Chapter 8

Travelling waves

1 Introduction

One of the cornerstones in the study of both linear and nonlinear PDEs is the wave propagation. A *wave* is a recognizable signal which is transferred from one part of the medium to another part with a recognizable speed of propagation. Energy is often transferred as the wave propagates, but matter may not be. We mention here a few areas where wave propagation is of fundamental importance.

Fluid mechanics (water waves, aerodynamics)

Acoustics (sound waves in air and liquids)

Elasticity (stress waves, earthquakes)

Electromagnetic theory (optics, electromagnetic waves)

Biology (epizootic waves)

Chemistry (combustion and detonation waves)

The simplest form of a mathematical wave is a function of the form

$$u(x,t) = f(x - ct). \tag{8.1.1}$$

We adopt the convention that $c > 0$. We have already seen such a wave as a solution of the constant coefficient transport equation

$$u_t + cu_x = 0,$$

it can, however, appear in many other contexts.

At $t = 0$ the wave has the form $f(x)$ which is the initial wave profile. Then $f(x - ct)$ represents the profile at time t, that is just the initial profile translated to the right by ct spatial units. Thus the constant c represents the speed of the wave and thus, evidently, (8.1.1) represents a wave travelling to the right with speed $c > 0$. Similarly,

$$u(x,t) = f(x + ct)$$

represents a wave travelling to the right with speed c. These waves propagate undistorted along the lines $x \pm ct = const$.

One of the fundamental questions in the theory of nonlinear PDEs is whether a given PDE admit such a travelling wave as a solution. This question is generally asked without regard to initial conditions so that the wave is assumed to have existed for all times. However, boundary conditions of the form

$$u(-\infty, t) = constant, \qquad u(+\infty, t) = constant \tag{8.1.2}$$

are usually imposed. A *wavefront type-solution* to a PDE is a solution of the form $u(x,t) = f(x \pm ct)$ subject to the condition (8.1.2) of being constant at $\pm\infty$ (this constant are not necessarily the same); the function f is assumed to have the requisite degree of smoothness defined by the PDE. If u approaches the same constant at both $\pm\infty$, then the wavefront is called a *pulse*.

2 Examples

We have already seen that the transport equation with constant c admits the travelling wave solution and it is the only solution this equation can have. To illustrate the technique of looking for travelling wave solutions on a simple example first, let us consider the wave equation.

Example 2.1 Find travelling wave solutions to the wave equation

$$u_{tt} - a^2 u_{xx} = 0.$$

According to definition, travelling wave solution is of the form $u(x,t) = f(x - ct)$. Inserting this into the equation, we find

$$u_{tt} - a^2 u_{xx} = c^2 f'' - a^2 f'' = f'' \cdot (c^2 - a^2) = 0$$

so that either $f(s) = A + Bs$ for some constants A, B or $c = \pm a$ and f arbitrary. In the first case we would have

$$u(x,t) = A + B(x \pm ct)$$

but the boundary conditions (8.1.2) cannot be satisfied unless $B = 0$. Thus, the only travelling wave solution in this case is constant. For the other case, we see that clearly for any twice differentiable function f such that

$$\lim_{s \to \pm\infty} = d_{\pm\infty}$$

the solution

$$u(x,t) = f(x \pm at)$$

is a travelling wave solution (a pulse if $d_{+\infty} = d_{-\infty}$).

In general, it follows that any solution to the wave equation can be obtained as a superposition of two travelling waves: one to the right and one to the left

$$u(x,t) = f(x - at) + g(x + at).$$

Not all equations admit travelling wave solutions, as demonstrated below.

Example 2.2 Consider the diffusion equation

$$u_t = D u_{xx}.$$

Substituting the travelling wave formula $u(x,t) = f(x - ct)$, we obtain

$$-cf' - Df'' = 0$$

that has the general solution

$$f(s) = a + b \exp\left(-\frac{cs}{D}\right).$$

It is clear that for f to be constant at both plus and minus infinity it is necessary that $b = 0$. Thus, there are no nonconstant travelling wave solutions to the diffusion equation.

We have already seen that the non-viscid Burger's equation

$$u_t + uu_x = 0$$

does not admit a travelling wave solution: any profile will either smooth out or form a shock wave (which can be considered as a generalized travelling wave - it is not continuous!). However, some amount of dissipation represented by a diffusion term allows to avoid shocks.

