SYMPLECTIC STRUCTURES
ON THE TANGENT BUNDLES
OF SYMPLECTIC AND COSYMPLECTIC MANIFOLDS
(short version)

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Abstract: We describe all symplectic structures on the tangent bundles of symplectic and cosymplectic manifolds.

0. Introduction

In [4], the authors classified all Hamiltonian type natural operators on the cotangent bundle T^*M. In [1], the author described all Hamiltonian type natural operators transforming a function f on a symplectic manifold (M, ω) into a vector field $V(\omega, f)$ on M. The present note deals with canonical constructions on symplectic structures, too. We describe all symplectic structures on the tangent bundles of symplectic and cosymplectic manifolds. This problem arises in the context of respective natural operators in the sense of [5], which are defined on symplectic (resp. cosymplectic) structures. Since homotheties are not symplectomorphisms (resp. cosymplectomorphisms), it is difficult to apply the homogeneous function theorem and the problem of classifications of the operators in question is more difficult than the one of natural operators defined on all 2–forms, [2], [3], [6], e.t.c.

Symplectic structures are involved in the equation of motion. That is why, the results of the paper are interesting with respect to the theoretical mechanics. They are also interesting with respect to the theory of natural operators.

We start with the problem how to construct canonically a symplectic manifold $(TM, \Lambda(\omega))$ for a given symplectic $2m$–manifold (M, ω), where ω is a closed 2–form with $\omega^m \neq 0$ for any point in M. This problem arises in the context of respective $\mathcal{M}f_{2m}$– natural operators Λ. The first main result of the present note is the following classification theorem.

Theorem 1. *Let Λ be a $\mathcal{M}f_{2m}$ –natural operator in question. Then there exist real numbers α and $\beta \neq 0$ such that*

$$\Lambda(\omega) = \alpha\pi^*\omega + \beta\tilde{\omega}^*\Omega \tag{1}$$

for any symplectic structure ω on M, where $\pi^*\omega$ is the vertical lifting of ω to the tangent bundle TM, $\pi : TM \to M$ is the tangent bundle projection,

[0]**Key words:** natural operators, symplectic structures, cosymplectic structures
AMS Classification: 58 A 20

$(\ )^*$ is the pull–back, Ω is the well–known canonical symplectic structure on the cotangent bundle T^*M and $\tilde{\omega}: TM \to T^*M$ is the standard isomorphism induced by ω, $\tilde{\omega}(v) = \omega_x(.,v)$, $v \in T_xM$, $x \in M$.

On the other hand for any real numbers α and $\beta \neq 0$ the operator $\Lambda(\omega)$ defined as in (1) *is a symplectic structure on TM.*

Theorem 1 is a consequence of the following more general fact.

Theorem 2. *Let Λ be a $\mathcal{M}f_{2m}$-natural operator transforming a symplectic structure ω on a $2m$-manifold M into 2-form $\Lambda(\omega)$ on TM. Then there exist real numbers α and β such that*

$$\Lambda(\omega) = \alpha\pi^*\omega + \beta\tilde{\omega}^*\Omega \tag{2}$$

for any symplectic structure ω on M, where Ω and $\tilde{\omega}$ are as in Theorem 1.

Using the isomorphism $\tilde{\omega}: TM \to T^*M$ induced by ω we can obtain respective versions of Theorems 1 and 2 for T^*M instead of TM.

Next, we study the problem how to construct canonically a symplectic manifold $(TM, \Lambda(\omega,\theta))$ for a given cosymplectic $2m+1$-manifold (M,ω,θ), where ω is a closed 2-form and θ is a closed 1-form with $\omega^m \wedge \theta \neq 0$ for any point in M. This problem arises in the context of respective $\mathcal{M}f_{2m+1}-$ natural operators Λ. The second main result of the present note is the following classification theorem.

Theorem 3. *Let Λ be a $\mathcal{M}f_{2m+1}$-natural operator in question. Then there exist an uniqely determined real number a and uniquely determined smooth maps $b, c: \mathbf{R} \to \mathbf{R}$ with $b(x) \neq 0$ and $b(x) + c(x) \neq 0$ for all $x \in M$ such that*

$$\Lambda(\omega,\theta) = a\pi^*\omega + (b\circ\theta)\varphi^*_{\omega,\theta}\Omega + (b'\circ\theta)\varphi^*_{\omega,\theta}\lambda \wedge d\theta + (c\circ\theta)\pi^*\theta\wedge d\theta \tag{3}$$

for any cosymplectic structure (ω,θ) on M, where λ is the standard Liouville 1-form on the cotangent bundle T^*M, $\Omega = -d\lambda$ is the well-known canonical symplectic structure on T^*M, $\pi: TM \to M$ is the tangent bundle projection, $\varphi_{\omega,\theta}: TM \to T^*M$ is the standard isomorphism induced by (ω,θ), $\varphi_{\omega,\theta}(v) = \omega_x(.,v) + \theta(v)\theta_x$, $v \in T_xM$, $x \in M$, $(\ )^*$ is the pull-back and $d\theta$ is the differential of $\theta: TM \to \mathbf{R}$.

On the other hand for any real number a and smooth maps $b, c: \mathbf{R} \to \mathbf{R}$ with $b(x) + c(x) \neq 0$ and $b(x) \neq 0$ for all $x \in M$ the operator $\Lambda(\omega,\theta)$ defined as in (3) *is a symplectic structure on TM.*

Theorem 2 is a consequence of the following more general Theorem 4.

Theorem 4. *Let Λ be a $\mathcal{M}f_{2m+1}$-natural operator transforming a cosymplectic structure (ω,θ) on a $2m+1$-manifold M into 2-form $\Lambda(\omega,\theta)$ on TM. Then there exist uniquely determined smooth maps $\alpha,\beta,\gamma,\delta,\epsilon$: $\mathbf{R} \to \mathbf{R}$ such that*

$$\Lambda(\omega,\theta) = (\alpha\circ\theta)\varphi^*_{\omega,\theta}\Omega + (\beta\circ\theta)\pi^*\omega + (\gamma\circ\theta)\varphi^*_{\omega,\theta}\lambda\wedge\pi^*\theta + \\ (\delta\circ\theta)\varphi^*_{\omega,\theta}\lambda\wedge d\theta + (\epsilon\circ\theta)\pi^*\theta\wedge d\theta \tag{4}$$

for any cosymplectic structure (ω, θ) *on* M, *where* λ, $\Omega = -d\lambda$, $\varphi_{\omega,\theta}$, π *and* $(\ )^*$ *are as in Theorem 3.*

Using the isomorphism $\varphi_{\omega,\theta} : TM \to T^*M$ induced by (ω, θ) we can obtain respective versions of Theorems 3 and 4 for T^*M instead of TM.

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