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Corollary 2.12. A linear operator \( A \) is the generator of a \( C_0 \) semigroup \( (G(t))_{t \geq 0} \) satisfying \( \|G(t)\| \leq e^{\omega t} \) if and only if

1. \( A \) is closed and \( \text{dom}(A) = X \);
2. \( \rho(A) \supset (\omega, \infty) \) and for such \( \lambda \)

\[
\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}.
\]

Proof. Follows from the contractive semigroup \( S(t) = e^{\omega t}G(t) \) being generated by \( A - \omega I \).

The full version of the Hille-Yosida theorem reads

Theorem 2.13. \( A \in \mathcal{G}(M, \omega) \) if and only if

1. \( A \) is closed and densely defined,
2. there exist \( M > 0, \omega \in \mathbb{R} \) such that \( (\omega, \infty) \subset \rho(A) \) and for all \( n \geq 1, \lambda > \omega \)

\[
\| (\lambda A)^{-n} \| \leq \frac{M}{(\lambda - \omega)^n}.
\]

2.2.3 Dissipative operators and the Lumer-Phillips theorem

Let \( X \) be a Banach space (real or complex) and \( X^* \) be its dual. From the Hahn–Banach theorem, Theorem 1.7 for every \( x \in X \) there exists \( x^* \in X^* \) satisfying

\[
\langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2.
\]

Therefore the duality set

\[
\mathcal{J}(x) = \{ x^* \in X^*; \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \}
\]

is nonempty for every \( x \in X \).

Definition 2.14. We say that an operator \( (A, \text{dom}(A)) \) is dissipative if for every \( x \in \text{dom}(A) \) there is \( x^* \in \mathcal{J}(x) \) such that

\[
\Re \langle x^*, Ax \rangle \leq 0.
\]

If \( X \) is a real space, then the real part in the above definition can be dropped.

Theorem 2.15. A linear operator \( A \) is dissipative if and only if for all \( \lambda > 0 \) and \( x \in \text{dom}(A) \),

\[
\| (\lambda I - A)x \| \geq \lambda \|x\|.
\]
Proof. Let $A$ be dissipative, $\lambda > 0$ and $x \in D(A)$. If $x^* \in \mathcal{L}$ and $\mathcal{R} < Ax, x^* \geq 0$, then
\[ \|Ax - Ax_0\| \geq |\langle Ax - Ax, x^* \rangle | \geq \mathcal{R} < Ax, x^* \geq \lambda \|x\|^2 \]
so that we get (2.65).

Conversely, let $x \in D(A)$ and $\lambda \|x\| \leq \|Ax - Ax_0\|$ for $\lambda > 0$. Consider $y^*_\lambda \in \mathcal{L}(Ax - Ax)$ and $x^*_\lambda = y^*_\lambda / \|y^*_\lambda\|$. Then
\[ \mathcal{R} < Ax, z^*_\lambda > = \mathcal{R} (\langle x, x^*_\lambda \rangle - \langle Ax, x^*_\lambda \rangle) \]
for every $\lambda > 0$. From this estimate we obtain that $\mathcal{R} < Ax, z^*_\lambda > \leq 0$ and, by $|\lambda| \geq \mathcal{R} \alpha$,
\[ \mathcal{R} < x, z^*_\lambda > \geq |\lambda| \|x\| - |\mathcal{R} < Ax, z^*_\lambda > | \geq \lambda \|x\| - \|Ax\| \]
or $\mathcal{R} < x, z^*_\lambda > \leq \|x\| - \lambda^{-1} \|Ax\|$. Now, the unit ball in $X^*$ is weakly-* compact and thus there is a sequence $(z^*_\lambda)_{\lambda \in \mathcal{R}}$ converging to $z^*$ with $\|z^*\| = 1$. From the above estimates, we get
\[ \|x\| |\langle x, x^* \rangle | \geq 0 \]
and $\mathcal{R} < x, z^* > \leq \|x\|$. Hence, also, $|\langle x, x^* \rangle | \geq \|x\|$ On the other hand, $\mathcal{R} < x, z^* > \leq |\langle x, x^* \rangle | \leq \|x\|$ and hence $\langle x, x^* \rangle = \|x\|$. Taking $x^* = z^*\|x\|$ we see that $x^* \in \mathcal{L}(x)$ and $\mathcal{R} < Ax, x^* \geq 0$ and thus $A$ is dissipative.

Theorem 2.16. Let $A$ be a linear operator with dense domain $D(A)$ in $X$.

(a) If $A$ is dissipative and there is $\lambda_0 > 0$ such that the range $\mathcal{R} \{\lambda_0 I - A\} = X$, then $A$ is the generator of a $C_0$-semigroup of contractions in $X$.

(b) If $A$ is the generator of a $C_0$ semigroup of contractions on $X$, then $\mathcal{R} \{\lambda I - A\} = X$ for all $\lambda > 0$ and $A$ is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in \mathcal{L}(x)$ we have $\mathcal{R} < Ax, x^* \geq 0$.