Example 2.3 Consider Burger's equation with viscosity

$$u_t + uu_x - \nu u_{xx} = 0, \qquad \nu > 0. \tag{8.2.3}$$

The term uu_x will have a shocking up effect that will cause waves to break and the term νu_{xx} is a diffusion like term. We attempt to find a travelling wave solution of (8.2.3) of the form

$$u(x,t) = f(x - ct).$$

Substituting this to (8.2.3) we obtain

$$-cf'(s) + f(s)f'(s) - \nu f''(s) = 0,$$

where $s = x - ct$. Noting that $ff' = \frac{1}{2}(f^2)'$ we re-write the above as

$$-cf' + \frac{1}{2}(f^2)' - \nu f'' = 0,$$

hence we can integrate getting

$$-cf + \frac{1}{2}f^2 - \nu f' = B$$

where B is a constant of integration. Hence

$$\frac{df}{ds} = \frac{1}{2\nu}\left(f^2 - 2cf - 2B\right). \tag{8.2.4}$$

Let us consider the case when the quadratic polynomial above factorizes into real linear factors, that is

$$\left(f^2 - 2cf - 2B\right) = (f - f_1)(f - f_2)$$

where

$$f_1 = c - \sqrt{c^2 + 2B}, \qquad f_2 = c + \sqrt{c^2 + 2B}. \tag{8.2.5}$$

This requires $c^2 > 2B$ and yields, in particular, $f_2 > f_1$. Eq. (9.4.26) can be easily solved by separating variables. First note that f_1 and f_2 are the only equilibrium points and $(f - f_1)(f - f_2) < 0$ for $f_1 < f < f_2$ so that any solution starting between f_1 and f_2 will stay there tending to f_1 as $s \to +\infty$ and to f_2 as $s \to -\infty$. Any solution starting above f_2 will tend to ∞ as $s \to +\infty$ and any one starting below f_1 will tend to $-\infty$ as $s \to -\infty$. Thus, the only non constant travelling wave solutions are possible for $f_1 < f < f_2$. For such f integration if (9.4.26) yields

$$\begin{aligned}\frac{s - s_0}{2\nu} &= \int \frac{df}{(f - f_1)(f - f_2)} = -\frac{1}{f_2 - f_1}\int\left(\frac{1}{f - f_1} + \frac{1}{f_2 - f}\right)df \\ &= \frac{1}{f_2 - f_1}\ln\frac{f_2 - f}{f - f_1}.\end{aligned}$$

Solving for f yields

$$f(s) = \frac{f_2 + f_1 e^{K(s - s_0)}}{1 + e^{K(s - s_0)}}, \tag{8.2.6}$$

where $K = \frac{1}{2\nu}(f_2 - f_1) > 0$. We see that for large positive s $f(s) \sim f_1$ whereas for negative values of s we obtain asymptotically $f(s) \sim f_2$. It is clear that the initial value s_0 is not essential so that we shall suppress it in the sequel. The derivative of f is

$$f'(s) = K\frac{e^{Ks}(f_1 - f_2)}{(1+e^{Ks})^2} < 0.$$

It is easy to see that for large $|s|$ the derivative $f(s)$ is close to zero so that f is almost flat. Moreover, the larger ν (so that the smaller K), the more flat is f as the derivative is closer to zero. Hence, for small ν we obtain a very steep wave front that is consistent with the fact for $\nu = 0$ we obtain inviscid Burger's equation that admits only discontinuous travelling waves.

The formula for travelling wave solution to (8.2.3) is then

$$u(x,t) = \frac{f_2 + f_1 e^{K(x-ct)}}{1 + e^{K(x-ct)}}, \tag{8.2.7}$$

where the speed of the wave is determined from (8.2.5) as

$$c = \frac{1}{2}(f_1 + f_2).$$

Graphically the travelling wave solution is the profile f moving to the right at speed c. This solution, because it resembles the actual profile of a shock wave, is called the *shock structure* solution; it joints the asymptotic states f_1 and f_2. Without the term νu_{xx} the solutions of (8.2.3) would shock up and tend to break. The presence of the diffusion term prevents this breaking effect by countering the nonlinearity. The result is competition and balance between the nonlinear term uu_x and the diffusion term $-\nu u_{xx}$, much the same as occurs in a real shock wave in the narrow region where the gradient is steep. In this context the $-\nu u_{xx}$ term could be interpreted as a viscosity term.

