Spectral Invariants for Symmetry and Asymmetry

Minicourse on the Functional Analysis of
Index, Eta-invariant, and Spectral Flow
of Elliptic Problems on Manifolds with Boundary
with Emphasis on the General Spectral Flow Formula

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Plan

I. Spectral Invariants of Dirac Type Operators and Partitioned Manifolds, Review
II. Topology of Self-adjoint Fredholm Operators
III. The General Spectral Flow Formula
IV. Weak Symplectic Functional Analysis

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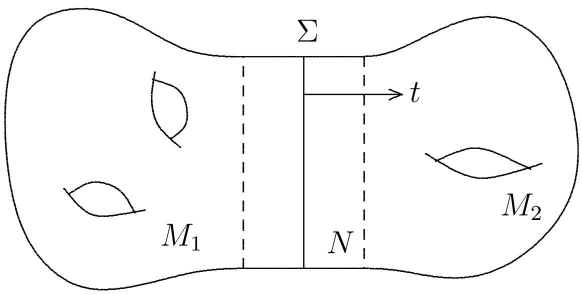

I. Spectral Invariants of Dirac Type Operators and Partitioned Manifolds, Review

Plan

- Analysis tools
  - General setting
  - Unique continuation property
  - Invertible extension
  - Poisson operator
  - Calderón projection and Cauchy data spaces (CD)
  - Twisted orthogonality of CD
  - Atiyah-Patodi-Singer projection
  - Elliptic boundary problems
- Spectral invariants
  - Aspects
    - A. Functional analysis
    - B. Perturbation
    - C. Product
    - D. Locality
    - E. Boundary reduction
    - F. Pasting / correction
    - G. Bojarski type formulas
  - Invariants
    - **Index:** Chiral asymmetry
    - **Spectral Flow:** Net sign change of eigenvalues
    - **η-Invariant:** Asymmetry of spectrum
    - **ζ-Determinant:** Anomaly measure
- References

II. The Topology of the Space of (Generally Unbounded) Self-adjoint Fredholm Operators

Plan

- The bounded case
  - Three connected components
  - Classifying space for the functor K^1
- The unbounded case
  - Gap topology
  - Riesz transformation
    - (i) Intuitive picture
    - (ii) Fuglede counter example
    - (iii) Some positive results
  - Cayley transformation

    Continuity: OK

    Spectral flow (SF): OK

    Connected image: Surprise!

    Uniqueness of SF: Lesch's Result
- Spectral flow of spectral-continuous curves
- Existence of self-adjoint Fredholm extensions
  - Symmetric generalized Dirac operators
  - Signature of symplectic form (normal symbol) on kernel of tangential operator
  - Calderón projection and a general cobordism theorem
- References

III. The General Spectral Flow Formula

Plan

- Goal: $\mathrm{sf}\{\cdot\} = \mathrm{Mas}\{\cdot,\cdot\}$
- History 1: Morse index theorem
- History 2: Dirac operators
- von Neumann correspondence for single operators
  - Self-adjoint extensions of A closed symmetric
  - Symplectic Hilbert space $\boldsymbol{\beta}(A) := \operatorname{dom} A_{\max} / \operatorname{dom} A_{\min}$
  - Lagrangian subspaces of $\boldsymbol{\beta}(A)$
- Difficulties to overcome for families
  - (i) *Varying* maximal domain
  - (ii) *Three* symplectic Hilbert spaces $\boldsymbol{\beta}$, $L^2(\Sigma)$, and $H^{1/2}(\Sigma)$
  - (iii) Only *weak* symplectic structure on $H^{1/2}(\Sigma)$
  - (iv) No canonical *symplectic decomposition*
  - (v) Basic operator of analysis *not* closed in L^2
  - (vi) *Regularity* requirements
- The general spectral flow formula (GSFF)
- Idea of proof
- Summary of results

GSFF: $\mathrm{sf}\{A_{s,P_s}\} = -\,\mathrm{Mas}\{\ker P_s, \operatorname{im} Q_s\}$

- A_s symm. gen. Dirac type operator, $s \in [0,1]$
- P_s pseudodifferential elliptic boundary condition
- Q_s Calderón projection

Bojarski: $\mathrm{sf}\{A_s\} = \mathrm{sf}\{A^1_{s,I-Q^2_s}\} = -\,\mathrm{Mas}\{\operatorname{im} Q^1_s, \operatorname{im} Q^2_s\}$

1st o. Ham. system: $\mathrm{sf}\{A_{s,W_s}\} = \mathrm{Mas}\{\mathfrak{G}(\Gamma_s(T)), W_s\}$

2d o. Ham. system: $\mathrm{sf}\{L_{s,W_s}\} = \mathrm{Mas}\{\mathfrak{G}\big(\Gamma_s(T)\big), W_s\}$

IV. Weak Symplectic Functional Analysis

Plan

Symplectic $\mathbb{C}$-vector spaces:
- Isotropic subspaces, symplectic decompositions
- Algebraic Fredholm pairs

Examples:
- Banach spaces
- Hilbert spaces
- Natural symplectic decomposition for generalized Dirac type operators
- Lifting of isotropic subspaces

Maslov index:
- h-unitary operators, spectrum
- Definition, strong symplectic case
- Dependence and independence of sympl. decomposition
- Lifting and regularity

Application:
- Families of symmetric operators
- Cauchy data spaces, self-adjoint continuations