Proof. Let $\lambda > 0$, then dissipativeness of $A$ implies $\|Ax - Ax_0\| \geq \lambda \|x\|$ for $x \in D(A), \lambda > 0$. This gives injectivity and, since by assumption, the $\mathcal{R} \{\lambda I - A\} D(A) = X$, $\lambda_0 I - A$ is a bounded everywhere defined operator and thus closed. But then $\lambda_0 I - A$, and hence $A$, are closed. We have to prove that $\mathcal{R} \{\lambda I - A\} D(A) = X$ for all $\lambda > 0$. Consider the set $\mathcal{A} = \{\lambda > 0; \mathcal{R} \{\lambda I - A\} D(A) = X\}$. Let $\lambda \in \mathcal{A}$. This means that $\lambda \in \rho(A)$ and, since $\rho(A)$ is open, $A$ is open in the induced topology. We have to prove that $A$ is closed in the induced topology. Assume $\lambda_n \to \lambda, \lambda > 0$. For every $y \in X$ there is $x_n \in D(A)$ such that
\[ \lambda_n x_n - Ax_n = y. \]
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From (2.2), \( \|x_n\| \leq \frac{1}{\lambda_n} \|y\| \leq C \) for some \( C > 0 \). Now

\[
\begin{align*}
\lambda_n \|x_n - x_m\| &\leq \|\lambda_n(x_n - x_m) - A(x_n - x_m)\| \\
&= \|\lambda_n x_n - \lambda_m x_n - (\lambda_n - \lambda_m)x_n - A(x_n - x_m)\| \\
&\leq \|\lambda_n - \lambda_m\| \|x_n\| \leq C|\lambda_n - \lambda_m|
\end{align*}
\]

Thus, \((x_n)_{n\in\mathbb{N}}\) is a Cauchy sequence. Let \(x_n \to x\), then \(Ax_n \to \lambda x - y\). Since \(A\) is closed, \(x \in D(A)\) and \(\lambda x - Ax = y\). Thus, for this \(\lambda\), \(\text{Im}(\lambda I - A)D(A) = X\) for all \(\lambda > 0\). Furthermore, if \(x \in D(A)\), \(x^* \in \mathcal{J}(x)\), then

\[
\langle G(t)x, x^* \rangle \leq \|G(t)x\| \|x^*\| \leq \|x\|^2
\]

and therefore

\[
\Re \langle G(t)x - x, x^* \rangle = \Re \langle G(t)x, x^* \rangle - \|x\|^2 \leq 0
\]

and, dividing the left hand side by \(t\) and passing with \(t \to \infty\), we obtain

\[
\Re \langle Ax, x^* \rangle \leq 0.
\]

Since this holds for every \(x^* \in \mathcal{J}(x)\), the proof is complete.

**Adjoint operators**

Before we move to an important corollary, let us recall the concept of the adjoint operator. If \(A \in \mathcal{L}(X, Y)\), then the adjoint operator \(A^*\) is defined as

\[
\langle y^*, Ax \rangle = \langle A^*y^*, x \rangle
\]

and it can be proved that it belongs to \(\mathcal{L}(Y^*, X^*)\) with \(\|A^*\| = \|A\|\). If \(A\) is an unbounded operator, then the situation is more complicated. In general, \(A^*\) may not exist as a single-valued operator. In other words, there may be many operators \(B\) satisfying

\[
\langle y^*, Ax \rangle = \langle By^*, x \rangle, \quad x \in D(A), \ y^* \in D(B). \tag{2.39}
\]

Operators \(A\) and \(B\) satisfying (2.39) are called **adjoint to each other**.

However, if \(D(A)\) is dense in \(X\), then there is a unique maximal operator \(A^*\) adjoint to \(A\); that is, any other \(B\) such that \(A\) and \(B\) are adjoint to each other, must satisfy \(B \subseteq A^*\). This \(A^*\) is called the **adjoint operator** to \(A\). It can be constructed in the following way. The domain \(D(A^*)\) consists of all elements \(y^*\) of \(Y^*\) for which there exists \(f^* \in X^*\) with the property

\[
\begin{align*}
\Delta u = \Delta u &\quad \omega_1^2(\mathcal{A}) \cap \omega_2^2(\mathcal{A}) \\
\sum_{\omega_1^2(\mathcal{A}) \cap \omega_2^2(\mathcal{A})} \Delta u = \sum_{\Delta \Delta u \in \mathcal{L}_1} \Delta u \in \mathcal{L}_1 \quad \left\{ \begin{array}{l}
\omega_1^2(\mathcal{A}) \ni u \in \mathcal{L}_1 \\
\Delta u \in \mathcal{L}_1 
\end{array} \right.
\end{align*}
\]
for any \( x \in D(A) \). Because \( D(A) \) is dense, such element \( f^* \) can be proved to be unique and therefore we can define \( A^* y^* = f^* \). Moreover, the assumption \( D(A) = X \) ensures that \( A^* \) is a closed operator though not necessarily densely defined. In reflexive spaces the situation is better: if both \( X \) and \( Y \) are reflexive, then \( A^* \) is closed and densely defined with \( A = (A^*)^* \). (2.41) see [105, Theorems III.5.28, III.5.29].