The last example related to travelling waves is concerned with the so called solitons that appear in solutions of numerous important partial differential equations. The simples equation producing solitons is the Korteweg-deVries equation that governs long waves in shallow water.

Example 2.4 Find travelling wave solutions of the KdV equation

$$u_t + uu_x + ku_{xxx} = 0, \tag{8.2.8}$$

where $k > 0$ is a constant. As before, we are looking for a solution of the form

$$u(x,t) = f(s), \qquad s = x - st,$$

where the waveform f and the wave speed c are to be determined. Substituting f into (8.2.8) we get

$$-cf' + \frac{1}{2}(f^2)' + kf''' = 0,$$

integration of which gives

$$-cf + \frac{1}{2}f^2 + kf'' = a$$

for some constant a. This is a second order equation that does not contain the independent variable, hence we can use substitution $f' = F(f)$, discussed in Subsection 3.4. Hence, $f''_s = F'_f f'_s = F'_f F$ and we can write

$$-cf + \frac{1}{2}f^2 + kF'_f F = -cf + \frac{1}{2}f^2 + \frac{k}{2}(F_f^2)' = a.$$

Integrating, we get

$$F^2 = \frac{1}{k}\left(cf^2 - \frac{1}{3}f^3 + 2af + 2b\right)$$

where b is a constant. Thus,

$$f' = \pm\sqrt{\frac{1}{3k}}\left(-f^3 + 3cf^2 + 6af + 6b\right)^{1/2}. \tag{8.2.9}$$

To fix attention we shall take the "+" sign. Denote by $\phi(f)$ the cubic polynomial on the right-hand side. We have the following possibilities:

(i) ϕ has one real root α;

(ii) ϕ has three distinct real roots $\gamma < \beta < \alpha$;

(iii) ϕ has three real roots satisfying $\gamma = \beta < \alpha$;

(iv) ϕ has three real roots satisfying $\gamma < \beta = \alpha$;

(v) ϕ has a triple root γ.

Since we are looking for travelling wave solutions that should be bounded at $\pm\infty$ and nonnegative, we can rule out most of the cases by qualitative analysis described in Section 2. Note first that the right-hand side of (8.2.9) is defined only where $\phi(f) > 0$. Then, in the case (i), α is a unique equilibrium point, $\phi > 0$ only for $f < \alpha$ and hence any solution converges to α as $t \to \infty$ and diverges to $+\infty$ as $t \to -\infty$. Similar argument rules out case (v).

If we have three distinct roots, then by the same argument, the only bounded solutions can exist in the cases (iii) and (ii) (in the case (iv) the bounded solutions could only appear where $\phi < 0$.) The case (ii) leads to the so-called *cnoidal* waves expressible through special functions called cn-functions and hence the name. We shall concentrate on case (iii) so that

$$\phi(f) = -f^3 + 3cf^2 + 6af + 6b = (\gamma - f)^2(\beta - f)$$

and, since $f > \gamma$

$$\sqrt{\phi(f)} = (f - \gamma)\sqrt{\alpha - f}.$$

Thus, the differential equation (8.2.9) can be written

$$\frac{s}{\sqrt{3k}} = \int \frac{df}{(f-\gamma)\sqrt{\alpha - f}}. \tag{8.2.10}$$

To integrate, we first denote $v = f - \gamma$ and $B = \alpha - \gamma$ (with $0 < v < b$), getting

$$\frac{s}{\sqrt{3k}} = \int \frac{dv}{v\sqrt{B - v}}.$$

Next, we substitute $w = \sqrt{B - v}$, hence $v = B - w^2$ and $dw = -ds/2\sqrt{B - v}$ so that the above will be transformed to

$$\frac{s}{\sqrt{3k}} = 2\int \frac{dw}{w^2 - B}.$$

This integral can be evaluated by partial fractions, so that, using $0 < w < \sqrt{B}$, we get

$$\frac{s}{\sqrt{3k}} = \frac{1}{\sqrt{B}}\int\left(\frac{dw}{w - \sqrt{B}} - \frac{dw}{w + \sqrt{B}}\right)dw = \frac{1}{\sqrt{B}}\ln\frac{\sqrt{B} - w}{\sqrt{B} + w}.$$

