Elliptic operators on singular manifolds and K-homology

A. Yu. Savin

Independent University of Moscow

In 1970s Atiyah showed that elliptic operators on a smooth closed manifold define cycles in K-theory. The relationship between elliptic theory and K-theory is even more precise: the group $\mathrm{Ell}(M)$ of stable homotopy classes of elliptic pseudodifferential operators on a manifold M is isomorphic to the even K-homology group of the manifold:

$$\mathrm{Ell}(M) \simeq K_0(M). \tag{1}$$

It turns out that a similar isomorphism holds in many situations, when the manifold is no longer smooth.

1. Manifolds with isolated singularities. Let M be a compact manifold with a finite number of isolated conical points. Denote by $\mathrm{Ell}(M)$ the group of stable homotopy classes of elliptic operators of order zero on M (see, e.g. [1]).

Theorem 1 *On a manifold with isolated conical singularities isomorphism* (1) *holds.*

2. Manifolds with edges of codimension one. Let $\widetilde{M}$ be a compact manifold with boundary and the boundary is represented as the total space of a covering $\pi : \partial\widetilde{M} \to X$ over some base X. The quotient space of the equivalence relation identifying the points in the fibers of the covering is called a manifold with edge X. Denote the quotient by M. Corresponding to the covering on the boundary there is a class of nonlocal operators on $\widetilde{M}$ generated by the usual pseudodifferential operators of order zero on M and operators induced by transpositions of leaves of the covering in a neighborhood of the boundary. We assume that the operators have symbols independent of covariables near the boundary. Denote by $\mathrm{Ell}(M)$ the group of stable homotopy classes of elliptic operators from the class just described.

Independent University of Moscow 121002, Bolshoi Vlasevsky Pereulok, Dom 11, RUSSIA

E-mail: antonsavin@mtu-net.ru

Theorem 2 *On a manifold with codimension one edges the isomorphism* (1) *holds.*

The proof of these results is based on Atiyahś functional-analytic description of K-homology as the group of stable homotopy classes of abstract elliptic operators and a generalization of the Atiyah–Singer difference construction to noncommutative algebras of symbols. We describe the former result for the case of conical points.

3. Difference construction on manifolds with conical points. Suppose that the manifold with conical points is obtained from a compact manifold M with boundary by identification of points on the boundary components of M. Then one can define the C^*-algebra

$$\mathcal{A}_{T^*M} = \left\{ \begin{array}{c} u \in C_0(B^*M \setminus S^*M), \\ v \in C_0([0,1), \overline{\Psi}_p(\partial M)) \end{array} \middle| \begin{array}{c} u|_{S_t} = \mathrm{smbl}(v(t)), \\ u|_{S_0} = v(0) \end{array} \right\},$$

where B^*M and S^*M are respectively bundles of unit balls and spheres in T^*M, $\overline{\Psi}_p(\partial M)$ is the closure of the algebra of parameter-dependent pseudodifferential operators on ∂M, while $S_t \subset B^*M|_{\partial M}$ is the bundle of spheres of radius $t \in [0,1]$.

Denote by $\overline{M}$ the manifold with conical points corresponding to M.

Theorem 3 $\mathrm{Ell}(\overline{M}) \simeq K_0(\mathcal{A}_{T^*M})$.

References

[1] V. Nazaikinskii and B. Sternin, Operator algebras on manifolds with isolated singularities. *Differential Equations*, 39(1), 2003.

[2] A. Savin and B. Sternin. Index defects in the theory of spectral boundary value problems. Preprint math.KT/0211177.