**Corollary 2.17.** Let \( A \) be a densely defined closed linear operator. If both \( A \) and \( A^* \) are dissipative, then \( A \) is the generator of a \( C_0 \)-semigroup of contractions on \( X \).

**Proof.** It suffices to prove that, e.g., \( Im(I - A) = X \). Since \( A \) is dissipative and closed, \( Im(I - A) \) is a closed subspace of \( X \). Indeed, if \( y_n \to y \), \( y_n \in \overline{Im(I - A)} \), then, by dissipativity, \( \|x_n - x_m\| \leq \|(x_n - x_m) - (Ax_n - Ax_m)\| = \|y_n - y_m\| \) and \( (x_n)_{n \in \mathbb{N}} \) converges. But then \( (Ax_n)_{n \in \mathbb{N}} \) converges and, by closedness, \( x \in D(A) \) and \( x - Ax = y \in Im(I - A) \). Assume \( Im(I - A) \neq X \), then by H-B theorem, there is \( 0 \neq x^* \in X^* \) such that \( \langle x^*, x - Ax \rangle = 0 \) for all \( x \in D(A) \). But then \( x^* \in D(A^*) \) and, by density of \( D(A) \), \( x^* - A^* x^* = 0 \) but dissipativeness of \( A^* \) gives \( x^* = 0 \).

**The Cauchy problem for the heat equation**

Let \( C = \Omega \times (0, \infty) \), \( \Sigma = \partial \Omega \times (0, \infty) \) where \( \Omega \) is an open set in \( \mathbb{R}^n \). We consider the problem

\[
\begin{align*}
\partial_t u &= \Delta u, \quad \text{in} \Omega \times [0, \infty], \\
u &= 0, \quad \text{on} \Sigma, \\
u &= u_0, \quad \text{on} \Omega.
\end{align*}
\] (2.42) (2.43) (2.44)

**Theorem 2.18.** Assume that \( u_0 \in L_2(\Omega) \) where \( \Omega \) is bounded and has a \( C^2 \) boundary. Then there exists a unique function \( u \) satisfying (2.44), (1.26) such that \( u \in C([0, \infty); L_2(\Omega)) \cap C([0, \infty); W_2^2(\Omega) \cap W_2^1(\Omega)) \).

**Proof.** The strategy is to consider (2.44–1.26) as the abstract Cauchy problem

\[
\begin{align*}
u' &= Au, \quad u(0) = u_0 \quad \text{in} \ X = L_2(\Omega) \quad \text{where} \ A \text{ is the unbounded operator defined by }
\end{align*}
\]

\[
Au = \Delta u
\] for
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First we observe that \( A \) is densely defined as \( C_0^\infty(\Omega) \subset W_0^1(\Omega) \) and \( \Delta C_0^\infty(\Omega) \subset L_2(\Omega) \). Next, \( A \) is dissipative. For \( u \in L_2(\Omega) \), \( J u = u \) and

\[
(A u, u) = -\int_\Omega |\nabla u|^2 \, dx \leq 0 \quad \implies \quad u - \Delta u \in L^1(\Omega) \quad \forall u \in \mathbb{D}(A)
\]

Further, we consider the variational problem associated with \( I - A \), that is, to find \( u \in W_0^1(\Omega) \)

\[
a(u, v) = \int_\Omega \nabla u \nabla v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx, \quad v \in W_0^1(\Omega) \quad \lambda > 0
\]

where \( f \in L_2(\Omega) \) is given. Clearly, \( a(u, u) = \| u \|_{1,\Omega}^2 \) and thus is coercive. Hence there is a unique solution \( u \in W_0^1(\Omega) \) which, by writing

\[
\int_\Omega \nabla u \nabla v \, dx = \int_\Omega fv \, dx - \int_\Omega uv \, dx, \quad v \in W_0^1(\Omega)
\]

can be shown to be in \( W_2^2(\Omega) \). This ends the proof of generation.

If we wanted to use the Hille-Yosida theorem instead, then to find the resolvent, we would have to solve

\[
a(u, v) = \int_\Omega \nabla u \nabla v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx, \quad v \in W_0^1(\Omega) \quad \lambda > 0
\]

for \( \lambda > 0 \). The procedure is the same and we get in particular for the solution

\[
|\nabla u_\lambda|^2_{0,\Omega} + \lambda \| u_\lambda \|^2_{0,\Omega} \leq \| f \|_{0,\Omega} \| u_\lambda \|_{0,\Omega}.
\]

Since \( u_\lambda = R(\lambda, A)f \) we obtain

\[
\| R(\lambda, A)f \|^2_{0,\Omega} \leq \lambda^{-1} \| f \|_{0,\Omega}.
\]

Closedness follows from continuous invertibility.