Solving with respect to w we obtain

$$\frac{\sqrt{B} - w}{\sqrt{B} + w} = \exp s\sqrt{\frac{B}{3k}}$$

and thus

$$w = \sqrt{B}\,\frac{1 - \exp s\sqrt{\frac{B}{3k}}}{1 + \exp s\sqrt{\frac{B}{3k}}} = -\sqrt{B}\,\frac{\sinh s\sqrt{\frac{B}{12k}}}{\cosh s\sqrt{\frac{B}{12k}}}.$$

Returning to the original variables $w^2 = B - v = \alpha - \gamma - f + \gamma = \alpha - f$ and using the hyperbolic identity $\cosh^2\theta - \sinh^2\theta = 1$, we get

$$\begin{aligned} f(s) &= \alpha - w^2(s) = \alpha - (\alpha-\gamma)\frac{\sinh^2 s\sqrt{\frac{\alpha-\gamma}{12k}}}{\cosh^2 s\sqrt{\frac{\alpha-\gamma}{12k}}} \\ &= \gamma + (\alpha-\gamma)\text{sech}^2\left(s\sqrt{\frac{\alpha-\gamma}{12k}}\right) \end{aligned}$$

Clearly, $f(s) \to \gamma$ as $s \to \pm\infty$ so that the travelling wave here is a pulse. It is instructive to write the roots α and γ in terms of the original parameters. To identify them we observe

$$\begin{aligned} \phi(f) &= -f^3 + 3cf^2 + 6af + 6b = (\gamma - f)^2(\beta - f) \\ &= -f^3 + f^2(\alpha + 2\gamma) + f(-\gamma^2 - 2\alpha\gamma) + \gamma^2\alpha. \end{aligned}$$

Thus, the wave speed is given by

$$c = \frac{\alpha + 2\gamma}{3} = \frac{\alpha - \gamma}{3} + \gamma.$$

Since γ is just the level of the wave at $\pm\infty$, by moving the coordinate system we can make it equal to zero. In this case we can write the travelling wave solution to the KdV equation as

$$u(x,t) = 3c\text{sech}^2\left(\sqrt{\frac{\sqrt{c}}{4k}}(x - ct)\right) \tag{8.2.11}$$

It is important to note that the velocity of this wave is proportional to its amplitude which makes it different from linear waves governed by, say the wave equation, where the wave velocity is the property of the medium rather than of the wave itself.

Example 2.5 Fisher's equation
Many natural processes involve mechanisms of both diffusion and reaction, and such problems are often modelled by so-called *reaction-diffusion equations* of the form

$$u_t - Du_{xx} = f(u), \tag{8.2.12}$$

where f is a given, usually nonlinear function of u. We introduced earlier the Fisher equation

$$u_t - Du_{xx} = ru\left(1 - \frac{u}{K}\right) \tag{8.2.13}$$

to model the diffusion of a species (e.g. insect population) when the reaction (or, at this instance) growth term is given by the logistic law. Here D is the diffusion constant, and r and K are the growth rate and carrying capacity, respectively.

We shall examine the Fisher equation and, in particular, we shall address the question of existence of a travelling wave solution.

Let us consider the Fisher equation in dimensionless form

$$u_t - u_{xx} = u(1-u) \tag{8.2.14}$$

and, as before, we shall look for solutions of the form

$$u(x,t) = U(s), \qquad s = x - ct, \tag{8.2.15}$$

where c is a positive constant and U has the property that it approaches constant values at $s \to \pm\infty$. The function U to be determined should be twice differentiable. The wave speed c is a priori unknown and must be determined as a part of the solution of the problem. Substituting (8.2.15) into (8.2.14) yields a second order ordinary differential equation for U:

$$-cU' - U'' = U(1-U). \tag{8.2.16}$$

Contrary to the previous cases, this equation cannot be solved in a closed form and the best approach to analyze it is to perform the phase plane analysis. In a standard way we write (8.2.16) as a simultaneous system of first order equations by defining $V = U'$. In this way we obtain

