \min_{y \in K} ||g - y||_{a} \]

However, expanding, this is the same as finding minimum of the function
\[ y \to a(g- y, g - y) = a(g, g) + a(y, y) - 2a(g, y) = a(y, y) - 2<\phi, y>_{H^* \times H} + a(g, g). \]

Taking into account that \( a(g, g) \) is a constant, we see that \( x \) is the unique minimizer of
\[ y \to \frac{1}{2} a(y, y) - <\phi, y>_{H^* \times H}. \]
Corollary 1.36. Assume that \( a(\cdot, \cdot) \) is a continuous coercive bilinear form on a Hilbert space \( H \). Then, given any \( \phi \in H^* \), there exists a unique element \( x \in H \) such that for any \( y \in H \)
\[
a(x, y) = \langle \phi, y \rangle_{H^* \times H}.
\] (1.47)

Moreover, if \( a \) is symmetric, then \( x \) is characterized by the property
\[
x \in H \quad \text{and} \quad \frac{1}{2} a(x, x) - \langle \phi, x \rangle_{H^* \times H} = \min_{y \in H} \frac{1}{2} a(y, y) - \langle \phi, y \rangle_{H^* \times H}.
\] (1.48)

Proof. We use the Stampacchia theorem with \( K = H \). Then there is a unique element \( x \in H \) satisfying
\[
a(x, y - x) \geq \langle \phi, y - x \rangle_{H^* \times H}.
\]
Using linearity, this must hold also for
\[
a(x, ty - x) \geq \langle \phi, ty - x \rangle_{H^* \times H}.
\]
for any \( t \in \mathbb{R}, y \in H \). Factoring out \( t \), we find
\[
a(x, y - x) \geq \langle \phi, y - x \rangle_{H^* \times H}.
\]
and passing with \( t \to \pm \infty \), we obtain
\[
a(x, y) \geq \langle \phi, y \rangle_{H^* \times H}, \quad a(x, y) \leq \langle \phi, y \rangle_{H^* \times H}. \]

1.3.6 Adjoint Operators

An important role in functional analysis is played by the operation of taking operator adjoint. If \( A \in \mathcal{L}(X, Y) \), then the adjoint operator \( A^* \) is defined as
\[
\langle y^*, Ax \rangle = \langle A^* y^*, x \rangle
\] (1.49)
and it can be proved that it belongs to \( \mathcal{L}(Y^*, X^*) \) with \( \|A^*\| = \|A\| \). If \( A \) is an unbounded operator, then the situation is more complicated. In general, \( A^* \) may not exist as a single-valued operator. In other words, there may be many operators \( B \) satisfying
\[
\langle y^*, Ax \rangle = \langle By^*, x \rangle,
\]
\( x \in D(A), y^* \in D(B) \). (1.50)

Operators \( A \) and \( B \) satisfying (1.50) are called \emph{adjoint to each other}.

However, if \( D(A) \) is dense in \( X \), then there is a unique maximal operator \( A^* \) adjoint to \( A \); that is, any other \( B \) such that \( A \) and \( B \) are adjoint to each other, must satisfy \( B \subset A^* \). This \( A^* \) is called the \emph{adjoint operator} to \( A \). It can be constructed in the following way. The domain \( D(A^*) \) consists of all elements \( y^* \) of \( Y^* \) for which there exists \( f^* \in X^* \) with the property