$$\begin{aligned} U' &= V, \\ V' &= -cV - U(1-U). \end{aligned} \tag{8.2.17}$$

We find equilibrium points of this system solving

$$\begin{aligned} 0 &= V, \\ 0 &= -cV - U(1-U), \end{aligned}$$

which gives two: $(0,0)$ and $(0,1)$. The Jacobi matrix of the system is

$$J(U,V) = \begin{pmatrix} 0 & 1 \\ 2U-1 & -c \end{pmatrix}$$

so that

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$$

with eigenvalues

$$\lambda_{\pm}^{0,0} = \frac{-c \pm \sqrt{c^2-4}}{2}$$

and

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

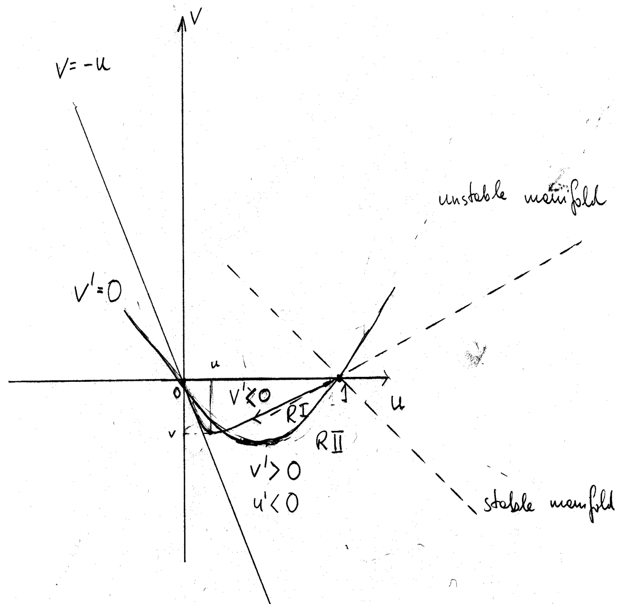

Fig. 5.1 Phase portrait of system (8.2.17).

with eigenvalues

$$\lambda_{\pm}^{1,0} = \frac{-c \pm \sqrt{c^2+4}}{2}.$$

It is easily seen that for any c, $\lambda_{\pm}^{1,0}$ are real and of opposite sign and therefore $(1,0)$ is a saddle. On the other hand, $\lambda_{\pm}^{0,0}$ are both real and negative if $c \geq 2$ and in this case $(0,0)$ is a stable node (for the linearized system), and are complex with negative real part if $0 < c < 2$ in which case $(0,0)$ is a stable focus.

Since the wave profile $U(s)$ must have finite limits as $s \to \pm\infty$ and since we know that the only limit points of solutions of autonomous systems are equilibrium points, search for travelling wave solutions of (8.2.16) is equivalent to looking for orbits of (8.2.17) joining equilibria, that is approaching them as $s \to \pm\infty$. Such orbits are called *heteroclinic* if they join different equilibrium points and *homoclinic* if the orbit returns to the same equilibrium point from which it started.

We shall use the Stable Manifold Theorem. According to it, there are two orbits giving rise, together with the equilibrium point $(1,0)$, to the unstable manifold defined at least in some neighbourhood of the saddle point $(1,0)$, such that each orbit $\phi(s) = (U(s), V(s))$ satisfies: $\phi(s)\ to(1,0)$ as $s \to -\infty$. Our aim is then to show that at least one of these orbits can be continued till $s \to \infty$ and reaches then $(0,0)$ in a monotonic way. Let us consider the orbit that moves into the fourth quadrant $U > 0, V < 0$. This quadrant is divided into to regions by the isocline $0 = V' = -cV - U(1-U)$: region I with $\frac{1}{c}(U^2 - U) < V < 0$ and region II where $V < \frac{1}{c}(U^2 - U)$, see Fig. 5. 1. In region I we have $V' < 0, U' < 0$ and in region II we have $V' > 0, U' < 0$. From the earlier theory we know that if a solution is not defined for all values of the argument s then it must blow up as $s \to s'$ where $(-\infty, s')$ is the maximal interval of existence of the solution. We note first that the selected orbit on our unstable manifold enters Region I so that $(U(s), V(s)) \in$ Region I for $-\infty < s \leq s_0$ for some s_0. In fact, the tangent of the isocline at $(1,0)$ is $1/c$ and the tangent of the unstable manifold is $\lambda_{+}^{1,0} = \frac{\sqrt{c^2+4}-c}{2}$. Denoting

$$\psi(c) = \frac{c(\sqrt{c^2+4} - c)}{2}$$

we see that $\psi(0) = 0$, $\lim_{c\to+\infty} \psi(s) = 1$ and

$$\psi'(c) = \frac{4}{\sqrt{c^2+4}(\sqrt{c^2+4}+c)^2} > 0,$$

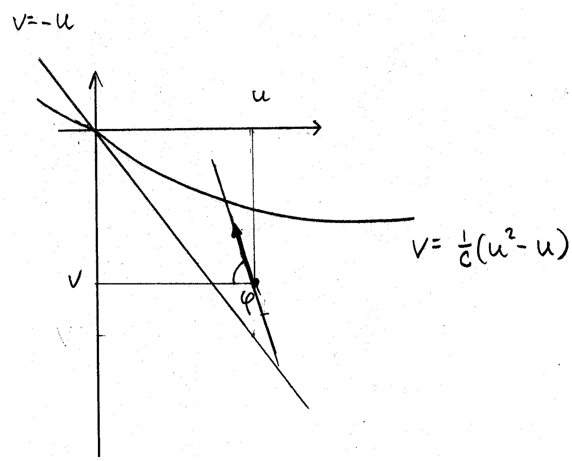

Fig. 5.2. The orbit in Region II.

thus $0 \leq \psi(c) \leq 1$ for all $c \geq 0$ and hence

$$\frac{1}{c} \geq \frac{\sqrt{c^2+4}-c}{2},$$

so that the slope of the isocline is larger than that of the orbit and the orbit must enter Region 1. Then, since $V' < 0$ there, $V(s) < V(s_0)$ as long as $V(s)$ stays in Region I. Hence, the orbit must leave Region I in finite time as there is no equilibrium point with strictly negative V coordinate. Of course, there cannot be any blow up as long as the orbit stays in Region I. However, as at the crossing point the sign of V' changes, this point is a local minimum for V so that the orbit starts moving up and continues to move left.

The slope of $V = \frac{1}{c}U(1-U)$ at the origin is $-1/c$ so for $c \geq 1$ (and in particular for $c \geq 2$) the parabola $V = \frac{1}{c}U(1-U)$ stays above the line $V = -U$ so that any orbit entering Region II from Region I must stay for some time above $V = -U$, that is, $V/U > -1$. Consider any point $(U, -U)$, $U > 0$, and estimate the slope of the vector field on the line, see Fig. 5.2, We have

$$\tan\phi = \frac{-cV - U(1-U)}{-V} = c + \frac{U}{V}(1-U) = c - 1 + U > c - 1$$

Considering the direction, we see that if $c \geq 2$, then the vector field points to the left from the line $U = -V$. Hence, the whole trajectory must stay between $U = -V$ and $V = \frac{1}{c}U(1-U)$. Hence, the orbit is bounded and therefore exists for all s and enters $(0,0)$ in a monotonic way, that is U is decreasing monotonically from 1 at $s = -\infty$ to 0 at $s = +\infty$ while $U' = V$ is non-positive and goes from zero at $s = -\infty$ through minimum back to 0 at $s = +\infty$.

Thus, summarizing, the orbit $(U(s), V(s))$ is globally defined for $-\infty < s < +\infty$ joining the equilibrium points $(1,0)$ and $(0,0)$. Thus, $U(s) \to 1$ as $s \to -\infty$ and $U(s) \to 0$ as $s \to \infty$. Moreover, as $0 > U'(s) = V(s) \to 0$ as $s \to \pm\infty$, U is monotonically decreasing and becomes flat at both "infinities" giving a travelling wavefront solution.

We note that for $c < 2$ the orbit no longer enters $(0,0)$ monotonically but, as suggested by linearization, spirals into the equilibrium point with U passing through positive and negative values. Thus, if we are interested only in positive values of U in order to have a physically realistic solution (e.g. if U is a population density), we should reject this case.